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PROJECTIVE DIMENSION OF COMPLEX BORDISM MODULES OF CW-SPECTRA, I

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Let $MU_*(\ )$ be the (reduced) complex bordism theory defined on the Boardman's stable category [4] of CW-spectra. Recall that MU_* $(\equiv MU_*(S^0))$ $\cong Z[x_1, x_2, \cdots]$, $\deg x_i = 2i$. In [3] Baas has constructed a tower of homology theories

$$MU_*(\ ) = MU\langle\infty\rangle_*(\ ) \to \cdots \to MU\langle n\rangle_*(\ ) \to \cdots \to MU\langle 0\rangle_*(\ ) \cong H_*(\ )$$

such that $MU\langle n\rangle_*$ $(\equiv MU\langle n\rangle_*(S^0)) \cong Z[x_1, \cdots, x_n]$, which factorizes the Thom homomorphism $\mu\colon MU_*(\ ) \to H_*(\ )$. When $Td(x_1)=1$ and $Td(x_j)=0$ for all $j \geqq 2$ (it is possible to choose ring generators x_i of MU_* with such properties), we shall write $MU_{Td}\langle n\rangle_*(\ )$ instead of $MU\langle n\rangle_*(\ )$ for emphasis. $MU_{Td}\langle 1\rangle_*(\ )$ can be identified with the connective homology K-theory $k_*(\ )$. Then the tower of homology theories

$$MU_*(\ ) \to \cdots \to MU_{Td}\langle n\rangle_*(\ ) \to \cdots \to MU_{Td}\langle 1\rangle_*(\ ) \cong k_*(\ )$$

factorizes the homomorphism $\zeta\colon MU_*(\ ) \to k_*(\ )$ lifting the Thom homomorphism $\mu_C\colon MU_*(\ ) \to K_*(\ )$.

Under the assumption that X is a finite CW-complex, Conner, Smith and Johnson ([6] and [9]) investigated conditions that the Thom homomorphism μ: $MU_*(X) \to H_*(X)$ is an epimorphism, and that the homomorphism $\zeta\colon MU_*(X) \to k_*(X)$ is an epimorphism. In the present paper we try to extend these results to a CW-spectrum.

In §1 we study some basic properties of CW-spectra and homology theories $MU\langle n\rangle_*(\ )$ for the sake of our later references.

Landweber [10] indicated that there exists a MU_*-resolution for a CW-spectrum as well as a finite CW-complex (Theorem 1). In §2 we construct two spectral sequences

i) $E\langle n\rangle^2_{p,q}(X) = \mathrm{Tor}^{MU_*}_{p,q}(MU\langle n\rangle_*, MU_*(X)) \Rightarrow MU\langle n\rangle_*(X)$
and

ii) $E_2^{p,q}[X]=\mathrm{Ext}_Z^{p,q}(K_*(X), Z)\Rightarrow K^*(X)$,
using a connective MU_*-resolution for a connective CW-spectrum X. The second spectral sequence yields the following universal coefficient sequence

$$0 \to \mathrm{Ext}(K_{*-1}(X), Z) \to K^*(X) \to \mathrm{Hom}(K_*(X), Z) \to 0$$

(Theorem 3).

In § 3 we give necessary and sufficient conditions that $\mu: MU_*(X)\to H_*(X)$ is an epimorphism (Theorems 4 and 5) and that $\zeta: MU_*(X)\to k_*(X)$ is an epimorphism (Theorem 7). Finally we give a new proof of Johnson's theorem [8] (Theorem 8).

In a subsequent paper with the same title we will discuss conditions under which $\mu\langle n\rangle: MU_*(X)\to MU\langle n\rangle_*(X)$ is an epimorphism for a general $n\geqq 0$.

1. Homology theories $MU\langle n\rangle_*(\ )$ of CW-spectra

1.1. Let $\mathcal{C}$ be the category of based CW-complexes and $\mathcal{S}$ the stable category of CW-spectra defined by Boardman [4] (and also see [11]). We may regard a based CW-complex as a CW-spectrum *via* the canonical inclusion functor $J: \mathcal{C}\to\mathcal{S}$. *A* CW-spectrum X is said to be *l-connected* if

$$\pi_i(X) = \{\Sigma^0, X\}_i \cong \{\Sigma^i, X\}_0 = 0 \qquad \text{for all } i\leqq l.$$

When a CW-spectrum X is l-connected for some l, we say X is *connective*. Notice that a based CW-complex is (-1)-connected.

Let X be a l-connected CW-spectrum. We define an additive cohomology theory on $\mathcal{C}$ by

$$h^p(B) = \{JB, X\}^p .$$

According to Brown's theorem [5] there exists an Ω-spectrum $\{Y_p\}$ such that $\{JB, X\}^p\cong[B, Y_p]$. Remark that Y_p is a $(l+p)$-connected CW-complex. Any n-connected CW-complex is homotopy equivalent to a certain CW-complex having no cells in dimensions $<n+1$ (except the base point). So we can assume that Y_p has no cells in dimensions $<l+p+1$. Let $Y=\cup J_pY_p$ be the CW-spectrum associated with the prespectrum $\{Y_p\}$. Since J_pY_p is a CW-spectrum without cells in dimensions $<l+1$, Y has no cells in dimensions $<l+1$. Furthermore the associated spectrum Y is homotopy equivalent to X [11, Theorem 14.4]. Thus we obtain the following proposition [4].

Proposition 1. *Let X be a l-connected CW-spectrum. Then there exists a CW-spectrum Y such that*
i) *Y has no cells in dimensions less than $l+1$ (except the base point), and*
ii) *Y is homotopy equivalent to X.*

Let X be a finite CW-spectrum and $\{X^p\}$ the skeleton filtration of X. By an induction process on p we shall construct the function dual $D(X^p)$ of X^p such that the number of n-cells in X^p coincides with that of $(-n)$-cells in $D(X^p)$. Assume that $D(X^{p-1})$ satisfies the required property. X^p/X^{p-1} is a finite wedge of p-spheres, i.e., $X^p/X^{p-1}=\vee\Sigma^p$. We can take $\vee\Sigma^{-p}$ as $D(X^p/X^{p-1})=D(\vee\Sigma^p)$, because $\{Z, \vee\Sigma^{-p}\}\cong\oplus\{Z, \Sigma^{-p}\}\cong\{\vee\Sigma^p Z, \Sigma^0\}\cong\{Z, F(\vee\Sigma^p, \Sigma^0)\}$ for arbitrary CW-spectra Z. So $D(X^p/X^{p-1})$ satisfies the required property. Let $\delta\colon \Sigma^{-1}D(X^{p-1})=D(\Sigma X^{p-1})\to D(X^p/X^{p-1})=\vee\Sigma^{-p}$ be the induced morphism of the boundary $X^p/X^{p-1}\to\Sigma X^{p-1}$. We define the function dual $D(X^p)$ of X^p as the mapping cone of δ. As is easily seen, there is a one to one correspondence between the set of n-cells in X^p and that of $(-n)$-cells in $D(X^p)$. By choosing a large enough skeleton of X we get

Lemma 2. *Let X be a finite CW-spectrum. The function dual DX of X can be taken as a finite CW-spectrum such that the number of n-cells in X coincides with that of $(-n)$-cells in DX.*

1.2. Let MU denote the unitary Thom spectrum. We recall that

$$\pi_*(MU)\cong Z[x_1, x_2, \cdots]$$

where $x_i\in\pi_{2i}(MU)$. Baas [3] has constructed a tower of CW-spectra

$$MU=MU\langle\infty\rangle\to\cdots\to MU\langle n\rangle\to\cdots\to MU\langle 0\rangle \tag{1.1}$$

such that

$$\pi_*(MU\langle n\rangle)\cong Z[x_1, \cdots, x_n]\,.$$

Denote by $\mu_{m,n}$, $0\leqq n<m\leqq\infty$, the canonical morphism $MU\langle m\rangle\to MU\langle n\rangle$.

