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r-FOLD ζ-SKEW-SYMMETRIC MULTILINEAR FORMS

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Let R be a commutative ring with identity 1, and for an integer $r \geqq 2$, ζ an element of R with $\zeta^r = 1$. For an R-module M and an r-fold multilinear map θ on M, we shall say that θ is ζ-skew symmetric, if $\theta(x_1, x_2, x_3, \cdots, x_r) = \zeta\,\theta(x_2, x_3, \cdots, x_r, x_1)$ holds for every elements $x_1, x_2, x_3, \cdots, x_r \in M$. In this paper, we investigate the R-module with r-fold ζ-skew symmetric multilinear map. In §1, we prove some fundamental properties on r-fold ζ-skew symmetric multilinear R-modules, which include ones on symmetric or cyclically-symmetric multilinear R-modules in [H_2] or [K_2]. In §2, we give two examples of r-fold ζ-skew-symmetric multilinear R-modules, one is the determinants of matrices, and another is a 3-fold trace form of an R-algebra. In §3, we shall show that a finitely generated ζ-skew symmetric multilinear R-module is characterized by an r-fold ζ-skew-symmetric matrix, which is an expansion of [K_1]. In §4, for a 3-fold 1-skew symmetric multilinear R-module $\langle [\boldsymbol{A}] \rangle$ defined by a 3-fold 1-skew symmetric matrix $\boldsymbol{A}$, we give some conditions for $\langle [\boldsymbol{A}] \rangle$ to be an associative R-algebra by some multiplication on $\langle [\boldsymbol{A}] \rangle$.

1. r-fold ζ-skew-symmetric multilinear R-module $(M, \theta; U)$

Let R be a commutative ring with identity 1, r a positive integer ($r \geqq 2$), ζ an element of R with $\zeta^r = 1$, and U a faithful R-module.

DEFINITION For an R-module M, we shall call $(M, \theta; U)$ an r-fold *ζ-skew-symmetric multilinear R-module*, simply r-fold *ζ-skew-symmetric R-module,* if θ: $M \times M \times \cdots \times M \to U$; $(x_1, x_2, x_3, \cdots, x_r) \rightsquigarrow \theta(x_1, x_2, x_3, \cdots, x_r)$ is an r-fold multilinear map of M into U satisfying $\theta(x_1, x_2, x_3, \cdots, x_r) = \zeta\,\theta(x_2, x_3, \cdots, x_r, x_1)$. If $\zeta = 1$, r-fold 1-skew-symmetric R-module is called an r-fold cyclically symmetric R-module. By θ^* and θ_*, one denotes the following R-homomorphisms:

$$\theta_*\colon M \to \mathrm{Hom}_R(\otimes_R^{r-1} M, U);\ x \rightsquigarrow \theta(x, -)\,, \quad \text{and}$$
$$\theta_*\colon \otimes_R^{r-1} M \to \mathrm{Hom}_R(M, U);\ x_1 \otimes \cdots \otimes x_{r-1} \rightsquigarrow \theta(-, x_1, \cdots, x_{r-1})\,,$$

where $\otimes_R^{r-1} M$ and $\theta(x, -)$ denote $\otimes_R^{r-1} M = M \otimes_R M \otimes_R \cdots \otimes_R M$: the tensor product of r-1-copies of M over R, and $\theta(x, -)$: $\otimes_R^{r-1} M \to U$; $x_2 \otimes \cdots \otimes x_r \rightsquigarrow \theta(x, x_2, \cdots, x_r)$. $(M, \theta; U)$ is said to be *regular*, if θ^* is injective. If θ^* is injective, and if θ_* is surjective, then $(M, \theta; U)$ is *nondegenerate*. Furthermore, $(M, \theta; U)$ is said to be finitely generated, projective, if M is finitely generated, projective over R, respectively. If $U=R$, $(M, \theta; R)$ is denoted by (M, θ).

Lemma 1. *Let $(M, \theta; U)$ be an r-fold ζ-skew-symmetric finitely generated projective R-module. Then, $(M, \theta; U)$ is nondegenerate if and only if θ_* is surjective. In particular, an r-fold ζ-skew-symmetric R-module (M, θ) is nondegenerate and finitely generated projective over R if and only if there exist $x_{2,j}, x_{3,j}, \cdots, x_{r,j}, z_j \in M$; $j=1, 2, \cdots, n$ with $x=\sum_{j=1}^{n} \theta(x, x_{2,j}, x_{3,j}, \cdots, x_{r,j}) z_j$ for all $x \in M$,* (cf. [H_2]; Lemma 1.1).

Proof. Let $(M, \theta; U)$ be an r-fold ζ-skew-symmetric finitely generated projective R-module. We shall show that if θ_* is surjective then θ^* is injective. Suppose θ^* is surjective and $x \in \operatorname{Ker} \theta^*$. Since M is finitely generated projective over R, there are $\psi_1, \psi_2, \cdots, \psi_m \in \operatorname{Hom}_R(M, R)$ and $y_1, y_2, \cdots, y_m \in M$ such that $x=\sum_{i=1}^{m} \psi_i(x) y_i$. For any $u \in U$, $\psi_k u=\psi_k(-) u$ is contained in $\operatorname{Hom}_R(M, U)$ $=\operatorname{Im} \theta_*$, hence there is a $\sum_i x_{i2} \otimes x_{i3} \otimes \cdots \otimes x_{ir} \in \otimes_R^{r-1} M$ with $\theta_*(\sum_i x_{i2} \otimes x_{i3} \otimes \cdots \otimes x_{ir})=\psi_k(-) u$. $\theta(x, -)=0$ implies that $\psi_k(x) u=\sum_{i=1}^{t} \theta(x, x_{i2}, \cdots, x_{ir})=0$ for all $u \in U$, so $\psi_k(x)=0$; $k=1, 2, \cdots, m$. Hence we get $x=\sum_{i=1}^{m} \psi_i(x) y_i=0$, and θ^* is injective. The second part of the lemma is easy.

For an r-fold ζ-skew-symmetric R-module $(M, \theta; U)$, we can define quite similar notions "*orthogonal sum*" and "*the center of* $(M, \theta; U)$" to ones in [H_2]. Let L and N be R-submodules of M. If $\theta(x, y, z_3, \cdots, z_r)=\theta(y, x, z_3, \cdots, z_r)=0$ holds for all $x \in L$, $y \in N$ and $z_3, \cdots, z_r \in M$, then L and N are said to be *orthogonal*, and $L+N$ is denoted by $L \perp N$, furthermore, $N^{\perp}$ denotes $\{x \in M \mid \theta(x, y, z_3, \cdots, z_r)=\theta(y, x, z_3, \cdots, z_r)=0; \forall y \in N, \forall z_3, \cdots, z_r \in M\}$. $\boldsymbol{Z}(M, \theta; U)$ $=\{f \in \operatorname{Hom}_R(M, M) \mid \theta(f(x_1), x_2, x_3, \cdots, x_t)=\theta(x_1, f(x_2), x_3, \cdots, x_r)$ for all $x_1, x_2, \cdots, x_r \in M\}$ is called the center of $(M, \theta; U)$.

Lemma 2. *Let $(M, \theta; U)$ be a regular r-fold ζ-skew-symmetric R-module with $r \geqq 3$.*
(1) (cf [H_2]; 2.2, 2.3, 2.4) *Let L and N be R-submodules of M such that $M=L \perp N$. Then, $(N, \theta|_N, U)$ is regular, $L \cap N=\{0\}$ and $L=N^{\perp}$ hold. If L' and N' are another R-submodules of M with $M=L' \perp N'$, then L' is decomposed as follows; $L'=(L' \cap L) \perp (L' \cap N)$. Therefore, if $(M, \theta; U)$ has an orthogonal decomposition of a finite number of indecomposable components, then the indecomposable components are uniquely determined up to isomorphisms. If $(M, \theta; U)$ is nondegenerate, so is $(N, \theta|_N, U)$.*
(2) (cf. [H_2]; 4.1) *$\boldsymbol{Z}(M, \theta; U)$ is a commutative R-algebra, and $(M, \theta; U)$ is orthogonally indecomposable if and only if $\boldsymbol{Z}(M, \theta; U)$ has no idempotents without* 0 *and* 1.
(3) *Let $(M', \theta'; U)$ another r-fold ζ-skew-symmetric R-module, f: $M \rightarrow M'$ an R-*

homomorphism satisfying $\theta'(f(x_1), f(x_2), f(x_3), \cdots, f(x_r))=\theta(x_1, x_2, x_3, \cdots, x_r)$ *for all* $x_1, x_2, x_3, \cdots, x_r \in M$. *If* $(M, \theta; U)$ *is regular, then f is injective.*

