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DISCONTINUOUS SUBGROUPS OF EXTENSIONS OF SEMI-SIMPLE LIE GROUPS

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Introduction

Let G be a semi-simple Lie group acting as a group of linear transformations on the vector space V, K a maximal compact subgroup of G, Γ a discrete subgroup of normalizing a lattice L in V and such that $\Gamma \setminus G$ has finite invariant measure. In this paper we investigate deformations of and cohomology groups attached to $\Gamma \cdot L \subset G \cdot V$, where \cdot donotes semi-direct product. In fact, we give a local description of the space of homomorphisms of $\Gamma \cdot L$ into $G \cdot V$ topologized by compact-open topology, and compute $H^1(\Gamma \cdot L, Ad)$ in certain cases. We also introduce the notion of $\Gamma \cdot L$ -invariant form à la Matsushima-Murakami and prove a type decomposition theorem for harmonic forms on $G \cdot V/K$. A special case of this theorem was first proved by Kuga [6].

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1. Deformations of $\Gamma \cdot L$

Let *H* be a Lie group and Λ discrete subgroup of *H*. By a *deformation* of Λ in *H* we mean a family r_t (*t* ranging over an open interval containing 0) of homomorphisms of Λ into *H* depending in a C^{∞} fashion on *t* and such that r_0 is the canonical injection of Λ into *H*. Let $R(\lambda)$ be the tangent vector to $r_t(\lambda)$ at t=0. Then $Z: \lambda \to R(\lambda)\lambda^{-1}$ is a crossed homomorphism of Λ into the Lie algebra \mathfrak{D} of *H* for the adjoint action of Λ on \mathfrak{D} . (\mathfrak{D} is identified with the tangent space to *H* at *e*.) In fact, differentiating the relation $r_t(\lambda \cdot \lambda') = r_t(\lambda)r_t(\lambda')$ we obtain $R(\lambda \cdot \lambda') = R(\lambda)\lambda' + \lambda R(\lambda')$. Upon right multiplication by $\lambda'^{-1}\lambda^{-1}$ we get:

$$R(\lambda \cdot \lambda')\lambda'^{-1}\lambda^{-1} = R(\lambda)\lambda^{-1} + \lambda(R(\lambda')\lambda'^{-1})\lambda^{-1}$$

Since $R(\lambda)\lambda^{-1} \in \mathfrak{Y}$ we have proved the assertion. The mapping $Z: \lambda \to R(\lambda)\lambda^{-1}$ is called the *crossed homomorphism tangent to* the deformation r_t of Λ .

Two deformations r_t and s_t of Λ are *equivalent* if there is a smooth curve h_t in H such that $r_t(\lambda) = h_t s_t(\lambda) h_t^{-1}$ for all $\lambda \in \Lambda$ and sufficiently small t.

Lemma 1. The crossed homomorphisms tangent to equivalent deformations differ by principal homomorphisms. Hence there is a natural map from equivalence classes of deformations of Λ into $H^{1}(\Lambda, Ad)$.

We call $H_0^1(\Lambda, Ad)$ the image of this map.

Proof. Suppose $r_t(\lambda) = h_t s_t(\lambda) h_t^{-1}$. Then upon differentiating we obtain

$$R(\lambda) = P \cdot \lambda + S(\lambda) - \lambda \cdot p$$

hence $R(\lambda)\lambda^{-1} = P - Ad(\lambda)P + S(\lambda)\lambda^{-1}$ where P is the tangent vector to h_t at e. q.e.d.

Let G be a connected semi-simple real Lie group without compact factors, ρ a representation of G on the finite dimensional real vector space V, Γ a finitely generated discrete subgroup of G such that G/Γ has finite invariant measure, L a lattice in V normalized by Γ , \mathfrak{G} and \mathfrak{V} Lie algebras of G and V, respectively. Clearly $\Gamma \cdot L$ is a discrete subgroup of the semi-direct product $G \cdot V$ of G and V defined via ρ .

Theorem 1. Under the assumption $H^1(\Gamma, V) = H^1(\Gamma, G) = 0$, every equivalence class of deformations of $\Gamma \cdot L$ in $G \cdot V$ is represented by a deformation of the form $s_t(\gamma, l) = (\gamma, l + S_t(l))$ where S_t is an element of $Z(\rho(G))$ depending differentially on $t \in (-\alpha, \alpha)$ and $S_0 = 0$, where $Z(\rho(G))$ is the centralizer of $\rho(G)$ in the full matrix algebra over V. In this way, one has a bijection between germs around 0 of C^{∞} maps $S: (-\alpha, \alpha) \rightarrow Z(\rho(G))$ such that $S_0 = 0$ and equivalence classes of deformations of $\Gamma \cdot L$ in $G \cdot V$.

Proof. We begin with a lemma

Lemma 2. Let r_t be a deformation of $\Gamma \cdot L$ in $G \cdot V$. Then $r_t(L) \subset V$ for t sufficiently small.

Proof of Lemma 2. Since $H^1(\Gamma, \mathfrak{G}) = H^1(\Gamma, V) = 0$ there is a curve (g_t, v_t) in $G \cdot V$ such that $r_t(\gamma) = (g_t, v_t)\gamma(g_t, v_t)^{-1}$ for every $\gamma \in \Gamma$ and sufficiently small t [12]. Put $s_t(\gamma, l) = (g_t, v_t)^{-1}r_t(\gamma, l)(g_t, v_t)$. Then one has $s_t(\gamma) = \gamma$. We can write

$$s_t(l) = (s_s^1(l), l+s_t^2(l))$$

It is enough to show that $s_t^1(l) = e$ for all $l \in L$ and small t. Since s_t is a homomorphism, $s_t(\gamma)s_t(l)s_t(\gamma)^{-1} = s_t(\rho(\gamma)l)$. Substituting $s_t(\gamma) = \gamma$ we obtain

$$\gamma s_t^1(l)\gamma^{-1} = s_t^1(\rho(\gamma)l) \qquad \gamma, l \in \Gamma, L$$

