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Author(s)	Ōshima, Hideaki
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ON F -PROJECTIVE HOMOTOPY OF SPHERES

HIDEAKI ŌSHIMA

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We write F for the real (R), complex (C) or quaternionic (H) numbers. Let FP^n be the F -projective space of n F -dimensions and

$$h_F: S^{(n+1)d-1} \rightarrow FP^n$$

the canonical fibration with fibre S^{d-1} , where $d = \dim_R F$. We work in the topological category of pointed spaces and pointed maps. Given a space X and a positive integer m , we define the F -projective homotopy sets

$$\pi_m^F(X) = \begin{cases} h_F^*[FP^n, X] & \text{if } m = (n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

and similarly the stable F -projective homotopy groups

$$\pi_m^{SF}(X) = \begin{cases} h_F^*[FP^n, X] & \text{if } m = (n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

here $\{X, Y\} = \varinjlim [S^r X, S^r Y]$, the limit maps being induced by suspension.

For small j , $\pi_{n+j}^{SF}(S^n)$ has been calculated by Bredon [6], Rees [11], Strutt [13] and Randall [10]. In this note we restrict our attention to the case $F=C$ or H . We calculate the Adams e -invariants of elements in $\pi_m^{SF}(S^{nd})$ in §1 and estimate the order of a *canonical* element in $\pi_{(k+n+1)d-1}^{SF}(S^{nd})$ for $n=1$ in §2 and $n \equiv 0(M_{k+1}(F))$ in §3 (see §§2, 3 for the definitions of “canonical” and $(k+1)$ -th F -James number $M_{k+1}(F)$). For example we show that under some assumptions on k and a prime p , if $n \equiv 0(M_{k+1}(F))$ and $\nu_p(n) = \nu_p(M_{k+1}(F))$, $\pi_{(k+n+1)d-1}^{SF}(S^{nd})$ ($\subset \pi_{(k+1)d-1}^S$, the stable $(k+1)d-1$ stem) contains an element of order $p^{\nu_p(k+1)+1}$, where $\nu_p(q)$ denotes the exponent of p in the prime factorization of q .

1. e -invariants of F -projective elements

It is clear that $\pi_{(m+1)d-1}^F(S^{nd}) = \pi_{(m+1)d-1}^{SF}(S^{nd}) = 0$ for $m < n$. For $m \geq n$, by cellularity

$$\pi_{(m+1)d-1}^F(S^{nd}) = \bar{h}_F^*[FP_n^m, S^{nd}]$$

and similarly for the stable case, here $FP_n^m = FP^m / FP^{m-1}$ and \bar{h}_F denotes the composition of h_F and the natural projection $FP^m \rightarrow FP_n^m$.

We introduce the following notations:

$$\phi_F(x) = \begin{cases} \exp(x) - 1 & \text{if } F = C \\ \left\{ 2 \operatorname{sh} \frac{\sqrt{x}}{2} \right\}^2 & \text{if } F = H \end{cases}$$

$(\operatorname{sh}(x) = \frac{\exp(x) - \exp(-x)}{2})$; the rational numbers $\alpha_F(n, j)$ defined by

$$\left\{ \frac{\phi_F^{-1}(x)}{x} \right\}^n = \sum_{j=0}^{\infty} \alpha_F(n, j) x^j$$

(ϕ_F^{-1} denotes the inverse function of ϕ_F); e, e_R' , the Adams complex and real e -invariants [1];

$$\deg: [FP_n^{k+n}, S^{nd}] \text{ (or } \{FP_n^{k+n}, S^{nd}\}) \rightarrow Z$$

maps f to the degree of $S^{nd} = FP_n^n \subset FP_n^{k+n} \xrightarrow{f} S^{nd}$; $\xi = \xi_F(m)$, the underlying complex vector bundle of the canonical F line bundle over FP^m ; $z = z_F(m) = \xi - \frac{d}{2} \in K(FP^m)$; $t = t_F(m) = (-1)^{d/2+1} c_{d/2}(\xi) \in H^d(FP^m; Z)$ ($d/2$ -th Chern class); $\beta = z_C(1) \in K(S^2)$, the Bott generator; $\psi^k: K(\quad) \rightarrow K(\quad)$, the Adams operation; $\operatorname{ch}: K(\quad) \rightarrow H^*(\quad; Q)$, the Chern character. Then the followings are well known.

$$\begin{aligned} K(FP^m) &= Z[z]/z^{m+1} \\ H^*(FP^m; Z) &= Z[t]/t^{m+1} \\ \operatorname{ch}(z) &= \phi_F(t). \end{aligned}$$

Now we prove the following.

