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ON THE IRREDUCIBILITY OF DIRICHLET FORMS ON DOMAINS IN INFINITE DIMENSIONAL SPACES

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1. Introduction

Let (B, H, μ) be an abstract Wiener space. Let $\phi : B \to \mathbb{R}$ be an H–continuous function and define $U := \{z \in B \mid \phi(z) > 0\}$. Assume $\mu(U) > 0$. Then U satisfies that for any $z \in U$, there exists an open set V_z in H such that $z + V_z \subset U$. Hence we may define an H–derivative for the function on U as in the Wiener space itself. In fact, Kusuoka [13, 14, 15] defined an H–derivative on U and define a Dirichlet form $\mathcal{E}_U$ on U and gave a criterion of the irreducibility of the Dirichlet form. Actually his assumption on U, namely H–connectivity and the regularity of ϕ, i.e., strong $C^\infty - C_0$ property implies a stronger property, uniform positivity improving property, (see [5] and Remark 11 in §2) than irreducibility. The author made use of his theorem to prove the irreducibility of the Dirichlet forms on loop spaces. The aim of this article is to prove the irreducibility of $\mathcal{E}_U$ without "strong $C^\infty - C_0$ property" and provide a simpler proof than Kusuoka's proof. Our proof does not use special properties of Gaussian measures and so our theorem may hold in more general situation (see Remark 10 in §2).

The organization of this paper is as follows. In §2, we will prove our main thorem and in §3, we will prove the irreducibility of the Dirichlet form on loop group using our method.

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2. Main Results

Let (B, H, μ) be an abstract Wiener space. Let a measurable function $\phi : B \to \mathbb{R}$ to be an H–continuous function, i.e. for any $z \in B$, $\phi(z + \cdot) : H \to \mathbb{R}$ is a continuous function. Let us set $U = \{z \in B \mid \phi(z) > 0\}$. Assume that $\mu(U) > 0$ and we denote $d\mu_U := d\mu|_U / \mu(U)$. Let us recall the definition of the Dirichlet form on U ([13]).

DEFINITION 1. A function u on U is in $\mathcal{D}_U$ if and only if the following holds:

(1) for any $v \in H$, define $V_v(z) = \{t \in \mathbb{R} \mid z + tv \in U\}$. Then there exists a measurable function u_v on U such that

$$u(z) = u_v(z) \qquad \mu\text{-a.s. } z.$$
$$t\ (\in V_v(z)) \to u_v(z+tv) \in \mathbb{R} \quad \text{is an absolutely continuous function.}$$

(2) there exists a measurable function $F \in L^2(U \to H, d\mu_U)$ such that for any $v \in H$

$$\lim_{t \to 0} \mu_U \left(z \in U \middle| z + tv \in U, z \in U \text{ and } \left| \frac{u(z+tv) - u(z)}{t} - (F(z), v)_H \right| > \varepsilon \right) = 0$$

For u in the above, we define $Du(z) := F(z)$

Now we are in a position to define our Dirichlet form $(\mathcal{E}_U, \mathcal{D}_U)$. For $u \in \mathcal{D}_U$, define

$$\mathcal{E}_U(u, u) := \int_U (Du(z), Du(z))_H d\mu_U(z).$$

The Markovian property of $\mathcal{E}_U$ is clear and the proof of the closedness can be found in [13]. Also see [7] and Remark 10 in this section. Our main theorem is concerned with the irreducibility of this Dirichlet form.

Theorem 2. *Let the above subset U to be H-connected, i.e. for any $z \in U$, $U(z) = \{h \in H \mid z + h \in U\}$ is a connected open set in H. Then $(\mathcal{E}_U, \mathcal{D}_U)$ is irreducible.*

Note that when U is open connected set in B, then U is H-connected. However in infinite dimensional space with measure, the topology is meaningless sometimes. The reader may think whether there are different examples. We will present an example.

EXAMPLE 3. Let $X(t, x, w)$ be the solution of a stochastic differential equation (= SDE) of elliptic type on a compact Riemannian manifold M. Let O be an open connected set in $C([0, 1] \to M; \gamma(0) = x)$. Let us consider the inverse image $X^{-1}(O)$. Then we can prove that there exists an H-connected measurable subset U_O such that $\mu(X^{-1}(O) \triangle U_O) = 0$ using the property of SDE. Note that U_O is not a connected open set in usual sense. In [4], these kind of results were applied to the Dirichlet forms on submanifolds and loop spaces. So the H-connectivity is well fitted in with the property of the solution of an SDE.

