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# RING STRUCTURES OF $K_U$-COHOMOLOGIES OF DOLD MANIFOLDS

Michikazu FUJII

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## Introduction

In [4] we determined the $K_U$-cohomologies of the Dold manifold $D(m, n)$ additively. But we could not determine the ring structures of them, because we could not find a generator of the 2-torsion part in $\tilde{K}_U^1(D(m, 2r+1))$. The purpose of this paper is to determine the ring structures of $K_U$-cohomologies of the Dold manifold $D(m, n)$. The stunted Dold manifold plays an important role in the present discussions.

Let $S^k$, $k\geqq 0$, denote the unit $k$-sphere in $R^{k+1}$, each point of which is represented by a sequence $(x_0,\cdots, x_k)$ of real numbers $x_i$ with $\sum x_i^2=1$, and $S^{2l+1}$, $l\geqq 0$, denote the unit $(2l+1)$-sphere in $C^{l+1}$, each point of which is represented by a sequence $(z_0,\cdots, z_l)$ of complex numbers $z_i$ with $\sum |z_i|^2=1$. Then the Dold manifold $D(k, l)$ is defined as the quotient space of the product space $S^k\times S^{2l+1}$ under the identification $(x, z)=(-x, \overline{\lambda z})$for $x\in S^k$, $z\in S^{2l+1}\subset C^{l+1}$ and all $\lambda\in C$ with $|\lambda|=1$. Let $[x_0,\cdots, x_k, z_0,\cdots, z_l]\in D(k, l)$ denote the class of $(x_0,\cdots, x_k, z_0,\cdots,z_l)\in S^k\times S^{2l+1}$. The manifold $D(k', l')$, $k'\leqq k$ and $l'\leqq l$, is naturally imbedded in $D(k, l)$ by identifying $[x_0,\cdots, x_{k'}, z_0,\cdots, z_{l'}]$ with $[x_0,\cdots, x_{k'}\ 0,\cdots, 0, z_0,\cdots, z_{l'}, 0,\cdots, 0]$.

Denote by $\xi$ the canonical real line bundle over the real projective $k$-space $RP(k)$, and $\xi_1=p^!\xi$ the induced bundle of $\xi$ by the projection $p:D(k, l)\rightarrow RP(k)$; and denote by $\eta_1$ the canonical real 2-plane bundle over $D(k, l)$ (cf. [4], § 2).

**Theorem 1.** *The Thom space $T(m\xi_1\oplus n\eta_1)$ and the stunted Dold manifold $D(k+m, l+n)/D(m-1, l+n)\cup D(k+m, n-1)$ are homeomorphic, where $m\xi_1$ and $n\eta_1$ are the m-fold and n-fold sum of $\xi_1$ and $\eta_1$ respectively.*

From this theorem we have the following

**Proposition 2.** *We have the following homeomorphisms*:

i) $h: D(k, n)/D(k, n-1) \approx S^n \wedge (RP(n+k)/RP(n-1))$,

ii) $D(m, l)/D(m-1, l) \approx S^m \wedge CP(l)^+$,

*where* $S^n \wedge (RP(n+k)/RP(n-1))$ *is the n-fold suspension of the stunted real projective space, and* $CP\,(l)^+$ *is the disjoint union of the complex projective l-space* $CP(l)$ *and a point.*

Let $g$ is the generator of $\tilde{K}_U^0(S^2)$ given by the reduced Hopf bundle and $g^{[r]}$ is the generator of $\tilde{K}_U^0(S^{2r})$ given by the external product $g \wedge \cdots \wedge g$. Also, let $\nu^{(r+1)}$ is the generator of $\tilde{K}_U^0(RP(2r+s)/RP(2r))$ (cf. [1], Theorem 7.3), then $g^{[r]}\nu^{(r+1)}$ is the generator of $\tilde{K}_U^{-2r}(RP(2r+s)/RP(2r))$. Now, using Proposition 2, i), we can define a generator $\omega$ of the 2–torsion part in $\tilde{K}_U^1(D(m, 2r+1))$ as follows: $\omega = \pi^! h^! g^{[r]}\nu^{(r+1)}$, where $\pi$ is the projection $D(m, 2r+1) \to D(m, 2r+1)/D(m, 2r)$, and determine the multiplicative structures of $\tilde{K}_U^*(D(m, n))$, namely

