Yamabe, Hidehiko
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# *Kernel Functions of Diffusion Equations II*

By Hidehiko YAMABE

The present paper is a continuation of the author's previous paper "Kernel Functions of Diffusion Equation I", Osaka Mathematical Journal Vol. 9, 1957, pp. 201-214. Some notations which were defined in the previous paper will be used without repeating the definitions.

2. Suppose that $D$ be regularly open and $\partial D$ be smooth. Then Theorem 1 of the previous paper holds and $K(x, y; t)$ is a well defined continuous non-negative function, which is smaller than $E_t(x, y)$. In this paper the dimension $d$ is assumed to be $\geqq 3$. Set

$$G(x, y) = \lim_{h\to 0}\int_h^\infty K(x, y; t)\,dt = \int_{+0}^\infty K(x, y; t)\,dt \tag{1}$$

**Lemma 2.1.** *$G(x, y)$ is the Green's function of the Laplacian over $D$ with zero boundary.*

Proof. Take a $C^2$-function $\varphi(y)$ and set $\varphi_s(y)=\int_D K(x, y; s)\varphi(y)\,dy$ over $D$. Then

$$\begin{aligned}\Delta_x\int_D G(x, y)\varphi_s(y)\,dy &= \int_D \Delta_x G(x, y)\varphi_s(y)\,dy \\ &= \int_D\int_{+0}^\infty \Delta_x K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D\int_{+0}^\infty \frac{\partial}{\partial t}K(x, y; t+s)\,dt\,\varphi(y)\,dy \\ &= \int_D \lim_{h\to 0} K(x, y; h+s)\varphi(y)\,dy \\ &= \lim\int_D K(x, y; h+s)\varphi(y)\,dy \\ &= \varphi_s(x)\,,\end{aligned} \tag{2}$$

Therefore by making $s$ towards 0, we have the required relation, which proves the lemma.

Now take an arbitrary bounded open set $D$ and consider an increasing sequence of bounded open sets $\{D_k\}$ with smooth boundaries converging to $D$. To each $D_k$ we can associate the kernel function $K_k(x, y; t)$ which forms an increasing sequence of non-negative functions.

Define for each $k$

$$\lim (E_{t/n} *_{D_k})^n = K_k(x, y;\ t)\,. \tag{3}$$

Here $*_{D_k}$ should be understood as a convolution over $D_k$. Evidently $K_k(x, y;\ t)$ is an increasing function in $k$ and

$$0 \leqq \int_{D_k} K_k(x, y;\ t)\,dy \leqq \int_D E_t(x, y)\,dy \leqq 1\,. \tag{4}$$

Hence everywhere in $D$

$$\lim_k K_k(x, y;\ t) = K(x, y;\ t)$$

exists.

**Lemma 2.3.** *Suppose that both $x$ and $y$ are in $D$. Then $K(x, y;\ t)$ is continuous in $x$ and in $y$ at least separately.*

Proof. Because of being the strong limit of $K_k$'s, $K(x, y;\ t)$ is non-negative and is a strong solution of a diffusion equation $\partial U/\partial t = \Delta U$. Hence $K$ is a genuine solution[1]. Therefore $k$ is continuous in $x$ and in $y$ at least separately.

Since

$$0 \leqq K(x, y;\ t) \leqq E_t(x, y)\,, \tag{5}$$

$$\begin{aligned} G(x, y) &= \lim_{h \to 0} \int_h^\infty K(x, y;\ t)\,dt \\ &= \int_{+0}^\infty K(x, y;\ t)\,dt \leqq \int_0^\infty E_t(x, y)\,dt \end{aligned} \tag{6}$$

is a well defined function unless $x = y$.

Clearly

$$\begin{aligned} G(x, y) &= \lim_{k \to \infty} G_k(x, y) \\ &= \int_{+0}^\infty K(x, y;\ t)\,dt \end{aligned} \tag{7}$$

when both $x$ and $y$ are in $D$. This $G(x, y)$ is called as a generalized Green's function[2] of the Laplacian over $D$.