Therem 2 is closely related to Theorem 6.1 in [15]. The difference is that in the above theorem we donot assume the strong $C^\infty - C_0$ property of ϕ. However the conclusion is weaker than Kusuoka's results (see Remark 11 in §2). Note that to prove the irreducibility of the Dirichlet form on loop space (see [4]), we need only the following weaker result which is an easy consequence of the above theorem because $\mathbb{D}_2^1(B) = \mathcal{D}_B$, where $\mathbb{D}_2^1(B)$ is the Sobolev space in the sense of Watanabe which consists of the L^2-functions whose first derivatives are also in L^2.

Corollary 4. *Let U be the domain in* Theorem 2. *Assume $u \in \mathbb{D}_2^1(B)$ satisfies that $Du(z) = 0$ μ-a.s. $z \in U$. Then u is a constant function on U μ-a.s.*

To prove our main theorem, we need the following result for functions in $\mathcal{D}_U$. This is a similar result to Kusuoka's Proposition 3.2 in [12].

Lemma 5. *Let $u \in \mathcal{D}_U$ and fix $v \in H$. Let $A \subset U$ be a measurable subset with $A + sv \subset U$ for any $0 \le s \le t$. Then there exists a subset $A_v \subset A$ with $\mu(A \setminus A_v) = 0$ and*

$$u_v(z + sv) = u_v(z) + \int_0^s (Du(z + \tau v), v)_H \, d\tau \quad \textit{for any } z \in A_v \textit{ and } 0 \le s \le t,$$

where u_v is a version in Definition 1(1).

Proof. Let us set

$$\Omega = \left\{ (z, s) \in A \times [0, t] \;\middle|\; \text{there exists the derivative } \frac{d}{ds} u_v(z + sv) \right\}$$

Then Ω is a measurable subset of $A \times [0, t]$. Also by the absolute continuity, $m\{s \in [0, t] \mid (z, s) \in \Omega\} = t$ where m denotes the Lebesgue measure. So by the Fubini theorem, Ω has full measure in $A \times [0, t]$. Hence again by the Fubini theorem, a.s. $s \in [0, t]$, there exists $A(s) \subset A$ with $\mu(A \setminus A(s)) = 0$ such that for any $z \in A(s)$, $d/ds u_v(z + sv)$ exists. On the other hand, by the Definition 1 (2) and quasi-invariance of μ, for any $s \in [0, t]$ there exists $\tilde{A}(s)$ with $\mu(A \setminus \tilde{A}(s)) = 0$ and $t_n \to 0$ such that for any $z \in \tilde{A}(s)$

$$\lim_{n \to \infty} \frac{u_v(z + sv + t_n v) - u_v(z + sv)}{t_n} = (F(z + sv), v)$$

Consequently a.s. $s \in [0, t]$, for any $z \in A(s) \cap \tilde{A}(s)$,

$$\frac{d}{ds} u_v(z + sv) = (F(z + sv), v). \tag{1}$$

Let

$$\tilde{\Omega} = \left\{(z, s) \in A \times [0, t] \;\Big|\; \frac{d}{ds} u_v(z + sv) = (F(z + sv), v)\right\}$$

Then $\tilde{\Omega}$ is measurable and by (1), $(\mu \otimes m)(A \times [0, t] \setminus \tilde{\Omega}) = 0$. Using the Fubini again, there exists A_v with $\mu(A \setminus A_v) = 0$ such that for any $z \in A_v$ a.s. $s \in [0, t]$,

$$\frac{d}{ds} u_v(z + sv) = (F(z + sv), v).$$

This completes the proof. □

Our proof of Theorem 2 is based on the ergodicity of μ.

Lemma 6 (Ergodicity of Wiener measure). *Let V be a countable dense subset of H. Let A_1, A_2 be measurable subsets of B with $\mu(A_i) > 0$ $(i = 1, 2)$. Then there exists $v \in V$ such that $\mu(\{A_1 + v\} \cap A_2) > 0$.*

We will prove the "arcwise connectedness of U" in Lemma 9 if U is H-connected using the ergodicity of μ. To this end, we will deduce some regularity property of ϕ. We fix a countable dense subset V in H and for $v \in V$, let C_v be a countable dense subset of $C([0, 1] \to H \mid h(0) = 0, h(1) = v)$ consisting of piecewise linear functions. We will set $C_V = \cup_{v \in V} C_v := \{h_n(s)\}_{n=1,2,\ldots}$. The infinite product topological space of real number $\mathbb{R}^\infty$ is a separable Fréchet space and the space of continuous functions $C([0, 1] \to \mathbb{R}^\infty)$ is itself a separable Fréchet space and homeomorphic to the infinite product space $C([0, 1] \to \mathbb{R})^\infty$ naturally. Let us consider the following Fréchet semi-norm $\| \ \|$ on $C([0, 1] \to \mathbb{R}^\infty)$,

$$\|\{x_n\}\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0 \le s \le 1} |x_n(s)|}{1 + \sup_{0 \le s \le 1} |x_n(s)|}.$$

Let us denote $B_r(\{y_n\}) = \{\{x_n\} \mid \|\{x_n\} - \{y_n\}\| < r\}$.

