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# ON $F$-PROJECTIVE HOMOTOPY OF SPHERES

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We write $F$ for the real ($R$), complex ($C$) or quaternoinic ($H$) numbers. Let $FP^n$ be the $F$-projective space of $n$ $F$-dimensions and

$$h_F\colon S^{(n+1)d-1} \to FP^n$$

the canonical fibration with fibre $S^{d-1}$, where $d=\dim_R F$. We work in the topological category of pointed spaces and pointed maps. Given a space $X$ and a positive integer $m$, we define the $F$-projective homotopy sets

$$\pi_m^F(X) = \begin{cases} h_F^*[FP^n, X] & \text{if } m=(n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

and similarly the stable $F$-projective homotopy groups

$$\pi_m^{SF}(X) = \begin{cases} h_F^*\{FP^n, X\} & \text{if } m=(n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

here $\{X, Y\} = \underrightarrow{\lim}\,[S^rX, S^rY]$, the limit maps being induced by suspension.

For small $j$, $\pi_{n+j}^{(S)F}(S^n)$ has been calculated by Bredon [6], Rees [11], Strutt [13] and Randall [10]. In this note we restrict our attention to the case $F=C$ or $H$. We calculate the Adams $e$-invariants of elements in $\pi_m^{(S)F}(S^{nd})$ in §1 and estimate the order of a *canonical* element in $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n=1$ in §2 and $n\equiv 0(M_{k+1}(F))$ in §3 (see §§2, 3 for the definitions of "canonical" and $(k+1)$-th $F$-James number $M_{k+1}(F)$). For example we show that under some assumptions on $k$ and a prime $p$, if $n\equiv 0(M_{k+1}(F))$ and $\nu_p(n)=\nu_p(M_{k+1}(F))$, $\pi_{(k+n+1)d-1}^{SF}(S^{nd})$ ($\subset \pi_{(k+1)d-1}^S$, the stable $(k+1)d-1$ stem) contains an element of order $p^{\nu_p(k+1)+1}$, where $\nu_p(q)$ denotes the exponent of $p$ in the prime factorization of $q$.

## 1. $e$-invariants of $F$-projective elements

It is clear that $\pi_{(m+1)d-1}^F(S^{nd}) = \pi_{(m+1)d-1}^{SF}(S^{nd}) = 0$ for $m<n$. For $m \geqq n$, by cellularity

$$\pi_{(m+1)d-1}^F(S^{nd}) = \bar{h}_F^*[FP_n^m, S^{nd}]$$

and similarly for the stable case, here $FP_n^m = FP^m/FP^{n-1}$ and $\bar{h}_F$ denotes the composition of $h_F$ and the natural projection $FP^m \to FP_n^m$.

We introduce the following notations:

$$\phi_F(x) = \begin{cases} \exp(x) - 1 & \text{if } F = C \\ \left\{2\mathrm{sh}\dfrac{\sqrt{x}}{2}\right\}^2 & \text{if } F = H \end{cases}$$

$\left(\mathrm{sh}(x) = \dfrac{\exp(x) - \exp(-x)}{2}\right)$; the rational numbers $\alpha_F(n, j)$ defined by

$$\left\{\frac{\phi_F^{-1}(x)}{x}\right\}^n = \sum_{j=0}^{\infty} \alpha_F(n, j) x^j$$

($\phi_F^{-1}$ denotes the inverse function of $\phi_F$); $e$, $e_R'$, the Adams complex and real $e$-invariants [1];

$$\deg\colon [FP_n^{k+n}, S^{nd}] \ (\text{or } \{FP_n^{k+n}, S^{nd}\}) \to Z$$

maps $f$ to the degree of $S^{nd} = FP_n^n \subset FP_n^{k+n} \xrightarrow{f} S^{nd}$; $\xi = \xi_F(m)$, the underlying complex vector bundle of the canonical $F$ line bundle over $FP^m$; $z = z_F(m) = \xi - \dfrac{d}{2} \in K(FP^m)$; $t = t_F(m) = (-1)^{d/2+1} c_{d/2}(\xi) \in H^d(FP^m; Z)$ ($d/2$-th Chern class); $\beta = z_C(1) \in K(S^2)$, the Bott generator; $\psi^k\colon K(\ ) \to K(\ )$, the Adams operation; ch: $K(\ ) \to H^*(\ ; Q)$, the Chern character. Then the followings are well known.

$$K(FP^m) = Z[z]/z^{m+1}$$
$$H^*(FP^m; Z) = Z[t]/t^{m+1}$$
$$\mathrm{ch}(z) = \phi_F(t)\,.$$

Now we prove the following.

