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Kernel Functions of Diffusion Equations II

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The present paper is a continuation of the author's previous paper "Kernel Functions of Diffusion Equation I", Osaka Mathematical Journal Vol. 9, 1957, pp. 201-214. Some notations which were defined in the previous paper will be used without repeating the definitions.

2. Suppose that D be regularly open and ∂D be smooth. Then Theorem 1 of the previous paper holds and $K(x, y; t)$ is a well defined continuous non-negative function, which is smaller than $E_t(x, y)$. In this paper the dimension d is assumed to be $\geqq 3$. Set

$$G(x, y) = \lim_{h\to 0}\int_h^\infty K(x, y; t)\,dt = \int_{+0}^\infty K(x, y; t)\,dt \tag{1}$$

Lemma 2.1. *$G(x, y)$ is the Green's function of the Laplacian over D with zero boundary.*

Proof. Take a C^2-function $\varphi(y)$ and set $\varphi_s(y) = \int_D K(x, y; s)\varphi(y)\,dy$ over D. Then

$$\begin{aligned}\Delta_x\int_D G(x, y)\varphi_s(y)\,dy &= \int_D \Delta_x G(x, y)\varphi_s(y)\,dy \\ &= \int_D\int_{+0}^\infty \Delta_x K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D\int_{+0}^\infty \frac{\partial}{\partial t}K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D \lim_{h\to 0} K(x, y; h+s)\varphi(y)\,dy \\ &= \lim\int_D K(x, y; h+s)\varphi(y)\,dy \\ &= \varphi_s(x)\,,\end{aligned} \tag{2}$$

Therefore by making s towards 0, we have the required relation, which proves the lemma.

Now take an arbitrary bounded open set D and consider an increasing sequence of bounded open sets $\{D_k\}$ with smooth boundaries converging to D. To each D_k we can associate the kernel function $K_k(x, y; t)$ which forms an increasing sequence of non-negative functions.

Define for each k

$$\lim (E_{t/n} *_{D_k})^n = K_k(x, y; t). \tag{3}$$

Here $*_{D_k}$ should be understood as a convolution over D_k. Evidently $K_k(x, y; t)$ is an increasing function in k and

$$0 \leqq \int_{D_k} K_k(x, y; t) dy \leqq \int_D E_t(x, y) dy \leqq 1. \tag{4}$$

Hence everywhere in D

$$\lim_k K_k(x, y; t) = K(x, y; t)$$

exists.

Lemma 2.3. *Suppose that both x and y are in D. Then $K(x, y; t)$ is continuous in x and in y at least separately.*

Proof. Because of being the strong limit of K_k's, $K(x, y; t)$ is non-negative and is a strong solution of a diffusion equation $\partial U/\partial t = \Delta U$. Hence K is a genuine solution[1]. Therefore k is continuous in x and in y at least separately.

Since

$$0 \leqq K(x, y; t) \leqq E_t(x, y), \tag{5}$$

$$\begin{aligned} G(x, y) &= \lim_{h \to 0} \int_h^\infty K(x, y; t) dt \\ &= \int_{+0}^\infty K(x, y; t) dt \leqq \int_0^\infty E_t(x, y) dt \end{aligned} \tag{6}$$

is a well defined function unless $x = y$.

Clearly

$$\begin{aligned} G(x, y) &= \lim_{k \to \infty} G_k(x, y) \\ &= \int_{+0}^\infty K(x, y; t) dt \end{aligned} \tag{7}$$

when both x and y are in D. This $G(x, y)$ is called as a generalized Green's function[2] of the Laplacian over D.

Suppose that there are given a point y on ∂D and a sequence of points $\{y_m\}$ in D convergent to y. We further assume that

$$\lim_{m \to \infty} G(x, y_m) = 0. \tag{8}$$

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1) See (15) of the previous paper I. There are other papers where this result or a more generalized one is given.

2) The author does not claim at all that the introduction of such a definition is original. Indeed, Bouligand, Kellogg and de la Vallée-Poussin already had introduced such definition. However, the author does not have any decisive information as to who was the first to have done it.

Remark: Kellogg's[3] result says that except for y on a set of capacity 0, (8) holds.

Lemma 2. 3.

$$\lim_{m\to\infty}\int_s^\infty K(x, y_m\,;\, t)\,dt = 0\,. \tag{9}$$

Proof.

$$\begin{aligned} 0 &\leqq \lim_{m\to\infty}\int_s^\infty K(x, y_m\,;\, t)\,dt \\ &\leqq \lim_{m}\int_{+0}^\infty K(x, y_m\,;\, t)\,dt \\ &= \lim_{m\to\infty} G(x, y_m) = 0\,. \end{aligned} \tag{10}$$

Hence the lemma is proved.

Lemma 2. 4. *The sequence*

$$\left\{\int_s^\infty K(x, y_m\,;\, t)\,dt\right\} \tag{11}$$

constitutes a family of equi-continuous functions over any compact set C contained in D.

Proof.

$$\begin{aligned} \int_s^\infty K(x, y_m\,;\, t)\,dt &= \int_s^\infty \int_D K(x, z\,;\, s/2)\,K(z, y_m\,;\, t-s/2)\,dz\,dt \\ &= \int_D \left[K(x, z\,;\, s/2)\int_{s/2}^\infty K(z, y_m\,;\, t)\,dt\right] dz\,. \end{aligned} \tag{12}$$

However

$$\begin{aligned} \int_{s/2}^\infty K(z, y_m\,;\, t)\,dt &\leqq \int_{s/2}^\infty E_t(x, y_m\,;\, t)\,dt \\ &\leqq \sqrt{2\pi}^{-d}\,\frac{d}{2}\left(\frac{s}{2}\right)^{(-d/2)+1} \\ &= \frac{sd}{4}\sqrt{s\pi}^{-d}\,. \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} &\left|\int_s^\infty K(x, y_m\,;\, t)\,dt - \int_s^\infty K(x', y_m\,;\, t)\,dt\right| \\ &\leqq \frac{sd}{4}\sqrt{s\pi}^{-d}\left|\int_D (K(x, z\,;\, s/2) - K(x', z\,;\, s/2))\,dz\right|. \end{aligned} \tag{14}$$

3) See (14) of the previous paper I.