Lemma 7. *Let $\phi : B \to \mathbb{R}$ be an H-continuous function. Then the map $\Phi_\phi : B \to C([0, 1] \to \mathbb{R}^\infty)$ such that*

$$\Phi_\phi(z)(s) = \{\phi(z + h_n(s))\}_{n=1,2,\ldots}$$

is a measurable map.

Proof. It suffices to prove that $\Phi_\phi^{-1}(B_r(\{y_n\}))$ is a measurable subset in B. Since

ϕ is H-continuous,

$$\Phi_\phi^{-1}(B_r(\{y_n\})) = \left\{ z \in B \;\middle|\; \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0\le s\le 1,\, s\in\mathbb{Q}} |\phi(z+h_n(s)) - y_n(s)|}{1+\sup_{0\le s\le 1, s\in\mathbb{Q}} |\phi(z+h_n(s)) - y_n(s)|} < r \right\}.$$

Cleary this is a measurable set. □

Lemma 8 (Lusin's theorem). *Let E be a separable Fréchet space and $F : B \to E$ be a measurable map. Then for any $\varepsilon > 0$, there exists a compact subset K_ε such that $\mu(K_\varepsilon^c) < \varepsilon$ and $F|_{K_\varepsilon} : K_\varepsilon \to E$ is a continuous map.*

The following is a main lemma.

Lemma 9 ("Arcwise connectedness"). *Let U be the domain in* Theorem 2. *Then the following property holds.*

For any measurable sets A_1, A_2 of positive measure in U, there exists a compact subset $K_1 \subset A_1$ with $\mu(K_1) > 0$ and $h \in C_V$ such that

$$K_1 + h(s) \subset U \text{ for any } s \in [0,1]$$
$$K_1 + h(1) \subset A_2.$$

Proof. By the ergodicity of μ, there exists $v \in V$ such that $\mu(\{A_1 + v\} \cap A_2) > 0$. Hence by Lemma 7 and Lemma 8, there exists a compact subset of positive measure $K \subset A_1$ such that
(1) $K + v \subset A_2$
(2) $\Phi_\phi : K \to C([0,1] \to \mathbb{R}^\infty)$ is continuous map.
Also there exists $z_0 \in K$ and for any neighbourhood $B(z_0)$, $\mu(B(z_0) \cap K) > 0$ holds. Since z_0, $z_0 + v \in U$, by the H-connectivity of U, there exists a continuous curve $h_l \in C_v$ such that $\min_{s\in[0,1]} \phi(z_0 + h_l(s)) > 0$ holds. By the continuity of Φ_ϕ on K, there exists a closed neighbourhood $B(z_0)$ such that for any $z \in B(z_0) \cap K$,

$$\min_{s\in[0,1]} \phi(z + h_l(s)) > 0.$$

This proves the theorem where $K_1 = B(z_0) \cap K$ and $h = h_l$. □

Proof of Theorem 2. Assume that u is not a constant function. Then there exist measurable subsets A_1 and A_2 of U with $\mu(A_i) > 0$ and positive number δ such that $\inf_{z\in A_1} u(z) - \sup_{z\in A_2} u(z) \ge \delta$. Let K_i be the subsets and $h(s)$ be the element in C_V as in Lemma 9. Since $h(s)$ is a piecewise linear function, there exists a finite partition $0 = t_0 < t_1 < \ldots < t_n = 1$ and for any $0 \le i \le n-1$,

$$\dot{h}(s) = v_i \quad s \in (t_i, t_{i+1}),\ v_i \in V$$

holds. We will apply Lemma 5 to the case where $A = K_1 + h(t_i)$, $v = v_i$, $t = t_{i+1} - t_i$. Let us denote $K_{1,i} = (K + h(t_i))_{v_i} - h(t_i)$. Note that $\mu(K \setminus K_{1,i}) = 0$. Consequently we have

$$(2) \quad u_{v_i}(z + h(t_{i+1})) = u_{v_i}(z + h(t_i)) + \int_{t_i}^{t_{i+1}} (Du(z + h(s)), v_i)_H \, ds. \quad \text{for } z \in K_{1,i}.$$