Furthermore, from $s_t(l+l^1) = s_t(l)s_t(l^1)$ we deduce $s_t^1(l+l^1) = s_t^1(l)s_t^1(l^1)$. The map $\phi: L \to G$ defined by $\phi(l) = s_t^1(l)$ is thus a homomorphism. Consider the algebraic hull M of $\phi(L)$ in G. Since $\phi(L)$ is invariant under conjugations by elements

of Γ , M is invariant under inner automorphisms by elements of Γ , and therefore by a theorem of Borel and Selberg Lie algebra of M is an ideal in G. Hence the connected component M_0 of M is a normal subgroup of G. Since L is abelian, M_0 is commutative contradicting semi-simplicity of G unless $M_0 = \{e\}$. Hence M is discrete and $s_t^1(l) = e$ for t sufficiently small and for all $l \in L$ thereby proving the lemma.

It follows from this lemma that every deformation r_t of $\Gamma \cdot L$ is equivalent to a deformation s_t of $\Gamma \cdot L$ of the form $s_t(\gamma, l) = (\gamma, l+s_t^2(l))$. Then we check easily that $s_0^2 = 0$ and $s_t^2(l+l^1) = s_t^2(l) + s_t^2(l^1)$. Hence s_t^2 can be extended to a linear endomorphism S_t of V. Denoting the full matrix algebra over V by M(V), we see that S_t can be regarded as a C^{∞} map from $(-\alpha, \alpha)$ into M(V) with $S_0 = 0$. It is then trivial to verify that s_t is a homomorphism of $\Gamma \cdot L$ if and only if

$$\rho(\gamma)S_t = S_t\rho(\gamma) \quad \text{for all } \gamma \in \Gamma$$

i.e., $S_t \in Z(\rho(\Gamma)) = Z(\rho(G))$. [2]

It is readily checked that if q_t is another deformation of the same type, viz., $q_t(\gamma, l) = (\gamma, l+Q_t(l))$, then s_t and q_t are equivalent if and only if $S_t = Q_t$ for tsufficiently small. Hence there is a natural injection from equivalence classes of $\Gamma \cdot L$ into germs around 0 of S_t . Conversely, for every C^{∞} map $S: (-\alpha, \alpha) \rightarrow Z(\rho(G))$ with $S_0 = 0$, set $s_t(\gamma, l) = (\gamma, l+S_t(l))$. Then it is readily verified that s_t is a homomorphism of $\Gamma \cdot L$ into $G \cdot V$ for $t \in (-\alpha, \alpha)$, thus showing that the above map is actually a bijection.

Corollary 1. $H_0^1(\Gamma \cdot L, Ad) \simeq Z(\rho(G)).$

Let $\Re(\Gamma \cdot L)$ denote the space of homomorphisms of $\Gamma \cdot L$ into $G \cdot V$ topologized by compact-open topology, and $i \in \Re(\Gamma \cdot L)$ the canonical injection of $\Gamma \cdot L$ into $G \cdot V$. Two homomorphisms of $\Gamma \cdot L$ into $G \cdot V$ are called *equivalent* if they differ by an inner automorphism of $G \cdot V$. Then by a similar argument as above we can actually prove the following result:⁽¹⁾

Theorem 1'. Under the assumption $H^1(\Gamma, G) = H^1(\Gamma, V) = 0$, there exists an open neighborhood U of i in $\Re(\Gamma \cdot L)$ such that every homomorphism $r \in U$ is equivalent to a homomorphism s of the form

$$s(\gamma, l) = (\gamma, l+S(l))$$

where S is in an open neighborhood of zero in $Z(\rho(G))$.

This implies that *i* has an arc-wise connected neighborhood.

⁽¹⁾ We first prove an analogue of lemma 2 by replacing " r_t " with "r" and "t sufficiently small" with "r sufficiently close to i."

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Corollary 2. Under the condition $H^1(\Gamma, G) = H^1(\Gamma, V) = 0$, *i* has a neighborhood in $\Re(\Gamma \cdot L)$ homeomorphic to \mathbb{R}^{k+n+m} where $k = \dim G$, $n = \dim V - \dim V_0$, $m = \dim Z(\rho(G))$ and $V_0 = \{v \in V \mid \rho(g)v = v \text{ for all } g \in G\}$.

Proof. Let $Z^{*}(\rho(G))$ denote the centralizer of $\rho(G)$ in Gl(V), and \tilde{G} be the semi-direct product of $Z^{*}(\rho(G))$ and $G \cdot V$, where the action of $Z(\rho(G))^{*}$ on $G \cdot V$ is given by R(g, v) = (g, R(v)) for $R \in Z^{*}(\rho(G))$. Then $\tilde{G} = Z^{*}(\rho(G)) \cdot G \cdot V$ acts on $G \cdot V$ as a group of automorphisms of $G \cdot V$ as follows:

(1)
$$(R, g, v)[(h, u)] = (g, v)[R(h, u)](g, v)^{-1} = (ghg^{-1}, v + \rho(g)R(u) - \rho(ghg^{-1})v)$$

for $(h, u) \in G \cdot V$. Denote this automorphism of $G \cdot V$ by $\sigma_{(R,g,v)}$.⁽²⁾

By choosing the neighborhood U of i provided by theorem 1' sufficiently small we can assume that $(I+S) \in Z^{*}(\rho(G))$. In fact, it suffices to show (I+S)(L)or equivalently r(L) is a lattice in V. In view of footnote (1) the latter condition is satisfied for all r sufficiently close to i. Since, in the notation of theorem 1', r and s differ by an inner automorphism of $G \cdot V$, there is (g_1, v_1) in $G \cdot V$ such that

(2)
$$r(\gamma, l) = (g_1, v_1)s(\gamma, l)(g_1, v_1)^{-1}$$

= $(g_1\gamma g_1^{-1}, v_1 + \rho(g_1)(l + S)(l) - \rho(g_1\gamma g_1^{-1})(v_1)).$

The group \tilde{G} acts on $\Re(\Gamma \cdot L)$ by $r \to \sigma_{(R,g,v)} \cdot r$. By virtue of (1), (2) and the fact that $I + S \in Z^{z}(\rho(G))$ if U is sufficiently small, the orbit $\tilde{G}(i)$ of *i* contains neighborhood of *i* in $\Re(\Gamma \cdot L)$. It follows that $\tilde{G}(i)$ is an open subset of $\Re(\Gamma \cdot L)$.