Let us denote by $MU_*(\ )$ $(=MU\langle\infty\rangle_*(\ ))$ and $MU\langle n\rangle_*(\ )$ the (reduced) homology theories represented by the spectra MU and $MU\langle n\rangle$ respectively. Proposition 1 implies that

$$MU\langle m\rangle_j(X)=0 \qquad \text{for } j\leqq l \text{ and } 0\leqq m\leqq\infty, \tag{1.2}$$

when X is l-connected. We have the following basic relation between $MU\langle n\rangle_*(\ )$ and $MU\langle n-1\rangle_*(\ )$ [3]: There is a natural exact sequence

$$\cdots\to MU\langle n\rangle_j(X)\xrightarrow{\cdot x_n} MU\langle n\rangle_{j+2n}(X)\xrightarrow{\tau\langle n\rangle} MU\langle n-1\rangle_{j+2n}(X)\to\cdots \tag{1.3}$$

for any CW-spectrum X where $\tau\langle n\rangle=(\mu_{n,n-1})_*$ and $\cdot x_n$ denotes the multiplication by x_n.

Let $K(Z)$ denote the Eilenberg-MacLane spectrum. The Thom map $\mu\colon MU\to K(Z)$ admits a factorization

$$MU \xrightarrow{\mu_{\infty,0}} MU\langle 0\rangle \xrightarrow{\nu_0} K(Z)\,.$$

ν_0 induces an isomorphism $(\nu_0)_*\colon \pi_*(MU\langle 0\rangle)\to\pi_*(K(Z))$, and hence ν_0 is a homotopy equivalence. Therefore $MU\langle 0\rangle_*(\ )$ becomes the ordinary (reduced) homology theory, i.e.,

$$MU\langle 0\rangle_*(\ )\cong H_*(\ )\,. \tag{1.4}$$

So we may regard $(\mu_{\infty,0})_*$ as the Thom homomorphism μ. Let us denote by $\mu\langle n\rangle$ and $\nu\langle n\rangle$ the homomorphisms $(\mu_{\infty,n})_*$ and $(\mu_{n,0})_*$ respectively.

Now we shall prove two lemmas using the exact sequenence (1.3).

Lemma 3. *Let X be a CW-spectrum such that $MU\langle n\rangle_i(X)$ is a torsion free abelian group for $i\leqq k$. Then $\tau\langle n\rangle\colon MU\langle n\rangle_j(X)\to MU\langle n-1\rangle_j(X)$ is an epimorphism for $j\leqq k+2n+1$.*

Proof. In the following commutative diagram

$$\begin{array}{ccccccc} MU\langle n\rangle_{i+2n+1}(X) & \xrightarrow{\tau\langle n\rangle} & MU\langle n-1\rangle_{i+2n+1}(X) & \to & MU\langle n\rangle_i(X) & \xrightarrow{\cdot x_n} & MU\langle n\rangle_{i+2n}(X) \\ & & & & \downarrow & & \downarrow \\ & & & & MU\langle n\rangle_i(X)\otimes Q & \xrightarrow{\cdot x_n} & MU\langle n\rangle_{i+2n}(X)\otimes Q \end{array}$$

for $i\leqq k$, the upper row is exact and $\cdot x_n\colon MU\langle n\rangle_*(X)\otimes Q\to MU\langle n\rangle_*(X)\otimes Q$ is a monomorphism by virtue of Dold's theorem [7]. Hence we get the required result immediately.

Lemma 4. *Let X be a connective CW-spectrum. If $\mu\langle n\rangle\colon MU_i(X)\to MU\langle n\rangle_i(X)$ are epimorphisms for all $i\leqq k$, then $\mu\langle n+1\rangle\colon MU_i(X)\to MU\langle n+1\rangle_i(X)$ are also so for the same i.*

Proof. By an induction on i, $i\leqq k$, we shall prove the lemma. For sufficiently small i, $\mu\langle n+1\rangle_i$ is an epimorphism because of (1.2). Next, assume that $\mu\langle n+1\rangle_i$ are epimorphisms for all i, $i\leqq j-1$ and $j\leqq k$. Consider the following commutative diagram

$$\begin{tikzcd} MU_{j-2n-2}(X) \arrow[r, "\cdot x_{n+1}"] \arrow[d, "\mu\langle n+1\rangle_{j-2n-2}"] & MU_j(X) \arrow[d, "\mu\langle n+1\rangle_j"] \arrow[dr, "\mu\langle n\rangle_j"] & \\ MU\langle n+1\rangle_{j-2n-2}(X) \arrow[r, "\cdot x_{n+1}"'] & MU\langle n+1\rangle_j(X) \arrow[r, "\tau\langle n+1\rangle_j"'] & MU\langle n\rangle_j(X) \end{tikzcd}$$

in which the bottom row is exact. $\mu\langle n+1\rangle_{j-2n-2}$ and $\mu\langle n\rangle_j$ are epimorphisms by the assumptions. By chasing the above diagram we see easily that $\mu\langle n+1\rangle_j$ is an epimorphism.

1.3. Let X be a connective CW-spectrum and $0 \leqq n < m \leqq \infty$. We observe the Atiyah-Hirzebruch spectral sequences $\{E\langle m\rangle^r(X)\}$ for $MU\langle m\rangle_*(X)$. Let $\{F_p MU\langle m\rangle_*(X)\}$ be the usual increasing filtration of $MU\langle m\rangle_*(X)$ defined by skeletons. Note that $F_j MU\langle m\rangle_*(X) = 0$ for sufficiently small j. As is well known, we have isomorphisms

$$E\langle m\rangle^2_{p,*}(X) \cong H_p(X) \otimes MU\langle m\rangle_*$$

and

$$E\langle m\rangle^\infty_{p,*}(X) \cong F_p MU\langle m\rangle_*(X)/F_{p-1} MU\langle m\rangle_*(X)$$

of $MU\langle m\rangle_*$-modules. Since the Atiyah-Hirzebruch spectral sequences for $MU\langle m\rangle_*(X) \otimes Q$ collapse, the differentials of $\{E\langle m\rangle^r(X)\}$ are torsion valued. As an elementary result we have that

$$\text{(1.5)} \qquad \begin{aligned} &E\langle m\rangle^2_{j,0}(X) \cong E\langle m\rangle^\infty_{j,0}(X) &&\text{for } \ j \leqq k+3, \text{ and} \\ &E\langle m\rangle^2_{p,q}(X) \cong E\langle m\rangle^\infty_{p,q}(X) &&\text{for } \ p \leqq k, \end{aligned}$$

provided X is a connective CW-spectrum such that $H_i(X)$ is torsion free abelian for $i \leqq k$.

Proposition 5. *Let X be a connective CW-spectrum and $0 \leqq n < m \leqq \infty$. If the Atiyah-Hirzebruch spectral sequence for $MU_*(X)$ collapses, then $(\mu_{m,n})_*$ induces an isomorphism*

$$(\tilde{\mu}_{m,n})_* \colon MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} MU\langle m\rangle_*(X) \to MU\langle n\rangle_*(X).$$

Proof. Since the spectral sequence $\{E^r(X) \equiv E\langle\infty\rangle^r(X)\}$ for $MU_*(X)$ collapses, $\{E\langle n\rangle^r(X)\}$ collapse for all $n \geqq 0$. On the other hand,

$$\operatorname{Tor}_1^{MU\langle m\rangle_*}(MU\langle n\rangle_*, H_p(X) \otimes MU\langle m\rangle_*) \cong \operatorname{Tor}_1^Z(MU\langle n\rangle_*, H_p(X)) = 0$$

for $0 \leqq n < m \leqq \infty$. Here we have the commutative diagram

$$\begin{array}{ccccc} 0 \to MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} F_{p-1} MU\langle m\rangle_*(X) & \to & MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} F_p MU\langle m\rangle_*(X) & & \\ \downarrow & & \downarrow & & \\ 0 \longrightarrow F_{p-1} MU\langle n\rangle_*(X) & \longrightarrow & F_p MU\langle n\rangle_*(X) & & \\ & & & \to MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} (H_p(X) \otimes MU\langle m\rangle_*) \to 0 & \\ & & & \downarrow & \\ & & & \longrightarrow H_p(X) \otimes MU\langle n\rangle_* \longrightarrow 0 & \end{array}$$

with exact rows. By an induction on p we can show that

$$MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} F_p MU\langle m\rangle_*(X) \to F_p MU\langle n\rangle_*(X)$$

are isomorphisms for all p. Passing to the direct limit, it follows that

$$(\tilde{\mu}_{m,n})_*: MU\langle n\rangle_* \underset{MU\langle m\rangle_*}{\otimes} MU\langle m\rangle_*(X) \to MU\langle n\rangle_*(X)$$

is an isomorphism.