Set for $0 \le i \le n-2$,

$$\tilde{S}_{1,i} = \left\{ z \in K_1 \mid u_{v_{i+1}}(z + h(t_{i+1})) = u_{v_i}(z + h(t_{i+1})) = u(z + h(t_{i+1})) \right\}$$

and

$$\begin{aligned} S_0 &= \left\{ z \in K_1 \mid u_{v_{n-1}}(z + h(1)) = u(z + h(1)), u_{v_0}(z) = u(z) \right\} \\ \tilde{K}_1 &= \cap_{i=0,\dots,n-1} K_{1,i} \cap \cap_{i=0,\dots,n-2} S_{1,i} \cap S_0 \end{aligned}$$

Then $\mu(K \setminus \tilde{K}_1) = 0$. Summing up the equalities (2) from $i = 0$ to $i = n-1$, we see that for $z \in \tilde{K}_1$,

$$\delta \le u(z + h(1)) - u(z) = \int_0^1 (Du(z + h(s)), \dot{h}(s)) ds = 0$$

This is a contradiction. □

REMARK 10. Our method can be applied to the Dirichlet forms which were studied by Albeverio and Röckner [7]. Let us recall their setting. Let E be a locally convex Hausdorff topological vector space with probability measure μ. We assume that there exists a Hilbert space H such that the embedding $E' \subset H \subset E$ is continuous and dense. Also we assume that μ is quasi-invariant in the direction of H. For $h \in H$, we denote

$$\frac{d\mu(\cdot + h)}{d\mu}(z) := a_h(z)$$

and assume that the limit

$$\beta_h(z) := \lim_{s \to 0} \frac{\sqrt{a_{sh}(z)} - 1}{s}$$

exists in $L^2(E, d\mu)$ and for any $t_1 < t_2$

$$(3) \qquad \int_{t_1}^{t_2} |\beta(z + tk)| dt < \infty \quad \mu\text{-a.s. } z.$$

Then exactly by the same argumet to [7], we can define the Dirichlet form $\mathcal{E}_U$ on $U \subset E$. In the case of abstract Wiener space, the definition coincides with the definition in

Definition 1. Also if we assume the ergodicity of μ in the direction of H, then our method proves the irreducibility of $\mathcal{E}_U$ if U is H-connected.

REMARK 11. Actually Kusuoka [15] proved a stronger statement under additional assumption, strong $C^\infty - C_0$ property of ϕ. Strong $C^\infty - C_0$ property is a kind of locally uniform continuity in the direction of H. Let us explain the results. Let $p(t, x, A)$ be the transition probability of the diffusion process which is defined by the Dirichlet form $\mathcal{E}_U$. Also let us denote the transition probability of usual Ornstein-Uhlenbeck process on B by $p_O(t, x, A)$. Then μ-a.s. $x \in U$, $p(t, x, dy)$ is absolutely continuous with respect to $p_O(t, x, dy)$. Let us denote the density function by

$$\frac{dp(t, x, dy)}{dp_O(t, x, dy)} = \rho(t, x, y).$$

Then Kusuoka proved that $\rho(t, x, y) > 0 \quad \mu_U \otimes \mu_U$-a.s. (x, y). This implies immediately the uniform positivity improving property (= UPIP) of the diffusion semigroup on U. Namely for any $\varepsilon > 0$,

$$\inf\left\{\int_A p(t, x, B) \mid \mu_U(A), \mu_U(B) \geq \varepsilon, A, B \text{ are measurable subsets}\right\} > 0.$$

Note that Hino [10] proved the UPIP is a necessary condition of the existence of spectral gap of the Dirichlet form. For the Dirichlet form which is defined in Remark 10, we may establish UPIP under the assumption that the original Dirichlet form on E has the spectral gap.

