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<th>Kernel functions of diffusion equations. II</th>
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<tr>
<td>Author(s)</td>
<td>Yamabe, Hidehiko</td>
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<tr>
<td>Citation</td>
<td>Osaka Mathematical Journal. 11(1) P.1-P.6</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1959</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6110">https://doi.org/10.18910/6110</a></td>
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<tr>
<td>DOI</td>
<td>10.18910/6110</td>
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Kernel Functions of Diffusion Equations II

By Hidehiko Yamabe

The present paper is a continuation of the author's previous paper "Kernel Functions of Diffusion Equation I", Osaka Mathematical Journal Vol. 9, 1957, pp. 201–214. Some notations which were defined in the previous paper will be used without repeating the definitions.

2. Suppose that $D$ be regularly open and $\partial D$ be smooth. Then Theorem 1 of the previous paper holds and $K(x, y; t)$ is a well defined continuous non-negative function, which is smaller than $E_t(x, y)$. In this paper the dimension $d$ is assumed to be $\geq 3$. Set

$$G(x, y) = \lim_{h \to 0} \int_0^\infty K(x, y; t) \, dt = \int_0^\infty K(x, y; t) \, dt$$

Lemma 2.1. $G(x, y)$ is the Green's function of the Laplacian over $D$ with zero boundary.

Proof. Take a $C^2$-function $\varphi(y)$ and set $\varphi_s(y) = \int_D K(x, y; s) \varphi(y) \, dy$ over $D$. Then

$$\Delta_x \int_D G(x, y) \varphi_s(y) \, dy = \int_D \Delta_x G(x, y) \varphi_s(y) \, dy$$

$$= \int_D \int_0^\infty \Delta_x K(x, y; t + s) \, dt \varphi(y) \, dy$$

$$= \int_D \int_0^\infty \frac{\partial}{\partial t} K(x, y; t + s) \, dt \varphi(y) \, dy$$

$$= \int_D \lim_{h \to 0} K(x, y; h + s) \varphi(y) \, dy$$

$$= \lim_{h \to 0} \int_D K(x, y; h + s) \varphi(y) \, dy$$

$$= \varphi_s(x),$$

Therefore by making $s$ towards 0, we have the required relation, which proves the lemma.

Now take an arbitrary bounded open set $D$ and consider an increasing sequence of bounded open sets $\{D_k\}$ with smooth boundaries converging to $D$. To each $D_k$ we can associate the kernel function $K_k(x, y; t)$ which forms an increasing sequence of non-negative functions.
Define for each $k$

$$\lim (E_{t/n}^* D_k)^n = K_k(x, y; t).$$

Here $^* D_k$ should be understood as a convolution over $D_k$. Evidently $K_k(x, y; t)$ is an increasing function in $k$ and

$$0 \leq \int_{D_k} K_k(x, y; t) dy \leq \int_D E_t(x, y) dy \leq 1.$$  

Hence everywhere in $D$

$$\lim_k K_k(x, y; t) = K(x, y; t)$$

exists.

**Lemma 2.3.** Suppose that both $x$ and $y$ are in $D$. Then $K(x, y; t)$ is continuous in $x$ and in $y$ at least separately.

Proof. Because of being the strong limit of $K_k$'s, $K(x, y; t)$ is non-negative and is a strong solution of a diffusion equation $\partial U/\partial t = \Delta U$. Hence $K$ is a genuine solution$^1$. Therefore $k$ is continuous in $x$ and in $y$ at least separately.

Since

$$0 \leq K(x, y; t) \leq E_t(x, y),$$  

$$G(x, y) = \lim_{k \to 0} \int_0^\infty K(x, y; t) dt$$

is a well defined function unless $x=y$.

Clearly

$$G(x, y) = \lim_{k \to \infty} G_k(x, y)$$

$$= \int_0^\infty K(x, y; t) dt$$

when both $x$ and $y$ are in $D$. This $G(x, y)$ is called as a generalized Green's function$^2$ of the Laplacian over $D$.