Let C(G) denote the center of G which is a discrete subgroup of G. Then we have

Lemma 3. The isotropy subgroup \tilde{G}_i of \tilde{G} at *i* is isomorphic to $C(G) \times V_0$.

Proof of Lemma 3. Setting $h=\gamma$ and u=l in (1), one sees that if $(R, g, v) \in \tilde{G}_i$ then $g\gamma g^{-1}=\gamma$ for all $\gamma \in \Gamma$ and therefore $g \in C(G)$. Then, setting l=0, we have $v-\rho(\gamma)v=0$ for all $\gamma \in \Gamma$ and hence $v=\rho(g)v$ for all $g \in G$, that is, $v \in V_0$. Finally, for $g \in C(G)$, $v \in V_0$, the condition $\sigma_{(R,g,v)}(\gamma, l)=(\gamma, l)$ implies $\rho(g)R(l)=l$ for all $l \in L$, i.e., $R=\rho(g)^{-1}$. Conversely, it is trivial to check that $(\rho(g)^{-1}, g, v) \in \tilde{G}_i$ if $g \in C(G)$ and $v \in V_0$.

REMARK. \tilde{G}_i coincides with the center of \tilde{G} .

We have thus constructed an injective continuous map

$$\{\sigma_{(R,g,v)}|(R,g,v)\in \widetilde{G}\}$$

⁽²⁾ One can actually prove that the connected component of the unit element of the full automorphism group of $G \cdot V$ is given by

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$$\psi \colon \tilde{G}/\tilde{G}_i \to \tilde{G}(i) \subset \Re(\Gamma \cdot L)$$

Suppose $\Gamma \cdot L$ has a set of p generators. Then $\mathcal{R}(\Gamma \cdot L)$ can be identified with the real subvariety of $(G \cdot V)^p$ defined by the set of relations between generators of $\Gamma \cdot L$. Hence $\mathcal{R}(\Gamma \cdot L)$ and therefore $\tilde{G}(i)$ is locally compact. This ensures that ψ is actually a homeomorphism of \tilde{G}/\tilde{G}_i onto $\tilde{G}(i)$ and completes the proof of corollary 2.

2. The cohomology group $H^{1}(\Gamma \cdot L, Ad)$

Theorem 2. Under the assumption $H^{1}(\Gamma, \mathfrak{G}) = H^{1}(\Gamma, \mathfrak{G}) = 0$, one has

$$H^1(\Gamma \cdot L, Ad) \simeq Z(\rho(G))$$

We first introduce some notation. Let Z be a crossed homomorphism of L into $\mathfrak{G}+\mathfrak{V}$. We say $Z \in \mathbb{Z}_{\Gamma}^1$ if there exists a crossed homomorphism X of Γ into $\mathfrak{G}+\mathfrak{V}$ such that

(3)
$$Z(\rho(\gamma)l) - Ad(\gamma)Z(l) = X(\gamma) - Ad(\rho(\gamma)l)X(\gamma)$$

for all γ , $l \in \Gamma$, L. Let $Z: L \to \mathfrak{G} + \mathfrak{B}$ be the principal homomorphism, i.e., Z(l) = R - Ad(l)R. Then it is readily checked that Z satisfies (3) with $X(\gamma) = R - Ad(\gamma)R$. Therefore the group of principal homomorphisms B^1 is contained in Z_{Γ}^1 and we set

$$H^1_{\Gamma}(L, \mathfrak{G} + \mathfrak{B}) = Z^1_{\Gamma}/B^1$$

Lemma 4. Let $X \in \mathfrak{G} + \mathfrak{V}$. Then $X - Ad(\gamma)X - Ad(l)X + Ad(l \cdot \gamma)X = 0$ for all γ , $l \in \Gamma$, L implies $X \in \mathfrak{G}_{\rho} + \mathfrak{V}$, where \mathfrak{G}_{ρ} denotes the Lie algebra of kernel of ρ .

Proof of Lemma 4. Let $l = \exp Y$, $Y \in \mathfrak{V}$, and $R \in \mathfrak{G} + \mathfrak{V}$. Then

$$Ad(l)R = R + [Y, R] + \cdots$$
 higher terms.

Since \mathfrak{V} is commutative and $[Y, R] \in \mathfrak{V}$, all higher terms vanish and we have

In particular, let $R = X - Ad(\gamma)X$. Then the hypothesis implies Ad(l)R = R, hence [Y, R] = 0 for all $Y \in \mathfrak{B}$. Let $X = X_1 + X_2$ with $X_1 \in \mathfrak{B}$ and $X_2 \in \mathfrak{B}$. Since \mathfrak{B} and \mathfrak{B} are invariant under the adjoint action of Γ , [Y, R] = 0 implies $[X_1 - Ad(\gamma)X_1, Y] = 0$ for all $Y \in \mathfrak{B}$. Since $[X_1, Y] = \rho(X_1)Y$, one has $\rho(X_1) = \rho(Ad(\gamma)X_1) = \rho(\gamma)\rho(X_1)\rho(\gamma)^{-1}$ for all $\gamma \in \Gamma$. By virtue of a theorem of Borel and Selberg [2] $\rho(X_1)$ belongs to the center of $\rho(G)$, that is, $\rho(X_1) = 0$ or $X_1 \in G_\rho$. Hence $X \in \mathfrak{G}_\rho + \mathfrak{B}$.