**Theorem 3.** *As for the ring structures of* $\tilde{K}_U^*(D(m, n))$ *we have the following relations:*

a) $\gamma^2 = g'^2 = \beta^2 = g'\beta = 0, \quad g'\alpha = 2\beta$,

b) $\alpha^{r+1} = 0, \quad \gamma\alpha^r = 0$ (*for* $n=2r$) or $\gamma\alpha^{r+1} = 0$ (*for* $n=2r+1$),
$\beta\alpha^r = 0$ (*for* $n=2r$) or $\beta\alpha^r = 2^t\omega$ (*for* $n=2r+1$),

c) $\alpha\nu_1 = \gamma\nu_1 = g'\nu_1 = \beta\nu_1 = 0, \quad \alpha\omega = \gamma\omega = g'\omega = \beta\omega = 0$,

d) $\nu_1^2 = -2\nu_1, \quad \omega\nu_1 = -2\omega, \quad \omega^2 = 0$,

*where* $\alpha, \gamma, g', \beta$ *and* $\nu_1$ *are the generators given in* [4], *Theorem* (3.14), *and* $\omega$ *is the generator of the* 2–*torsion part in* $\tilde{K}_U^1(D(m, 2r+1))$ *given by the above formula.*

## 1. Proof of Theorem 1

The total space $E(m\xi_1 \oplus n\eta_1)$ of $m\xi_1 \oplus n\eta_1$ is the quotient space of the product space $S^k \times S^{2l+1} \times R^m \times C^n$ under the identification $((x, z), (u, v)) = ((-x, \overline{\lambda z}), (-u, \overline{\lambda v}))$ for $x \in S^k$, $z \in S^{2l+1} \subset C^{l+1}$, $u \in R^m$, $v \in C^n$ and all $\lambda \in C$ with $|\lambda| = 1$. Moreover, the associated unit disk bundle $D(m\xi_1 \oplus n\eta_1)$ is homeomorphic to the quotient space of the product apace $S^k \times S^{2l+1} \times D^m \times D^{2n}$ under the identification $((x, z), (u, v)) = ((-x, \overline{\lambda z}), (-u, \overline{\lambda v}))$, where $x \in S^k$, $z \in S^{2l+1} \subset C^{l+1}$, $u \in D^m$, $v \in D^{2n} \subset C^n$ and $\lambda$ is as above. Let $[(x, z), (u, v)]$ denote the class of $((x, z), (u, v))$ in $D(m\xi_1 \oplus n\eta_1)$. Then $[(x, z), (u, v)]$ is an element of the associated unit sphere bundle $S(m\xi_1 \oplus n\eta_1)$ if and only if $||u|| = 1$ or $||v|| = 1$.

We define a map

$$f: S^k \times S^{2l+1} \times D^m \times D^{2n} \to S^{k+m} \times S^{2l+2n+1}$$

by

$$f((x, z), (u, v)) = ((u, \sqrt{1-\|u\|^2}x), (v, \sqrt{1-\|v\|^2}z)).$$

Since

$$f((-x, \overline{\lambda z}), (-u, \overline{\lambda v})) = (-(u, \sqrt{1-\|u\|^2}x), \overline{\lambda(v, \sqrt{1-\|v\|^2}z)}),$$

the map $f$ defines a map

$$g\colon D(m\,\xi_1 \oplus n\eta_1) \to D(k+m, l+n)$$

such that $g(S(m\xi_1 \oplus n\eta_1)) \subset D(m-1, l+n) \cup D(k+m, n-1)$. The map

$$g\colon D(m\xi_1 \oplus n\eta_1) - S(m\xi_1 \oplus n\eta_1) \to D(k+m, l+n) - D(m-1, l+n) \cup D(k+m, n-1)$$

is a homeomorphism. Therefore, the map $g$ defines a quotient map

$$h\colon T(m\,\xi_1 \oplus n\eta_1) \to D(k+m, l+n)/D(m-1, l+n) \cup D(k+m, n-1)$$

which is a homeomorphism.