**Theorem 1.1.** *For $f \in [FP_n^{k+n}, S^{nd}]$ (or $f \in \{FP_n^{k+n}, S^{nd}\}$), we have*

$$e(\bar{h}_F^*(f)) = -\deg(f)\, \alpha_F(n, k+1)\,.$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
S^{(k+n+1)d-1} & \xrightarrow{\bar{h}_F} & FP_n^{k+n} & \longrightarrow & FP_n^{k+n+1} & \longrightarrow & S^{(k+n+1)d} \\
\downarrow{=} & & \downarrow{f} & & \downarrow{\tilde{f}} & & \downarrow{=} \\
S^{(k+n+1)d-1} & \xrightarrow{\tilde{f}} & S^{nd} & \xrightarrow{i} & C_{\tilde{f}} & \xrightarrow{j} & S^{(k+n+1)d}
\end{array}$$

where the horizontal sequences are cofibrations. Then we have the commutative diagram of the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \tilde{K}(FP_n^{k+n}) & \longleftarrow & \tilde{K}(FP_n^{k+n+1}) & \longleftarrow & \tilde{K}(S^{(k+n+1)d}) & \longleftarrow & 0 \\ & & \uparrow f^* & & \uparrow \bar{f}^* & & \Big\| & & \\ 0 & \longleftarrow & \tilde{K}(S^{nd}) & \overset{i^*}{\longleftarrow} & \tilde{K}(C_{\bar{f}}) & \overset{j^*}{\longleftarrow} & \tilde{K}(S^{(k+n+1)d}) & \longleftarrow & 0\,. \end{array}$$

Let $a \in K(C_{\bar{f}})$ be such that $i^*(a)=\beta^{nd/2}$. Let $b=j^*(\beta^{(k+n+1)d/2})$. Then

$$\psi^2(a) = d^n a + \lambda b \quad \text{for some } \lambda \in Z\,,$$

and

$$e(\bar{f}) = \frac{\lambda}{d^n(d^{k+1}-1)} \in Q/Z\,.$$

Let

$$\bar{f}^*(a) = \sum_{i=0}^{k+1} a_i z^{i+n}\,.$$

Then

$$\psi^2 \bar{f}^*(a) = \sum_{i=0}^{k+1} a_i(\psi^2(z))^{i+n} = \sum_{i=0}^{k+1} a_i(z^2+dz)^{i+n}$$
$$= \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} a_i \binom{n+i}{j-i} d^{n+2i-j} z^{n+j}$$

and this equals

$$\bar{f}^*\psi^2(a) = \bar{f}^*(d^n a+\lambda b) = d^n \sum_{i=0}^{k+1} a_i z^{n+i} + \lambda z^{k+n+1}\,,$$

so that comparing the coefficients of $z^{k+n+1}$ we have

$$\lambda = \sum_{i=0}^{k} a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)} + d^n(d^{k+1}-1)a_{k+1}$$

and so

$$e(\bar{f}) = \frac{\sum_{i=0}^{k} a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)}}{d^n(d^{k+1}-1)}\,. \tag{1.2}$$

Consider the commutative diagram

$$\begin{array}{ccc} K(FP_n^{k+n}) & \overset{f^*}{\longleftarrow} & K(S^{nd}) \\ \Big\downarrow \mathrm{ch} & & \Big\downarrow \mathrm{ch} \\ H^*(FP_n^{k+n}; Q) & \overset{f^*}{\longleftarrow} & H^*(S^{nd}; Q)\,. \end{array}$$