Lemma 5. Every $Z \in Z_{\Gamma}^1$ is cohomologous to a unique $Y \in Z_{\Gamma}^1$ satisfying

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(5)
$$Y(\rho(\gamma)l) = Ad(\gamma)Y(l)$$

for all γ , $l \in \Gamma$, L.

Proof of Lemma 5. Since $Z \in Z_{\Gamma}^{1}$ there is a crossed homomorphism X of Γ into $\mathfrak{G} + \mathfrak{V}$ such that (3) is satisfied. In view of vanishing of $H^{1}(\Gamma, \mathfrak{G} + \mathfrak{V})$ one may write $X(\gamma) = R - Ad(\gamma)R$. Define $Y: L \to \mathfrak{G} + \mathfrak{V}$ by Y(l) = Z(l) - R + Ad(l)R and substitute in (3) to obtain $Y(\rho(\gamma)l) = Ad(\gamma)Y(l)$.

To prove uniqueness let Y' be another candidate. Then Y - Y' is a coboundary, so (Y - Y')(l) = R - Ad(l)R for some $R \in G + V$. Since Y and Y' satisfy (5) one has $(Y - Y')(l) = Ad(\gamma)(Y - Y')(\rho(\gamma)^{-1}l)$ and so

$$R - Ad(l)R = Ad(\gamma)R - Ad(l \cdot \gamma)R.$$

By Lemma 4, $R \in \mathfrak{G}_{\rho} + \mathfrak{V}$ and hence (Y - Y')(l) = R - Ad(l)R = 0. In fact, if $l = \exp T$, $T \in V$, then by (4) R - Ad(l)R = [R, T] = 0 since $R \in G_{\rho} + V$. q.e.d.

Lemma 6. $H^1_{\Gamma}(L, \mathfrak{G}+\mathfrak{V}) \simeq H^1(\Gamma \cdot L, \mathfrak{G}+\mathfrak{V}).$

Proof of Lemma 6. Let Z be a representative of an element of $H^{1}(\Gamma \cdot L, \mathfrak{G} + \mathfrak{B})$, then $Z|_{L} \in Z_{\Gamma}^{1}$. In fact, $Z(\gamma) + Ad(\gamma)Z(l) = Z(\gamma \cdot l) = Z(\rho(\gamma)l \cdot \gamma) = Z(\rho(\gamma)l) + Ad(\rho(\gamma)l)Z(\gamma)$ so that (3) is satisfied with $X = Z|_{L}$. It is clear that $Z \sim 0$ implies $Z|_{L} \sim 0$. Therefore $Z \rightarrow Z|_{L}$ induces a homomorphism $\beta: H^{1}(\Gamma \cdot L, \mathfrak{G} + \mathfrak{B}) \rightarrow H^{1}_{\Gamma}(L, \mathfrak{G} + \mathfrak{B}).$

To prove injectivity of β , let $Z|_L \sim 0$. Since $H^1(\Gamma, \mathfrak{G} + \mathfrak{B}) = 0, Z|_{\Gamma}$ is also a principal homomorphism. Hence suppose $Z|_{\Gamma}(\gamma) = R - Ad(\gamma)R$ and $Z|_L(l) = R' - Ad(\gamma)R'$. Then

$$Z(l \cdot \gamma) = R' - Ad(l)R' + Ad(l)(R - Ad(\gamma)R)$$

We saw earlier that (3) is satisfied by every crossed homomorphism Z of $\Gamma \cdot L$ into $\mathfrak{G} + \mathfrak{B}$ with $X = Z|_{\Gamma}$. In our case, it reads as follows:

$$\begin{aligned} R' - Ad(\rho(\gamma)l)R' - Ad(\gamma)R' + Ad(\gamma)Ad(l)R' \\ = R - Ad(\gamma)R - Ad(\rho(\gamma)l)R + Ad(\rho(\gamma)l)R \end{aligned}$$

for all γ , $l \in \Gamma$, L. Replacing l by $\rho(\gamma)^{-1}l$ and setting R'' = R - R' we obtain

$$R'' - Ad(\gamma)R'' - Ad(l)R'' + Ad(l \cdot \gamma)R'' = 0$$

In view of Lemma 4, $R'' \in G_{\rho} + V$. Since L acts trivially on $\mathfrak{G}_{\rho} + \mathfrak{B}$ by adjoint representation, one has R'' - Ad(l)R'' = 0 and hence

$$R' - Ad(l)R' = R - Ad(l)R$$

Therefore one has $Z(l \cdot \gamma) = R - Ad(l \cdot \gamma)R$ proving $Z \sim 0$. Hence β is an injective homomorphism,

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To prove surjectivity of β , let $Z: L \to \mathfrak{G} + \mathfrak{V}$ be representative of a cohomology class in $H^1_{\Gamma}(L, \mathfrak{G} + \mathfrak{V})$. Then by lemma 5, there is a unique $Y \sim Z$ such that $Y(\rho(\gamma)l) = Ad(\gamma)Y(l)$ for all $\gamma, l \in \Gamma, L$. Now for $l \cdot \gamma \in \Gamma \cdot L$ set $\tilde{Y}(l \cdot \gamma) = Y(l)$. Then

$$egin{aligned} Y&((l\cdot\gamma)(l'\cdot\gamma')) = ilde{Y}((l+
ho(\gamma)l')(\gamma\gamma')) \ &= Y(l+
ho(\gamma)l') \ &= Y(l)+Ad(l)Y(
ho(\gamma)l') \ &= Y(l)+Ad(l\cdot\gamma)Y(l') \ &= ilde{Y}(l\cdot\gamma)+Ad(l\cdot\gamma) ilde{Y}(l'\cdot\gamma') \end{aligned}$$

Hence \tilde{Y} is a crossed homomorphism. Clearly $\beta({\tilde{Y}}) = {Y}$ where by ${\tilde{Y}} {(Y)}$ we mean cohomology class of $\tilde{Y}(Y)$ respectively, therefore β is surjective and proof of lemma 6 is complete.