Theorem 1.1. *For $f \in [FP_n^{k+n}, S^{nd}]$ (or $f \in \{FP_n^{k+n}, S^{nd}\}$), we have*

$$e(\bar{h}_F^*(f)) = -\deg(f) \alpha_F(n, k+1).$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc} S^{(k+n+1)d-1} & \xrightarrow{\bar{h}_F} & FP_n^{k+n} & \longrightarrow & FP_n^{k+n+1} & \longrightarrow & S^{(k+n+1)d} \\ \downarrow = & \downarrow \bar{f} & \downarrow f & & \downarrow \bar{f} & & \downarrow = \\ S^{(k+n+1)d-1} & \xrightarrow{\bar{f}} & S^{nd} & \xrightarrow{i} & C_{\bar{f}} & \xrightarrow{j} & S^{(k+n+1)d} \end{array}$$

where the horizontal sequences are cofibrations. Then we have the commutative diagram of the short exact sequences

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \tilde{K}(FP_n^{k+n}) & \longleftarrow & \tilde{K}(FP_n^{k+n+1}) & \longleftarrow & \tilde{K}(S^{(k+n+1)d}) \longleftarrow 0 \\
 & & \uparrow f^* & & \uparrow f^* & & \uparrow = \\
 0 & \longleftarrow & \tilde{K}(S^{nd}) & \xleftarrow{i^*} & \tilde{K}(C_{\tilde{f}}) & \xleftarrow{j^*} & \tilde{K}(S^{(k+n+1)d}) \longleftarrow 0.
 \end{array}$$

Let $a \in K(C_{\tilde{f}})$ be such that $i^*(a) = \beta^{nd/2}$. Let $b = j^*(\beta^{(k+n+1)d/2})$. Then

$$\psi^2(a) = d^n a + \lambda b \text{ for some } \lambda \in \mathbb{Z},$$

and

$$e(\tilde{f}) = \frac{\lambda}{d^n(d^{k+1}-1)} \in \mathbb{Q}/\mathbb{Z}.$$

Let

$$\tilde{f}^*(a) = \sum_{i=0}^{k+1} a_i z^{i+n}.$$

Then

$$\begin{aligned}
 \psi^2 \tilde{f}^*(a) &= \sum_{i=0}^{k+1} a_i (\psi^2(z))^{i+n} = \sum_{i=0}^{k+1} a_i (z^2 + dz)^{i+n} \\
 &= \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} a_i \binom{n+i}{j-i} d^{n+2i-j} z^{n+j}
 \end{aligned}$$

and this equals

$$\tilde{f}^* \psi^2(a) = \tilde{f}^*(d^n a + \lambda b) = d^n \sum_{i=0}^{k+1} a_i z^{n+i} + \lambda z^{k+n+1},$$

so that comparing the coefficients of z^{k+n+1} we have

$$\lambda = \sum_{i=0}^k a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)} + d^n (d^{k+1}-1) a_{k+1}$$

and so

$$(1.2) \quad e(\tilde{f}) = \frac{\sum_{i=0}^k a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)}}{d^n (d^{k+1}-1)}.$$

Consider the commutative diagram

$$\begin{array}{ccc}
 \tilde{K}(FP_n^{k+n}) & \xleftarrow{f^*} & \tilde{K}(S^{nd}) \\
 \downarrow \text{ch} & & \downarrow \text{ch} \\
 H^*(FP_n^{k+n}; \mathbb{Q}) & \xleftarrow{f^*} & H^*(S^{nd}; \mathbb{Q}).
 \end{array}$$