## 2. Proof of Proposition 2

i). By taking $m=l=0$ in Theorem 1, we have the homeomorphism

$$T(n\eta_1) \approx D(k, n)/D(k, n-1).$$

Since $\eta_1$ over $D(k, 0)$ is the 2-plane bundle $1 \oplus \xi_1$ (cf. [4], Theorem (2.2)), we have

$$T(n\eta_1) = T(n \oplus n\xi_1) \approx S^n \wedge T(n\xi_1).$$

If we identify $D(k, 0)$ with $RP(k)$, the line bundle $\xi_1$ is the canonical line bundle $\xi$ over $RP(k)$. Therefore we have the homeomorphism

$$T(n\xi_1) \approx RP(n+k)/RP(n-1).$$

Combining the above three homeomorphisms, we have the homeomorphism

$$h\colon D(k, n)/D(k, n-1) \approx S^n \wedge (RP(n+k)/RP(n-1)).$$

ii). By taking $n=k=0$ in Theorem 1, we have the homeomorphism

$$T(m\xi_1) \approx D(m, l)/D(m-1, l).$$

Since $\xi_1$ over $D(0, l)$ is the trivial line bundle, if we identify $D(0, l)$ with $CP(l)$, we have

$$T(m\xi_1) = T(m) \approx S^m \wedge CP(l)^+.$$

Therefore we have the homeomorphism

$$D(m, l)/D(m-1, l) \approx S^m \wedge CP(l)^+ .$$

## 3. Proof of Theorem 3

Firstly we show that $\omega$ is a generator of the 2-torsion part in $\tilde{K}_U^1(D(m, 2r+1))$. Consider the exact sequence of the pair $(D(2t, 2r+1), D(2t, 2r))$

$$\tilde{K}_U^1(D(2t, 2r+1)/ D(2t, 2r)) \to \tilde{K}_U^1(D(2t, 2r+1)) \to \tilde{K}_U^1(D(2t, 2r)) .$$

According to [4], Theorem (3.14), we have $\tilde{K}_U^1(D(2t, 2r))=0$ and $\tilde{K}_U^1(D(2t, 2r+1))=Z_{2^t}$. Also, in virtue of Proposition 2, i), and [1], Theorem 7.3, we have

$$\tilde{K}_U^1(D(2t, 2r+1)/D(2t, 2r)) \cong \tilde{K}_U^{-2r}(RP(2t+2r+1)/RP(2r)) \cong Z_{2^t} ,$$

whose generator is $g^{[r]}\nu^{(r+1)}$. Therefore, $\omega$ is the generator of $\tilde{K}_U^1(D(2t, 2r+1))$.

Using the exact sequence of the pair $(D(2t+1, 2r+1), D(2t+1, 2r))$

$$\tilde{K}_U^1(D(2t+1, 2r+1)/D(2t+1, 2r)) \to \tilde{K}_U^1(D(2t+1, 2r+1)) \to \tilde{K}_U^1(D(2t+1, 2r)),$$

it is easy to see that $\omega$ is the generator of the 2-torsion part $Z_{2^{t+1}}$ of $\tilde{K}_U^1(D(2t+1, 2r+1))$ in the same way as the above case.

Next we show the relations. Since $(g^{[k]})^2=0$ in $\tilde{K}_U^0(S^{2k})$, the relations $\gamma^2=g'^2=\beta^2=g'\beta=0$ and $\omega^2=0$ follow from $g'=(sf)^! g^{[t+1]}$, $\beta=(sf)^! g^{[t+1]}\mu$, $\gamma=f^! g^{[t]}\mu$ and $\omega=\pi^! h^! g^{[r]}\nu^{(r+1)}$. The relation $\nu_1^2=-2\nu_1$ follows from the relation $\nu^2=-2\nu$ in $\tilde{K}_U^0(RP(m))$.

Since $\tilde{K}_U^1(D(2t+1, 2r))$ has no torsion, Chern character $ch$: $\tilde{K}_U^1(D(2t+1, 2r))\to H^*(D(2t+1, 2r); Q)$ is monomorphic. Therefore the relations $g'\alpha=2\beta$ and $\beta\alpha^r=0$ follow from

$$\mathrm{ch}\, g'\alpha = 2b(a/2!+\cdots+a^r/(2r)!) = 2\,\mathrm{ch}\,\beta \quad \text{and} \quad \mathrm{ch}\,\beta\alpha^r = 0$$

respectively. The relation $g'\nu_1=\beta\nu_1=0$ is trivial for $n=2r$.