Then

$$f^*(\beta^{nd/2}) = \sum_{i=0}^{k} a_i z^{n+i}$$

and

$$\deg(f)t^n = f^*\mathrm{ch}(\beta^{nd/2}) = \mathrm{ch} f^*(\beta^{nd/2}) = \sum_{i=0}^{k} a_i(\mathrm{ch}(z))^{n+i} \tag{1.3}$$
$$= \sum_{i=0}^{k} a_i\phi_F(t)^{n+1}.$$

By definition

$$(\phi_F^{-1}(x))^n = \sum_{j=0}^{\infty} \alpha_F(n, j)x^{n+j}$$
$$x = \phi_F^{-1}\phi_F(x)$$

so that

$$t^n = \sum_{j=0}^{\infty} \alpha_F(n, j)\phi_F(t)^{n+j}.$$

Then by (1.3)

$$a_i = \deg(f)\alpha_F(n, i) \quad \text{for } 0 \leqq i \leqq k,$$

so that by (1.2)

$$e(\tilde{f}) = \frac{\deg(f)\sum_{j=0}^{k} \alpha_F(n, j)\binom{n+j}{k+1-j}d^{n+2j-(k+1)}}{d^n(d^{k+1}-1)}. \tag{1.4}$$

Next we observe that the function $\phi_F^{-1}$ satisfies the equation

$$\phi_F^{-1}(x^2+dx) = d\phi_F^{-1}(x).$$

Then

$$(\phi_F^{-1}(x^2+dx))^n = \sum_{j=0}^{\infty} \alpha_F(n, j)(x^2+dx)^{n+j}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_F(n, j)\binom{n+j}{i-j}d^{n+2j-i}x^{n+i}$$

equals

$$(d\phi_F^{-1}(x))^n = d^n \sum_{i=0}^{\infty} \alpha_F(n, i)x^{n+i}$$

so that comparing the coefficients of $x^{k+n+1}$, we have

$$\sum_{j=0}^{k} \alpha_F(n, j)\binom{n+j}{k+1-j}d^{n+2j-(k+1)} = d^n(1-d^{k+1})\alpha_F(n, k+1)$$

and then by (1.4)

$$e(\tilde{f}) = -\deg(f)\alpha_F(n, k+1).$$

This completes the proof of Theorem 1.1.

Using $KO^*$-theory, we can obtain lower bounds of $\deg(f)$ (e.g. [8], [9]), but now we need upper bounds and unfortunately we have not sharp estimation with the exception of the two special cases $n=1$ and $n\equiv 0(M_{k+1}(F))$. In the following two sections we will study these two cases.

## 2. $\pi^{(S)F}_{(k+n+1)d-1}(S^{nd})$ for $n=1$

For a positive integer $q$, it is well known that the order of the composition

$$S^{2d-1}\xrightarrow{h_F} FP^1=S^d\xrightarrow{q} S^d$$

is infinite, so that

$$\deg(f)=0 \quad \text{for } f\in[FP^{k+1}, S^d]\ (k>0)$$

and so by Theorem 1.1

$$e=0\colon \pi^F_{(k+2)d-1}(S^d)\longrightarrow Q/Z\ (k>0)\,.$$

By induction on $k$ we know that the rank of $\{FP^{k+n}_n, S^{nd}\}$ is one. We will call a generator of this free part (and its image by $\bar{h}^*_F$) a *canonical* element. Let $f\in\{FP^{k+n}_n, S^{nd}\}$ be a canonical element, then (take $-f$ if necessary)

$$\deg(f)=k_s(FP^{k+n}_n, S^{nd})$$

where the right hand side has been defined in [8] and called the stable James number of the pair $(FP^{k+n}_n, S^{nd})$. In particuler we have used the notation

$$d_F(k+1)=k_s(FP^{k+1}, S^d)$$

and this has been estimated in [7], [8] and [9].