Suppose that there are given a point $y$ on $\partial D$ and a sequence of points $\{y_m\}$ in $D$ convergent to $y$. We further assume that

$$\lim_{m \to \infty} G(x, y_m) = 0\,. \tag{8}$$

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1) See (15) of the previous paper I. There are other papers where this result or a more generalized one is given.

2) The author does not claim at all that the introduction of such a definition is original. Indeed, Bouligand, Kellogg and de la Vallée-Poussin already had introduced such definition. However, the author does not have any decisive information as to who was the first to have done it.

**Remark**: Kellogg's[3] result says that except for $y$ on a set of capacity 0, (8) holds.

**Lemma 2.3.**

$$\lim_{m\to\infty}\int_s^\infty K(x, y_m\,;\ t)\,dt = 0\,. \tag{9}$$

Proof.

$$\begin{aligned} 0 &\leqq \lim_{m\to\infty}\int_s^\infty K(x, y_m\,;\ t)\,dt \\ &\leqq \lim_m \int_{+0}^\infty K(x, y_m\,;\ t)\,dt \\ &= \lim_{m\to\infty} G(x, y_m) = 0\,. \end{aligned} \tag{10}$$

Hence the lemma is proved.

**Lemma 2.4.** *The sequence*

$$\left\{\int_s^\infty K(x, y_m\,;\ t)\,dt\right\} \tag{11}$$

*constitutes a family of equi-continuous functions over any compact set C contained in D.*

Proof.

$$\begin{aligned} \int_s^\infty K(x, y_m\,;\ t)\,dt &= \int_s^\infty \int_D K(x, z\,;\ s/2)K(z, y_m\,;\ t-s/2)\,dz\,dt \\ &= \int_D \left[K(x, z\,;\ s/2)\int_{s/2}^\infty K(z, y_m\,;\ t)\,dt\right]dz\,. \end{aligned} \tag{12}$$

However

$$\begin{aligned} \int_{s/2}^\infty K(z, y_m\,;\ t)\,dt &\leqq \int_{s/2}^\infty E_t(x, y_m\,;\ t)\,dt \\ &\leqq \sqrt{2\pi}^{-d}\frac{d}{2}\left(\frac{s}{2}\right)^{(-d/2)+1} \\ &= \frac{sd}{4}\sqrt{s\pi}^{-d}\,. \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} &\left|\int_s^\infty K(x, y_m\,;\ t)\,dt - \int_s^\infty K(x', y_m\,;\ t)\,dt\right| \\ &\leqq \frac{sd}{4}\sqrt{s\pi}^{-d}\left|\int_D (K(x, z\,;\ s/2) - K(x', z\,;\ s/2))\,dz\right|. \end{aligned} \tag{14}$$

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3) See (14) of the previous paper I.

The right hand side can be made arbitrarily small if $x$ and $x'$ are sufficiently near to each other, because the function $\int_D K(x, z; s/2) dz$ itself is a solution of the diffusion equation, and is therefore uniformly continuous over $C$. This proves the lemma.

**Lemma 2.5.**

$$\lim_{m\to\infty}\int_D\left(\int_s^\infty K(x, y_m; t)\,dt\right)^2 dx = 0. \tag{15}$$

Proof. Given small positive $\varepsilon$, there exists a compact subset $C$ of $D$ such that

$$\begin{aligned}&\int_{D-C}\left(\int_s^\infty K(x, y_m; t)\,dt\right)^2 dx\\ &\leqq \int_{D-C}\left(\frac{sd}{4}\right)^2 (s\pi)^{-d}\,dx\\ &= \left(\frac{sd}{4}\right)^2 (s\pi)^{-d}\,\mathrm{meas}\,(D-C)\\ &\leqq \varepsilon.\end{aligned} \tag{16}$$

For this $C$ there exists a large $m_C$ such that if $m \geqq m_C$, then

$$\int_s^\infty K(x, y_m; t)\,dt \leqq \sqrt{\varepsilon} \tag{17}$$

uniformly over $C$ because of Lemmas 2.3. and 2.4.