In case of $n=2r-1$, since the elements $\alpha$, $\nu_1$, $g'$ and $\beta$ of $\tilde{K}_U^*(D(2t+1, 2r-1))$ are induced from the elements $\alpha$, $\nu_1$, $g'$ and $\beta$ of $\tilde{K}_U^*(D(2t+1, 2r))$ by the inclusion map $i$: $D(2t+1, 2r-1)\subset D(2t+1, 2r)$, multiplicativity of the homomorphism $i^!$ shows the relations $g'\nu_1=\beta\nu_1=0$ and $g'\alpha=2\beta$ for $n=2r-1$. Also, the element $\beta\alpha^{r-1}\in\tilde{K}_U^1(D(2t+1, 2r-1))$ is the image of $\beta\alpha^{r-1}\in\tilde{K}_U^1(D(2t+1, 2r))$ by $i^!$. On the other hand, consider the exact sequence

$$\tilde{K}_U^1(D(2t+1, 2r)) \xrightarrow{i^!} \tilde{K}_U^1(D(2t+1, 2r-1)) \to \tilde{K}_U^2(D(2t+1, 2r)/D(2t+1, 2r-1)) .$$

In virtue of Proposition 2, i), and [1], Theorem 7.3, we have

$$\tilde{K}_U^2(D(2t+1, 2r)/D(2t+1, 2r-1))\cong\tilde{K}_U^{-2r+2}(RP(2t+2r+1)/RP(2r-1))\cong Z+Z_{2^t} ,$$

so that we have $i^!\beta\alpha^{r-1}=2^t\omega$ for $\beta\alpha^{r-1}\in\tilde{K}_U^1(D(2t+1, 2r)$. Therefore we have the relation $\beta\alpha^{r-1}=2^t\omega$ in $\tilde{K}_U^1(D(2t+1, 2r-1))$.

Since $\mathrm{ch}\,\alpha^{r+1}=0$ and $\mathrm{ch}\,\gamma\alpha^r=0$ (for $n=2r$) ($\mathrm{ch}\,\gamma\alpha^{r+1}=0$ (for $n=2r+1$)), the elements $\alpha^{r+1}$ and $\gamma\alpha^r$ (for $n=2r$) ($\gamma\alpha^{r+1}$ (for $n=2r+1$)) lie in $p^!\tilde{K}_U^0(RP(m))$. Therefore the relation $r^!\alpha=0$ implies $\alpha^{r+1}=p^!r^!\alpha^{r+1}=0$ and $\gamma\alpha^r=p^!r^!(\gamma\alpha^r)$ $=p^!((r^!\gamma)(r^!\alpha^r))=0$ (for $n=2r$) ($\gamma\alpha^{r+1}=p^!r^!(\gamma\alpha^{r+1})=0$ (for $n=2r+1$)), where $r$ is the cross section defined in [4], Lemma (3.4).

Since $\gamma\nu_1\in p^!\tilde{K}_U^0(RP(2t))$ and $r^!\gamma=0$, we have $\gamma\nu_1=p^!r^!(\gamma\nu_1)=p^!((r^!\gamma)$ $(r^!\nu_1))=0$. The relation $\alpha\nu_1=0$ was showed in [4].

The elements $g'\omega$ and $\beta\omega$ lie in $p^!\tilde{K}_U^0(RP(2t+1))$. Since the diagram

$$\begin{tikzcd}
\tilde{K}_U^1(D(2t+1, 2r+1)/D(2t+1,2r)) \arrow[r, "\pi^!"] \arrow[d, shift right, "r^!"'] & \tilde{K}_U^1(D(2t+1, 2r+1)) \arrow[d, shift right, "r^!"'] \\
\tilde{K}_U^1(*) \arrow[r, "\pi^!"] \arrow[u, shift right, "p^!"'] & \tilde{K}_U^1(RP(2t+1)) \arrow[u, shift right, "p^!"']
\end{tikzcd}$$

is commutative, we have $r^!\omega=\pi^!r^!(h^!g^{[r]}\nu^{(r+1)})=0$. Therefore we have $g'\omega$ $=p^!r^!(g'\omega)=p^!((r^!g')(r^!\omega))=0$ and $\beta\omega=p^!r^!(\beta\omega)=p^!((r^!\beta)(r^!\omega))=0$.

Finally we show the relations $\omega\alpha=0$, $\omega\gamma=0$ and $\omega\nu_1=-2\omega$ in $\tilde{K}_U^1(D(m, 2r+1))$. For simplicity we put $Y_1=RP(m+2r+1)$, $Y_2=RP(2r)$, $X_1=D(m, 2r+1)$, $X_2=D(m, 2r)$ and $Z=D(m+2r+1, 2r+1)$.