Proof. Some parts of this lemma are similarly proved to the proof of [H_2]. (1): Suppose $M=L\perp N$. First, we show $M=L\oplus N$. For any $x\in L\cap N$ and $y_2, y_3, \cdots, y_r\in M$, we have $y_2=y_2'+y_2''$ for some $y_2'\in L$ and $y_2''\in N$, and $\theta(x, y_2, y_3, \cdots, y_r)=\theta(x, y_2', y_3, \cdots, y_r)+\theta(x, y_2'', y_3, \cdots, y_r)=0$, so $x=0$ and $L\cap N=\{0\}$. To see that $(N, \theta|_N; U)$ is regular, suppose $x\in \mathrm{Ker}(\theta|_N)^*$. For any $y_i=y_i'+y_i''\in M$ with $y_i'\in L$ and $y_i''\in N$; $i=2, 3, \cdots, r$, $\theta(x, y_2, y_3, \cdots, y_r)=\theta(x, y_2', y_3, \cdots, y_t)+\theta(x, y_2'', y_3, \cdots, y_r)=\theta(x, y_2'', y_3, \cdots, y_r)=\zeta\,\theta(y_2'', y_3, \cdots, y_r, x)=\zeta\,\theta(y_2'', y_3', \cdots, y_r, x)+\zeta\,\theta(y_2'', y_3'', \cdots, y_r, x)=\zeta\,\theta(y_2'', y_3'', \cdots, y_r, x)=\cdots=\zeta^{r-1}\theta(y_r'', x, y_2'', y_3'', \cdots, y_{r-1}'')=\theta(x, y_2'', y_3'', \cdots, y_r'')=0$, hence $x=0$. To see $N^{\perp}=L$, suppose $x\in N^{\perp}$ and $x=x'+x''$ with $x'\in L$, $x''\in N$. For any $y_i=y_i'+y_i''\in M$ with $y_i'\in L$, $y_i''\in N$; $i=2, 3, \cdots, r$, we have $\theta(x'', y_2, y_3, \cdots, y_r)=\theta(x'', y_2'', y_3, \cdots, y_r)=\theta(x, y_2'', y_3, \cdots, y_r)=0$, hence $x''=0$, that is, $x=x'\in L$. Suppose L' and N' are another R-submodules of M with $M=L'\perp N'$. Then, from the above statement, we get $N'=L'^{\perp}$ and $L'=N'^{\perp}$. To see $L'=(L'\cap L)+(L'\cap N)$, suppose x is any element in L', and $x=x'+x''$ with $x'\in L$, $x''\in N$. For any $y\in N'$ and $z_i\in M$ written as $y=y'+y''$ and $z_i=z_i'+z_i''$ for $y', z_i'\in L$ and $y'', z_i''\in N$, $i=3, \cdots, r$, we have $\theta(x', y, z_3, \cdots, z_r)=\theta(x', y'+y'', z_3'+z_3'', \cdots, z_r'+z_r'')=\theta(x', y', z_3'+z_3'', \cdots, z_r'+z_r'')=\zeta\,\theta(y', z_3'+z_3'', \cdots, z_r'+z_r'', x')=\zeta\,\theta(y', z_3', \cdots, z_r'+z_r'', x')=\cdots=\theta(x', y', z_3', \cdots, z_r')=\theta(x', y', z_3', \cdots, z_r')+\theta(x'', y', z_3', \cdots, z_r')+\theta(x', y'', z_3', \cdots, z_r')+\theta(x'', y'', z_3', \cdots, z_r')=\theta(x, y, z_3', \cdots, z_r')=0$, since $\theta(x'', y', z_3', \cdots, z_r')=(x', y'', z_3', \cdots, z_r')=\zeta\,\theta(y'', z_3', \cdots, z_r', x'')=0$. Hence x' is in $N'^{\perp}(=L')$, that is, $x'\in L'\cap L$. Therefore, $x''=x-x'$ is also in $L'\cap N$, and we get $L'=(L'\cap L)\perp(L'\cap N)$. In the last, we suppose that $(M, \theta; U)$ is nondegenerate. $M=L\oplus N$ means that for any $f\in \mathrm{Hom}_R(N, U)$, there is an $F\in \mathrm{Hom}_R(M, U)$ such that $F|_N=f$. There exists an element $\sum_i x_{i,2}\otimes x_{i,3}\otimes\cdots\otimes x_{i,r}$ in $\otimes_R^{r-1}M$ such that $F(x)=\sum_i \theta(x, x_{i,2}\, x_{i,1}, \cdots, x_{i,r})$ for every $x\in M$. If $x\in N$ and $x_{i,2}=x_{i,2}'+x_{i,2}''$ for $x_{i,2}'\in L$, $x_{i,2}''\in N$, then $f(x)=\sum_i \theta(x, x_{i,2}, x_{i,3}, \cdots, x_{i,r})=\sum_i \theta(x, x_{i,2}'', x_{i,3}, \cdots, x_{i,r})=\sum_i \zeta\,\theta(x_{i,2}'', x_{i,3}, \cdots, x_{i,r}, x)=\cdots=\sum_i \theta(x, x_{1,2}'', x_{1,3}'', \cdots, x_{1,r}'')=(\theta|_N)_*(\sum_i x_{1,2}''\otimes x_{i,3}''\otimes\cdots\otimes x_{i,r}'')(x)$, and $\sum_i x_{i,2}''\otimes x_{i,3}''\otimes\cdots\otimes x_{i,r}''\in\otimes_R^{r-1}N$. Hence $(N, \theta|_N, U)$ is nondegenerate. (2): For any $f, g\in \boldsymbol{Z}(M, \theta; U)$, $\theta(f(g(x_1)), x_2, x_3, \cdots, x_r)$ is computed as follows: $\theta(f(g(x_1)), x_2, x_3, \cdots, x_r)=\theta(g(x_1), f(x_2), x_3, \cdots, x_r)=\theta(x_1, g(f(x_2)), x_3, \cdots, x_r)$ and $\theta(g(x_1), f(x_2), x_3, \cdots, x_r)=\zeta\,\theta(f(x_2), x_3, \cdots, x_r, g(x_1))=\zeta\,\theta(x_2, f(x_3), \cdots, x_r, g(x_1))=\zeta^2\,\theta(f(x_3), \cdots, x_r, g(x_1), x_2)=\cdots=\zeta^{r-1}\theta(f(x_r), g(x_1), x_2, \cdots, x_{r-1})=\theta(g(x_1), x_2, x_3, \cdots, x_{r-1}, f(x_r))=\theta(x_1, g(x_2), x_3, \cdots, x_{r-1}, f(x_r))=\zeta^{-1}\theta(f(x_r), x_1, g(x_2), x_3, \cdots, x_{r-1})=\zeta^{-1}\theta(x_r, f(x_1), g(x_2), x_3, \cdots, x_{r-1})=\theta(f(x_1), g(x_2), x_3, \cdots, x_{r-1}, x_r)=\theta(g(f(x_1)), x_2, x_3, \cdots, x_{r-1}, x_r)$. Hence, $fg=gf$, and fg is contained in $\boldsymbol{Z}(M, \theta; U)$. If $(M, \theta; U)$ has non trivial orthogonal decomposition $M=L\perp N$, the projection $e\colon N\to M$ is a non trivial idempotent in $\boldsymbol{Z}(M, \theta; U)$. Conversely, if $\boldsymbol{Z}(M, \theta; U)$ has an idempotent e different from 0 and 1, then we get $M=e(M)\perp(1-e)(M)$.

(3): $f(x)=0$ implies that $\theta(x, x_2, x_3, \cdots, x_r)=\theta(f(x), f(x_2), f(x_3), \cdots, f(x_r))=0$ for all $x_2, x_3, \cdots, x_{r-1}\in M$, that is, $x=0$, since θ is regular.

2. Examples

EXAMPLE 1. Let θ: $M\times M\times\cdots\times M\to U$; $(x_1, x_2, \cdots, x_r)\rightsquigarrow\theta(x_1, x_2, \cdots, x_r)$ be an r-fold alternative multilinear map, that is, $\theta(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_r)=-\theta(x_1, x_2, \cdots, x_{i+1}, x_i, \cdots, x_r)$ holds for $i=1, 2, \cdots, r-1$. Then, for $\zeta=(-1)^{r-1}$, $(M, \theta; U)$ is an r-fold ζ-skew-symmetric R-module.