Then

$$f^*(\beta^{nd/2}) = \sum_{i=0}^k a_i z^{n+i}$$

and

$$(1.3) \quad \begin{aligned} \deg(f)t^n &= f^* \operatorname{ch}(\beta^{nd/2}) = \operatorname{ch} f^*(\beta^{nd/2}) = \sum_{i=0}^k a_i (\operatorname{ch}(z))^{n+i} \\ &= \sum_{i=0}^k a_i \phi_F(t)^{n+i}. \end{aligned}$$

By definition

$$\begin{aligned} (\phi_F^{-1}(x))^n &= \sum_{j=0}^{\infty} \alpha_F(n, j) x^{n+j} \\ x &= \phi_F^{-1} \phi_F(x) \end{aligned}$$

so that

$$t^n = \sum_{j=0}^{\infty} \alpha_F(n, j) \phi_F(t)^{n+j}.$$

Then by (1.3)

$$a_i = \deg(f) \alpha_F(n, i) \quad \text{for } 0 \leq i \leq k,$$

so that by (1.2)

$$(1.4) \quad e(\tilde{f}) = \frac{\deg(f) \sum_{j=0}^k \alpha_F(n, j) \binom{n+j}{k+1-j} d^{n+2j-(k+1)}}{d^n (d^{k+1}-1)}.$$

Next we observe that the function ϕ_F^{-1} satisfies the equation

$$\phi_F^{-1}(x^2 + dx) = d \phi_F^{-1}(x).$$

Then

$$\begin{aligned} (\phi_F^{-1}(x^2 + dx))^n &= \sum_{j=0}^{\infty} \alpha_F(n, j) (x^2 + dx)^{n+j} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \alpha_F(n, j) \binom{n+j}{i-j} d^{n+2j-i} x^{n+i} \end{aligned}$$

equals

$$(d \phi_F^{-1}(x))^n = d^n \sum_{i=0}^{\infty} \alpha_F(n, i) x^{n+i}$$

so that comparing the coefficients of x^{k+n+1} , we have

$$\sum_{j=0}^k \alpha_F(n, j) \binom{n+j}{k+1-j} d^{n+2j-(k+1)} = d^n (1 - d^{k+1}) \alpha_F(n, k+1)$$

and then by (1.4)

$$e(\tilde{f}) = -\deg(f) \alpha_F(n, k+1).$$

This completes the proof of Theorem 1.1.

Using KO^* -theory, we can obtain lower bounds of $\deg(f)$ (e.g. [8], [9]), but

now we need upper bounds and unfortunately we have not sharp estimation with the exception of the two special cases $n=1$ and $n\equiv 0(M_{k+1}(F))$. In the following two sections we will study these two cases.

2. $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n=1$

For a positive integer q , it is well known that the order of the composition

$$S^{2d-1} \xrightarrow{h_F} FP^1 = S^d \xrightarrow{q} S^d$$

is infinite, so that

$$\deg(f) = 0 \quad \text{for } f \in [FP^{k+1}, S^d] \ (k > 0)$$

and so by Theorem 1.1

$$e = 0: \pi_{(k+2)d-1}^F(S^d) \longrightarrow Q/Z \ (k > 0).$$

By induction on k we know that the rank of $\{FP_n^{k+n}, S^{nd}\}$ is one. We will call a generator of this free part (and its image by \bar{h}_F^*) a *canonical* element. Let $f \in \{FP_n^{k+n}, S^{nd}\}$ be a canonical element, then (take $-f$ if necessary)

$$\deg(f) = k_s(FP_n^{k+n}, S^{nd})$$

where the right hand side has been defined in [8] and called the stable James number of the pair (FP_n^{k+n}, S^{nd}) . In particular we have used the notation

$$d_F(k+1) = k_s(FP^{k+1}, S^d)$$

and this has been estimated in [7], [8] and [9].