Lemma 7. $H^1_{\Gamma}(L, \mathfrak{G} + \mathfrak{V}) \simeq Z(\rho(G)).$

Proof of Lemma 7. In view of lemma 5

 $H^{1}_{\Gamma}(L, \mathfrak{G}+\mathfrak{V}) = \{ \text{crossed homomorphisms } Y: L \to \mathfrak{G}+\mathfrak{V} \text{ satisfying (5)} \}$

Write $Y=Y_1+Y_2$ where $Y_i=\pi_i \cdot Y$, π_i being projection of $\mathfrak{G}+\mathfrak{V}$ onto its i^{th} (i=1,2) summand. Identifying \mathfrak{V} and V via the exponential map, we can write using (4)

$$\begin{aligned} Ad(l) Y(l') &= Y(l') - [Y(l'), l] \\ &= Y(l') - [Y_1(l'), l] \\ &= Y(l') - \rho(Y_1(l'))l. \end{aligned}$$

Therefore we have

$$\begin{split} Y_1(l+l') + Y_2(l+l') &= Y(l+l') \\ &= Y(l) + Ad(l) Y(l') \\ &= Y_1(l) + Y_1(l') + Y_2(l) + Y_2(l') - \rho(Y_1(l'))l \end{split}$$

Hence

$$\left\{ \begin{array}{l} Y_{\rm i}(l\!+\!l') = Y_{\rm i}(l)\!+\!Y_{\rm i}(l') \ , \\ Y_{\rm 2}(l\!+\!l') = Y_{\rm 2}(l)\!+\!Y_{\rm 2}(l')\!-\!\rho(Y_{\rm i}(l'))l \ . \end{array} \right.$$

Therefore Y_1 can be extended to a linear map of V into \mathfrak{G} such that $Y_1(\rho(\gamma)v) = Ad(\gamma)Y_1(v)$ for all $\gamma, v \in \Gamma, V$. By a theorem of Borel and Selberg $Y(\rho_1(g)v) = Ad(g)Y_1(v)$ and consequently $Y_1(\rho(X)v) = ad(X)Y_1(v)$ for all $X \in G$. Therefore $Y_1 \in \operatorname{Hom}_G(V, \mathfrak{G})$ and im $Y_1 = \mathfrak{G}_1$ is an ideal in \mathfrak{G} . Now we have

$$Y_{1}(\rho(Y_{1}(l'))l) = ad(Y_{1})l')Y_{1}(l)$$

= [Y_{1}(l'), Y_{1}(l)]

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which is anti-symmetric in l and l'. But $\rho(Y_1(l'))l = -Y_2(l+l') + Y_2(l) + Y_2(l')$ is symmetric in l and l', therefore $Y_1(\rho(Y_1(l'))l) = [Y_1(l'), Y_1(l)] = 0$, that is, $[\mathfrak{G}_1, \mathfrak{G}_1] = 0$. Since \mathfrak{G}_1 is an ideal of semi-simple Lie algebra, commutativity of \mathfrak{G}_1 implies $\mathfrak{G}_1 = 0$. Hence $Y_1 = 0$ and

$$Y_2 = Y \in \operatorname{Hom}_{\Gamma}(V, V) = \operatorname{Hom}_{G}(V, V) = Z(\rho(G))$$
q.e.d.

Lemmas 6 and 7 complete the proof of Theorem 2.

In view of Theorems 1 and 2 every crossed homomorphism of $\Gamma \cdot L$ into $\mathfrak{G} + \mathfrak{V}$ is tangent to a deformation of $\Gamma \cdot L$ in $G \cdot V$ and we have

Corollary. $H^{1}_{0}(\Gamma \cdot L, Ad) = H^{1}(\Gamma \cdot L, Ad).$

3. Cohomology of $\Gamma \cdot L$ -invariant forms

Let be Γ a uniform discrete subgroup of a connected semi-simple real Lie group G, K a maximal compact subgroup of G, ρ and ψ finite dimensional real representations of G on V and W, respectively, and L a lattice in V normalized by Γ . It is convenient to assume Γ has no elements of finite order, that is, $\Gamma \cap gKg^{-1} = \{e\}$ for all $g \in G$. Since Γ and therefore $\Gamma \cdot L$ has no elements of finite order, $\Gamma \cdot L$ acts freely on $\tilde{X} = G \cdot V/K$, that is, $(\gamma, l)(\tilde{x}) = (\gamma', l')(\tilde{x})$ for some $\tilde{x} \in \tilde{X}$ implies $(\gamma, l) = (\gamma', l')$, and the quotient space becomes a real analytic variety. $\Gamma \cdot L \setminus \tilde{X}$ has a natural structure of a fibre bundle over $\Gamma \setminus X (X = G/K)$ with fibre isomorphic to $L \setminus V$. We have canonical isomorphisms $L_{l\cdot\gamma}: T^*_{\tilde{x}}(\tilde{X}) \to$ $T^*_{G \cdot \gamma \cap \tilde{x}}(\tilde{X})$ induced by the free action of $\Gamma \cdot L$ on $G \cdot V/K$.

We can choose an inner product (,) on W such that $\psi(K)$ is contained in the orthogonal group on W relative to (,). Now for each $x \in X$ define an inner product $(,)_x$ on W by $(u, v)_x = (\psi(g)u, \psi(g)v)$, where x = gK. It is easily verified that $(,)_x$ is independent of the choice of coset representative g, and its dependence on x is smooth. We similarly define an inner product $[,]_x$ on Vfor each $x \in X$. Then $[,]_x$ induces a flat Riemannian metric B(x) on V (and therefore on $L \setminus V$) which depends smoothly on x. Finally, we define an invariant Riemannian metric on the vector bundle $T(X) \times T(V) \rightarrow \tilde{X} \approx X \times V$ by

$$ds^2 = ds_0^2 + B(x)$$

where ds_0^2 is an invariant metric on X. Note that one has $\tilde{X} \approx X \times V$ by the correspondence $\tilde{x} = (g, v) \leftrightarrow (gK, \rho(g)v)$.