**Lemma 1.** *We have the homotopy-commutative diagram*

$$\begin{tikzcd}
X_1\wedge X_1 \arrow[r, "\pi\wedge 1"] & (X_1/X_2)\wedge X_1 \arrow[r, "h\wedge 1", "\approx"'] & S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1 \arrow[r, "1\wedge 1\wedge i"] & S^{2r+1}\wedge(Y_1/Y_2)\wedge Z \\
 & & S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m) \arrow[u, "1\wedge 1\wedge r"'] & S^{2r+1}\wedge(Y_1/Y_2)\wedge Y_1 \arrow[u, "1\wedge 1\wedge r"'] \\
X_1 \arrow[r, "\pi"] \arrow[uu, "d_1"] & X_1/X_2 \arrow[r, "h", "\approx"'] \arrow[uu, "\bar{d}_1"] & S^{2r+1}\wedge(Y_1/Y_2) \arrow[r, equal] \arrow[u, "q"'] & S^{2r+1}\wedge(Y_1/Y_2), \arrow[u, "1\wedge d_2"']
\end{tikzcd}$$

*where $i$ is the inclusion map $X_1\subset Z$, $h$ is the homeomorphism of Proposition* 2, i), *$d_1$ is the diagonal map, $\bar{d}_1$ and $d_2$ are the maps induced by the diagonal maps, $r$ is the cross section of* [4], *Lemma* (3.4), *and $q$ is the map given by*

$$q([a]\wedge[b, \sqrt{1-||b||^2}x]) = [a]\wedge[b, \sqrt{1-||b||^2}x]\wedge[x]\,.$$

Proof. It is sufficient to show the followings:

i) the maps $u=(1\wedge 1\wedge i)\circ(h\wedge 1)\circ\bar{d}_1\circ h^{-1}$ and $v=(1\wedge 1\wedge r)\circ(1\wedge d_2)$ are homotopic,

ii) the map $q$ is well defined and the maps $\bar{u}=(h\wedge 1)\circ\bar{d}_1\circ h^{-1}$ and $w=(1\wedge 1\wedge r)\circ q$ are homotopic.

For this purpose we investigate the details of the homeomorphism $h$. If we identify $D(m,\ 0)$ with $RP(m)$, the canonical real 2-plane bundle $\eta_1$ over $D(m,\ 0)$ is the 2-plane bundle $1\oplus\xi$ over $RP(m)$. The homeomorphism

$$h_1\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)$$

is induced from the map

$$\begin{aligned} f_1\colon (S^m\times D^{2r+1}\times D^{2r+1},\ S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ \to (D^{2r+1}\times S^{m+2r+1},\ S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r}) \end{aligned}$$

given by

$$f_1(x,\ a,\ b) = (a,\ (b,\ \sqrt{1-\|b\|^2}x)),$$

and the homeomorphism

$$h_2\colon T((2r+1)\eta_1) \to X_1/X_2$$

is induced from the map

$$\begin{aligned} f_2\colon (S^m\times S^1\times D^{2(2r+1)},\ S^m\times S^1\times S^{2(2r)+1})\\ \to (S^m\times S^{2(2r+1)+1},\ S^m\times S^{2(2r)+1}) \end{aligned}$$

given by

$$f_2(x,\ z,\ v) = (x,\ (v,\ \sqrt{1-\|v\|^2}z)),$$

where $D^{2r+1}$ and $D^{2(2r+1)}$ are unit disks of $R^{2r+1}$ and $C^{2r+1}$ respectively.

We define a map

$$\begin{aligned} \phi\colon (S^m\times D^{2r+1}\times D^{2r+1},\ S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ \to (S^m\times S^1\times D^{2(2r+1)},\ S^m\times S^1\times S^{2(2r)+1}) \end{aligned}$$

by

$$\phi(x,\ a,\ b) = (x,\ 1,\ \theta(a,\ b)),$$

where $\theta$ is the standard homeomorphism $D^{2r+1}\times D^{2r+1}\to D^{2(2r+1)}$ given by

$$\theta(a,\ b) = \max(\|a\|,\ \|b\|)(\|a\|^2+\|b\|^2)^{-1/2}(a+bi).$$

Since

$$\phi(-x,\ a,\ -b) = (-x,\ 1,\ \overline{\theta(a,\ b)}),$$

the map $\phi$ defines a quotient map

$$\psi\colon T((2r+1)\oplus(2r+1)\xi) \to T((2r+1)\eta_1)$$

which is a homeomorphism. The homeomorphism $h$ is the composition $h_1\circ\psi^{-1}\circ h_2^{-1}$ of the three homeomorphisms.