For example, for $n\geqq r$, let R^n be free R-module of rank n consisting of n-rows $(a_1, a_2, \cdots, a_n)$ for all $a_i\in R$. For $\boldsymbol{a}_i=(a_{i1}, a_{i2}, \cdots, a_{in})\in R^n$; $i=1, 2, \cdots, r$, let $\boldsymbol{A}=(a_{ij})$ be an $r\times n$-matrix with (i, j)-entry a_{ij} for $i=1, 2, \cdots, r$ and $j=1, 2, \cdots, n$. Let L be a non-empty set of r-rows $(k_1, k_2, \cdots, k_r)$ of integers with $1\leqq k_1<k_2<\cdots<k_r\leqq n$. For a $(k_1, k_2, \cdots, k_r)\in L$, we denote by $\det(\boldsymbol{A}(k_1, k_2, \cdots, k_r))$ the determinant of an $r\times r$-submatrix $\boldsymbol{A}(k_1, k_2, \cdots, k_r)=(a_{i,k_j})$ of $\boldsymbol{A}$ consisting of k_1-column, k_2-column, $\cdots$, k_r-column of $\boldsymbol{A}$. Then, the sum $\Sigma_L \det(\boldsymbol{A}(k_1, k_2, \cdots, k_r))$ of $\det(\boldsymbol{A}(k_1, k_2, \cdots, k_r))$ for all $(k_1, k_2, \cdots, k_r)\in L$ defines an r-fold multilinear form D_L: $R^n\times R^n\times\cdots\times R^n\to R$; $(\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_r)\rightsquigarrow\Sigma_L \det(\boldsymbol{A}(k_1, k_2, \cdots, k_r))$. Then, (R^n, D_L) is an r-fold ζ-skew-symmetric R-module. If for every i with $1\leqq i\leqq n$, there is a unique element $(k_1, k_2, \cdots, k_r)$ in L with $i=k_j$ for some $1\leqq j\leqq r$, (necessarily, n is a multiple of r), then (R^n, D_L) is nondegnerate. Because, for each i-th projection p_i: $R^n\to R$; $(a_1, a_2, \cdots, a_n)\rightsquigarrow a_i$, if $(k_1, k_2, \cdots, k_r)$ is unique element of L with $i=k_j$, $(-1)^{j+1}\boldsymbol{e}(k_1)\otimes\cdots\otimes\boldsymbol{e}(k_{j-1})\otimes\boldsymbol{e}(k_{j+1})\otimes\cdots\otimes\boldsymbol{e}(k_r)$ $(\in\otimes_R^{r-1} R^n)$ satisfies $D_L(\boldsymbol{a}, (-1)^{j+1}\boldsymbol{e}(k_1), \cdots, \boldsymbol{e}(k_{j-1}), \boldsymbol{e}(k_{j+1}), \cdots, \boldsymbol{e}(k_r))=a_i$ for all $\boldsymbol{a}=(a_1, a_2, \cdots, a_n)\in R^n$. Hence, we get $p_i=(D_L)_*((-1)^{j+1}\boldsymbol{e}(k_1)\otimes\cdots\otimes\boldsymbol{e}(k_{j-1})\otimes\boldsymbol{e}(k_{j+1})\otimes\cdots\otimes\boldsymbol{e}(k_r))$, where $\boldsymbol{e}(1)=(1, 0, \cdots, 0)$, $\boldsymbol{e}(2)=(0, 1, 0, \cdots, 0)$, $\cdots$, $\boldsymbol{e}(n)=(0, \cdots, 0, 1)$ $(\in R^n)$. Therefore, $(D_L)_*$: $\otimes_R^{r-1} R^n\to\mathrm{Hom}_R(R^n, R)$ is surjective, so by Lemma 1 (R^n, D_L) is nondegenerate. Particularly, if $n=r$, (R^r, D) is nodegenerate.

EXAMPLE 2. Let A be a non commutative R-algebra with identity 1 such that A is a finitely generated projective R-module with a projective dual basis $\{b_i\in A$ and $\psi_i\in\mathrm{Hom}_R(A, R)$; $i=1, 2, \cdots, n\}$, i.e. $x=\Sigma_{i=1}^n\psi_i(x)\, b_i$ for all $x\in A$. The trace map $\mathrm{Tr}_{A/R}$ of A is defined by $\mathrm{Tr}_{B/R}$: $A\to R$; $x\rightsquigarrow\Sigma_{i=1}^n\psi_i(xb_i)$. Then, one reminds that the bilinear form B_A: $A\times A\to R$: $(x, y)\rightsquigarrow\mathrm{Tr}_{A/R}(xy)$ is symmetric, and it does not depend on choice of projective dual basis. The symmetric bilinear R-module (A, B_A) is denoted by $\langle A\rangle$. Furthermore, the 3-fold multilinear form Γ_A: $A\times A\times A\to R$; $(x, y, z)\rightsquigarrow\mathrm{Tr}_{A/R}(xyz)$ defines a 3-fold cyclically symmetric R-module (A, Γ_A) which is denoted by $\langle\!\langle A\rangle\!\rangle$.

Proposition 1. *Let A be an R-algebra with identity 1 such that A is finitely generated and projective over R.*

(1) *$\langle A\rangle$ is regular if and only if $\langle\!\langle A\rangle\!\rangle$ is regular.*

(2) *The following conditions are equivalent:*

(1) *There exists a* $\Sigma_i a_i \otimes b_i \in A \otimes_R A$ *such that* $\Sigma_i b_i a_i = 1$ *and* $\Sigma_i x a_i \otimes b_i = \Sigma_i b_i \otimes a_i x$ *for all* $x \in A$ *hold,*

(2) *There exists a* $\Sigma_i a_i \otimes b_i \in A \otimes_R A$ *such that* $\Sigma_i b_i a_i = \Sigma_i a_i b_i = 1$ *and* $\Sigma_i x a_i \otimes b_i = \Sigma_i a_i \otimes b_i x$ *for all* $x \in A$ *hold,*

(3) $\langle A \rangle = (A, B_A)$ *is nondegenerate,*

(4) $\langle\!\langle A \rangle\!\rangle = (A, \Gamma_A)$ *is nondegenerate.*

(3) *If* $\langle\!\langle A \rangle\!\rangle$ *is regular, then the center* $\boldsymbol{Z}(\langle\!\langle A \rangle\!\rangle)$ *of* $\langle\!\langle A \rangle\!\rangle$ *coincides with* $\{f_a \colon A \to A;\ x \rightsquigarrow xa \mid a \in \boldsymbol{Z}(A)\}$, *where* $\boldsymbol{Z}(A)$ *denotes the center of algebra A.*

(4) (cf. [W]; Theorem 3) *Let B be an another R-algebra with identity* 1 *which is finitely generated projective over R, and* $f \colon A \to B$ *a surjective and additive R-homomorphism satisfying* $\Gamma_B(f(x), f(y), f(z)) = \Gamma_A(x, y, z)$ *for all* $x, y, z \in A$. *If* $\langle\!\langle A \rangle\!\rangle$ *or* $\langle\!\langle B \rangle\!\rangle$ *is regular, then* $f(1)$ *is an inversible element in* $\boldsymbol{Z}(B)$, *and a map* $g \colon A \to B;\ a \rightsquigarrow f(a) f(1)^{-1}$ *is an R-algebra homomorphism. In particular, if* $f(1) = 1$, *then* $f \colon A \to B$ *is an R-algebra homomorphism.*