**Proposition 2.1.** *For an odd prime $p$ and an integer $l\geqq 1$, e-invariant of a canonical element in $\pi^{SC}_{2pl-1}(S^2)$ (or $\pi^{SH}_{2p+1}(S^4)$) is of order $p$ (or a multiple of $p$).*

Proof. (i) $F=C$. We have

$$\frac{\phi_C^{-1}(x)}{x}=\frac{\log(1+x)}{x}=\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}x^i$$

so that

$$\alpha_C(1, k+1)=\frac{(-1)^{k+1}}{k+2}$$

and then for a canonical element $f\in\{CP^{k+1}, S^2\}$

$$e(h^*_C(f))=(-1)^k\frac{d_C(k+1)}{k+2}\,.$$

Suppose that $k+2=uv$, where $u$ and $v$ are relatively prime integers and not one. Then by [8], $u$, $v$ and hence $uv$ devide $d_C(k+1)$. Therefore $e(h^*_C(f))=0$. In case with $k+2=2^w$ for $w\geqq 2$, $2^w$ devides $d_C(2^w-1)$ [8] and hence $e(h_C^*(f))=0$. If $k+2=p^l$ for an odd prime $p$ and a positive integer $l$, [8] says that $\nu_p(d_C(p^l-1))=l-1$ so that the order of $e(h_C^*(f))$ is $p$. This completes the proof of Proposition 2.1 for $F=C$.

(ii) $F=H$. We have

$$\frac{\phi_H^{-1}(x)}{x}=\left(\frac{\mathrm{sh}^{-1}\frac{\sqrt{x}}{2}}{\frac{\sqrt{x}}{2}}\right)^2=\sum_{i=0}^{\infty}\frac{(-1)^i}{2^{4i}}\sum_{u+v=i}\frac{(2u)!(2v)!}{(u!)^2(v!)^2(2u+1)(2v+1)}x^i$$

so that

$$\alpha_H(1, k+1)=\frac{(-1)^{k+1}}{2^{4k+4}}\sum_{i+j=k+1}\frac{(2i)!(2j)!}{(i!)^2(j!)^2(2i+1)(2j+1)}.$$

Therefore if $2k+3=p$, a prime,

$$\nu_p(\alpha_H(1, k+1))=-1.$$

On the other hand by [9]

$$d_H(k+1)\,|\,(2k+2)!(2k)!\cdots 4!$$

so that by Theorem 1.1 for a canonical element $f\in\{HP^{k+1}, S^4\}$

$$\nu_p(e(h_H^*(f))=-1.$$

This completes the proof of Proposition 2.1.

## 3. $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n\equiv 0(M_{k+1}(F))$

First we repeat the basic relations of the James number $M_{k+1}(F)$, $\alpha_F(n, j)$ and the coreducibility of $FP_n^{k+n}$ as given in Adams-Walker [2], Atiyah [4] [5], Atiyah-Todd [3] and Sigrist-Suter [12].

Let $M_{k+1}(F)$ be the order of $J(\xi)$ in the $J$-group $J(FP^k)$ [4].

**Lemma 3.1.** ([2], [12]) *For a prime $p$, we have*

(i) $\nu_p(M_{k+1}(C))=\begin{cases}\max(r+\nu_p(r)),\ 1\leqq r\leqq \dfrac{k}{p-1} & \textit{if } p\leqq k+1\\ 0 & \textit{if } p>k+1.\end{cases}$

(ii) $\nu_2(M_{k+1}(H))=\max(2k+1, 2r+\nu_2(r)),\ 1\leqq r\leqq k,$
$\nu_p(M_{k+1}(H))=\nu_p(M_{2k+2}(C))$ *if $p$ odd.*

**Lemma 3.2.** ([5, p. 143], [3], [12]) *The following three statements are equivalent.*

(i) $n\equiv 0(M_{k+1}(F))$

(ii) *for* $0 \leqq j \leqq k$, $\alpha_F(n, j) \in \begin{cases} Z & \textit{if } F=C \textit{ or } F=H \textit{ and } j \textit{ even} \\ 2Z & \textit{if } F=H \textit{ and } j \textit{ odd} \end{cases}$

(iii) $FP_n^{k+n}$ *is coreducible, that is, there exists a retraction* $FP_n^{k+n} \to S^{nd}$.

When above equivalent conditions are satisfied, for a retraction $f$: $FP_n^{k+n} \to S^{nd}$ we have

$$e(\bar{h}_F^*(f)) = -\alpha_F(n, k+1)\,. \tag{3.3}$$

Therefore next we have to compute $\alpha_F(n, k+1)$. Remark that $f$ represents a canonical element in the stable category.