Suppose that there are given a point $y$ on $\partial D$ and a sequence of points $\{y_m\}$ in $D$ convergent to $y$. We further assume that

$$\lim_{m \to \infty} G(x, y_m) = 0.$$  

1) See (15) of the previous paper I. There are other papers where this result or a more generalized one is given.

2) The author does not claim at all that the introduction of such a definition is original. Indeed, Bouligand, Kellogg and de la Vallee-Poussin already had introduced such definition. However, the author does not have any decisive information as to who was the first to have done it.
Remark: Kellogg's result says that except for \( y \) on a set of capacity 0, (8) holds.

Lemma 2.3.

(9) \[ \lim_{m \to \infty} \int_{s}^{\infty} K(x, y_m; t) dt = 0. \]

Proof.

(10) \[ 0 \leq \lim_{m \to \infty} \int_{s}^{\infty} K(x, y_m; t) dt \leq \lim_{m \to \infty} \int_{s}^{\infty} K(x, y_m; t) dt = \lim_{m \to \infty} G(x, y_m) = 0. \]

Hence the lemma is proved.

Lemma 2.4. The sequence

(11) \[ \left\{ \int_{s}^{\infty} K(x, y_m; t) dt \right\} \]

constitutes a family of equi-continuous functions over any compact set \( C \) contained in \( D \).

Proof.

(12) \[ \int_{s}^{\infty} K(x, y_m; t) dt = \int_{s}^{\infty} \int_{D} K(x, z; s/2) K(z, y_m; t-s/2) dz dt \]

\[ = \int_{D} \left[ K(x, z; s/2) \int_{s/2}^{\infty} K(z, y_m; t) dt \right] dz. \]

However

(13) \[ \int_{s/2}^{\infty} K(z, y_m; t) dt \leq \int_{s/2}^{\infty} E_r(x, y_m; t) dt \leq \sqrt{2\pi}^{-d} d \left( \frac{s}{2} \right)^{(-d/2)+1} \]

\[ = \frac{sd}{4} \sqrt{\frac{s}{\pi}}^{-d}. \]

Therefore

(14) \[ \left| \int_{s}^{\infty} K(x, y_m; t) dt - \int_{s}^{\infty} K(x', y_m; t) dt \right| \leq \frac{sd}{4} \sqrt{\frac{s}{\pi}}^{-d} \int_{D} \left( K(x, z; s/2) - K(x', z; s/2) \right) dz. \]

3) See (14) of the previous paper I.
The right hand side can be made arbitrarily small if \( x \) and \( x' \) are sufficiently near to each other, because the function \( \int_D K(x, z; s/2) \, dz \) itself is a solution of the diffusion equation, and is therefore uniformly continuous over \( C \). This proves the lemma.

**Lemma 2.5.**

\[
\lim_{m \to \infty} \int_D \left( \int_0^\infty K(x, y_m; t) \, dt \right)^2 \, dx = 0.
\]

Proof. Given small positive \( \varepsilon \), there exists a compact subset \( C \) of \( D \) such that

\[
\int_{D \setminus C} \left( \int_0^\infty K(x, y_m; t) \, dt \right)^2 \, dx \\
\leq \int_{D \setminus C} \left( \frac{s \delta^2}{4} \right) (s\pi)^{-d} \, dx \\
= \left( \frac{s \delta^2}{4} \right) (s\pi)^{-d} \text{meas}(D - C) \\
\leq \varepsilon.
\]

For this \( C \) there exists a large \( m_c \) such that if \( m \geq m_c \), then

\[
\int_0^\infty K(x, y_m; t) \, dt \leq \sqrt{\varepsilon}
\]

uniformly over \( C \) because of Lemmas 2.3. and 2.4.

Hence

\[
\int_D \left( \int_0^\infty K(x, y_m; t) \, dt \right)^2 \, dx \leq \varepsilon + \varepsilon \text{meas}
\]

\( C \)

\( \leq \varepsilon (1 + \text{meas } D) \),

if \( m \geq m_c \). This proves the lemma.