Proof. (1) is obvious: $\langle\!\langle A \rangle\!\rangle$ is regular if and only if $\mathrm{Tr}_{A/R}(x \cdot -) = 0$ implies $x = 0$, that is, $\langle A \rangle$ is regular. (2): (1)$\Rightarrow$(2): Since $\Sigma_i x a_i \otimes b_i = \Sigma_i b_i \otimes a_i x$ in $A \otimes_R A$ holds for all $x \in A$, we get $\Sigma_i a_i \otimes b_i = \Sigma_i b_i \otimes a_i$, $\Sigma_i a_i b_i = \Sigma_i b_i a_i (= 1)$ and $\Sigma_i x a_i \otimes b_i = \Sigma_i b_i \otimes a_i x = (\Sigma_i b_i \otimes a_i)(1 \otimes x) = (\Sigma_i a_i \otimes b_i)(1 \otimes x) = \Sigma_i a_i \otimes b_i x$ for any $x \in A$. (2)$\Rightarrow$(3): The condition that $\Sigma_i x a_i \otimes b_i = \Sigma_i a_i \otimes b_i x$ in $A \otimes_R A$ holds for every $x \in A$, means that $\Sigma_i a_i \mathrm{Tr}_{A/R}(b_i x) = x(\Sigma_i b_i a_i)$ holds for every $x \in A$. Because, $\Sigma_i a_i \mathrm{Tr}_{A/R}(b_i x) = \Sigma_{i,j} a_i \psi_j(b_i x b_j)$, and $\Sigma_{i,j} x b_j a_i \otimes \psi_j(b_i) = \Sigma_{i,j} a_i \otimes \psi_j (b_i x b_j)$ in $A \otimes_R A$ implies $\Sigma_{i,j} a_i \psi_j(b_i x b_j) = \Sigma_{i,j} x b_j a_i \psi_j(b_i) = \Sigma_{i,j} x \psi_j(b_i) b_j a_i = x(\Sigma_j b_i a_i)$, Since $\Sigma_j b_i a_i = 1$, we get $x = \Sigma_j a_j \mathrm{Tr}_{A/R}(b_j x)$ and $\psi_i(x) = \psi_i(\Sigma_j a_j \mathrm{Tr}_{A/R}(b_j x)) = \Sigma_j \psi_i(a_j) \mathrm{Tr}_{A/R}(b_j x) = \mathrm{Tr}_{A/R}((\Sigma_j \psi_i(a_j) b_j) \cdot x) = B_A((\Sigma_j \psi_i(a_j) b_j), x)$ for all $x \in A$, so $(B_A)_* \colon A \to \mathrm{Hom}_R(A, R) \colon x \rightsquigarrow B_A(-, x)$ is surjective, that is, $\langle A \rangle$ is nondegenerate. (3)$\Rightarrow$(1): Since $(B_A)_* \colon A \to \mathrm{Hom}_R(A, R)$ is surjective, there is an $a_i \in A$ with $\psi_i(-) = \mathrm{Tr}_{A/R}(a_i \cdot -)$, and $x = \Sigma_j \mathrm{Tr}_{A/R}(x \cdot a_j) b_j$ hold for any $x \in A$. In particular, we have $1 = \Sigma_i \mathrm{Tr}_{A/R}(a_i) b_i = \Sigma_{i,j} \psi_j(a_i b_j) b_i = \Sigma_{i,j} \mathrm{Tr}_{A/R}(a_i b_j a_j) b_i = \Sigma_{i,j} \mathrm{Tr}_{A/R}(b_j a_j a_i) b_i = \Sigma_j b_j a_j$. On the other hand, we have $\Sigma_i x a_i \otimes b_i = \Sigma_{i,j} \mathrm{Tr}_{A/R}(x a_i \cdot a_j) b_j \otimes b_j = \Sigma_{i,j} b_j \otimes \mathrm{Tr}_{A/R}(x a_i \cdot a_j) b_i = \Sigma_{i,j} b_j \otimes \mathrm{Tr}_{A/R}(a_j x \cdot a_i) b_i = \Sigma_j b_j \otimes a_j x$ for any $x \in A$. (3)$\Leftrightarrow$(4): Since $(B_A)_* \colon A \to \mathrm{Hom}_R(A, R) \colon x \rightsquigarrow \mathrm{Tr}_{A/R}(- \cdot x)$ is surjective if and only if $(\Gamma_A)_* \colon A \otimes_R A \to \mathrm{Hom}_R(A, R);\ x \otimes y \rightsquigarrow \mathrm{Tr}_{A/R}(- \cdot xy)$ is surjective, using (1) we get that $\langle A \rangle = (A, B_A)$ is nondegnerate if and only if $\langle\!\langle A \rangle\!\rangle = (A, \Gamma_A)$ is nondegnerate. (3): Suppose that $\langle\!\langle A \rangle\!\rangle$ is regular and $f \in \boldsymbol{Z}(\langle\!\langle A \rangle\!\rangle)$. Since $\Gamma_A(f(x), y, z) = \Gamma_A(x, f(y), z)$ holds for all $x, y, z \in A$, f satisfies $\mathrm{Tr}_{A/R}(f(xy) zw) = \mathrm{Tr}_{A/R}(x y f(z) w) = \mathrm{Tr}_{A/R}(y f(z) w x) = \mathrm{Tr}_{A/R}(f(y) z w x) = \mathrm{Tr}_{A/R}(x f(y) z w)$ and $\Gamma_A(f(xy) - x f(y), z, w) = 0$ for all $x, y, z, w \in A$, that is, $f(xy) = x f(y)$. Therefore, $f(x) = x f(1)$ for every $x \in A$. Put $f(1) = a$, then $f = f_a$. Therefore, we have $\Gamma_A(ay, z, x) = \mathrm{Tr}_{A/R}(ayzx) = \mathrm{Tr}_{A/R}(xayz) = \Gamma_A(xa, y, z) = \Gamma_A(f(x), y, z) = \Gamma_A(x, f(y), z) = \Gamma_A(x, ya, z) = \mathrm{Tr}_{A/R}(xyaz) = \mathrm{Tr}_{A/R}(yazx) = \Gamma_A(ya, z, x)$ for every $x, y, z \in A$, so $ay = ya$ for all $y \in A$, hence $a \in \boldsymbol{Z}(A)$. The converse is easy. (4): Let $f: A\to B$ be a surjective and additive R-homomorphism satisfying $\Gamma_B(f(x), f(y), f(z))=\Gamma_A(x, y, z)$ for all $x, y, z\in A$. There is an element e in A such that $f(e)=1$. Then, we have $\mathrm{Tr}_{B/R}(f(xy)f(z))=\mathrm{Tr}_{B/R}(f(e)f(xy)f(z))$ $=\Gamma_B(f(e), f(xy), f(z))=\Gamma_A(e, xy, z)=\mathrm{Tr}_{A/R}(exyz)=\Gamma_A(ex, y, z)=\Gamma_B(f(ex), f(y), f(z))=\mathrm{Tr}_{B/R}(f(ex)f(y)f(z))$, so $\mathrm{Tr}_{B/R}(\{f(ex)f(y)-f(xy)\}\ b)=0$ for all $b\in B$. If $\langle\langle B\rangle\rangle$ is regular, then so is $\langle B\rangle$, and we have $f(xy)=f(ex)f(y)$. Similarly, $\mathrm{Tr}_{B/R}(f(xy)f(z))=\mathrm{Tr}_{B/R}(f(xy)f(e)f(z))=\Gamma_B(f(xy), f(e), f(z))=\Gamma_A(xy, e, z)=\mathrm{Tr}_{A/R}(xyez)=\Gamma_A(x, ye, z)=\Gamma_B(f(x), f(ye), f(z))=\mathrm{Tr}_{B/R}(f(x)f(ye)f(z))$, we have $f(xy)=f(x)f(ye)$. Hence, we get $f(e^2)f(z)=f(z)f(e^2)$ and $f(xy)=f(x)f(y)f(e^2)$ for any $x, y, z\in A$, so $f(1)^{-1}=f(e^2)\in \boldsymbol{Z}(A)$ and $f(xy)f(e^2)=f(x)f(e^2)f(y)f(e^2)$ hold for any $x, y\in A$. Therefore, $g: A\to B: a\rightsquigarrow f(a)f(1)^{-1}$ is an algebra homomorphism. If $\langle\langle A\rangle\rangle$ is regular, then by Lemma 2: (3), $f: \langle\langle A\rangle\rangle\to\langle\langle B\rangle\rangle$ is an isomorphism, so $\langle\langle B\rangle\rangle$ is regular. By the above statement, $g: A\to B: a\rightsquigarrow f(a) f(1)^{-1}$ is an algebra isomorphism.

REMARK I. 1) The conditions in (2) of Proposition 1 mean that A is strongly separable over R in the meaning of [K$_2$], which is equivalent to that A is separable over R and $A=\boldsymbol{Z}(A)\oplus[A, A]$, where $[A, A]=\{\Sigma_i(a_i b_i-b_i a_i)\,|\, a_i, b_i\in A\}$. 2) For symmetric algebras A and B over a field, Watanabe [W] proved (4) in Proposition 1.

3. Matrix representation of ζ-skew-symmetric multilinear R-module

For any positive integer m, U^m(or R^m) denotes an R-module consisting of m-rows $(u_1, u_2, \cdots, u_m)$ with $u_i\in U$, (or $u_i\in R$).