Now, the homeomorphism $h$ is given by

$$h^{-1}([a]\wedge[b, \sqrt{1-\|b\|^2}x]) = [x, (\theta(a, b), \sqrt{1-\|\theta(a, b)\|^2})] .$$

Therefore we have

$$u([a]\wedge[b, \sqrt{1-\|b\|^2}x]) = [a]\wedge[b, \sqrt{1-\|b\|^2}x]\wedge[(x, 0), (\theta(a, b), \sqrt{1-\|\theta(a, b)\|^2})]$$

and

$$v([a]\wedge[b, \sqrt{1-\|b\|^2}x]) = [a]\wedge[b, \sqrt{1-\|b\|^2}x]\wedge[(b, \sqrt{1-\|b\|^2}x), (1, 0)] .$$

We define two maps $F_t^1$ and $F_t^2$, for $0\leq t\leq 1$,

$$\begin{aligned}(S^m\times D^{2r+1}\times D^{2r+1}, S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ \to (D^{2r+1}\times S^{m+2r+1}\times(S^{m+2r+1}\times S^{2(2r+1)+1}), (S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r})\\ \times(S^{m+2r+1}\times S^{2(2r+1)+1}))\end{aligned}$$

by

$$F_t^1(x, a, b) = (a, (b, \sqrt{1-\|b\|^2}x))\times((x, 0), (t\theta(a, b), \sqrt{1-\|t\theta(a, b)\|^2}))$$

and

$$F_t^2(x, a, b) = (a, (b, \sqrt{1-\|b\|^2}x)\times((tb,\sqrt{1-\|tb\|^2}x), (1, 0)) .$$

Then the maps $F_t^1$ and $F_t^2$ are compatible with the identification, so that they define maps $G_t^1$ and $G_t^2$ respectively

$$\begin{aligned}(D((2r+1)\oplus(2r+1)\xi), S((2r+1)\oplus(2r+1)\xi))\\ \to (D^{2r+1}\times RP(m+2r+1)\times D(m+2r+1, 2r+1), (S^{2r}\times RP(m+2r+1)\\ \cup D^{2r+1}\times RP(2r))\times D(m+2r+1, 2r+1)) .\end{aligned}$$

Therefore, they define quotient maps $H_t^1$ and $H_t^2$ respectively

$$T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)\wedge Z ,$$

and we have

$$u = H_1^1\circ h_1^{-1} \quad\text{and}\quad v = H_1^2\circ h_1^{-1} .$$

Since the maps $F_0^1$ and $F_0^2$ are homotopic, the maps $H_0^1$ and $H_0^2$ are homotopic. Therefore the maps $H_1^1$ and $H_1^2$ are homotopic, so that the maps $u$ and $v$ are homotopic. This shows i).

The map $q$ is defined as follows: We define a map

$$\begin{aligned}f\colon (S^m\times D^{2r+1}\times D^{2r+1}, S^m\times(S^{2r}\times D^{2r+1}\cup D^{2r+1}\times S^{2r}))\\ \to (D^{2r+1}\times S^{m+2r+1}\times S^m, (S^{2r}\times S^{m+2r+1}\cup D^{2r+1}\times S^{2r})\times S^m)\end{aligned}$$

by

$$f(x, a, b) = (a, (b, \sqrt{1-\|b\|^2}x), x)\,.$$

Since

$$f(-x, a, -b) = (a, (-b, -\sqrt{1-\|b\|^2}x), -x)\,,$$

the map $f$ defines a quotient map

$$g\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m)\,,$$

and we have $q=g\circ h_1^{-1}$.

Now, we can define a map, for $0\leq t\leq 1$,

$$H_t\colon T((2r+1)\oplus(2r+1)\xi) \to S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1$$

by

$$H_t([x, a, b]) = [a]\wedge[b, \sqrt{1-\|b\|^2}x]\wedge[x, (t\theta(a, b), \sqrt{1-\|t\theta(a, b)\|^2}]\,,$$

and we have $\bar{u}=H_1\circ h_1^{-1}$. Since the maps $w$ and $H_0\circ h_1^{-1}$ are homotopic, the maps $\bar{u}$ and $w$ are homotopic. This shows ii).