Hence

$$\begin{aligned}\int_D\left(\int_s^\infty K(x, y_m; t)\,dt\right)^2 dx &\leqq \varepsilon + \varepsilon\,\mathrm{meas}\,C\\ &\leqq \varepsilon\,(1+\mathrm{meas}\,D),\end{aligned} \tag{18}$$

if $m \geqq m_C$. This proves the lemma.

**Lemma 2.6.**

$$\int_D\left(\int_{s/2}^\infty K(x, y; t)\,dt\right)^2 dx \geqq \frac{s}{2}\int_D (K(x, y; s))^2\,dx. \tag{19}$$

Proof. In order to prove this lemma, the Fourier expansion with respect to $\Delta$ with 0 boundary condition will be employed.

Since $D_k$'s are bounded domains with smooth boundaries, this type of Fourier expansion is available. Namely there exist eigenvalues $-\lambda_i^{(k)}$'s and nomalized eigenfunctions $\theta_i^{(k)}$ of $\Delta$ satisfying:

$$\Delta\theta_i^{(k)} = -\lambda_i^{(k)}\theta_i^{(k)} \tag{20}$$

for $i=1, 2, \cdots$, and $k=1, 2, \cdots$

(21) $$\lambda_i^{(k)} > 0 \quad \text{and}$$

(21)′ $$\lim_{i\to\infty} \lambda_i^{(k)} = \infty .$$

Then, in the $L^2$-sense

(22) $$\int_{s/2}^{\infty} K_k(x, y; t)\,dt = \sum_{i=1}^{\infty} \exp(-\lambda_i^{(k)} s/2)/\lambda_i^{(k)} \cdot \theta_i^{(k)}(x)\,\theta_i^{(k)}(y) .$$

Therefore

(23) $$\begin{aligned} &\int_{D_k} \left( \int_{s/2}^{\infty} K_k(x, y; t)\,dt \right)^2 dx \\ &= \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s/2)/\lambda_i^{(k)})^2 (\theta_i^{(k)}(y))^2 \\ &= \sum_{i=1}^{\infty} \left( \exp\left(\lambda_i^{(k)} \frac{s}{2}\right)/\lambda_i^{(k)} \right)^2 (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ &\geqq \frac{s}{2} \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ &= \frac{s}{2} \int_{D_k} (K_k(x, y; s))^2 dx . \end{aligned}$$

By making $k$ tend to infinity we have

(24) $$\int_D \left( \int_{s/2}^{\infty} K(x, y; t)dt \right)^2 dx \geqq \frac{s}{2} \int_D (K(x, y; s))^2 dx$$

which proves the lemma.

Immediately from Lemma 2.5 and Lemma 2.6,

**Lemma 2.7.**

(25) $$\lim_{m\to\infty} \int_D (K(x, y_m; s))^2 dx = 0$$

Now we are going to prove that

**Theorem 2.** *If* $\lim_{m\to\infty} G(x, y_m) = 0$, *then*

(26) $$\lim_{m\to\infty} K(x, y_m; s) = 0$$

*for any positive* $s$.

Proof.

$$0 \leqq K(x, y_m ; s) \leqq \int_D K\left(x, z ; \frac{s}{2}\right) K\left(z, y_m ; \frac{s}{2}\right) dz \tag{27}$$
$$\leqq \left( \int_D \left( K\left(x, z ; \frac{s}{2}\right)\right)^2 dz \right)^{1/2} \left( \int_D \left( K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz \right)^{1/2}$$
$$\leqq \sqrt{2\pi \frac{s}{2}}^{-d} (\text{meas } D) \left( \int_D K\left(z, y_m ; \frac{s}{2}\right)\right)^2 dz \Big)^{1/2}$$

where the right hand side will go to 0 as $m$ goes to infinity because of Lemmas 2.5, 2.6 and 2.7. Hence the theorem is proved.

**Remark:** Throughout this paper $\Delta$ does not have to be the Laplacian, but has only to be a Laplace-Beltrami operator with respect to a $C^2$-Riemannian structure which is continuous on the boundary $\partial D$.

Hidehiko Yamabe,
Department of Mathematics
Osaka University
and Department of Mathematics
Institute of Technology
University of Minnesota.

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