Proposition 2.1. *For an odd prime p and an integer $l \geq 1$, e -invariant of a canonical element in $\pi_{2pl-1}^{SC}(S^2)$ (or $\pi_{2p+1}^{SH}(S^4)$) is of order p (or a multiple of p).*

Proof. (i) $F=C$. We have

$$\frac{\phi_C^{-1}(x)}{x} = \frac{\log(1+x)}{x} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^i$$

so that

$$\alpha_C(1, k+1) = \frac{(-1)^{k+1}}{k+2}$$

and then for a canonical element $f \in \{CP^{k+1}, S^2\}$

$$e(h_C^*(f)) = (-1)^k \frac{d_C(k+1)}{k+2}.$$

Suppose that $k+2=uv$, where u and v are relatively prime integers and not one. Then by [8], u, v and hence uv devide $d_C(k+1)$. Therefore $e(h_C^*(f))=0$. In

case with $k+2=2^w$ for $w \geq 2$, 2^w devides $d_c(2^w-1)$ [8] and hence $e(h_C^*(f))=0$. If $k+2=p^l$ for an odd prime p and a positive integer l , [8] says that $\nu_p(d_c(p^l-1))=l-1$ so that the order of $e(h_C^*(f))$ is p . This completes the proof of Proposition 2.1 for $F=C$.

(ii) $F=H$. We have

$$\frac{\phi_H^{-1}(x)}{x} = \left(\frac{\operatorname{sh}^{-1} \frac{\sqrt{x}}{2}}{\frac{\sqrt{x}}{2}} \right)^2 = \sum_{i=0}^{\infty} \frac{(-1)^i}{2^{4i}} \sum_{u+v=i} \frac{(2u)!(2v)!}{(u!)^2(v!)^2(2u+1)(2v+1)} x^i$$

so that

$$\alpha_H(1, k+1) = \frac{(-1)^{k+1}}{2^{4k+4}} \sum_{i+j=k+1} \frac{(2i)!(2j)!}{(i!)^2(j!)^2(2i+1)(2j+1)}.$$

Therefore if $2k+3=p$, a prime,

$$\nu_p(\alpha_H(1, k+1)) = -1.$$

On the other hand by [9]

$$d_H(k+1) \mid (2k+2)!(2k)!\cdots 4!$$

so that by Theorem 1.1 for a canonical element $f \in \{HP^{k+1}, S^4\}$

$$\nu_p(e(h_H^*(f))) = -1.$$

This completes the proof of Proposition 2.1.

3. $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n \equiv 0(M_{k+1}(F))$

First we repeat the basic relations of the James number $M_{k+1}(F)$, $\alpha_F(n, j)$ and the coreducibility of FP_n^{k+n} as given in Adams-Walker [2], Atiyah [4] [5], Atiyah-Todd [3] and Sigrist-Suter [12].

Let $M_{k+1}(F)$ be the order of $J(\xi)$ in the J -group $J(FP^k)$ [4].

Lemma 3.1. ([2], [12]) *For a prime p , we have*

- (i) $\nu_p(M_{k+1}(C)) = \begin{cases} \max(r + \nu_p(r)), & 1 \leq r \leq \frac{k}{p-1} \text{ if } p \leq k+1 \\ 0 & \text{if } p > k+1. \end{cases}$
- (ii) $\nu_2(M_{k+1}(H)) = \max(2k+1, 2r + \nu_2(r)), 1 \leq r \leq k,$
 $\nu_p(M_{k+1}(H)) = \nu_p(M_{2k+2}(C)) \text{ if } p \text{ odd.}$

Lemma 3.2. ([5, p. 143], [3], [12]) *The following three statements are equivalent.*

- (i) $n \equiv 0(M_{k+1}(F))$

- (ii) for $0 \leq j \leq k$, $\alpha_F(n, j) \in \begin{cases} Z & \text{if } F=C \text{ or } F=H \text{ and } j \text{ even} \\ 2Z & \text{if } F=H \text{ and } j \text{ odd} \end{cases}$
- (iii) FP_n^{k+n} is coreducible, that is, there exists a retraction $FP_n^{k+n} \rightarrow S^{nd}$.

When above equivalent conditions are satisfied, for a retraction $f: FP_n^{k+n} \rightarrow S^{nd}$ we have

$$(3.3) \quad e(\bar{h}_F^*(f)) = -\alpha_F(n, k+1).$$

Therefore next we have to compute $\alpha_F(n, k+1)$. Remark that f represents a canonical element in the stable category.