**Lemma 3.4.** ([3], [12]) *Let $n$ be a positive integer, $k$ a non negative integer and $p$ a prime (an odd prime if $F=H$). Then we have*

(i) $\nu_p(\alpha_F(n, j)) \geqq 0$ *for* $0 \leqq j \leqq k$ *if and only if* $\nu_p(n) \geqq \nu_p(M_{k+1}(F))$,

(ii) $\nu_2(\alpha_H(n, j)) \geqq \begin{cases} 0 & j \textit{ even} \\ 1 & j \textit{ odd} \end{cases}$ *for* $0 \leqq j \leqq k$ *if and only if* $\nu_2(n) \geqq \nu_2(M_{k+1}(H))$,

(iii) *if* $\nu_2(n) \geqq 2j-1$, $\nu_2(n) = 2j + \nu_2(j) + \nu_2(\alpha_H(n, j))$.

In §1 we defined the coefficients $\alpha_C(n, j)$ by the formula

$$\sum_{j=0}^{\infty} \alpha_C(n, j)x^j = \left(\frac{\phi_C^{-1}(x)}{x}\right)^n = \left(\frac{\log(1+x)}{x}\right)^n = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^i\right)^n.$$

Using the multinomial expansion we find

$$\begin{aligned} \alpha_C(n, j) &= (-1)^j \sum_{s} \frac{n!}{s_0! s_1! \cdots s_j!} \prod_{i=0}^{j} \frac{1}{(i+1)^{s_i}} \\ &= (-1)^j \sum_{s} T(n, j, s), \text{ say,} \end{aligned} \tag{3.5}$$

where the summation extends over all ordered sets $s=(s_0, s_1, \cdots, s_j)$ of non negative integers such that $\sum s_i = n$, $\sum i s_i = j$.

**Lemma 3.6.** ([3, 6.5]) *Let $p$ be a prime and $k$ a non negative integer. Suppose that $\nu_p(\alpha_C(n, j)) \geqq 0$ for $0 \leqq j \leqq k$. Then*

*$\nu_p(T(n, k+1, s)) \geqq 0$ for all sequences $s$ in (3.5), with the following possible exception: if $k+1 = s(p-1)$ with $s$ integral, and if $s$ is the sequence in which $s_0 = n-s$, $s_{p-1} = s$, and all other $s_i$ are zero, we have*

$$\nu_p(T(n, k+1, s)) = \nu_p(n) - \nu_p(s) - s\,.$$

**Lemma 3.7.** (i) *Let $p$ be a prime (an odd prime if $F=H$), $n$ and $k$ non negative integers. Suppose that $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\frac{(k+1)d}{2} = s(p-1)$ for some integer $s$ and*

$$\nu_p(\alpha_F(n, k+1)) = \nu_p(n) - \nu_p(M_{k+2}(F))\,.$$

(ii) *If* $\nu_2(M_{k+1}(H))\leqq\nu_2(n)$, $\nu_2(\alpha_H(n, k+1))=\nu_2(n)-2(k+1)-\nu_2(k+1)$.

Proof. By (3.1)

$$\nu_2(M_{k+1}(H))\geqq 2k+1$$

so that (ii) follows from (3.4).

(i) for $F=C$ follows from (3.1), (3.5) and (3.6) immediately.

We define the rational numbers $d_i(n)$ by

$$\sum_{i=0}^{\infty} d_i(n)y^i=\left(\frac{\mathrm{sh}^{-1}y}{y}\right)^{2n}$$

then

$$d_{2i}(n)=2^{2i}\alpha_H(n, i),\ d_{2i+1}=0\,. \tag{3.8}$$

Recall that $\mathrm{sh}^{-1}y=\log(y+\sqrt{1+y^2})$. The power series of $y+\sqrt{1+y^2}$ is of the form $1+g(y)$, where $g(y)$ has the inverse $g^{-1}(x)=x-\dfrac{1}{2}\sum_{i=2}^{\infty}(-1)^i x^i$. We have

$$\sum_{i=0}^{\infty} d_i(n)y^{i+2n}=(\mathrm{sh}^{-1}y)^{2n}=(\log(1+g(y)))^{2n}=\sum_{i=0}^{\infty}\alpha_C(2n, i)g(y)^{i+2n}\,.$$