Proposition 6. *Let X be a connective CW-spectrum and $1 \leqq m \leqq \infty$.*

I) *The following conditions are equivalent:*

0) *$H_*(X)$ is a free abelian group;*

i)$_m$ *$MU\langle m\rangle_*(X)$ is a free $MU\langle m\rangle_*$-module;*

ii)$_m$ *$MU\langle m\rangle_*(X)$ is a projective $MU\langle m\rangle_*$-module.*

II) *The following conditions are equivalent:*

0)$'$ *$H_*(X)$ is a torsion free abelian group;*

iii)$_m$ *$MU\langle m\rangle_*(X)$ is a flat $MU\langle m\rangle_*$-module.*

Proof. 0) $\to$ i)$_m$ and 0)$'$ $\to$ iii)$_m$: Since the spectral sequences $\{E\langle m\rangle^r(X)\}$ for $MU\langle m\rangle_*(X)$ collapse, there exist exact sequences

$$0 \to F_{p-1}MU\langle m\rangle_*(X) \to F_pMU\langle m\rangle_*(X) \to H_p(X)\otimes MU\langle m\rangle_* \to 0$$

of $MU\langle m\rangle_*$-modules for all p. On the other hand, we note that

$$\mathrm{Tor}_k^{MU\langle m\rangle_*}(H_p(X)\otimes MU\langle m\rangle_*, C) \cong \mathrm{Tor}_k^Z(H_p(X), C), \qquad k \geqq 0,$$

for any $MU\langle m\rangle_*$ -module C. Then 0) $\to$ i)$_m$ and 0)$'$ $\to$ iii)$_m$ follow immediately.

iii)$_m$ $\to$ 0)$'$: By an induction on p we shall show that $H_p(X)$ is torsion free abelian. Assume that $H_j(X)$ is torsion free abelian for $j \leqq p-1$. Because of (1.5) $\nu\langle m\rangle$: $MU\langle m\rangle_i(X) \to H_i(X)$ is an epimorphism for $i \leqq p+2$. Consider the following commutative square

$$\begin{array}{ccc} (Z \underset{MU\langle m\rangle_*}{\otimes} MU\langle m\rangle_*(X))_p & \to & (Q \underset{MU\langle m\rangle_*}{\otimes} MU\langle m\rangle_*(X))_p \\ \downarrow & & \downarrow \\ H_p(X) & \longrightarrow & H_p(X)\otimes Q\,. \end{array}$$

The upper horizontal map is a monomorphism and the right vertical one is an isomorphism (Proposition 5). So we find that the bottom horizontal map is a monomorphism, i.e., $H_p(X)$ is torsion free abelian.

i)$_m$ $\to$ ii)$_m$ is obvious.

ii)$_m$ $\to$ 0): $H_*(X)$ is torsion free abelian because a projective $MU\langle m\rangle_*$-module is flat. Making use of Proposition 5 we get an isomorphism

$$\bar{\nu}\langle m\rangle: Z \underset{MU\langle m\rangle_*}{\otimes} MU\langle m\rangle_*(X) \to H_*(X)\,.$$

Then the projectivity of $MU\langle m\rangle_*(X)$ implies that $H_*(X)$ is free abelian.

2. Spectral sequences arising from MU_*-resolutions

2.1. First we introduce (connective) $MU\langle m\rangle_*$-resolutions, $0\leqq m\leqq\infty$, for a (connective) CW-spectrum X.

I) *A partial (connective) $MU\langle m\rangle_*$-resolution of X of length* 1 is a cofibration of (connective) CW-spectra

$$W \xrightarrow{f} X \subset Y$$

such that

i) $MU\langle m\rangle_*(W)$ is a projective $MU\langle m\rangle_*$-module, and

ii) $f_*: MU\langle m\rangle_*(W)\to MU\langle m\rangle_*(X)$ is an epimorphism.

II) *A $MU\langle m\rangle_*$-resolution of X* is a diagram consisting of CW-spectra and morphisms

$$\begin{array}{ccccccccc} X= & X_0 & \subset X_1 & \subset\cdots\subset & X_k & \subset X_{k+1}\subset\cdots \\ & \nwarrow & \nwarrow & & \nwarrow & \\ & W_0 & W_1 & & W_k & \end{array}$$

such that $W_k\to X_k\subset X_{k+1}$ is a partial $MU\langle m\rangle_*$-resolution of X_k (of length 1) for each $k\geqq 0$.

It is said to be *connective* if W_k, X_k and the union $X_\infty=\cup X_k$ of X_k are all connective.

III) We say that a (connective) $MU\langle m\rangle_*$ -resolution $\{X_k, W_k\}$ of X has *length l* when $MU\langle m\rangle_*(X_l)$ is a projective $MU\langle m\rangle_*$-module.

Note that a $MU\langle m\rangle_*$-resolution $\{X_k, W_k\}$ of X yields a projective $MU\langle m\rangle_*$-resolution

$$(2.1)\qquad \to MU\langle m\rangle_{*+k}(W_k)\to\cdots\to MU\langle m\rangle_{*+1}(W_1)\to MU\langle m\rangle_*(W_0)$$
$$\to MU\langle m\rangle_*(X)\to 0$$

of $MU\langle m\rangle_*(X)$.

Let X be a connective CW-spectrum and $W(X)=\{X_k, W_k\}$ a connective MU_*-resolution of X. The union $X_\infty(\equiv X_\infty^{W(X)})$ of X_k has the following property.

Lemma 7. *X_∞ is contractible.*

Proof. Let X be l-connected. First we shall show by an induction on k that

$$\mu: MU_j(X_k)\to H_j(X_k)$$

is an epimorphism for each $j\leqq l+3k$. Assume that $\mu: MU_j(X_k)\to H_j(X_k)$ is an epimorphism for $j\leqq l+3k$. Then $H_j(X_{k+1})$ is free abelian for the same j. (1.5) implies that $\mu: MU_i(X_{k+1})\to H_i(X_{k+1})$ is an epimorphism for $i\leqq l+3(k+1)$. This means that $H_j(X_k)\to H_j(X_{k+1})$ is a zero map for $j\leqq l+3k$. Therefore we get

$$H_*(X_\infty) \cong \varinjlim H_*(X_k) = 0 .$$

Consider the Atiyah-Hirzebruch spectral sequence $\{E^r\}$ for $\pi_*(X_\infty)$. Since $E^2 = H_*(X_\infty; \pi_*) = 0$ and X_∞ is connective, we can easily see that $\pi_*(X_\infty) = 0$ and hence X_∞ is contractible.

Next we discuss the existence of (connective) MU_*-resolutions. The following result was given by Landweber [10] (and also see [1]).

Proposition 8. *Let X be a (l-connected) CW-spectrum. Then there exists a partial (l-connected) MU_*-resolution*

$$W \to X \subset Y$$

of X of length 1. In particular, W can be taken as a wedge sum of finite CW-spectra.

Proof. Assume that X is l-connected. By Proposition 1 we may assume that X has no cells in dimensions $< l+1$. Take any element $x \in MU_p(X) \cong \{\Sigma^p, X \wedge MU\}$, $p > l$. Then there exists finite CW-subspectra X' and E_x of X and MU respectively and x is factorized in the form

$$\Sigma^p \xrightarrow{x'} X' \wedge E_x \subset X \wedge MU .$$

In virtue of Lemma 2 we may insist that the function dual DX' of X' has no cells in dimensions $> -(l+1)$. So $\Sigma^p \wedge DX'$ is a finite CW-spectrum of dimension $\leqq p-l-1$. Since $MU^{p-l-1} = \cup J_{2n}MU(n)^{p-l-1+2n}$, we can choose E_x to be in the form $J_{2n}MU(n)^{p-l-1+2n}$. Putting $W_x = \Sigma^p \wedge DE_x$, it is a finite CW-spectrum having no cells in dimensions $< l+1$, and hence l-connected. Since $H_*(MU(n))$ is free abelian, Proposition 6 implies that $MU_*(W_x)$ is a free MU_*-module. Let f_x: $W_x = \Sigma^p \wedge DE_x \to X' \subset X$ be the dual morphism of x'. By construction we see that

$$x \in \operatorname{Im} \{(f_x)_* \colon MU_*(W_x) \to MU_*(X)\} .$$

Put $W = \vee W_x$ and $f = \vee f_x$: $W = \vee W_x \to X$ where x runs over a set of generators for $MU_*(X)$. As is easily seen,

i) W is a l-connected CW-spectrum such that $MU_*(W)$ is a free MU_*-module, and

ii) f_*: $MU_*(W) \to MU_*(X)$ is an epimorphism. Consequently, the cofibration $W \xrightarrow{f} X \subset Y$ forms a partial l-connected MU_*-resolution of X.