Lemma 3.4. ([3], [12]) *Let n be a positive integer, k a non negative integer and p a prime (an odd prime if $F=H$). Then we have*

- (i) $\nu_p(\alpha_F(n, j)) \geq 0$ for $0 \leq j \leq k$ if and only if $\nu_p(n) \geq \nu_p(M_{k+1}(F))$,
- (ii) $\nu_2(\alpha_H(n, j)) \geq \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases}$ for $0 \leq j \leq k$ if and only if $\nu_2(n) \geq \nu_2(M_{k+1}(H))$,
- (iii) if $\nu_2(n) \geq 2j-1$, $\nu_2(n) = 2j + \nu_2(j) + \nu_2(\alpha_H(n, j))$.

In §1 we defined the coefficients $\alpha_C(n, j)$ by the formula

$$\sum_{j=0}^{\infty} \alpha_C(n, j) x^j = \left(\frac{\phi^{-1}(x)}{x} \right)^n = \left(\frac{\log(1+x)}{x} \right)^n = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^i \right)^n.$$

Using the multinomial expansion we find

$$(3.5) \quad \begin{aligned} \alpha_C(n, j) &= (-1)^j \sum_s \frac{n!}{s_0! s_1! \dots s_j!} \prod_{i=0}^j \frac{1}{(i+1)^{s_i}} \\ &= (-1)^j \sum_s T(n, j, s), \text{ say,} \end{aligned}$$

where the summation extends over all ordered sets $s = (s_0, s_1, \dots, s_j)$ of non negative integers such that $\sum s_i = n$, $\sum i s_i = j$.

Lemma 3.6. ([3, 6.5]) *Let p be a prime and k a non negative integer. Suppose that $\nu_p(\alpha_C(n, j)) \geq 0$ for $0 \leq j \leq k$. Then*

$\nu_p(T(n, k+1, s)) \geq 0$ for all sequences s in (3.5), with the following possible exception: if $k+1 = s(p-1)$ with s integral, and if s is the sequence in which $s_0 = n - s$, $s_{p-1} = s$, and all other s_i are zero, we have

$$\nu_p(T(n, k+1, s)) = \nu_p(n) - \nu_p(s) - s.$$

Lemma 3.7. (i) *Let p be a prime (an odd prime if $F=H$), n and k non negative integers. Suppose that $\nu_p(M_{k+1}(F)) \leq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\frac{(k+1)d}{2} = s(p-1)$ for some integer s and*

$$\nu_p(\alpha_F(n, k+1)) = \nu_p(n) - \nu_p(M_{k+2}(F)).$$

(ii) If $\nu_2(M_{k+1}(H)) \leq \nu_2(n)$, $\nu_2(\alpha_H(n, k+1)) = \nu_2(n) - 2(k+1) - \nu_2(k+1)$.

Proof. By (3.1)

$$\nu_2(M_{k+1}(H)) \geq 2k+1$$

so that (ii) follows from (3.4).

(i) for $F=C$ follows from (3.1), (3.5) and (3.6) immediately.

We define the rational numbers $d_i(n)$ by

$$\sum_{i=0}^{\infty} d_i(n) y^i = \left(\frac{\operatorname{sh}^{-1} y}{y} \right)^{2n}$$

then

$$(3.8) \quad d_{2i}(n) = 2^{2i} \alpha_H(n, i), \quad d_{2i+1} = 0.$$

Recall that $\operatorname{sh}^{-1} y = \log(y + \sqrt{1+y^2})$. The power series of $y + \sqrt{1+y^2}$ is of the form $1 + g(y)$, where $g(y)$ has the inverse $g^{-1}(x) = x - \frac{1}{2} \sum_{i=2}^{\infty} (-1)^i x^i$. We have

$$\sum_{i=0}^{\infty} d_i(n) y^{i+2n} = (\operatorname{sh}^{-1} y)^{2n} = (\log(1 + g(y)))^{2n} = \sum_{i=0}^{\infty} \alpha_C(2n, i) g(y)^{i+2n}.$$