3. Irreducibility on loop group

In the previous section, we establish a criterion of the irreducibility of the Dirichlet form on a domain in Wiener space. However the proof can be carried out on path space over Lie group too. Let G be a compact Lie group with an Ad-invariant Riemannian metric on its Lie algebra $\mathfrak{g}$. The Levi-Civita Laplacian defines the Brownian motion measure μ_e on the path group

$$P_e(G) = C([0, 1] \to G \mid \gamma(0) = e)$$

pinned Brownian motion measure ν_e on the based loop group

$$L_e(G) = \{\gamma \in P_e(G) \mid \gamma(1) = e\}.$$

By the pointwise multiplication such that $(\gamma \cdot \tilde{\gamma})(t) = \gamma(t) \cdot \tilde{\gamma}(t)$, $P_e(G)$, $L_e(G)$ are themselves also groups. Let us denote the energy of the path $c \in P_e(G)$ by $\|c\|_{\mathcal{P}}^2 = \int_0^1 |\dot{c}(t)|_{T_{c(t)}G}^2 dt$ if it is finite. Let us denote the subgroups of $P_e(G)$, $L_e(G)$ which consist of the energy finite paths by $\mathcal{P}$, $\mathcal{L}$ respectively. These spaces correspond to the

Cameron-Martin subspaces in the case of Wiener spaces. Namely under the multiplication by the finite energy paths, the measures μ_e, ν_e are quasi-invariant. In fact,

Lemma 12. *For any $c \in \mathcal{P}$, there exists a positive measurable function $\rho_c(\gamma)$ on $P_e(G)$ such that for any bounded measurable function f*

$$\int_{P_e(G)} f(c \cdot \gamma) d\mu_e(\gamma) = \int_{P_e(G)} f(\gamma)\rho_c(\gamma) d\mu_e(\gamma)$$

and for any $p \geq 1$

$$\|\rho_c\|_{L^p} = \exp\left(\frac{p-1}{2}\|c\|_{\mathcal{P}}^2\right).$$

Let

$$\begin{aligned} H &:= H^1([0,1] \to \mathfrak{g} \mid h(0) = 0) \\ H_0 &:= H^1([0,1] \to \mathfrak{g} \mid h(0) = 0, h(1) = 0) \end{aligned}$$

Let us denote the function spaces of the smooth cylindrical functions on $P_e(G)$, $L_e(G)$ by the same notation $\mathcal{F}C_b^\infty$. For a function $u \in \mathcal{F}C_b^\infty$ on $P_e(G)$ and $h \in H$, define

$$(\nabla u(\gamma), h)_H := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(u(e^{\varepsilon h} \cdot \gamma) - u(\gamma)\right),$$

where $e^{\varepsilon h}(t) = \exp(\varepsilon h(t))$ and the definition of the derivative on $L_e(G)$ is given in similar way replacing H by H_0. The gradient operators with the domains $(\nabla, \mathcal{F}C_b^\infty)$ define the closable symmetric Markovian forms on $P_e(G)$, $L_e(G)$. We denote the Markovian extensions by $(\mathcal{E}_{\text{path}}, \mathcal{D}_{\text{path}})$, $(\mathcal{E}_{\text{loop}}, \mathcal{D}_{\text{loop}})$ for the path group and the loop group respectively. It is already known that the closed extension of these forms are unique respectively (see Acosta [1] in the case of path group, [2] in the case of loop group). Namely the essential selfadjointness of the generator on the domain $\mathcal{F}C_b^\infty$ holds. We will denote the Sobolev spaces by

$$\begin{aligned} &\mathcal{D}_1^r := \text{the completion of } \mathcal{F}C_b^\infty \text{ with norm} \\ &\|u\|_{1,r}^r = \int_{P_e(G)} u(\gamma)^r d\mu_e(\gamma) + \int_{P_e(G)} |Du(\gamma)|^r d\mu_e(\gamma) \end{aligned}$$

By the quasi-homeomorphism property of Shigekawa's results [19], the irreducibility of the Dirichlet form $(\mathcal{E}_{\text{path}}, \mathcal{D}_{\text{path}})$ follows from the irreducibility of the usual Ornstein-Uhlenbeck process in Wiener space. Our main theorem is the following.

Theorem 13. *Let G be a simply connected Lie group. Then $(\mathcal{E}_{\text{loop}}, \mathcal{D}_{\text{loop}})$ is irreducible.*

Let us recall the history of the irreducibility problem of the Dirichlet form on loop group. It is natural to guess that the Dirichlet form $\mathcal{E}_{\mathrm{loop}}$ is irreducible if G is a simply connected Lie group. The first proof of the irreducibility of the Dirichlet form was given by Gross [9]. Actually he obtained a deeper results, namely, a characterization of the L^2 function on $P_e(G)$ of the form $f(\gamma(1))$ in terms of the (roughly speaking) universal enveloping algebra of the Lie algebra, where f is a function on G. This work leads the very interesring by-product, a noncommutative version of the celebrated isomomorphism between L^2-space with Gaussian measure and the Boson Fock space. However the proof is not easy. After Gross' proof, Sadasue [18] gave a short and simple proof. Our proof of the irreducibility of the Dirichlet form in §2 has the same spirit as his proof in the sense that the main idea is to reduce the problem to the ergodicity of the measure on the whole space. Also [3] and Léandre [16] proved the irreducibility of Dirichlet forms in the case of homogeneous spaces. Actually the Dirichlet form on loop space can be defined according to each torsion skew symmetric connection. It is proved in [4] and [17] that the irreducibility of the Dirichlet form holds for the Levi-Civita connection. In [6], the author proved the irreducibility for any torsion skew symmetric connection.