By an iterated application of Proposition 8 we have the following result which is the extension of Conner-Smith's theorem [6].

Theorem 1. *Let X be a (connective) CW-spectrum. Then there exists a (connective) MU_*-resolution of X.*

2.2. Let X be a connective CW-spectrum and $W(X)=\{X_k, W_k\}_{k\geq 0}$ a connective MU_*-resolution of X. By setting $\bar{X}_k=X_{k+1}/X_0$ and $\bar{X}_\infty=X_\infty/X_0$ we define an increasing filtration $\{\bar{X}_k\}$ of $\bar{X}_\infty$. Fix n, $0\leqq n<\infty$, and observe the spectral sequence $\{E\langle n\rangle^r(W(X))\}$ of $MU\langle n\rangle_*(X)$ associated with the filtration $\{\bar{X}_k\}$ (see [1] and [6]). Making use of Lemma 7 we define an increasing filtration of $MU\langle n\rangle_*(X)$ by

$$\begin{aligned}&F_p^W MU\langle n\rangle_k(X)\\ &\qquad = \mathrm{Im}\,\{MU\langle n\rangle_{k+1}(X_{p+1}/X_0)\to MU\langle n\rangle_{k+1}(X_\infty/X_0)\cong MU\langle n\rangle_k(X)\}\,.\end{aligned}$$

By definition of the spectral sequence we have

$$\begin{aligned}D\langle n\rangle^1_{p,q} &= MU\langle n\rangle_{p+q+1}(X_{p+1}/X_0)\\ E\langle n\rangle^1_{p,q} &= MU\langle n\rangle_{p+q+1}(X_{p+1}/X_p)\cong MU\langle n\rangle_{p+q}(W_p)\end{aligned}\tag{2.2}$$

and

$$\varinjlim_{r>p} E\langle n\rangle^r_{p,q}\cong E\langle n\rangle^\infty_{p,q}\cong F_p^W MU\langle n\rangle_{p+q}(X)/F_{p-1}^W MU\langle n\rangle_{p+q}(X)\,.\tag{2.3}$$

The differential operator $d\langle n\rangle^1$ is defined as the composition $E\langle n\rangle^1_{p,q}\to D\langle n\rangle^1_{p-1,q}\to E\langle n\rangle^1_{p-1,q}$. Since the following diagram

$$\begin{tikzcd}
E\langle n\rangle^1_{p,q} \arrow[r] & D\langle n\rangle^1_{p-1,q} \arrow[r] & E\langle n\rangle^1_{p-1,q}\\
MU\langle n\rangle_{p+q}(W_p) \arrow[r] \arrow[u, "\cong"] & MU\langle n\rangle_{p+q}(X_p) \arrow[r] \arrow[u] & MU\langle n\rangle_{p+q-1}(W_{p-1}) \arrow[u, "\cong"]
\end{tikzcd}$$

is commutative, the E^2-term is the homology of the complex

$$\to MU\langle n\rangle_{*+p}(W_p)\to\cdots\to MU\langle n\rangle_{*+1}(W_1)\to MU\langle n\rangle_*(W_0)\to 0\,.$$

By proposition 5 $\mu\langle n\rangle$ induces an isomorphism

$$\tilde{\mu}\langle n\rangle\colon MU\langle n\rangle_*\underset{MU_*}{\otimes} MU_*(W_p)\to MU\langle n\rangle_*(W_p)$$

for each $p\geqq 0$. On the other hand, we recall that

$$\to MU_{*+p}(W_p)\to\cdots\to MU_*(W_0)\to MU_*(X)\to 0$$

is a free MU_*-resolution of $MU_*(X)$. At present it follows immediately that

$$E\langle n\rangle^2_{p,q}\cong \mathrm{Tor}^{MU_*}_{p,q}(MU\langle n\rangle_*, MU_*(X))\,.$$

Considering the commutative square

$$\begin{tikzcd}
MU\langle n\rangle_*\underset{MU_*}{\otimes} MU_*(W_0) \arrow[r] \arrow[d] & MU\langle n\rangle_*\underset{MU_*}{\otimes} MU_*(X_0) \arrow[d]\\
MU\langle n\rangle_*(W_0) \arrow[r] & MU\langle n\rangle_*(X_0)\,,
\end{tikzcd}$$

it is trivial that

$$F_0^W MU\langle n\rangle_*(X) \cong \operatorname{Im}\{\tilde{\mu}\langle n\rangle\colon MU\langle n\rangle_* \underset{MU_*}{\otimes} MU_*(X) \to MU\langle n\rangle_*(X)\}\ .$$

And the edge map of the spectral sequence coincides with the reduced map $\tilde{\mu}\langle n\rangle\colon MU\langle n\rangle_* \underset{MU_*}{\otimes} MU_*(X) \to MU\langle n\rangle_*(X)$.

Next we shall show that the spectral sequence $\{E\langle n\rangle^r(W(X))\}$ is independent of the choice of a connective MU_*-resolution $W(X)$ of X.

Let $W(X)=\{X_k, W_k\}$ and $V(Y)=\{Y_k, V_k\}$ be connective MU_*-resolutions of X and Y respectively, and $f\colon X\to Y$ be a morphism of CW-spectra. Then there exist a connective MU_*-resolution $U(Y)=\{Z_k, U_k\}$ of Y and morphisms $\phi\colon W(X)\to U(Y)$ and $\psi\colon V(Y)\to U(Y)$ of connective MU_*-resolutions which lift f and 1_Y respectively. Moreover we can take as U_k CW-spectra of the form $W_k\vee V_k\vee U'_k$. Thus we have a family $\{Z_k, U'_k, \phi_k, \psi_k\}_{k\geqq 0}$ of connective CW-spectra and morphisms such that

i) $U_k=W_k\vee V_k\vee U'_k\to Z_k\subset Z_{k+1}$ is a partial connective MU_*-resolution of Z_k of length 1, and

ii) the following diagram

$$\begin{array}{ccccc} & W_k & \longrightarrow & X_k & \subset & X_{k+1} \\ & \downarrow & & \downarrow{\scriptstyle \phi_k} & & \downarrow{\scriptstyle \phi_{k+1}} \\ W_k\vee V_k\vee U'_k = & U_k & \longrightarrow & Z_k & \subset & Z_{k+1} \\ & \uparrow & & \uparrow{\scriptstyle \psi_k} & & \uparrow{\scriptstyle \psi_{k+1}} \\ & V_k & \longrightarrow & Y_k & \subset & Y_{k+1} \end{array}$$

is commutative where $Z_0=Y$, $\phi_0=f$ and $\psi_0=1_Y$.

In fact, we shall construct the desired family $\{Z_k, U'_k, \phi_k, \psi_k\}$ by an induction process. Assume that there is a family $\{Z_j, U'_{j-1}, \phi_j, \psi_j\}_{0\leqq j\leqq k}$ with the required properties. By Proposition 8 there exists a partial connective MU_*-resolution

$$U'_k \xrightarrow{\theta'_k} Z_k \subset Z'_{k+1}$$

of Z_k. Let $\theta_k\colon U_k=W_k\vee V_k\vee U'_k\to Z_k$ be the morphism induced by ϕ_k, ψ_k and θ'_k. We define a CW-spectrum Z_{k+1} as the mapping cone of θ_k. Clearly

$$U_k \xrightarrow{\theta_k} Z_k \subset Z_{k+1}$$

is a partial connective MU_*-resolution of Z_k. Besides we see that ϕ_k and ψ_k induce the desired morphisms ϕ_{k+1} and ψ_{k+1} respectively.