Put $y = g^{-1}(x)$. Then for non negative integer j we have

$$(3.9) \quad \sum_{i=0}^j d_i(n) \sum_{s} \frac{(i+2n)!}{s_1! s_2! \dots} \frac{(-1)^{i+j}}{2^{i+2n-s}} = \alpha_C(2n, j)$$

where the summation \sum_s extends over all ordered sets $s = (s_1, s_2, \dots)$ of non negative integers such that $\sum s_u = i+2n$, $\sum u s_u = j+2n$. Hence for an odd prime p and a positive integer m we have

$$(3.10) \quad \begin{aligned} \nu_p(d_i(n)) &\geq 0 \quad \text{for } 0 \leq i \leq m \quad \text{if and only if} \\ \nu_p(\alpha_C(2n, j)) &\geq 0 \quad \text{for } 0 \leq j \leq m. \end{aligned}$$

If these equivalent conditions are satisfied, (3.9) says that $\nu_p(d_{m+1}(n))$ or $\nu_p(\alpha_C(2n, m+1)) < 0$ implies $\nu_p(d_{m+1}(n)) = \nu_p(\alpha_C(2n, m+1))$. Therefore

$$(3.11) \quad \begin{aligned} \text{if } \nu_p(\alpha_C(2n, j)) &\geq 0 \quad \text{for } 0 \leq j \leq 2k+1 \text{ and } \nu_p(\alpha_C(2n, 2k+2)) \\ &< 0, \text{ then } \nu_p(\alpha_H(n, k+1)) = \nu_p(d_{2k+2}(n)) = \nu_p(\alpha_C(2n, 2k+2)). \end{aligned}$$

Suppose that $\nu_p(M_{k+1}(H)) \leq \nu_p(n) < \nu_p(M_{k+2}(H))$ for an odd prime p . Then by (3.4)

$$\nu_p(\alpha_H(n, j)) \geq 0 \quad \text{for } 0 \leq j \leq k$$

and by (3.8)

$$\nu_p(d_j(n)) \geq 0 \quad \text{for } 0 \leq j \leq 2k+1$$

and by (3.10)

$$\nu_p(\alpha_c(2n, j)) \geq 0 \quad \text{for } 0 \leq j \leq 2k+1$$

so that by (3.1) and (3.11) we know that $2k+2=s(p-1)$ with s integral and

$$\nu_p(\alpha_c(2n, 2k+2)) = \nu_p(2n) - \nu_p(s) - s = \nu_p(n) - \nu_p(M_{k+2}(H)) < 0.$$

This implies (i) for $F=H$ and completes the proof of Lemma 3.7.

Now we will estimate the order of the e -invariant of a canonical element. Let $\#a$ denote the order of an element a of a module.

Proposition 3.12. *Suppose that $n \equiv 0(M_{k+1}(F))^*$ and let $f: FP_n^{k+n} \rightarrow S^{nd}$ be a retraction.*

(i) *Let p be a prime (an odd prime if $F=H$) and suppose that $\nu_p(M_{k+1}(F)) \leq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then*

$$\nu_p(\#e(\bar{h}_F^*(f))) = \nu_p(M_{k+2}(F)) - \nu_p(n).$$

Moreover, in case $k \equiv 1 \pmod{4}$ and $(F, p) = (C, 2)$, considering f as a stable map (or if $n \equiv 0 \pmod{4}$), we have

$$\nu_2(\#e_R'(\bar{h}_C^*(f))) = \nu_2(M_{k+2}(C)) - \nu_2(n) + 1.$$

(ii) *If $\nu_2(M_{k+1}(H)) \leq \nu_2(n) < 2(k+1) + \nu_2(k+1)$,*

$$\nu_2(\#e(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n).$$

Moreover in case $k \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{2}$, we have

$$\nu_2(\#e_R'(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n) + 1.$$

Proof. (3.3), (3.7) and the fact

$$e = 2e_R': \pi_{8q+r}(S^{8q}) \rightarrow Q/Z \quad \text{if } r \equiv 3(8) \quad [1, 7, 14]$$

imply Proposition 3.12.