**Lemma 2.6.**

\[
\int_D \left( \int_{s/2}^\infty K(x, y; t) \, dt \right)^2 \, dx \geq \frac{s}{2} \int_D (K(x, y; s))^2 \, dx.
\]

Proof. In order to prove this lemma, the Fourier expansion with respect to \( \Delta \) with 0 boundary condition will be employed.

Since \( D_i \)'s are bounded domains with smooth boundaries, this type of Fourier expansion is available. Namely there exist eigenvalues \(-\lambda_i^{(k)}\)'s and normalized eigenfunctions \( \theta_i^{(k)} \) of \( \Delta \) satisfying:

\[
\Delta \theta_i^{(k)} = -\lambda_i^{(k)} \theta_i^{(k)}
\]

for \( i = 1, 2, \ldots \), and \( k = 1, 2, \ldots \).
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(21) \[ \lambda_i^{(k)} > 0 \quad \text{and} \]
(21') \[ \lim_{i \to \infty} \lambda_i^{(k)} = \infty. \]

Then, in the \(L^2\)-sense

(22) \[
\int_{s/2}^{\infty} K_k(x, y; t) \, dt
= \sum_{i=1}^{\infty} \exp\left(-\lambda_i^{(k)} s/2 \right)/\lambda_i^{(k)} \cdot \theta_i^{(k)}(x) \theta_i^{(k)}(y).
\]

Therefore

(23) \[
\int_{D_k} \left( \int_{s/2}^{\infty} K_k(x, y; t) \, dt \right)^2 \, dx
= \sum_{i=1}^{\infty} \left( \exp\left(-\lambda_i^{(k)} s/2 \right)/\lambda_i^{(k)} \right)^2 \left( \exp\left(-\lambda_i^{(k)} s\right)/(\theta_i^{(k)}(y))^2 \right)
= \frac{s}{2} \sum_{i=1}^{\infty} \left( \exp\left(-\lambda_i^{(k)} s\right)/(\theta_i^{(k)}(y))^2 \right)
= \frac{s}{2} \int_{D_k} (K_k(x, y; s))^2 \, dx.
\]

By making \(k\) tend to infinity we have

(24) \[
\int_D \left( \int_{s/2}^{\infty} K(x, y; t) \, dt \right)^2 \, dx \geq \frac{s}{2} \int_D (K(x, y; s))^2 \, dx
\]

which proves the lemma.

Immediately from Lemma 2.5 and Lemma 2.6,

Lemma 2.7.

(25) \[
\lim_{n \to \infty} \int_D (K(x, y_n; s))^2 \, dx = 0
\]

Now we are going to prove that

Theorem 2. \text{If} \lim_{n \to \infty} G(x, y_n) = 0,

then

(26) \[
\lim_{n \to \infty} K(x, y_n; s) = 0
\]

for any positive \(s\).
Proof.

\[(27) \quad 0 \leq K(x, y_m ; s) \leq \int_{D} K\left(x, z ; \frac{s}{2}\right)K\left(z, y_m ; \frac{s}{2}\right)dz \]
\[
\leq \left(\int_{D} \left(K\left(x, z ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \left(\int_{D} \left(K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \]
\[
\leq \sqrt{\frac{2\pi}{s}} \left(\text{meas } D\right) \left(\int_{D} \left(K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \]

where the right hand side will go to 0 as \(m\) goes to infinity because of Lemmas 2.5, 2.6 and 2.7. Hence the theorem is proved.

**Remark:** Throughout this paper \(\Delta\) does not have to be the Laplacian, but has only to be a Laplace-Beltrami operator with respect to a \(C^2\)-Riemannian structure which is continuous on the boundary \(\partial D\).

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This work was done with supports of A. Sloan Foundation and National Science Foundation of U.S.A.

*(Received February 23, 1959)*