The morphisms $\phi\colon W(X)\to U(Y)$ and $\psi\colon V(Y)\to U(Y)$ of connective MU_*-resolutions yield morphisms

$$\{E\langle n\rangle^r(W(X))\} \xrightarrow{\{\phi^r_*\}} \{E\langle n\rangle^r(U(Y))\} \xleftarrow{\{\psi^r_*\}} \{E\langle n\rangle^r(V(Y))\}$$

of spectral sequences and

$$F_p^W MU\langle n\rangle_*(X) \xrightarrow{F_p\phi_*} F_p^U MU\langle n\rangle_*(Y) \xleftarrow{F_p\psi_*} F_p^V MU\langle n\rangle_*(Y)$$

of the increasing filtrations. Note that $F_p\phi_*$ and $F_p\psi_*$ coincide with f_* and id respectively. From the identification of the E^2-terms we find that

$$\phi^2_* = \mathrm{Tor}^{MU^*}(MU\langle n\rangle_*, f_*) \quad \text{and} \quad \psi^2_* = \mathrm{Tor}^{MU_*}(MU\langle n\rangle_*, id)\,.$$

So ψ^r_* are isomorphisms for all r, $2 \leqq r \leqq \infty$. From the bijectivity of ψ^∞_* it follows immediately that $F_p^V MU\langle n\rangle_*(Y) = F_p^U MU\langle n\rangle_*(Y)$.

Putting $X=Y$ and $f=1_X$, we obtain that

$$E\langle n\rangle^r(W(X)) \cong E\langle n\rangle^r(V(X)) \qquad \text{for all} \quad r,\ 2\leqq r\leqq\infty$$

and

$$F_p^W MU\langle n\rangle_*(X) = F_p^V MU\langle n\rangle_*(X) \qquad \text{for each} \quad p\geqq 0.$$

Thus the spectral sequence $\{E\langle n\rangle^r(W(X))\}$ is independent of the choice of a connective MU_*-resolution $W(X)$.

In addition the above discussion shows the naturality of our spectral sequence.

Theorem 2. *Let X be a connective CW-spectrum and $0\leqq n<\infty$. Then there exists a natural spectral sequence $\{E\langle n\rangle^r(X)\}$ associated with $MU\langle n\rangle_*(X)$ such that*

$$E\langle n\rangle^2_{p,q}(X) = \mathrm{Tor}^{MU_*}_{p,q}(MU\langle n\rangle_*, MU_*(X))\,.$$

As an immediate corollary of Theorem 2 we have

Corollary 9. *Let X be a connective CW-spectrum and $0\leqq n<\infty$. If $\mathrm{Tor}^{MU_*}_{p,*}(MU\langle n\rangle_*, MU_*(X))=0$ for all $p\geqq 1$, then*

$$\tilde{\mu}\langle n\rangle\colon MU\langle n\rangle_* \underset{MU_*}{\otimes} MU_*(X) \to MU\langle n\rangle_*(X)$$

is an isomorphism.

2.3. Let K_* and K^* denote the complex homology and cohomology K-theories, i.e., the Z_2-graded (reduced) homology and cohomology theories represented by the BU-spectrum. Now we discuss the duality between $K_*(X)$ and $K^*(X)$ for a connective CW-spectrum X. The Kronecker index gives a natural homomorphism

$$\kappa\colon K^*(X) \to \mathrm{Hom}(K_*(X), Z)\,. \tag{2.4}$$

First we shall need the following special case [1].

Lemma 10. *Let X be a connective CW-spectrum with $H_*(X)$ free abelian. Then $\kappa\colon K^*(X)\to \operatorname{Hom}(K_*(X), Z)$ is an isomorphism.*

Proof. Let $\{E^r(X)\}$ and $\{E_r(X)\}$ be the Atiyah-Hirzebruch spectral sequences for $K_*(X)$ and $K^*(X)$ respectively. The duality homomorphism $\kappa\colon K^*(X)\to \operatorname{Hom}(K_*(X), Z)$ yields morphisms

$$\kappa_r\colon E_r(X) \to \operatorname{Hom}(E^r(X), Z)$$

for $2\leqq r\leqq\infty$. Since $H_*(X)$ is free abelian, the spectral sequence $\{E^r(X)\}$ collapses and moreover

$$\kappa_2\colon H^*(X) \to \operatorname{Hom}(H_*(X), Z)$$

is an isomorphism. This implies that the spectral sequence $\{E_r(X)\}$ collapses, and then it is strongly convergent [2, Proposition 9]. Thus

$$E_2^{p,*}(X) \cong F^pK^*(X)/F^{p+1}K^*(X) \quad \text{and} \quad \cap F^pK^*(X) = \{0\}\ ,$$

where $\{F^pK^*(X)\}$ is the usual decreasing filtration of $K^*(X)$ defined by skeletons. Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \to & H^p(X; Z) & \to & K^*(X)/F^{p+1}K^*(X) & \to & K^*(X)/F^pK^*(X) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \operatorname{Hom}(H_p(X; Z), Z) & \to & \operatorname{Hom}(F_pK_*(X), Z) & \to & \operatorname{Hom}(F_{p-1}K_*(X), Z) & \to 0
\end{array}$$

with exact rows. We can show by an induction on p that

$$K^*(X)/F^{p+1}K^*(X) \to \operatorname{Hom}(F_pK_*(X), Z)$$

are isomorphisms for all p. Remark that $K^*(X)\cong \varprojlim K^*(X)/F^{p+1}K^*(X)$ [2, (3.5) and (3.6)] and $\operatorname{Hom}(K_*(X), Z)\cong \varprojlim \operatorname{Hom}(F_pK_*(X), Z)$. We pass to inverse limits and get that

$$\kappa\colon K^*(X) \to \operatorname{Hom}(K_*(X), Z)$$

is an isomorphism.

By $MU_{**}(\ )$ we mean that $MU_*(\ )$ is treated as Z_2-graded by its even and odd components. The homomorphism of coefficients

$$\mu_C\colon MU_{**} \to Z$$

induced by the Thom map $\mu_C\colon MU\to BU$ may be identified (up to sign) with the classical Todd genus. $\mu_C=Td$ makes Z into a Z_2-graded MU_{**}-module, and then denote it by Z_{Td}.

There exist a CW-spectrum of the form $A=\vee A_\alpha$ and a morphism $f\colon A\to X$ such that

i) A_α is a finite CW-spectrum with $H_*(A_\alpha)$ free abelian, and
ii) $f_*: K_*(A) \to K_*(X)$ is an epimorphism.
(Cf., Proposition 8). On the other hand, a similar discussion to Proposition 5 shows that μ_C induces an isomorphism

$$\tilde{\mu}_C: Z_{Td} \underset{MU_{**}}{\otimes} MU_{**}(B) \to K_*(B) \tag{2.5}$$

for any connective CW-spectrum B with $H_*(B)$ free abelian. Therefore we find immediately that

$$\mu_C: MU_{**}(X) \to K_*(X) \tag{2.6}$$

is an epimorphism.

Let X be a connective CW-spectrum and $W(X) = \{X_k, W_k\}_{k \geqq 0}$ a connective MU_*-resolution of X. Since $\mu_C: MU_{**}(X_k) \to K_*(X_k)$ is an epimorphism and $K_*(W_k)$ is free abelian, the sequence

$$\to K_{*+k}(W_k) \to \cdots \to K_{*+1}(W_1) \to K_*(W_0) \to K_*(X) \to 0 \tag{2.7}$$

becomes a free Z-resolution of $K_*(X)$. Associated with the increasing filtration $\{\bar{X}_k = X_{k+1}/X_0\}$ we have the spectral sequence $\{E_r[X]\}$ of $K^*(X)$ such that

$$D_1^{p,q}[X] = K^{p+q+1}(X_{p+1}/X_0)$$
$$E_1^{p,q}[X] = K^{p+q+1}(X_{p+1}/X_p) \cong K^{p+q}(W_p)\,.$$

The E_2-term is the homology of the complex

$$0 \to K^*(W_0) \to K^{*+1}(W_1) \to \cdots \to K^{*+p}(W_p) \to \cdots\,.$$

By virtue of Lemma 10 the E_2-term is the homology of the complex

$$0 \to \mathrm{Hom}(K_*(W_0), Z) \to \mathrm{Hom}(K_{*+1}(W_1), Z) \to \cdots\,.$$

Hence it follows that

$$E_2^{p,q}[X] \cong \mathrm{Ext}^{p,q}(K_*(X), Z)\,.$$

The usual argument (cf., Theorem 2) shows that our spectral sequence is independent of the choice of a connective MU_*-resolution and it is natural.