DEFINITION. For integers n and $r\ (\geqq 2)$, let $F(r, n)$ be the set of all mappings of $\{1, 2, \cdots, r\}$ into $\{1, 2, \cdots, n\}$. Then, a set $\boldsymbol{A}=(a_f)_{f\in F(r,n)}=(a_{(f(1),\cdots,f(r))})_{f\in F(r,n)}$ of elements $a_f\in U$ which suffixed by elements $f=(f(1), \cdots, f(r))$ of $F(r, n)$, is called an *r-fold matrix of degree n*, or simply say *n^r-matrix, over U*, (in the case $U=R$, it was defined in [K, W]). We shall say that $\boldsymbol{A}=(a_f)_{f\in F(r,n)}$ is *ζ-skew-symmetric*, if it satisfies $a_{(f(1),f(2),f(3),\cdots,f(r))}=\zeta\, a_{f(2),f(3),\cdots,f(r),f(1))}$ for every $f=(f(1), \cdots, f(r))\in F(r, n)$. If $\zeta=1$, "1-skew-symmetric" will be said "cyclically symmetric". Let $\boldsymbol{A}=(a_f)_{f\in F(r,n)}$ be an n^r-matrix, and let $\boldsymbol{b}=(b_1, b_2, \cdots, b_n)$ be any element in R^n. For $1\leqq k\leqq r$, $\boldsymbol{b}_{(\dot{k})}\boldsymbol{A}$ denotes an n^{r-1}-matrix $(c_g)_{g\in F(r-1,n)}$ with $c_g=\Sigma_{i=1}^n b_i\, a_{(g(1),\cdots,g(k-1),i,g(k),\cdots,g(r-1))}$, and $\boldsymbol{b}_{(\dot{1})}\boldsymbol{A}$ is denoted by $\boldsymbol{bA}$. If $\boldsymbol{A}$ is regarded as an ordinary $h\times n^{r-1}$-matrix, $\boldsymbol{bA}$ is an element of $U^{n^{r-1}}$. $R^n\boldsymbol{A}=\{\boldsymbol{bA}\,|\,\boldsymbol{b}\in R^n\}$ becomes a finitely generated R-submodule of $U^{n^{r-1}}$. We note that for any $\boldsymbol{b}_i=(b_{i1}, b_{i2}, \cdots, b_{in})\in R^n$ $(i=1, 2, \cdots, r)$ and an n^r-marrix $\boldsymbol{A}=(a_f)_{f\in F(r,n)}$, we can define a product $\boldsymbol{b}_{1(\dot{1})}(\boldsymbol{b}_{2(\dot{2})}(\cdots(\boldsymbol{b}_{r(\dot{r})}\boldsymbol{A})))=\Sigma_{f\in F(r,n)}\, b_{1f(2)}\, b_{2f(2)} \cdots b_{rf(r)}\, a_f$.

For a given ζ-skew-symmetric n^r-matrix $\boldsymbol{A}=(a_f)_{f\in F(r,n)}$ over U, we can define a ζ-skew-symmetric multilinear map θ_A: $R^nA\times R^nA\times\cdots\times R^nA\to U$ as follows: For $(b_1A, b_2A, \cdots, b_rA)\in R^nA\times R^nA\times\cdots\times R^nA$, $\theta_A(b_1A, b_2A, \cdots, b_rA) = b_{1(\dot{1})}(b_{2(\dot{2})}(\cdots b_{r(\dot{r})}A))) = \Sigma_{f\in F(r,n)} b_{1f(1)} b_{2f(2)}\cdots b_{rf(r)} a_f (= \Sigma_{i,j,\cdots,k=1}^{n} b_{1i} b_{2j}\cdots b_{rk}\cdot a_{(i,j,\cdots,k)})$. This is well defined. Because, if $b_kA=b'_kA$ for $b_k=(b_{k1}, b_{k2}, \cdots, b_{kn})$ and $b'_k=(b'_{k1}, b'_{k2}, \cdots, b'_{kn})$ in R^n, then $\Sigma_{i=1}^n b_{k,i}\, a_{(i,g(k+1),\cdots,g(r),g(1),\cdots,g(k-1))} = \Sigma_{i=1}^n b'_{k,i}\, a_{(i,g(k+1)),\cdots,g(r),g(1),\cdots,g(k-1))}$ for every $g\in F(r, n)$, hence we get $b_{1(\dot{1})}(b_{2(\dot{2})}(\cdots(b_{k(\dot{k})}(\cdots b_{r(\dot{r})}A)))=$

$\Sigma_{g\in F(r,n)} b_{1g(1)} b_{2g(2)}\cdots b_{kg(k)}\cdots b_{rg(r)} a_{(g(1),\cdots,g(k-1),g(k),\cdots,g(r))}$

$= \zeta^{k-1}\Sigma_{g\in F(r,n)} b_{1g(1)} b_{2g(2)}\cdots b_{kg(k)}\cdots b_{rg(r)} a_{(g(k),\cdots,g(r),g(1),\cdots,g(k-1))}$

$= \zeta^{k-1}\Sigma_{g\in F(r,n)} b_{1g(1)} b_{2g(2)}\cdots b'_{kg(k)}\cdots b_{rg(r)} a_{(g(k),\cdots,g(r),g(1),\cdots,g(k-1))}$

$= \Sigma_{g\in F(r,n)} b_{1g(1)} b_{2g(2)}\cdots b'_{kg(k)}\cdots b_{rg(r)} a_{(g(1),-,g(k-1),g(l),\cdots,g(r))}$

$= b_{1(\dot{1})}(b_{2(\dot{2})}(\cdots(b'_{k(\dot{k})}(\cdots b_{r(\dot{r})}A)))$.

The r-fold ζ-skew-symmetric R-modlue $(R^nA, \theta_A; U)$ defined by a ζ-skew-symmetric n^r-matrix A will be denoted by $\langle[A]\rangle$.

Lemma 3. *For any ζ-skew-symmetric n^r-matrix A over U, $\langle[A]\rangle$ is always regular.*

Proof. To show that $(\theta_A)^*: R^n A\to\mathrm{Hom}_R(\otimes_R^{r-1} R^n A, U)$; $bA\leadsto\theta_A(bA, -)$ is injective, suppose $bA\in\mathrm{Ker}(\theta_A)^*$, that is, $\zeta\, b_{1(\dot{1})}(b_{2(\dot{2})}(\cdots b_{r-1(\dot{r-1})}(b_{(\dot{r})}A)))=0$ for all $b_j\in R^n$; $i=1, 2, \cdots, r-1$. We can check that for any n^k-matrix $H=(u_f)_{f\in F(k,n)}$, $cH=O$ for every $c\in R^n$ implies $H=O$, that is, $u_f=0$ for every $f\in F(k, n)$. Therefore, $b_{k(\dot{k})}(b_{k+1(\dot{k+1})}\cdots(b_{(\dot{r})}A))=O$ for every $b_k\in R^n$ implies $b_{k+1(\dot{k+1})}\cdots(b_{(\dot{r})}A)=0$. Hence, we get $bA=\zeta\, b_{(\dot{r})}A=O$.

Let $(M, \theta; U)$ be any finitely generated r-fold ζ-skew-symmetric R-module with $M=\Sigma_{i=1}^n Rm_i$. $B=(\theta(m_{f(1)}, m_{f(2)}, \cdots, m_{f(r)}))_{f\in F(r,n)}$ is a ζ-skew-symmetric n^r-matrix over U. We consider a relation between r-fold ζ-skew-symmetric R-modules $(M, \theta; U)$ and $\langle[B]\rangle$. For any $x=\Sigma_{i=1}^n c_i m_i\in M$, $(\theta(x, m_{f(1)}, \cdots, m_{f(r-1)}))_{f\in F(r-1,n)}=cB\in R^n B$ holds, where $c=(c_1, c_2, \cdots, c_n)\in R^n$. Hence, we can define an R-epimorphism

$$\Psi: M\twoheadrightarrow R^n B: x\leadsto(\theta(x, m_{f(1)}, \cdots, m_{f(r-1)}))_{f\in F(r-1,n)}\,.$$

Then, Ψ becomes a morphism of ζ-skew-symmetric R-modules of $(M, \theta; U)$ onto $\langle[B]\rangle=(R^nB, \theta_B; U)$, that is, for any $x_i=\Sigma_{j=1}^n c_{ij}m_j\in M$; $i=1, 2, \cdots, r$, $\theta_B(\Psi(x_1), \Psi(x_2), \cdots, \Psi(x_r))=\theta_B(c_1B, c_2B, \cdots, c_rB)=c_{1(\dot{1})}(c_{2(\dot{2})}(\cdots c_{r(\dot{r})}B)))=\theta(x_1, x_2, \cdots, x_r)$, where $c_i=(c_{i1}, c_{i2}, \cdots, c_{ir})\in R^n$. On the other hand, if one regards $B=(b_f)_{f\in F(r,n)}$ as an $n^{r-1}\times n$-matrix, then for any n^{r-1}-row $c=(c_g)_{g\in F(r-1,n)}\in R^{n^{r-1}}$, $c\cdot B=(\Sigma_{g\in F(r-1,n)} c_g b_{(g-1)}, \cdots, \Sigma_{g\in F(r-1,n)} c_g b_{(g,n)})\in U^n$, so $R^{n^{r-1}}\cdot B=\{c\cdot B\,|\,c\in R^{n^{r-1}}\}$ is an R-submodule of U^n. If $x_1\otimes x_2\otimes\cdots\otimes x_{r-1}(\in\otimes_R^{r-1}M)$ is expressed as $\Sigma_{f\in F(r-1,n)} c_f\, m_{f(1)}\otimes m_{f(2)}\otimes\cdots\otimes m_{f(r-1)}$ for $c_f\in R$, then $(\theta(x_1, \cdots, x_{r-1}, m_1), \theta(x_1, \cdots, x_{r-1}, m_2), \cdots, \theta(x_1, \cdots, x_{r-1}, m_n))$ can be expressed as $c\cdot B$ with $c=(c_f)_{f\in F(r-1,n)}$. Hence, Ψ is surjective.