Note that the irreducibility and the uniqueness of the closed extension of the Markovian form implies the ergodicity of the measure μ_e and ν_e. To see it, we will prove the following lemma.

Lemma 14. *Let $\tilde{\mathcal{D}}$ be the space of the functions u on $P_e(G)$ such that*

(1) *for any $h \in H$, there exists a measurable function $u_h(t, \gamma)$ such that $u_h(t, \gamma) = u(e^{th} \cdot \gamma)$ μ_e-a.s. γ for all t and the function t $(\in [0, 1]) \to u_h(t, \gamma)$ is absolutely continuous.*

(2) *there exists a map $F \in L^2(P_e(G) \to H, d\mu_e)$ such that*

$$\lim_{t\downarrow 0} \frac{u(e^{th} \cdot \gamma) - u(\gamma)}{t} = (F(\gamma), h)_H$$

in probability.

We will denote $Du(\gamma) := F(\gamma)$. Let us define for $u \in \tilde{\mathcal{D}}$

$$\tilde{\mathcal{E}}(u, u) = \int_{P_e(G)} |Du(\gamma)|_H^2 d\mu_e(\gamma)$$

Then the symmetric form $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ is a Dirichlet form.

Proof. Assume that $\{u_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{D}}$ and $u_n \to u$ in $\tilde{\mathcal{E}}_1$ sense. By the same argument as in Lemma 5, for any h and $t \in [0, \infty)$

$$(u_n)_h(t, \gamma) = (u_n)_h(0, \gamma) + \int_0^t ((Du_n)(e^{sh} \cdot \gamma), h)_H ds \quad \mu_e\text{-a.s. } \gamma.$$

Letting $n \to \infty$, using the quasi-invariance of μ_e and $\rho_h(\gamma) := \rho_{e^h}(\gamma) \in \cap_{p>1} L^p(P_e(G))$ we have for $1 < p < 2$,

$$\begin{aligned}
&\lim_{n\to\infty} \int_{P_e(G)} |u(e^{th}\cdot\gamma) - u_n(e^{th}\cdot\gamma)|^p d\mu_e(\gamma) \\
&= \lim_{n\to\infty} \int_{P_e(G)} |u(\gamma) - u_n(\gamma)|^p \rho_{th}(\gamma) d\mu_e(\gamma) \\
&\le \lim_{n\to\infty} \|u - u_n\|_2^p \cdot \left\{ \int_{P_e(G)} \rho_{th}(\gamma)^{2/2-p} d\mu_e(\gamma) \right\}^{(2-p)/2} = 0
\end{aligned}$$

and also by the similar argument, we see

$$\lim_{n\to\infty} \int_{P_e(G)} \left| \int_0^t (Du(e^{sh}\cdot\gamma), h) ds - \int_0^t (Du_n(e^{sh}\cdot\gamma), h) ds \right|^p d\mu_e(\gamma) = 0$$

Hence we have

$$u(e^{th}\cdot\gamma) = u(\gamma) + \int_0^t ((Du)(e^{sh}\cdot\gamma), h)_H ds \quad \mu_e\text{-a.s. } \gamma.$$

So the right hand side gives the absolutely continuous version of $u(e^{th}\cdot\gamma)$. Next we will prove that for any $1 < p < 2$,

$$\lim_{t\downarrow 0} \int_{P_e(G)} \left| \frac{u(e^{th}\cdot\gamma) - u(\gamma)}{t} - (Du(\gamma), h) \right|^p d\mu_e(\gamma) = 0$$

which implies the validity of (2). By the denseness of $\mathcal{F}C_b^\infty \otimes H \subset L^2(P_e(G) \to H, d\mu_e)$, for any $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{F}C_b^\infty \otimes H$ such that $\|f_\varepsilon - Du\|_2 \le \varepsilon$. Again using Lemma 12, we have