Since $E_2^{p,q}[X] = 0$ for $p \neq 0, 1$, our spectral sequence $\{E_r[X]\}$ collapses, and it is strongly convergent [2]. From an elementary discussion about spectral sequences we obtain a universal coefficient sequence relating K_* and K^*.

Theorem 3. *Let X be a connective CW-spectrum. Then there exists a natural exact sequence*

$$0 \to \mathrm{Ext}(K_{*-1}(X), Z) \to K^*(X) \to \mathrm{Hom}(K_*(X), Z) \to 0\,.$$

3. CW-spectra with low MU_*-projective dimension

3.1. Let X be a CW-spectrum and $0 \leqq n < m \leqq \infty$. Making use of Dold's theorem we have

$$\begin{aligned} & \operatorname{Tor}_{p,*}^{MU\langle m\rangle_*}(MU\langle n\rangle_* \otimes Q, MU\langle m\rangle_*(X)) \\ \cong\ & \operatorname{Tor}_{p,*}^{MU\langle m\rangle_*}(MU\langle n\rangle_*, MU\langle m\rangle_*(X) \otimes Q) \\ \cong\ & \operatorname{Tor}_{p,*}^{MU\langle m\rangle_*}(MU\langle n\rangle_*, H_*(X; Q) \otimes MU\langle m\rangle_*) \\ \cong\ & \operatorname{Top}_{p,*}^{Z}(MU\langle n\rangle_*, H_*(X; Q)) \cong 0 \end{aligned}$$

for all $p \geqq 1$. This yields that

$$\begin{aligned} & \operatorname{Tor}_{p,*}^{MU\langle m\rangle_*}(MU\langle n\rangle_*, MU\langle m\rangle_*(X)) \\ \cong\ & \operatorname{Tor}_{p+1,*}^{MU\langle m\rangle_*}(MU\langle n\rangle_* \otimes Q/Z, MU\langle m\rangle_*(X)) \end{aligned} \tag{3.1}$$

for all $p \geqq 1$.

We denote by hom $\dim_{MU\langle m\rangle_*} MU\langle m\rangle_*(X)$ the projective demension of $MU\langle m\rangle_*(X)$ as a $MU\langle m\rangle_*$-module. Now Conner-Smith's theorem [6] is extended to a connective CW-spectrum as follows (cf., [10]).

Theorem 4. *Let X be a connective CW-spectrum. Then the following conditions are equivalent*:

0) hom $\dim_{MU_*} MU_*(X) \leqq 1$;

I) *the Thom homomorphism μ: $MU_*(X) \to H_*(X)$ is an epimorphism*;

II) *the Thom homomorphism μ induces an isomorphism $\tilde{\mu}$: $Z \underset{MU_*}{\otimes} MU_*(X) \to H_*(X)$*;

III) $\operatorname{Tor}_{p,*}^{MU_*}(Z, MU_*(X)) = 0$ *for all $p \geqq 1$.*

Proof. We prove in the order: III)$\to$II)$\to$I)$\to$0)$\to$III). "II)$\to$I)" is trivial. "III)$\to$II)" and "0)$\to$III)" follow immediately from Corollary 9 and (3.1).

I)$\to$0): Let $W \to X \subset Y$ be a partial connective MU_*-resolution of X. By the surjectivity of μ: $MU_*(X) \to H_*(X)$, $W \to X \subset Y$ forms a (partial) connective H_*-resolution of X of length 1. Therefore $MU_*(Y)$ is a free MU_*-module by Proposition 6, so

$$\text{hom} \dim_{MU_*} MU_*(X) \leqq 1 .$$

Let X be a connective CW-spectrum with hom $\dim_{MU_*} MU_*(X) \leqq 1$. Then, by Theorem 4 and Lemma 4, $\mu\langle n\rangle$: $MU_*(X) \to MU\langle n\rangle_*(X)$ is an epimorphism for each $n \geqq 0$. This implies that a connective MU_*-resolution of X of length 1 forms a connective $MU\langle n\rangle_*$-resolution of X of length 1. Thus

(3.2) *X admits a connective $MU\langle n\rangle_*$-resolution of length* 1,

and hence

(3.3) $$\text{hom dim}_{MU\langle n\rangle_*} MU\langle n\rangle_*(X) \leqq 1,$$

provided $\text{hom dim}_{MU_*} MU_*(X) \leqq 1$.

The exact sequence $0 \to MU\langle n\rangle_* \xrightarrow{\cdot x_n} MU\langle n\rangle_* \to MU\langle n-1\rangle_* \to 0$ of $MU\langle n\rangle_*$-modules, $1 \leqq n < \infty$, yields an exact sequence

$$0 \to \mathrm{Tor}_{1,*}^{MU\langle n\rangle_*}(MU\langle n-1\rangle_*, MU\langle n\rangle_*(X)) \to MU\langle n\rangle_*(X)$$
$$\xrightarrow{\cdot x_n} MU\langle n\rangle_*(X) \to MU\langle n-1\rangle_* \underset{MU\langle n\rangle_*}{\otimes} MU\langle n\rangle_*(X) \to 0\,.$$

Combining this with (1.3) we get a natural exact sequence

(3.4) $$0 \to MU\langle n-1\rangle_* \underset{MU\langle n\rangle_*}{\otimes} MU\langle n\rangle_*(X) \xrightarrow{\tilde{\tau}\langle n\rangle} MU\langle n-1\rangle_*(X)$$
$$\to \mathrm{Tor}_{1,*-1}^{MU\langle n\rangle_*}(MU\langle n-1\rangle_*, MU\langle n\rangle_*(X)) \to 0$$

[3, Theorem 5.3].

Let M be a $MU\langle n\rangle_*$-module and N and L $MU\langle n-1\rangle_*$-modules. Every $MU\langle n-1\rangle_*$-module may be treated as a $MU\langle n\rangle_*$-module *via* the map $\tau\langle n\rangle$: $MU\langle n\rangle_* \to MU\langle n-1\rangle_*$. We have two strongly convergent spectral sequences $\{E_r\}$ and $\{\bar{E}_r\}$ associated with the same graded $MU\langle n-1\rangle_*$-module such that

$$E_2^{p,q} = \mathrm{Ext}^p_{MU\langle n\rangle_*}(M, \mathrm{Ext}^q_{MU\langle n-1\rangle_*}(N, L))$$

and

$$\bar{E}_2^{p,q} = \mathrm{Ext}^p_{MU\langle n-1\rangle_*}(\mathrm{Tor}_q^{MU\langle n\rangle_*}(M, N), L)$$

(cf., [12, (1.7)]). Replacing M and N by $MU\langle n\rangle_*(X)$ and $MU\langle n-1\rangle_*$ respectively, we find that

(3.5) *there exists a strongly convergent spectral sequence* $\{\bar{E}_r\}$ *associated with* $\mathrm{Ext}^*_{MU\langle n\rangle_*}(MU\langle n\rangle_*(X), L)$ *such that*

$$\bar{E}_2^{p,q} = \mathrm{Ext}^p_{MU\langle n-1\rangle_*}(\mathrm{Tor}_q^{MU\langle n\rangle_*}(MU\langle n\rangle_*(X), MU\langle n-1\rangle_*), L)\,.$$

Proposition 11. *Let X be a CW-spectrum and $1 \leqq n < \infty$. If* $\text{hom dim}_{MU\langle n\rangle_*} MU\langle n\rangle_*(X) \leqq 1$, *then*
i) $\tilde{\tau}\langle n\rangle$: $MU\langle n-1\rangle_* \underset{MU\langle n\rangle_*}{\otimes} MU\langle n\rangle_*(X) \to MU\langle n-1\rangle_*(X)$ *is an isomorphism, and*
ii) $\text{hom dim}_{MU\langle n-1\rangle_*} MU\langle n-1\rangle_*(X) \leqq 1$.