Lemma 4. *For a generator $\{m_i; i=1, 2, \cdots, n\}$ of M, one can define R-homomorphisms*
∇: $\otimes_R^{r-1} M \to U^n$ *and* Δ: $\mathrm{Hom}_R(M, U) \to U^n$ *as follows*:
∇: $\otimes_R^{r-1} M \to U^n$; $x_1 \otimes x_2 \otimes \cdots \otimes x_{r-1} \rightsquigarrow (\theta(m_1, x_1, \cdots, x_{r-1}), \theta(m_2, x_1, \cdots, x_{r-1}), \cdots, \theta(m_n, x_1, \cdots, x_{r-1}))$, *and* Δ: $\mathrm{Hom}_R(M, U) \to U^n$; $f \rightsquigarrow (f(m_1), \cdots, f(m_r))$.
Their images are $\mathrm{Im}\,\nabla = R^{n^{r-1}} \cdot \boldsymbol{B}$ *and* $\mathrm{Im}\,\Delta = \{(u_1, u_2, \cdots, u_n) \in U^n \mid \Sigma_{i=1}^n c_i u_i = 0$ *for all* $(c_1, c_2, \cdots, c_n) \in \mathrm{Rel}(\{m_i\})$, *where* $\mathrm{Rel}(\{m_i\}) = \{(c_1, c_2, \cdots, c_n) \in R^n \mid \Sigma_{i=1}^n c_i m_i = 0\}$. *Furthermore,* Δ *is injective, and the following diagram is commutative*:

$$(\sharp) \qquad \begin{array}{ccc} \otimes_R^{r-1} M & \xrightarrow{\nabla} & U^n \\ \downarrow \theta_* & & \| \\ \mathrm{Hom}_R(M, U) & \xrightarrow{\Delta} & U^n . \end{array}$$

Proof. One has an exact sequence $0 \to \mathrm{Rel}(\{m_i\}) \to R^n \to M \to 0$, so $\mathrm{Im}\,\Delta = \mathrm{Ker}(U^n \to \mathrm{Hom}_R(\mathrm{Rel}(\{m_i\}), U))$ follows from that $0 \to \mathrm{Hom}_R(M, U) \to \mathrm{Hom}_R(R^n, U) = U^n \to \mathrm{Hom}_R(\mathrm{Rel}(\{m_i\}), U)$ is exact. Since $\Delta \cdot \theta_*(x_1 \otimes x_2 \otimes \cdots \otimes x_{r-1}) = \Delta(\theta(-, x_1, x_2, \cdots, x_{r-1})) = \nabla(x_1 \otimes x_2 \otimes \cdots \otimes x_{r-1})$ hold for any $x_1 \otimes x_2 \otimes \cdots \otimes x_{r-1} \in \otimes_R^{r-1} M$, the diagram ($\sharp$) is commutative.

Proposition 2. *Let (M, θ, U) be an r-fold ζ-skew-symmetric R-module with a generator $\{m_1, m_2, \cdots, m_n\}$ as an R-module, i.e. $M = \Sigma_{i=1}^n Rm_i$, and let $B = (\theta(m_{f(1)}, m_{f(2)}, \cdots, m_{f(r)}))_{f \in F(r,n)}$. Then the following statements fold*:

1) *$(M, \theta; U)$ is regular if and only if Ψ: $M \to R^n\, \boldsymbol{B}$ is bijective.*
2) *θ_* is surjective if and only if* $\mathrm{Im}\,\Delta = \mathrm{Im}\,\nabla$.

Proof. 1) follows from that θ^* is injective if and only if Ψ is injective. 2) immediately follows from the diagram ($\sharp$).

DEFINITION. By $U_{n,m}$ (or $R_{n,m}$), we denote the set of all $n \times m$-matirces with entries in U (or R). Let $\boldsymbol{A} = (a_f)_{f \in F(r,n)}$ be an r-fold ζ-skew-symmetric n^r-matrix over U, and $\boldsymbol{B} = (b_{ij}) (\in R_{n,n})$ an ordinary $n \times n$-matrix over R. When one regards $\boldsymbol{A}$ as an $n \times n^{r-1}$-matrix over U, a subset Ann ($\boldsymbol{A}$) of $R_{n,n}$ and a subset Ann ($\boldsymbol{B}$) of $U_{n,n}$ are defined as follows; $\mathrm{Ann}(\boldsymbol{A}) := \{\boldsymbol{D} \in R_{n,n} \mid \boldsymbol{D} \cdot \boldsymbol{A} = \boldsymbol{O}\}$ and $\mathrm{Ann}(\boldsymbol{B}) := \{\boldsymbol{V} \in U_{n,n} \mid \boldsymbol{B} \cdot \boldsymbol{V} = \boldsymbol{O}\}$, where $\boldsymbol{D} \cdot \boldsymbol{A}$ or $\boldsymbol{B} \cdot \boldsymbol{V}$ means an ordinary product of matrices. For a subset $\mathfrak{b} \subseteq R_{n,n}$, Ann ($\mathfrak{b}$) denotes the intersection of Ann ($\boldsymbol{B}$) for all $\boldsymbol{B} \in \mathfrak{b}$. On the other nand, one car regard $\boldsymbol{A}$ as an $n^{r-1} \times n$-matrix over U, then for any $n \times n^{r-1}$-matrix $\boldsymbol{C}$ over R, the ordinary product $\boldsymbol{C} \cdot \boldsymbol{A}$ is an $n \times n$-matrix over U. We put $R_{n,n^{r-1}} \cdot \boldsymbol{A} = \{\boldsymbol{C} \cdot \boldsymbol{A} \in U_{n,n} \mid \boldsymbol{C} \in R_{n,n^{r-1}}\}$. For a set $\mathfrak{a}$ of $U_{n,n}$, ${}^t\mathfrak{a}$ denotes the set of transpose matrices ${}^t\boldsymbol{H}$'s for all $\boldsymbol{H} \in \mathfrak{a}$.

Proposition 3. *Let $\boldsymbol{A} = (a_f)_{f \in F(r,n)}$ be an ζ-skew-symmetric n^r-matrix over U. Then $\langle [\boldsymbol{A}] \rangle$ is nondegenerate if and only if* ${}^t(\mathrm{Ann}(\mathrm{Ann})\boldsymbol{A}))) = R_{n,n^{r-1}} \cdot \boldsymbol{A}$

holds.

Proof. Let $e_1=(1, 0, \cdots, 0)$, $e_2=(0, 1, 0, \cdots, 0)$, $\cdots$, $e_n=(0, \cdots, 0, 1)$ be elements of R^n. $R^n A$ is generated by $\{e_i A; i=1, 2, \cdots, n\}$ as an R-module. For the R-homomorphisms ∇ and Δ defined by the generator $\{e_i A; i=1, 2, \cdots, n\}$ in Lemma 4, we have $\mathrm{Im}\,\nabla=R^{n^{r-1}}\cdot A$, because of $\theta_A(e_{f(1)} A, e_{f(2)} A, \cdots, e_{f(r)} A)= e_{f(1)(\dot{1})}(e_{f(2)(\dot{2})}(\cdots e_{f(r)(\dot{r})} A)))=a_f$ for every $f\in F(r, n)$. On the other hand, it follows that $\mathrm{Rel}\,(\{e_i A\})=\{b=(b_1, b_2, \cdots, b_n)\in R^n \mid \Sigma_{i=1}^n b_i e_i A=bA=O\}$ and $\mathrm{Im}\,\Delta$ is the set of elements $(u_1, u_2, \cdots, u_n)\in U^n$ such that $\Sigma_{i=1}^n b_i u_i=0$ holds for all $b=(b_1, b_2, \cdots, b_n)\in R^n$ with $bA=O$. Hence, $\mathrm{Im}\,\nabla\supseteq \mathrm{Im}\,\Delta$, (or $\mathrm{Im}\,\nabla\subseteq\mathrm{Im}\,\Delta$), holds if and only if $R^{n^{r-1}}\cdot A\supseteq$, (or $\subseteq$), $\{(u_1, u_2, \cdots, u_n)\in U^n \mid \Sigma_{i=1}^n b_i u_i=0$ for all $b=(b_1, b_2, \cdots, b_n)\in R^n$ with $bA=O\}$. The latter condition is equivalent to that $R_{n,n^{r-1}}\cdot A\supseteq$, (or $\subseteq$), $\{(u_{ij})\in U_{n,n} \mid \Sigma_{j=1}^n b_{ij} u_{kj}=0; i, k=1, 2, \cdots, n,$ for all $B=(b_{ij})\in R_n$ with $B\cdot A=O\}={}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$. Hence, By Proposition 2, $(\theta_A)_*$ is surjective if and only if $R_{n,n^{r-1}}\cdot A={}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$. Since $\langle[A]\rangle$ is regular, the proof finished.