$$\begin{aligned}
&\lim_{t\downarrow 0} \int_{P_e(G)} \left| \frac{u(e^{th}\cdot\gamma) - u(\gamma)}{t} - (Du(\gamma), h) \right|^p d\mu_e(\gamma) \\
&\le \lim_{t\downarrow 0} \frac{C_p}{t} \|h\|^p \int_0^t \int_{P_e(G)} \left| Du(e^{sh}\cdot\gamma) - f_\varepsilon(e^{sh}\cdot\gamma) \right|^p d\mu_e(\gamma) ds \\
&\quad + \lim_{t\downarrow 0} \frac{C_p}{t} \|h\|^p \int_0^t \int_{P_e(G)} \left| f_\varepsilon(e^{sh}\cdot\gamma) - f_\varepsilon(\gamma) \right|^p d\mu_e(\gamma) ds \\
&\quad + C_p \|h\|^p \int_{P_e(G)} |f_\varepsilon(\gamma) - Du(\gamma)|^p d\mu_e(\gamma) \\
&\le C_p \varepsilon^p \|h\|^p \sup_{0\le t\le 1} \left\{ \int_{P_e(G)} \rho_{th}(\gamma)^{2/2-p} d\mu_e(\gamma) \right\}^{(2-p)/2} + C_p \varepsilon^p \|h\|^p
\end{aligned}$$

which completes the proof. □

Lemma 15. *If u satisfies that $u(e^h \cdot \gamma) = u(\gamma)$ μ_e-a.s. for any $h \in H$, then $u \in \mathcal{D}_{\text{path}}$ and $Du(\gamma) = 0$, i.e., u is a constant function.*

Proof. There exists $\Omega \subset P_e(G)$ with $\mu_e(P_e(G) \setminus \Omega) = 0$ and $u_h(t, \gamma)$ for any $t \in [0, \infty) \cap \mathbb{Q}$ such that

$$\begin{aligned} u_h(t, \gamma) &= u(e^{th} \cdot \gamma) \quad \mu_e\text{-a.s.}\gamma \\ u_h(t, \gamma) &= u_h(s, \gamma) \quad \text{for any } t, s \in \mathbb{Q} \text{ and } \gamma \in \Omega \end{aligned}$$

Therefore we can define the value $u_h(t, \gamma)$ for any $\gamma \in \Omega$ and $t \in \mathbb{R} \setminus \mathbb{Q}$ as follows

$$u_h(t, \gamma) = \lim_{s(\in\mathbb{Q})\to t} u_h(s, \gamma).$$

Then cleary the function $t \to u_h(t, \gamma)$ is constant function for each γ, in particular, absolutely continuous and $\lim_{t\to 0}\{u(e^{th} \cdot \gamma) - u(\gamma)\}/t = 0$. By the uniqueness of the closed extension of $(\mathcal{E}, \mathcal{F}C_b^\infty)$, this implies $u \in \tilde{\mathcal{D}} = \mathcal{D}_{\text{path}}$ and by the irreducibility of $\mathcal{E}_{\text{path}}$, u should be a constant function. □

As a corollary of this lemma, we see that the ergodicity of the measure μ_e holds as in Lemma 6 if we replace the addition $A_1 + v$ by the multiplication from the left side $v \cdot A_1$. Note that in the present situation, V is a countable dense subset in $\mathcal{P}$. As in §1, let us introduce the spaces C_v, C_V replacing H by $\mathcal{P}$ and 0 by $\mathbf{e}$ which is a constant map such that $\mathbf{e}(t) \equiv e$. Here C_v is a dense countable set of $\{c \in \mathcal{P} \mid c(0) = \mathbf{e},\ c(1) = v\}$ and set $C_V := \cup_{v\in V} C_v := \{h_n(t) \mid n = 1, \ldots, \}$. As in the Wiener space, let us introduce the $\mathcal{P}$-continuous function.

DEFINITION 16. A measurable function $\phi : P_e(G) \to \mathbb{R}$ is a $\mathcal{P}$-continuous function if the map $c\ (\in \mathcal{P}) \to \phi(c \cdot \gamma)$ is continuous.