Proof. Using (3.1) we get that

$$\mathrm{Tor}_{p,*}^{MU\langle n\rangle_*}(MU\langle n-1\rangle_*, MU\langle n\rangle_*(X))$$
$$\cong \mathrm{Tor}_{p+1,*}^{MU\langle n\rangle_*}(MU\langle n-1\rangle_* \otimes Q/Z, MU\langle n\rangle_*(X)) = 0$$

for all $p \geqq 1$, and by means of (3.4) that

$$\tilde{\tau}\langle n\rangle\colon MU\langle n-1\rangle_* \underset{MU\langle n\rangle_*}{\otimes} MU\langle n\rangle_*(X) \to MU\langle n-1\rangle_*(X)$$

is an isomorphism. In the spectral sequence $\{\bar{E}_r\}$ of (3.5) we have

$$\bar{E}_2^{p,0} = \mathrm{Ext}^p_{MU\langle n-1\rangle_*}(MU\langle n-1\rangle_*(X), L) \quad \text{and} \quad \bar{E}_2^{p,q} = 0$$

for $q \neq 0$. This implies that

$$\mathrm{Ext}^{p,*}_{MU\langle n-1\rangle_*}(MU\langle n-1\rangle_*(X), L) \cong \mathrm{Ext}^{p,*}_{MU\langle n\rangle_*}(MU\langle n\rangle_*(X), L) = 0$$

for all $p \geqq 2$. So hom $\dim_{MU\langle n-1\rangle_*} MU\langle n-1\rangle_*(X) \leqq 1$,

3.2. Let bu denote the connective BU-spectrum. The Thom map $\mu_C\colon MU \to BU$ is lifted to a morphism

$$\zeta\colon MU \to bu$$

of ring spectra. The usual morphism $\mu\colon MU \to K(Z)$ coincides with the composition $MU \xrightarrow{\zeta} bu \xrightarrow{\eta} K(Z)$. Let us denote by k_* the connective homology K-theory represented by bu.

Using the Stong-Hattori theorem we obtain

Proposition 12. *Let X be a connective CW-spectrum. $\mu\colon MU_*(X) \to H_*(X)$ is an epimorphism if and only if $\eta\colon k_*(X) \to H_*(X)$ is an epimorphism.*

REMARK. Looking carefully at the proof given in [9] we can show that $\mu\colon MU_j(X) \to H_j(X)$ are epimorphisms for all $j \leqq k$ if and only if $\eta\colon k_j(X) \to H_j(X)$ are so for the same j. (Or use Lemma 13).

As generators of the polynomial algebra MU_* we can choose $y_i \in \pi_{2i}(MU)$ such that

$$T_d(y_1) = 1 \quad \text{and} \quad T_d(y_j) = 0 \qquad \text{for } j \geqq 2.$$

Whenever we restrict our interest to the CW-spectra $MU\langle n\rangle$ with $MU\langle n\rangle_* \cong Z[y_1, \cdots, y_n]$, we denote them by $MU_{Td}\langle n\rangle$. The morphism $\zeta\colon MU \to bu$ lifting $\mu_C\colon MU \to BU$ admits a factorization

$$MU \xrightarrow{\mu_{\infty,1}} MU_{Td}\langle 1\rangle \xrightarrow{\lambda_1} bu\,.$$

Since λ_1 induces an isomorphism in the homotopy groups, λ_1 is a homotopy equivalence. Hence

$$MU_{Td}\langle 1\rangle_*(\ ) \cong k_*(\ )\,. \tag{3.6}$$

Then $(\mu_{\infty,1})_*$ may be regarded as the homomorphism ζ.

Making use of Theorem 4, Propositions 11 and 12 and (3.3) we obtain

Theorem 5. *Let X be a connective CW-spectrum and $1 \leqq n < \infty$. The following conditions are equivalent*:

0) $\operatorname{hom\,dim}_{MU_*} MU_*(X) \leqq 1$;

$0)_n$ $\operatorname{hom\,dim}_{MU_{Td}\langle n\rangle_*} MU_{Td}\langle n\rangle_*(X) \leqq 1$;

0)′ $\operatorname{hom\,dim}_{k_*} k_*(X) \leqq 1$;

I) μ: $MU_*(X) \to H_*(X)$ *is an epimorphism*;

I)′ η: $k_*(X) \to H_*(X)$ *is an epimorphism.*

Conner-Smith [6, Theorem 9.1 and Proposition 9.5] proved the following theorem for a finite CW-complex. Therefore we can show by taking the direct limits that it is also true for any CW-spectrum. Neverthless we shall directly prove it along the line of [6].

Theorem 6. *Let X be a CW-spectrum. Then μ_C induces an isomorphism*

$$\tilde{\mu}_C\colon Z_{Td} \underset{MU_{**}}{\otimes} MU_{**}(X) \to K_*(X)$$

and

$$\operatorname{Tor}_{p,*}^{MU_{**}}(Z_{Td}, MU_{**}(X)) = 0 \qquad \textit{for all } p \geqq 1.$$

Proof. Take a partial MU_*-resolution $W \to X \subset Y$ of X. By Proposition 8 we may assume that W is a wedge sum of finite CW-spectra. Consider the following commutative diagram

$$\begin{array}{l} 0 \to \operatorname{Tor}_{1,*}^{MU_{**}}(Z_{Td}, MU_{**}(X)) \to Z_{Td} \underset{MU_{**}}{\otimes} MU_{**+1}(Y) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \to K_{*+1}(Y) \qquad\qquad \to \\ \qquad\qquad\qquad\qquad Z_{Td} \underset{MU_{**}}{\otimes} MU_{**}(W) \to Z_{Td} \underset{MU_{**}}{\otimes} MU_{**}(X) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\quad \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\quad K_*(W) \qquad \to \qquad K_*(X) \qquad \to \qquad 0. \end{array}$$

The vertical maps are all epimorphisms by (2.6), and in particular the center is an isomorphism because of (2.5). Hence the bottom row becomes exact. Of course the upper row is exact. With an application of "four lemma" we see that the right vertical map is an isomorphism. Thus

$$\tilde{\mu}_C\colon Z_{Td} \underset{MU_{**}}{\otimes} MU_{**}(X) \to K_*(X)$$

is an isomorphism for any CW-spectrum X. Since this means that the left vertical map is also an isomorphism, we get

$$\operatorname{Tor}_{1,*}^{MU_{**}}(Z_{Td}, MU_{**}(X)) = 0\,.$$

And a routine discussion involving an induction shows that

$$\operatorname{Tor}_{p,*}^{MU_{**}}(Z_{Td}, MU_{**}(X)) = 0 \qquad \text{for all } p \geqq 1.$$

The following theorem is the extension of [9, Theorem 2] to a connective CW-spectrum.

Theorem 7. *Let X be a connective CW-spectrum. The following conditions are equivalent:*
0) $\quad \hom\dim_{MU_*} MU_*(X) \leqq 2$;
I) $\quad \zeta\colon MU_*(X) \to k_*(X)$ *is an epimorphism*;
II) $\quad \zeta$ *induces an isomorphism* $\bar{\zeta}\colon k_* \underset{MU_*}{\otimes} MU_*(X) \to k_*(X)$;
III) $\quad \mathrm{Tor}_{p,*}^{MU_*}(k_*, MU_*(X))=0$ *for all* $p \geqq 1$;
IV) $\quad \mathrm{Tor}_{p+1,*}^{MU_*}(Z, MU_*(X))=0$ *for all* $p \geqq 1$.

Proof. We prove in the order: IV)$\to$III)$\to$II)$\to$I)$\to$0)$\to$IV). "II)$\to$I)" is trivial, and "III)$\to$II)" and "0)$\to$IV)" follow from Corollary 9 and (3.1).