REMARK II. 1) In the above proof, we showed that $R_{n,n^{r-1}}\cdot A\subseteq{}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$ holds for any ζ-skew-symmetric n^r-matrix A over U, since the commutative diagram (♯) in Lemma 4 means $\mathrm{Im}\,\nabla\subseteq\mathrm{Im}\,\Delta$.
2) If U is an inversible R-module, that is, U is finitely generated projective and rank 1 over R. Then for any $f, g\in\mathrm{Hom}_R(U, R)$, $f(x)g(y)=f(y)g(x)$ holds for every $x, y\in U$, so $f(x)y=f(y)x$ for all $x, y\in U$.

DEFINITION. Any element D in $\mathrm{Hom}_R(U^{n^{r-1}}, R^n)$ will be able to regard as an $n^{r-1}\times n$-matrix $(d_{i,j})$ with (i, j)-entry $d_{i,j}\in U^*=\mathrm{Hom}_R(U, R)$. For an $n^{r-1}\times n$-matrix $A=(a_{i,j})$ over U and $D=(d_{i,j})\in\mathrm{Hom}_R(U^{n^{r-1}}, R^n)$, AD means an $n\times n$-matrix with (i, j)-entry $\Sigma_{k=1}^{n^{r-1}} d_{k,j}(a_{i,k})$ $(\in R)$.

Lemma 5. *Let U be an inversible R-module, and A a ζ-skew-symmetric n^r-matrix over U. If there exists a $D\in\mathrm{Hom}_R(U^{n^{r-1}}, R^n)$ such that $(AD)\cdot A=A$ regarding A as $n\times n^{r-1}$-matrix, then the $n^{r-1}\times n$-matrix A satisfies the condition $R_{n,n^{r-1}}\cdot A={}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$, hence $\langle[A]\rangle$ is nondegenerate and R-projective.*

Proof. By 1) in Remark II, $R_{n,n^{r-1}}\cdot A\subseteq{}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$ always holds. Since $(AD)\cdot A=A$, if I_n denotes the identity matrix in $R_{n,n}$, $(AD-I_n)\cdot A=O$ and $AD-I_n\in\mathrm{Ann}\,(A)$ hold. Hence, $H\in{}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$ implies $(AD-I_n)\cdot{}^tH=O$, so $(AD)\cdot{}^tH={}^tH$ holds. By 2) in Remark II, we get $H=H\cdot{}^t(AD)=\zeta(H\,{}^tD)\cdot A$. Because, if $h_{i,j}$(or $a_{i,j}$) is (i, j)-entry of H (or A), then $(AD)\cdot{}^tH={}^tH$ implies that $h_{j,i}=\Sigma_h(\Sigma_k d_{k,h}(a_{i,k}))\,h_{j,k}=\Sigma_k(\Sigma_h d_{k,h}(h_{j,h}))\,a_{i,k}=\zeta\,\Sigma_k(\Sigma_h d_{k,h}(h_{j,h}))$ $a_{k,i}$ is (j, i)-entry of $\zeta\,(H\,{}^tD)\cdot A$. Hence, we get $H\in R_{n,n^{r-1}}\cdot A$ and $R_{n,n^{r-1}}\cdot A={}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(A)))$. By Proposition 3, $\langle[A]\rangle$ is nondegenerate, and using Proposition A in Appendix, we get the R-projectity of $\langle[A]\rangle$.

Proposition 4. *Let $U=R$, and let $\boldsymbol{A}$ be a ζ-skew-symmetric n^r-matrix over R. Then, $\langle[\boldsymbol{A}]\rangle$ is nondegenerate and R-projective if and only if there is an $n^{r-1}\times n$-matrix $\boldsymbol{D}$ over R such that $\boldsymbol{A}\cdot\boldsymbol{D}\cdot\boldsymbol{A}=\boldsymbol{A}$ holds, where the product $\cdot$ means an ordinary product of matrices regarding $\boldsymbol{A}$ as an $n\times n^{r-1}$-matrix.*

Proof. The "if" part is obtained from Lemma 4. Suppose $\langle[\boldsymbol{A}]\rangle$ is nondegenerate and R-projective. By Lemma 5, $R_{n,n^{r-1}}\cdot\boldsymbol{A}={}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(\boldsymbol{A})))$ holds. By Proposition A in Appendix, there is an $n^{r-1}\times n$-matrix $\boldsymbol{F}$ over an injective hull of R as an R-molule such that every entry of the product $\boldsymbol{A}\cdot\boldsymbol{F}$ is in R and $(\boldsymbol{A}\cdot\boldsymbol{F})\cdot\boldsymbol{A}=\boldsymbol{A}$ holds. Since $\boldsymbol{B}\cdot(\boldsymbol{A}\cdot\boldsymbol{F})=(\boldsymbol{B}\cdot\boldsymbol{A})\cdot\boldsymbol{F}=\boldsymbol{O}\cdot\boldsymbol{F}=\boldsymbol{O}$ hold for all $\boldsymbol{B}\in\mathrm{Ann}\,(\boldsymbol{A})$, ${}^t(\boldsymbol{A}\cdot\boldsymbol{F})$ is contained in ${}^t(\mathrm{Ann}\,(\mathrm{Ann}\,(\boldsymbol{A})))=R_{n,n^{r-1}}\cdot\boldsymbol{A}$, that is, ${}^t(\boldsymbol{A}\cdot\boldsymbol{F})=\boldsymbol{D}\cdot\boldsymbol{A}$ for some $\boldsymbol{D}\in R_{n,n^{r-1}}$. Since $\boldsymbol{A}\cdot\boldsymbol{F}={}^t(\boldsymbol{D}\cdot\boldsymbol{A})=({}^t\boldsymbol{A})\cdot({}^t\boldsymbol{D})$ and ${}^t\boldsymbol{A}=\zeta\,\boldsymbol{A}$, we get that there is an $n^{r+1}\times n$-matrix $\zeta\,{}^t\boldsymbol{D}$ satisfying $\boldsymbol{A}\cdot(\zeta\,{}^t\boldsymbol{D})\cdot\boldsymbol{A}=\boldsymbol{A}$.

From Lemma 5 and Proposition 4, we get the follwing theorem:

Theorem 1. *Let (M,θ) be a finitely generated ζ-skew-symmetric R-module, and $M=\sum_{i=1}^n Rm_i$. $\boldsymbol{A}=\theta\,(m_{f(1)}, m_{f(2)}, \cdots, m_{f(r)})_{f\in F(r,n)}$ is a ζ-skew-symmtric n^r-matrix over R. The following conditions are equivalent:*

1) *(M,θ) is non degenerate and R-projective,*

2) *$\Psi:(M,\theta)\to\langle[\boldsymbol{A}]\rangle$ is an isomorphism, aud there is an $n^{r-1}\times n$-matrix $\boldsymbol{D}$ over R such that $\boldsymbol{A}\cdot\boldsymbol{D}\cdot\boldsymbol{A}=\boldsymbol{A}$ holds as a product of matrices $n^{r-1}\times n$-matrix $\boldsymbol{D}$ and $n^{r-1}\times n^{r-1}$-matrix $\boldsymbol{A}$.*