By a W-valued p-form on \tilde{X} we mean a C^{∞} section ω of the vector bundle $W \otimes \wedge^{p} T^{*}(\tilde{X}) \rightarrow \tilde{X}$. We say ω is $\Gamma \cdot L$ -invariant if

$$\omega \cdot (\gamma, l) = (\psi(\gamma) \otimes (\wedge^{p} \cdot L_{(\gamma, l)})) \omega$$

We denote the space of W-valued $\Gamma \cdot L$ -invariant p-forms on \tilde{X} by $\Omega^{p}(\tilde{X}, \Gamma \cdot L, W, \psi)$ or simply $\Omega^{p}(W)$ if no confusion arises.

For any vector spaces E and F one has canonical isomorphism

$$\sum_{i+j=r} \wedge^{i} E \otimes \wedge^{j} F \simeq \wedge^{r} (E \oplus F) .$$

Since $T^*(\tilde{X}) = T^*(X) \oplus T^*(V)$, it follows that we have a decomposition

$$\Omega^{p}(W) = \sum_{a+b=p} \Omega^{a,b}(W)$$

where $\Omega^{a,b}(W)$ is the subspace of $\Omega^{p}(W)$ consisting of C^{∞} sections of $W \otimes \wedge^{a}T^{*}(X) \otimes \wedge^{b}T^{*}(V) \to \tilde{X}$ and $\wedge^{a}T^{*}(X) \otimes \wedge^{b}T^{*}(V)$ is identified with a subspace of $\wedge^{p}T^{*}(\tilde{X})$ by the isomorphism mentioned above.

The operator d (exterior differentiation) makes the diagram

$$\cdots \to \Omega^{p}(W) \xrightarrow{d} \Omega^{p+1}(W) \to \cdots$$

into a cochain complex. The cohomology groups of this cochain complex will be denoted by $H^{p}(\tilde{X}, \Gamma \cdot L, W, \psi)$. Since \tilde{X} is contractible, it is known that $H^{p}(\tilde{X}, \Gamma \cdot L, W, \psi)$ is isomorphic to the cohomology group $H^{p}(\Gamma \cdot L, W)$ defined algebraically [4].

Let U be an open set in X=G/K on which one has a positively oriented orthonormal basis $\{L_1, \dots, L_k\}$ for $T_x(X)$, $x \in U$, depending smoothly on x. Then on $\tilde{U}=\pi^{-1}(U)$ (π denoting the canonical projection $\tilde{X} \to X$), one can obtain a positively oriented orthonormal basis $\{L_1, \dots, L_k, L_{k+1}, \dots, L_{k+n}\}$, or more precisely $\{L_1(\tilde{x}), \dots, L_{k+n}(\tilde{x})\}$, for $T_{\tilde{x}}(\tilde{X})$, $\tilde{x}=(x, v)$, relative to metric $ds_0^2+B(x)$, depending smoothly on \tilde{x} such that $\{L_1, \dots, L_k\}$ and $\{L_{k+1}, \dots, L_{k+n}\}$ span $T_x(X)$ and $T_v(V)$, respectively. Denote the basis dual to $\{L_1, \dots, L_{k+n}\}$ by $\{w^1(x), \dots, w^{k+n}(x)\}$ or simply $\{w^1, \dots, w^{k+n}\}$. When one identifies $T_v(V)$ with V, one may assume that L_1, \dots, L_{k+n} are independent of v and depend smoothly on x. On \tilde{U} , an element $\omega \in \Omega^{a,b}(W)$ has an expression of the form

$$(6) \qquad \omega(\tilde{\mathbf{x}}) = \sum_{i_1, \cdots, i_d} \sum_{j_1, \cdots, j_b} u_{i_1 \cdots i_d j_1 \cdots j_b}(\tilde{\mathbf{x}}) \otimes w^{i_1} \wedge \cdots \wedge w^{i_d} \otimes w^{j_1} \wedge \cdots \wedge w^{j_b}$$

where $u_{i_1\cdots i_d j_1\cdots j_b}(\tilde{x}) \in W$, i's and j's range over $\{1, \dots, k\}$ and $\{k+1, \dots, k+n\}$, respectively. The operator * is defined by

$$(7) \qquad (*\omega)(\tilde{x}) = \sum_{i_1, \cdots, i_d} \sum_{j_1, \cdots, j_b} \\ \delta^{1\cdots k+n}_{i_1\cdots i_d j_1\cdots j_b i_1}\cdots i_k - a j_1'\cdots j_{n-b}} u_{i_1\cdots i_d j_1\cdots j_b}(\tilde{x}) \\ \otimes w^{i_1} \wedge \cdots \wedge w^{i_{k-d}} \otimes w^{j_1'} \wedge \cdots \wedge w^{j_{n-b}'}$$

where $\{i_1, \dots, i_a, i'_1, \dots, i'_{k-a}\}$ (resp. $\{j_1, \dots, j_b, j'_1, \dots, j'_{n-b}\}$) coincide set-theoretically with $\{1, \dots, k\}$ (resp. $\{k+1, \dots, k+n\}$). One can show that the operator * is independent of the choice of orthonormal basis $\{L_1, \dots, L_{k+n}\}$. The inner product $(,)_x$ on W defines an isomorphism $W \to W^*$ which induces an isomorphism $\sharp: \Omega^{a,b}(W) \to \Omega^{a,b}(W^*) = \Omega^{b,a}(\tilde{X}, \Gamma \cdot L, W^*, {}^t\psi^{-1})$. Now let ∂ be the unique map which makes the following diagram commutative:

We set for $\omega \in \Omega^p(W)$

$$\delta\omega = (-1)^{(k+n)(p+1)+1} * \partial * \omega$$

and

 $\Delta = d\delta + \delta d$

A *p*-form ω is *harmonic* if $\Delta \omega = 0$. It can be shown that harmonicity of ω is equivalent to vanishing of $d\omega$ and $\delta\omega$. By Baily's generalization of Hodge's theorem

$$H^{p}(ilde{X},\,\Gamma\!\cdot\!L,\,W,\,\psi)\simeq \mathscr{H}^{p}(ilde{X},\,\Gamma\!\cdot\!L,\,W,\,\psi)$$

where $\mathcal{H}^{p}(\tilde{X}, \Gamma \cdot L, W, \psi)(\mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W\psi))$ denotes the subspace of harmonic forms in $\Omega^{p}(\tilde{X}, \Gamma \cdot L, W, \psi)(\Omega^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi))$ [1].