Put $y=g^{-1}(x)$. Then for non negative integer $j$ we have

$$\sum_{i=0}^{j} d_i(n)\sum_{s}\frac{(i+2n)!}{s_1!s_2!\cdots}\frac{(-1)^{i+j}}{2^{i+2n-s_1}}=\alpha_C(2n, j) \tag{3.9}$$

where the summation $\sum_{s}$ extends over all ordered sets $s=(s_1, s_2, \cdots)$ of non negative integers such that $\sum s_u=i+2n$, $\sum us_u=j+2n$. Hence for an odd prime $p$ and a positive integer $m$ we have

$$\begin{aligned}&\nu_p(d_i(n))\geqq 0\quad \textit{for } 0\leqq i\leqq m\quad \textit{if and only if}\\ &\nu_p(\alpha_C(2n, j)\geqq 0\quad \textit{for } 0\leqq j\leqq m\,.\end{aligned} \tag{3.10}$$

If these equivalent conditions are satisfied, (3.9) says that $\nu_p(d_{m+1}(n))$ or $\nu_p(\alpha_C(2n, m+1))<0$ implies $\nu_p(d_{m+1}(n))=\nu_p(\alpha_C(2n, m+1))$. Therefore

(3.11) if $\nu_p(\alpha_C(2n, j))\geqq 0$ for $0\leqq j\leqq 2k+1$ and $\nu_p(\alpha_C(2n, 2k+2))<0$, then $\nu_p(\alpha_H(n, k+1))=\nu_p(d_{2k+2}(n))=\nu_p(\alpha_C(2n, 2k+2))$.

Suppose that $\nu_p(M_{k+1}(H))\leqq\nu_p(n)<\nu_p(M_{k+2}(H))$ for an odd prime $p$. Then by (3.4)

$$\nu_p(\alpha_H(n, j))\geqq 0\quad \text{for } 0\leqq j\leqq k$$

and by (3.8)

$$\nu_p(d_j(n))\geqq 0\quad \text{for } 0\leqq j\leqq 2k+1$$

and by (3.10)

$$\nu_p(\alpha_C(2n, j)) \geqq 0 \quad \text{for } 0 \leqq j \leqq 2k+1$$

so that by (3.1) and (3.11) we know that $2k+2=s(p-1)$ with $s$ integral and

$$\nu_p(\alpha_C(2n, 2k+2)) = \nu_p(2n) - \nu_p(s) - s = \nu_p(n) - \nu_p(M_{k+2}(H)) < 0 .$$

This implies (i) for $F=H$ and completes the proof of Lemma 3.7.

Now we will estimate the order of the $e$-inavriant of a canonical element. Let $\sharp a$ denote the order of an element $a$ of a module.

**Proposition 3.12.** *Suppose that* $n \equiv 0(M_{k+1}(F))$*) *and let* $f\colon FP_n^{k+n} \to S^{nd}$ *be a retraction.*

(i) *Let* $p$ *be a prime (an odd prime if* $F=H$*) and suppose that* $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. *Then*

$$\nu_p(\sharp e(\bar{h}_F^*(f))) = \nu_p(M_{k+2}(F)) - \nu_p(n) .$$

*Moreover, in case* $k \equiv 1\ (4)$ *and* $(F, p)=(C, 2)$, *considering* $f$ *as a stable map (or if* $n \equiv 0\ (4)$*), we have*

$$\nu_2(\sharp e_R'(\bar{h}_C^*(f))) = \nu_2(M_{k+2}(C)) - \nu_2(n) + 1 .$$

(ii) *If* $\nu_2(M_{k+1}(H)) \leqq \nu_2(n) < 2(k+1) + \nu_2(k+1)$,

$$\nu_2(\sharp e(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n) .$$

*Moreover in case* $k \equiv 0\ (2)$ *and* $n \equiv 0\ (2)$, *we have*

$$\nu_2(\sharp e_R'(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n) + 1 .$$

Proof. (3.3), (3.7) and the fact

$$e = 2e_R'\colon \pi_{8q+r}(S^{8q}) \to Q/Z \quad \text{if } r \equiv 3(8) \quad [1, 7.\ 14]$$

imply Proposition 3.12.