The right hand side can be made arbitrarily small if x and x' are sufficiently near to each other, because the function $\int_D K(x, z; s/2)dz$ itself is a solution of the diffusion equation, and is therefore uniformly continuous over C. This proves the lemma.

Lemma 2. 5.

$$\lim_{m\to\infty}\int_D\left(\int_s^\infty K(x, y_m; t)dt\right)^2 dx = 0. \tag{15}$$

Proof. Given small positive ε, there exists a compact subset C of D such that

$$\begin{aligned}&\int_{D-C}\left(\int_s^\infty K(x, y_m; t)dt\right)^2 dx\\ &\leqq\int_{D-C}\left(\frac{sd}{4}\right)^2(s\pi)^{-d}dx\\ &=\left(\frac{sd}{4}\right)^2(s\pi)^{-d}\text{ meas }(D-C)\\ &\leqq\varepsilon.\end{aligned} \tag{16}$$

For this C there exists a large m_C such that if $m\geqq m_C$, then

$$\int_s^\infty K(x, y_m; t)dt\leqq\sqrt{\varepsilon} \tag{17}$$

uniformly over C because of Lemmas 2. 3. and 2. 4.

Hence

$$\begin{aligned}\int_D\left(\int_s^\infty K(x, y_m; t)dt\right)^2 dx &\leqq\varepsilon+\varepsilon\text{ meas } C\\ &\leqq\varepsilon\ (1+\text{meas } D),\end{aligned} \tag{18}$$

if $m\geqq m_C$. This proves the lemma.

Lemma 2. 6.

$$\int_D\left(\int_{s/2}^\infty K(x, y; t)dt\right)^2 dx\geqq\frac{s}{2}\int_D(K(x, y; s))^2 dx. \tag{19}$$

Proof. In order to prove this lemma, the Fourier expansion with respect to Δ with 0 boundary condition will be employed.

Since D_k's are bounded domains with smooth boundaries, this type of Fourier expansion is available. Namely there exist eigenvalues $-\lambda_i^{(k)}$'s and nomalized eigenfunctions $\theta_i^{(k)}$ of Δ satisfying:

$$\Delta\theta_i^{(k)} = -\lambda_i^{(k)}\theta_i^{(k)} \tag{20}$$

for $i=1, 2, \cdots$, and $k=1, 2, \cdots$

$$\lambda_i^{(k)} > 0 \quad \text{and} \tag{21}$$

$$\lim_{i\to\infty} \lambda_i^{(k)} = \infty . \tag{21'}$$

Then, in the L^2-sense

$$\begin{aligned}\int_{s/2}^{\infty} K_k(x, y;\ t)\,dt \\ = \sum_{i=1}^{\infty} \exp(-\lambda_i^{(k)} s/2)/\lambda_i^{(k)} \cdot \theta_i^{(k)}(x)\theta_i^{(k)}(y) .\end{aligned} \tag{22}$$

Therefore

$$\begin{aligned}&\int_{D_k}\left(\int_{s/2}^{\infty} K_k(x, y;\ t)\,dt\right)^2 dx \\ &= \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s/2)/\lambda_i^{(k)})^2 (\theta_i^{(k)}(y))^2 \\ &= \sum_{i=1}^{\infty} \left(\exp\left(\lambda_i^{(k)} \frac{s}{2}\right)/\lambda_i^{(k)}\right)^2 (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ &\geqq \frac{s}{2} \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ &= \frac{s}{2} \int_{D_k} (K_k(x, y;\ s))^2 dx .\end{aligned} \tag{23}$$

By making k tend to infinity we have

$$\int_{D}\left(\int_{s/2}^{\infty} K(x, y;\ t)dt\right)^2 dx \geqq \frac{s}{2} \int_{D} (K(x, y;\ s))^2 dx \tag{24}$$

which proves the lemma.

Immediately from Lemma 2.5 and Lemma 2.6,

Lemma 2.7.

$$\lim_{m\to\infty} \int_{D} (K(x, y_m;\ s))^2 dx = 0 \tag{25}$$

Now we are going to prove that

Theorem 2. *If* $\lim_{m\to\infty} G(x, y_m) = 0$,

then

$$\lim_{m\to\infty} K(x, y_m;\ s) = 0 \tag{26}$$

for any positive s.

Proof.

$$0 \leqq K(x, y_m ; s) \leqq \int_D K\left(x, z ; \frac{s}{2}\right) K\left(z, y_m ; \frac{s}{2}\right) dz \tag{27}$$
$$\leqq \left(\int_D \left(K\left(x, z ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \left(\int_D \left(K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2}$$
$$\leqq \sqrt{2\pi \frac{s}{2}}^{-d} (\text{meas } D) \left(\int_D K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2}$$

where the right hand side will go to 0 as m goes to infinity because of Lemmas 2.5, 2.6 and 2.7. Hence the theorem is proved.

Remark: Throughout this paper Δ does not have to be the Laplacian, but has only to be a Laplace-Beltrami operator with respect to a C^2-Riemannian structure which is continuous on the boundary ∂D.

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