4. Decomposition of space of harmonic forms

Theorem 3. We have a direct sum decomposition

$$\mathcal{H}^{p}(\tilde{X}, \Gamma \cdot L, W, \psi) = \sum_{a+b=p} \mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$$

Proof. Let $K^{a,b}$ denote the subspace of $\Omega^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$ consisting of forms constant along the fibres of $\tilde{X} \to X$, and $\Omega^{a}(b)$ or more precisely $\Omega^{a}(X, \Gamma, W \otimes \wedge^{b}V^{*}, \psi \otimes (\wedge^{b} \cdot \rho)^{*})$ the space of Γ -invariant *a*-forms on X with values in $W \otimes \wedge^{b}V^{*}$ for the representation $\psi \otimes (\wedge^{b} \cdot \rho)^{*}$. An element $\theta \in \Omega^{a}(b)$ has locally (i.e., in U) an expression of the form

$$\theta(x) = \sum_{i_1, \cdots, i_a} v_{i_1 \cdots i_a}(x) \otimes w^{i_1} \wedge \cdots \wedge w^{i_a}$$

where $v_{i_1\cdots i_a}(x) \in W \otimes \wedge^b V^*$, i.e., $v_{i_1\cdots i_a}(x) = \sum_{i_1,\cdots,i_b} a_{i_1,\cdots,i_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b}$ where $a_{i_1\cdots i_a j_1\cdots j_b}$: $X \to W$ is a C^{∞} function and $\{w^{k+1}, \cdots, w^{k+n}\}$ is the orthonormal basis of V^* introduced above. Hence

$$(8) \qquad \theta(x) = \sum_{i_1\cdots i_a} \sum_{j_1\cdots j_b} a_{i_1\cdots i_a} j_{j_1\cdots j_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b} \otimes w^{i_1} \wedge \cdots \wedge w^{i_a}$$

An (a, b) form $\omega \in \Omega^{a,b}(W)$ is constant along the fibres of $\tilde{X} \to X$ if and only if

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the functions $u_{i_1\cdots i_a j_1\cdots j_b}$ in (6) depend only on x. Furthermore, an expression of the form (6) with $u_{i_1\cdots i_a j_1\cdots j_b}$ depending only on x, is an element of $\Omega^{a,b}(W)$ if and only if

$$\omega(\gamma \tilde{x}) = \sum_{i_1, \cdots, i_a} \sum_{j_1, \cdots, j_b} \psi(\gamma) u_{i_1, \cdots i_a \ j_1 \cdots j_b}(x) \otimes (\wedge^a \cdot L_{\gamma}) w^{i_1} \wedge \cdots \wedge^{i_a} \otimes \wedge^b \cdot \rho)^* w^{j_1} \wedge \cdots \wedge w^{j_b}$$

Similarly, an expression of the form (8) is an element of $\Omega^{a}(b)$ is and only if

$$\theta(\gamma x) = \sum_{i_1, \cdots, i_a} \sum_{j_1, \cdots, j_b} \psi(\gamma) a_{i_1 \cdots i_a j_1 \cdots j_b}(x) \otimes \\ (\wedge^b \cdot \rho)^*(\gamma) w^{j_1} \wedge \cdots \wedge w^{j_b} \otimes (\wedge^a \cdot L_{\gamma}) w^{i_1} \wedge \cdots \wedge w^{i_a}.$$

We have therefore proved

Lemma 8. There is a natural isomorphism

$$\wedge : K^{a,b} \to \Omega^{a}(b)$$

(In the above notation, one defines

$$\hat{\omega}(x) = \sum_{i_1, \cdots, i_a} \sum_{j_1, \cdots, j_b} u_{i_1 \cdots i_a j_1 \cdots j_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b} \otimes w^{i_1} \wedge \cdots \wedge w^{i_a}).$$

The mapping $w^{j'_2} \wedge \cdots \wedge w^{j'_{n-b}} \mapsto \delta^{k+1\cdots k+n}_{j_1\cdots j_b j_1'\cdots j_1'_{n-b}} w^{j_1} \wedge \cdots \wedge w^{j_b}$ extends by linearity to an isomorphism $\wedge^{n-b} V^* \cong \wedge^b V^*$ and thus induces isomorphism

$$\alpha: \ \Omega^a(n-b) \to \Omega^a(b)$$

Lemma 9. The following diagram commutes up to sign

Proof of Lemma 9. The operator * on $K^{a,b}$ is given by (7) and on $\Omega^{a}(b)$ by

$$(9) \qquad (*\theta)(x) = \sum_{i_1,\cdots,i_a} \delta^{1\cdots k}_{i_1\cdots i_a j'_1\cdots j'_{k-a}} \sum_{j_1,\cdots,j_b} a_{i_1\cdots i_a j_1\cdots j_b}(x) \otimes w^{j_1} \wedge \cdots \wedge w^{j_b} \otimes w^{i_1'} \wedge \cdots \wedge w^{i'_{k-a}}.$$

Since

$$\delta_{i_1\cdots i_a j_1\cdots j_b i_1}^{i_1\cdots k_n} \cdots i_{k-a}^{j_1'\cdots j_{n-b}'} = (-1)^{b(k-a)} \delta_{i_1\cdots i_a i_1\cdots i_{k-a}}^{i_1\cdots k} \delta_{j_1\cdots j_b j_1'\cdots j_{n-b}}^{k+1\cdots k+n}$$

the assertion of the lemma follows from (7), (9) and definition of α .