Corresponding to Corollary 4, we see

Theorem 17. *Let $\phi : P_e(G) \to \mathbb{R}$ be a $\mathcal{P}$-continuous function. Assume $U := \{\gamma \in P_e(G) \mid \phi(\gamma) > 0\}$ is $\mathcal{P}$-connected, i.e. $U(\gamma) = \{c \in \mathcal{P} \mid c \cdot \gamma \in U\}$ is a connected open set in $\mathcal{P}$. Then for $u \in \mathcal{D}_1^r$ $(r > 1)$ with $Du(\gamma) = 0$ μ_e-a.s. $\gamma \in U$, u is a constant function μ_e-a.s. $\gamma \in U$.*

Proof. Assume u is not a constant function in U. Then there exist two subsets A_1, A_2 with $\mu_e(A_i) > 0$ and $\delta > 0$ such that $\inf_{z\in A_1} u(z) - \sup_{z\in A_2} u(z) \geq \delta$. Since ergodicity holds in $(P_e(G), \mu_e)$, by the argument similar to Lemma 8, we see that there exists a compact subset $K_1 \subset A_1$ with $\mu(K_1) > 0$ and $h \in C_V$ such that

$$h(s) \cdot K_1 \subset U \quad \text{for any } s \in [0, 1]$$

$$h(1) \cdot K_1 \subset A_2$$

Since $u \in \mathcal{D}_1^r$, there exist $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F}\mathcal{C}_b^{\infty}$ such that $\lim_{n\to\infty} \|u - u_n\|_{1,r} = 0$. For u_n, we have

$$u_n(h(t) \cdot \gamma) = u_n(\gamma) + \int_0^t \left(Du_n(h(s) \cdot \gamma), \dot{h}(s)\right)_H ds$$

Letting $n \to \infty$, we get μ_e-a.s. $\gamma \in K_1$

$$\delta \leq u(h(1) \cdot \gamma) - u(\gamma) = \int_0^1 \left(Du(h(s) \cdot \gamma), \dot{h}(s)\right)_H ds = 0$$

This is a contradiction. □

As an application of Theorem 17, we will prove Theorem 13 using the next lemma.

Lemma 18. *Let G be a simply connected Lie group. Let $B(\varepsilon)$ be the ε-ball centered at the origin in T_eG and assume that the exponential map* $\exp : B(\varepsilon) \to V \subset G$ *is diffeomorphism. Let $\mathcal{U} := \{\gamma \in P_e(G) \mid \gamma(1) \in V\}$. Then $\mathcal{U}$ is $\mathcal{P}$-connected.*

Proof. For $g \in V$, let

$$\eta_g(t) = \exp(-t \exp^{-1}(g))$$

Assume $\gamma \in \mathcal{U}$ and define for $0 \leq s,\ t \leq 1$

$$\Phi_s(t) = \eta_{\gamma(1)}(st)\gamma(t).$$

Then the continuous curve $\{\Phi_s\}_{0\leq s\leq 1}$ is in $\mathcal{U}$ and $\Phi_0 = \gamma$, $\Phi_1 \in L_e(G)$. So it suffices to prove that for any two γ, $c \cdot \gamma \in L_e(G)$, where $c \in \mathcal{L}$, we can find that continuous curve $l : [0, 1] \to \mathcal{L}$ with $l(0) = \mathbf{e}$, $l(1) = c$. This is true because of the simply connectivity of G. □

Now we are in a position to prove Theorem 13.

Proof of Theorem 13. Let $V' := \exp(B(\varepsilon/2))$ and set

$$\varphi(x) = \begin{cases} 1 & (x \in \exp(B(\varepsilon/2))) \\ 0 & (x \in \exp(B(\varepsilon))^c) \end{cases}$$

$$\mathcal{U}' = \{\gamma \mid \gamma(1) \in V'\}$$

Let us define for $u : \mathcal{L} \to \mathbb{R}$,

$$Tu(\gamma) = u(\eta_{\gamma(1)}\gamma)\varphi(\gamma(1))$$

We have

$$\int_{\mathcal{U}'} |Tu(\gamma)|^p d\mu_e(\gamma) = \int_{\mathcal{L}} |u(\gamma)|^p Z(\gamma) d\nu_e(\gamma) \tag{4}$$

where $Z(\gamma) = \int_V \rho_{\eta_\xi}(\gamma) p(1,\xi)\varphi(\xi) d\xi$ and $p(t,\xi)$ is the heat kernel on G solving the equation $\partial u/\partial t(t,x) = (1/2)\Delta u(t,x)$, $u(0,\xi) = \delta_e$, where Δ is the Laplace-Beltrami operator. By the result in Gross [8], if $u \in \mathcal{D}_{\text{loop}}$ then $Tu \in \mathcal{D}_1^r$ where $1 < r < 2$ and

$$D(Tu)(\gamma) = 0 \quad \mu_e\text{-a.s. } \gamma \in \mathcal{U}'.$$

Hence by Theorem 17 and Lemma 18, we see that Tu is constant function in $\mathcal{U}'$. By (4), this implies u is a constant function. □

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