Suppose that $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\dfrac{(k+1)d}{2} = s(p-1)$ with $s$ integral as seen before. Put $s=p^l u$, $u \not\equiv 0(p)$ for integers $l$, $u$. Then by (3.1)

$$\nu_p(M_{k+2}(F)) - \nu_p(n) \leqq \nu_p(M_{k+2}(F)) - \nu_p(M_{k+1}(F)) \leqq \begin{cases} l+1 & \text{if } (F, p) \neq (H, 2) \\ \max(l+1, 2) & \text{if } (F, p) = (H, 2) . \end{cases}$$

*) Using $S$-duality and a theorem of Sigrist (Ill. J. Math. 13 (1969), 198–201), we can show that this hypothesis can be removed but then $f$ must be canonical. The same remark is valid for the next corollary.

In the following Corollary 3.13, we will give a condition that implies

$$\nu_p(M_{k+2}(F))-\nu_p(n)=\begin{cases} l+1 & \text{if } (F,p)\neq(H,2) \\ \max(l+1,2) & \text{if } (F,p)=(H,2)\,. \end{cases}$$

**Corollary 3.13.** *Let $p$ be a prime. Suppose that $n\equiv 0(M_{k+1}(F))$ and $\nu_p(n)=\nu_p(M_{k+1}(F))$. Let $f\colon FP_n^{k+n}\to S^{nd}$ be a retraction.*

(i) *If $(F,p)\neq(H,2)$ and $k$ satisfies*

$$\frac{(k+1)d}{2}=p^l u(p-1),\ u\not\equiv 0(p),\quad \begin{matrix} u<p^{l+1} & (p \text{ odd}) \\ u<2^l & (p=2) \end{matrix}$$

*for some integers $u$ and $l$, then*

$$\nu_p(\# e(\bar{h}_F^*(f)))=l+1\,.$$

(ii) *If $k$ satisfies*

$$k+1=2^l u,\ u\not\equiv 0(2),\ u<2^{l+2}$$

*then*

$$\nu_2(\# e(\bar{h}_H^*(f)))=\begin{cases} l+1 & \text{if } l\geqq 1 \\ 2 & \text{if } l=0 \end{cases}$$

*and moreover in case $k=0$ or $2$ and $n\equiv 0\ (2)$ we have*

$$\nu_2(\# e_R'(\bar{h}_H^*(f)))=3$$

Proof. Using (3.1) and the fact [3]

$$M_{2k+1}(C)=M_{2k+2}(C)\quad \text{for } k\geqq 1$$

we can prove this Corollary by elementary calculation, so we omit the proof.

Remark. If $n\equiv 0(M_{k+1}(F))$, we have

$$\pi^F_{(k+n+1)d-1}(S^{nd})\xrightarrow[\cong]{}\pi^{SF}_{(k+n+1(d-1}(S^{nd})$$

with the exception of $(F,k,n)=(C,0,1)$, $(C,1,2)$ or $(H,0,1)$. For these three cases, we list up the results without proof.

**Proposition 3.14.**

$$\pi_3^C(S^2)=\{k^2\eta;\ k\in Z\}$$

$$\pi_7^C(S^4)=\left\{k^2\nu+\frac{k(k-1)}{2}\delta+6l\delta;\ k\in Z,\ l=0 \text{ or } 1\right\}$$

$$\pi_7^H(S^4)=\left\{k^2\nu+\frac{k(k-1)}{2}\delta;\ k\in Z\right\}$$

$$\pi_3^{SC}(S^2) = \pi_1^s = Z_2$$
$$\pi_7^{SC}(S^4) = \pi_7^{SH}(S^4) = \pi_3^s = Z_{24}.$$

*where* $\pi_3(S_2)=Z=\{\eta\}$ *and* $\pi_7(S^4)=Z\oplus Z_{12}=\{\nu\}\oplus\{\delta\}$.

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