This completes the proof of Lemma 1.

**Lemma 2.** *We have the commutative diagram*

$$\begin{tikzcd}
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2))\otimes\tilde{K}^0_U(Z) \arrow[r, "1\otimes i^!"] \arrow[d] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2))\otimes\tilde{K}^0_U(X_1) \arrow[r, "h^!\otimes 1"] \arrow[d] & {} \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge Z) \arrow[r, "(1\wedge i)^!"] \arrow[d, "(1\wedge r)^!"] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge X_1) \arrow[r, "(h\wedge 1)^!"] \arrow[d, "(1\wedge r)^!"] & {} \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge Y_1) \arrow[d, "(1\wedge d_2)^!"] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)\wedge RP(m)) \arrow[d, "q^!"] & \\
\tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)) \arrow[r, equal] & \tilde{K}^1_U(S^{2r+1}\wedge(Y_1/Y_2)) \arrow[r, "h^!"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{K}^1_U(X_1/X_2)\otimes\tilde{K}^0_U(X_1) \arrow[r, "\pi^!\otimes 1"] \arrow[d] & \tilde{K}^1_U(X_1)\otimes\tilde{K}^0_U(X_1) \arrow[d] \\
\tilde{K}^1_U((X_1/X_2)\wedge X_1) \arrow[r, "(\pi\wedge 1)^!"] \arrow[d, "\bar{d}_1^!"] & \tilde{K}^1_U(X_1\wedge X_1) \arrow[d, "d_1^!"] \\
\tilde{K}^1_U(X_1/X_2) \arrow[r, "\pi^!"] & \tilde{K}^1_U(X_1)\,.
\end{tikzcd}$$

Proof. It follows from Lemma 1 by naturality.

**Proposition.** *We have the relations* $\omega\alpha=0$, $\omega\gamma=0$ *and* $\omega\nu_1=-2\omega$ *in* $\tilde{K}_U^1(D(m, 2r+1))$.

Proof. Since $r^! \alpha = r^! \gamma = 0$ in $\tilde{K}_U^0(RP(m))$, we have $q^!(1\wedge r)^!(g^{[r]}\nu^{(r+1)}\wedge\alpha) = q^!(g^{[r]}\nu^{(r+1)}\wedge r^!\alpha)=0$ and $q^!(1\wedge r)^!(g^{[r]}\nu^{(r+1)}\wedge\gamma)=q^!(g^{[r]}\nu^{(r+1)}\wedge r^!\gamma)=0$ in $\tilde{K}_U^1(S^{2r+1}\wedge(Y_1/Y_2))$. In virtue of definition of $\omega$ and Lemma 2, these show $\omega\alpha=0$ and $\omega\gamma=0$ in $\tilde{K}_U^1(X_1)$.

Since the element $\nu_1$ of $\tilde{K}_U^0(X_1)$ is induced from the element $\nu_1$ of $\tilde{K}_U^0(Z)$ by the inclusion map $X_1\subset Z$, in order to show the relation $\omega\nu_1=-2\omega$ in $\tilde{K}_U^1(X_1)$, in virtue of definition of $\omega$ and Lemma 2, it is sufficient to show that we have the relation $\nu^{(r+1)}\cdot r^!\nu_1=-2\nu^{(r+1)}$ in $\tilde{K}_U^0(Y_1/Y_2)$ for $\nu_1$ of $\tilde{K}_U^0(Z)$.

Since $r^!\nu_1=\nu$ in $\tilde{K}_U^0(Y_1)$, we have the relation $\nu^{r+1}\cdot r^!\nu_1=-2\nu^{r+1}$ in $\tilde{K}_U^0(Y_1)$. The homomorphism, induced by the projection $j$: $Y_1\to Y_1/Y_2$,

$$j^!: \tilde{K}_U^0(Y_1/Y_2)\to\tilde{K}_U^0(Y_1)$$

is monomorphism, so that we have $\nu^{(r+1)}\cdot r^!\nu_1=-2\nu^{(r+1)}$ in $\tilde{K}_U^0(Y_1/Y_2)$. The proof is complete.

This completes the proof of Theorem 3.

OSAKA CITY UNIVERSITY

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