I)$\to$0): Let $W \to X \subset Y$ be a partial connective MU_*-resolution of X. The surjectivity of $\zeta\colon MU_*(X) \to k_*(X)$ implies that $W \to X \subset Y$ is a partial connective k_*-resolution of X. Remark that $k_*(Y)$ is free abelian. By the aid of Lemma 3, Proposition 12 and Theorem 4 we see that $\hom\dim_{MU_*} MU_*(Y) \leqq 1$, and hence

$$\hom\dim_{MU_*} MU_*(X) \leqq 2\,.$$

IV)$\to$III): The proof is due to [6]. From the exact sequence $0 \to k_{**} \xrightarrow{\cdot(1-x_1)} k_{**} \to Z_{Td} \to 0$ and Theorem 6 we obtain an isomorphism

$$\cdot(1-x_1)\colon \mathrm{Tor}_{p,**}^{MU_{**}}(k_{**}, MU_{**}(X)) \to \mathrm{Tor}_{p,**}^{MU_{**}}(k_{**}, MU_{**}(X))$$

for each $p \geqq 1$. Take any $\alpha \in \mathrm{Tor}_{p,q}^{MU_*}(k_*, MU_*(X))$, $p \geqq 1$. Then there exists $\beta = \{\beta_{q+2i}\} \in \sum_i \mathrm{Tor}_{p,q+2i}^{MU_*}(k_*, MU_*(X))$ such that $(1-x_1)\cdot\beta = \alpha$. Since $\beta_{q-2N} = \beta_{q+2N} = 0$ for large N, $\beta_{q+2N} = x_1^N \cdot \alpha = 0$. However our assumption yields that

$$\cdot x_1\colon \mathrm{Tor}_{p,*}^{MU_*}(k_*, MU_*(X)) \to \mathrm{Tor}_{p,*+2}^{MU_*}(k_*, MU_*(X))$$

is a monomorphism for each $p \geqq 1$. So $\alpha = 0$, i.e.,

$$\mathrm{Tor}_{p,*}^{MU_*}(k_*, MU_*(X)) = 0 \qquad \text{for all } p \geqq 1.$$

3.3. Let X be a connective CW-spectrum and $\{X^p\}$ the skeleton filtration of X. As is easily seen, we have that

$$\begin{aligned} &MU\langle m\rangle_j(X^p) \cong MU\langle m\rangle_j(X) \qquad \text{for } j \leqq p-1 \text{ and } 0 \leqq m \leqq \infty, \text{ and} \\ &H_j(X^p) = 0 \qquad \text{for } j \geqq p+1. \end{aligned} \tag{3.7}$$

Moreover we get that

$$MU\langle 1\rangle_{p+\varepsilon}(X^p) \cong MU\langle 1\rangle_{p+2j+\varepsilon}(X^p) \quad \text{for } j \geqq 0 \text{ and } \varepsilon = 0 \text{ or } -1, \tag{3.8}$$

making use of the exact sequence

$$H_{p+\varepsilon+2j+3}(X^p) \to MU\langle 1\rangle_{p+\varepsilon+2j}(X^p) \xrightarrow{\cdot x_1} MU\langle 1\rangle_{p+\varepsilon+2j+2}(X^p) \to H_{p+\varepsilon+2j+2}(X^p).$$

Under the condition that $n=0$ or 1,

(3.9) *$MU\langle n\rangle_*(X)$ is a (torsion) free abelian group if and only if $MU\langle n\rangle_*(X^p)$ are so for all p.*

Proof. Assume that $MU\langle 1\rangle_*(X)$ is (torsion) free abelian. By means of (3.7) $MU\langle 1\rangle_j(X^p)$ is (torsion) free abelian for $j\leqq p-1$. In the exact sequence $0\to MU\langle 1\rangle_p(X^{p-1})\to MU\langle 1\rangle_p(X^p)\to MU\langle 1\rangle_p(X^p/X^{p-1})$,

$$MU\langle 1\rangle_p(X^{p-1}) \cong MU\langle 1\rangle_{p-2}(X^{p-1}) \cong MU\langle 1\rangle_{p-2}(X)$$

and $MU\langle 1\rangle_p(X^p/X^{p-1})$ is free abelian. So $MU\langle 1\rangle_p(X^p)$ is (torsion) free abelian. Making use of (3.8) again we find that $MU\langle 1\rangle_*(X^p)$ is (torsion) free abelian.

The other cases are evident.

Lemma 13. *Let X be a connective CW-spectrum, $n=0$ or 1, and $n<m\leqq\infty$. Then $(\mu_{m,n})_*\colon MU\langle m\rangle_j(X)\to MU\langle n\rangle_j(X)$ is an epimorphism for each $j\leqq p$ if and only if $(\mu_{m,n})_*\colon MU\langle m\rangle_*(X^p)\to MU\langle n\rangle_*(X^p)$ is an epimorphism.*

Proof. The "if" part is immediate.

The "only if" part: Because of (3.7) $(\mu_{m,n})_*\colon MU\langle m\rangle_j(X^p)\to MU\langle n\rangle_j(X^p)$ is an epimorphism for $j\leqq p-1$. Consider the following commutative diagram

$$\begin{array}{ccccccc}
MU\langle m\rangle_{p+1}(X/X^p) & \to & MU\langle m\rangle_p(X^p) & \to & MU\langle m\rangle_p(X) & \to & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
MU\langle n\rangle_{p+1}(X/X^p) & \to & MU\langle n\rangle_p(X^p) & \to & MU\langle n\rangle_p(X) & \to & 0
\end{array}$$

with exact rows. The right vertical map is an epimorphism by the assumption. And the left one is so as is easily seen. With an application of "four lemma" we see that the central map is an epimorphism.

In the $n=0$ case we recall that $H_i(X^p)=0$ for $i\geqq p+1$. Consequently we obtain that $MU\langle m\rangle_*(X^p)\to H_*(X^p)$ is an epimorphism. In the $n=1$ case we have the commutative square

$$\begin{array}{ccc}
MU\langle m\rangle_{p+\varepsilon}(X^p) & \xrightarrow{\cdot x_1^j} & MU\langle m\rangle_{p+2j+\varepsilon}(X^p)\\
\downarrow & & \downarrow\\
MU\langle 1\rangle_{p+\varepsilon}(X^p) & \xrightarrow{\cdot x_1^j} & MU\langle 1\rangle_{p+2j+\varepsilon}(X^p)
\end{array}$$

where $\varepsilon=0$ or -1 and $j\geqq 1$. The left vertical map is an epimorphism and the bottom horizontal map is an isomorphism by (3.6). Therefore we get that $MU\langle m\rangle_*(X^p)\rightarrow MU\langle 1\rangle_*(X^p)$ is an epimorphism.

Combining (3.9) with Proposition 6 and Lemma 13 with Theorems 4 and 7 we obtain the following theorem (cf., [8]).

Theorem 8. *Let X be a connective CW-spectrum and $n=0$, 1 or 2. Then* hom $\dim_{MU_*}MU_*(X)\leqq n$ *if and only if* hom $\dim_{MU_*}MU_*(X^p)\leqq n$ *for all p.*

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References

[1] J.F. Adams: *Lectures on generalized cohomology theories,* Lecture notes in Math. 99, Springer Verlag, 1969, 1–138.

[2] S. Araki and Z. Yosimura: *A spectral sequence associated with a cohomology theory of infinite* CW-*complexes,* Osaka J. Math. **9** (1972), 351–365.

[3] N.A. Baas: *On bordism theory of manifolds with singularities,* Preprint series 31, Aarhus Univ. (1969/70).

[4] J.M. Boardman: *Stable homotopy theory,* mimeographed notes, Warwick Univ., 1965.

[5] E.H. Brown: *Cohomology theories,* Ann. of Math. **75** (1962), 467–484 and **78** (1963), 201.

[6] P.F. Conner and L. Smith: *On the complex bordism of finite complexes,* Inst. Hautes Etudes Sci. Publ. Math. **37** (1970), 117–221.

[7] A. Dold: *Relations between ordinary and extraordinary homology,* Colloq. on Algebraic Topology, Aarhus Univ. (1962), 2–9.

[8] D.C. Johnson: *Skeleta of complexes with low* MU$_*$ *projective dimension,* Proc. Amer. Math. Soc. **32** (1972), 599–604.

[9] D.C. Johnson and L. Smith: *On the relation of complex cobordism to connective* K-*theory,* Proc. Cambridge Philos. Soc. **70** (1971), 19–22.

[10] P.S. Landweber: *Complex bordism of classifying spaces,* Proc. Amer. Math. Soc. **27** (1971), 175–179.

[11] R. Vogt: *Boardman's stable homotopy category,* Lecture notes series 21, Aarhus Univ., 1970.

[12] Z. Yosimura: *On cohomology theories of infinite* CW-*complexes,* II, Publ. Res. Inst. Math. Sci. **8** (1972/73), 483-508.