It is clear that restrictions of d and δ to $\sum_{a,b} K^{a,b}$ are operators of bidegrees (1, 0) and (-1, 0), respectively. Similarly, d and δ are operators of degrees

+1 and -1 on $\sum_{a} \Omega^{a}(b)$, respectively.

Let C(x) be the matrix representation for $(,)_x$ with respect to some basis, i.e., for $w_1, w_2 \in W$, $(w_1, w_2)_x = {}^t w_1 C(x) w_2$. Then the isomorphisms # can be given the following explicit description:

$$\omega \to (C(x) \otimes 1 \otimes 1) \omega \qquad \omega \in \Omega^{a,b}(w) \subset K^{a,b},$$

$$\theta \to (C(x) \otimes 1 \otimes 1) \theta \qquad \theta \in \Omega^{a}(b).$$

Lemma 10. For $\omega \in K^{a,b}$, $d\omega = 0$ ($\delta \omega = 0$) if and only if $d \omega = 0$, ($\delta \omega = 0$).

Proof of Lemma 10. Equivalence of $d\omega = 0$ and $d\Delta = 0$ is obvious. To prove equivalence of $\delta\Delta = 0$ and $\delta\Delta = 0$, we note that $\delta\Delta = 0$ if and only if $d^*(C(x)\otimes 1\otimes 1)\Delta = 0$. It follows from the description of # above, definition of * and Lemma 9 that the latter is equivalent to $d(*\#\omega)=0$, that is, $\delta\omega = 0$. q.e.d.

It is clear that $H^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi) \subset K^{a,b}$ [6], and by Lemma 10 we can conclude

(10)
$$H^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi) \simeq H^{a}(X, \Gamma, W \otimes \wedge^{b} V^{*}, \psi \otimes (\wedge^{b} \cdot \rho)^{*})$$

Let $\Gamma \cdot L = G$, L = H and W = A in the Hochschild-Serre spectral sequence⁽³⁾. Then the $E_2^{a,b}$ term of the spectral sequence is $H^a(\Gamma, H^b(L, W))$, and E_{∞} term is a grading of $H^b(\Gamma \cdot L, W)$. Therefore

(11)
$$\dim H^{p}(\Gamma \cdot L, W) \leq \dim \sum H^{a}(\Gamma, H^{b}(L, W))$$

We know that $H^{b}(L, W) \simeq \wedge^{b} V^{*} \otimes W \simeq \operatorname{Hom}(\wedge^{b} V, W)$ and action of Γ on $H^{b}(L, W)$ is as follows:

To $f \in \text{Hom}(\wedge^{b}V, W)$ corresponds the homogeneous cochain c_{f} given by

$$c_f(u_0, \cdots, u_b) = f(u_1 - u_0 \wedge \cdots \wedge u_b - u_0) \qquad u_i \in L.$$

Then

an element of $H^{j}(H, A)$ and $x \in G$ a representative of $\bar{x} \in G/H$. Set

$$(\bar{x}f)(y_1,\dots,y_j) = x(f(x^{-1}y_1x,\dots,x^{-1}y_jx))$$

Then $f \to (\bar{x}f)$ induces a homomorphism of $H^j(H, A)$ into itself, and this defines action of G/H on $H^j(H, A)$. Furthermore, there is a spectral sequence with E_2 term

$$E_{2}^{i,j} = H^{i}(G/H, H^{j}(H, A))$$

converging to a certain grading of $H^{i+j}(G, A)$. For more details see e.g. [7].

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⁽³⁾ We give a brief description of a special case of Hochschild-Serre spectral sequence which is sufficient for our purposes. Let G be an abstract group, H an abelian normal subgroup of G, A an abelian group and ρ a homomorphism of G into the group of automorphisms of A. Suppose that the restriction of ρ to H is trivial. Let $f: H \times \cdots \times H \to A$ be a representative of j-times

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$$(\gamma c_f)(u_0, \dots, u_b) = \psi(\gamma)f(\gamma^{-1}(u_1 - u_0)\gamma \wedge \dots \wedge \gamma^{-1}(u_b - u_0)\gamma)$$

= $\psi(\gamma)f(\rho(\gamma)^{-1}(u_1 - u_0) \wedge \dots \wedge \rho(\gamma)^{-1}(u_b - u_0)),$

that is, Γ acts on $H^b(L, W) = W \otimes \wedge^b V^*$ via $\psi \otimes (\wedge^b \cdot \rho)^*$. Now we have

$$\dim H^{\mathfrak{p}}(\tilde{X}, \Gamma \cdot L, W, \psi) \geq \dim \sum_{a+b=\mathfrak{p}} \mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$$

$$= \dim \sum_{a+b=\mathfrak{p}} \mathcal{H}^{a}(X, \Gamma, W \otimes \wedge^{b}V^{*}, \psi \otimes (\wedge^{b} \cdot \rho)^{*}) \quad [by (10)]$$

$$= \dim \sum_{a+b=\mathfrak{p}} H^{a}(\Gamma, W \otimes \wedge^{b}V^{*}, \psi \otimes (\wedge^{b} \cdot \rho)^{*})$$

$$= \dim \sum_{a+b=\mathfrak{p}} H^{a}(\Gamma, H^{b}(L, W))$$

$$\geq \dim H^{\mathfrak{p}}(\Gamma \cdot L, W) \qquad [by (11)]$$

$$= \dim \mathcal{H}^{\mathfrak{p}}(\tilde{X}, \Gamma \cdot L, W, \psi).$$

Since the space of harmonic forms $\mathcal{H}^p(\tilde{X}, \Gamma \cdot L, W, \psi)$ is finite dimensional, we have shown

$$\mathcal{H}^{p}(\tilde{X}, \Gamma \cdot L, W, \psi) = \sum_{a+b=p} \mathcal{H}^{a,b}(\tilde{X}, \Gamma \cdot L, W, \psi)$$

completing the proof of Theorem 3.

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