Suppose that $\nu_p(M_{k+1}(F)) \leq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\frac{(k+1)d}{2} = s(p-1)$ with s integral as seen before. Put $s = p'u$, $u \not\equiv 0(p)$ for integers l, u . Then by (3.1)

$$\nu_p(M_{k+2}(F)) - \nu_p(n) \leq \nu_p(M_{k+2}(F)) - \nu_p(M_{k+1}(F)) \leq \begin{cases} l+1 & \text{if } (F, p) \neq (H, 2) \\ \max(l+1, 2) & \text{if } (F, p) = (H, 2) \end{cases}$$

^{*)} Using S -duality and a theorem of Sigrist (Ill. J. Math. 13 (1969), 198–201), we can show that this hypothesis can be removed but then f must be canonical. The same remark is valid for the next corollary.

In the following Corollary 3.13, we will give a condition that implies

$$\nu_p(M_{k+2}(F)) - \nu_p(n) = \begin{cases} l+1 & \text{if } (F, p) \neq (H, 2) \\ \max(l+1, 2) & \text{if } (F, p) = (H, 2). \end{cases}$$

Corollary 3.13. *Let p be a prime. Suppose that $n \equiv 0(M_{k+1}(F))$ and $\nu_p(n) = \nu_p(M_{k+1}(F))$. Let $f: FP_n^{k+n} \rightarrow S^{nd}$ be a retraction.*

(i) *If $(F, p) \neq (H, 2)$ and k satisfies*

$$\frac{(k+1)d}{2} = p^l u(p-1), \quad u \not\equiv 0(p), \quad \begin{cases} u < p^{l+1} & (p \text{ odd}) \\ u < 2^l & (p = 2) \end{cases}$$

for some integers u and l , then

$$\nu_p(\#e(\bar{h}_F^*(f))) = l+1.$$

(ii) *If k satisfies*

$$k+1 = 2^l u, \quad u \not\equiv 0(2), \quad u < 2^{l+2}$$

then

$$\nu_2(\#e(\bar{h}_H^*(f))) = \begin{cases} l+1 & \text{if } l \geq 1 \\ 2 & \text{if } l=0 \end{cases}$$

and moreover in case $k=0$ or 2 and $n \equiv 0(2)$ we have

$$\nu_2(\#e_R'(\bar{h}_H^*(f))) = 3$$

Proof. Using (3.1) and the fact [3]

$$M_{2k+1}(C) = M_{2k+2}(C) \quad \text{for } k \geq 1$$

we can prove this Corollary by elementary calculation, so we omit the proof.

REMARK. If $n \equiv 0(M_{k+1}(F))$, we have

$$\pi_{(k+n+1)d-1}^F(S^{nd}) \xrightarrow{\cong} \pi_{(k+n+1)d-1}^{SF}(S^{nd})$$

with the exception of $(F, k, n) = (C, 0, 1)$, $(C, 1, 2)$ or $(H, 0, 1)$. For these three cases, we list up the results without proof.

Proposition 3.14.

$$\begin{aligned} \pi_3^C(S^2) &= \{k^2\eta; k \in \mathbb{Z}\} \\ \pi_7^C(S^4) &= \left\{ k^2\nu + \frac{k(k-1)}{2}\delta + 6l\delta; k \in \mathbb{Z}, l = 0 \text{ or } 1 \right\} \\ \pi_7^H(S^4) &= \left\{ k^2\nu + \frac{k(k-1)}{2}\delta; k \in \mathbb{Z} \right\} \end{aligned}$$

$$\begin{aligned}\pi_3^{SC}(S^2) &= \pi_1^s = Z_2 \\ \pi_7^{SC}(S^4) &= \pi_7^{SH}(S^4) = \pi_3^s = Z_{24}.\end{aligned}$$

where $\pi_3(S_2) = Z = \{\eta\}$ and $\pi_7(S^4) = Z \oplus Z_{12} = \{\nu\} \oplus \{\delta\}$.

OSAKA CITY UNIVERSITY

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