REMARK III. Let R be a field or a Von Neumann regular ring, and $\boldsymbol{A}$ any n^r-matrix over R. One can show that there exists an $n^{r-1}\times n$-matrix $\boldsymbol{D}$ over R such that $\boldsymbol{A}\cdot\boldsymbol{D}\cdot\boldsymbol{A}=\boldsymbol{A}$ holds, regarding $\boldsymbol{A}$ as an $n\times n^{r-1}$-matrix. Let $\boldsymbol{A}$ regard as an $n\times n^{r-1}$-matrix, and for an $n(n^{r-1}-1)\times n^{r-1}$-zero matrix $\boldsymbol{O}$,

$$\text{put}\quad \boldsymbol{B}:=\begin{pmatrix}\boldsymbol{A}\\ \boldsymbol{O}\end{pmatrix}:n^{r-1}\times n^{r-1}\text{-matrix.}$$

Since the $n^{r-1}\times n^{r-1}$-matrix ring $R_{n^{r-1}}$ over R is a Von Neumann regular ring, there is an $n^{r-1}\times n^{r-1}$-matrix $\boldsymbol{D}$ with $\boldsymbol{B}\,\boldsymbol{D}\,\boldsymbol{B}=\boldsymbol{B}$. Let $\boldsymbol{D}_1$ be an $n^{r-1}\times n$-matrix and $\boldsymbol{D}_2$ an $n^{r-1}\times n\,(n^{r-2}-1)$-martix satisfying $\boldsymbol{D}=(\boldsymbol{D}_1,\boldsymbol{D}_2)$. By a computation, $\boldsymbol{A}\cdot\boldsymbol{D}_1\cdot\boldsymbol{A}=\boldsymbol{A}$ follows.

Corollary 1. *Let R be a field or a Von Neumann regular commutative ring. If $\boldsymbol{A}$ is a ζ-skew-symmetric n^r-matrix over R, then $\langle[\boldsymbol{A}]\rangle$ is always nondegenerate.*

4. 3-fold cyclically symmetric R-modules

Let $(M,\theta;U)$ be a 3-fold cyclically symmetric R-module, that is, $\theta(x,y,z)=\theta(y,z,x)$ holds for all $x,y,z\in M$.

DEFINITION. For $e\in M$, e is called a *regular element* of $(M,\theta;U)$, if

$\theta(-, -, e)$: $M\times M\rightarrow U$; $(x, y)\rightsquigarrow\theta(x, y, e)$ is a nondegenerate symmetric bilinear form.

REMARK IV; If there is a reglar element e of $(M, \theta; U)$, then $(M, \theta; U)$ is nondegenerate, and a multiplication $M\times M\rightarrow M$; $(x, y)\rightsquigarrow x\cdot y$, satisfying $\theta(x, y, z)=\theta(x\cdot y, z, e)$ for all $x, y, z\in M$, is defined on M, and M becomes a non commutative and non associative R-algebra with identity e, this R-lagebra denote by $((M, \theta; U), \cdot; e)$. If θ is symmetric and $U=R$ is a field, these was defined in [H_1].

Proposition 5. *Let $(M, \theta; U)$ be a cyclically symmetric R-module, and e and e' regular elements of $(M, \theta; U)$. For R-algebras $((M, \theta; U), \cdot; e)$ and $((M, \theta; U), *; e')$ defined by e ahd e', if $((M, \theta; U), \cdot; e)$ is an associative algebra, then the following statements hold:*

(1) *$(x*y)\cdot e'=x\cdot y$ and $(x\cdot y)*e=x*y$ hold every $x, y\in M$.*

(2) *e' is an inversible element in the center $\boldsymbol{Z}((M, \theta; U), \cdot; e)$ of $((M, \theta; U), \cdot; e)$, and $e'\cdot(e*e)=e$ holds. e is inversible in $\boldsymbol{Z}((M, \theta; U), *; e')$ and $e*(e'\cdot e')=e'$.*

(3) *ψ: $M\rightarrow M$; $x\rightsquigarrow x\cdot e'$ is a bijection with the inverse ϕ: $M\rightarrow M$; $x\rightsquigarrow x*e$, and satisfies $\phi(x\cdot y)=x*y$ for all $x, y\in M$.*

(4) *$(x\cdot y)*z=x*(y\cdot z)$ holds for all $x, y, z\in M$.*

(5) *$\psi(x\cdot y)=\psi(x)*\psi(y)$ holds for all $x, y\in M$, so ψ: $((M, \theta; U), \cdot; e)\rightarrow((M, \theta; U), *; e')$ is an R-algebra isomorphism. $((M, \theta; U), *; e)$ is also an associative algebra.*

Proof. (1): From the definition of multiplications $\cdot$ and $*$, it follows that $\theta((x*y)\cdot e', z, e)=\theta((x*y), e', z)=\theta(z, (x*y), e')=\theta(x, y, z)=\theta(x\cdot y, z, e)$ imply $(x*y)\cdot e'=x\cdot y$. Similarly, we get $(x\cdot y)*e=x*y$. (2): For any $x\in M$, $e'*x=x$ implies $x\cdot e'=(e'*x)\cdot e'=e'\cdot x$, and $(e*e)\cdot e'=e\cdot e=e$, hence $e'\in\boldsymbol{Z}((M, \theta; U), \cdot; e)$. Similarly, $e\in\boldsymbol{Z}((M, \theta; U), *; e')$ and $e*(e'\cdot e')=e'$. (3): From (1), we have $\psi(\phi(x))=(x*e)\cdot e'=x\cdot e=x$, and $\phi(\psi(x))=(x\cdot e')*e=x*e'=x$ and $\phi(x\cdot y)=(x\cdot y)*e=x*y$ hold for all $x, y\in M$. (4): Since $((M, \theta; U), \cdot; e)$ is associative, $(x\cdot y)*z=\phi((x\cdot y)\cdot z)=\phi(x\cdot(y\cdot z))=x*(y\cdot z)$ hold for all $x, y, z\in M$. (5): Using (4) and $e'\in\boldsymbol{Z}((M, \theta; U), \cdot; e)$, we get $\psi(x)*\psi(y)=(x\cdot e')*(y\cdot e')=((x\cdot e')\cdot y)*e')$ $=(e'\cdot x)\cdot y=e'\cdot(x\cdot y)=(x\cdot y)\cdot e'=\psi(x\cdot y)$.

DEFINITION. Let $(M, \theta; U)$ be a 3-fold cyclically symmetric R-module. If there is a regular element e of $(M, \theta; U)$ such that $((M, \theta; U), \cdot; e)$ is an associative algebra, then we shall say that $(M, \theta; U)$ is *associative*.

In the following, we consider the case $U=R$.

DEFINITION. Let $\boldsymbol{A}=(a_{i,j,k}; 1\leqq i, j, k\leqq n)$ be a cyclically symmetric n^3-matrix over R, and $\boldsymbol{e}=(e_1, e_2, \cdots, e_n)$ an element in R^n. We shall say that $\boldsymbol{e}$ is *regular with respect* to $\boldsymbol{A}$, if for any $\boldsymbol{x}=(x_1, \cdots, x_n)\in R^n$, $\boldsymbol{x}_{(1)}(\boldsymbol{e}_{(3)}\boldsymbol{A})=(\sum_{i,k=1}^n x_i e_k a_{i,k,k}; 1\leqq j\leqq n)=\boldsymbol{0}$ implies $\boldsymbol{x}\boldsymbol{A}=(\sum_{i=1}^n x_i a_{i,j,k}; 1\leqq j, k\leqq n)=\boldsymbol{0}$.

REMARK V. If $\boldsymbol{eA}=(\sum_{k=1}^{n} e_k a_{k,i,j};\ 1\leqq i,j\leqq n)$ is an inversible $n\times n$-matrix, then $\boldsymbol{e}$ is regular with respect to $\boldsymbol{A}$.

Theorem 2. *Let* $\boldsymbol{A}=(a_{i,j,k};\ 1\leqq i,j,k\leqq n)$ *be a cyclically symmetric* n^3-*matrix, and* $\boldsymbol{e}=(e_1, e_2, \cdots, e_n)$ *be regular with respect to* $\boldsymbol{A}$.
(1) $\langle[\boldsymbol{A}]\rangle$ *is R-projective and* $\boldsymbol{eA}$ *is a regular element of* $\langle[\boldsymbol{A}]\rangle$ *if and only if* $\boldsymbol{eA}=(\sum_{k=1}^{n} e_k a_{k,i,j};\ 1\leqq i,j\leqq n)$ *is a symmetric and Von Neumann regular* $n\times n$-*matrix, i.e. there is an* $n\times n$-*matrix* $\boldsymbol{C}=(c_{i,j};\ 1\leqq i,j\leqq n)$ *with* $(\boldsymbol{eA})\cdot\boldsymbol{C}\cdot(\boldsymbol{eA})=\boldsymbol{eA}$.
(2) *Assume the latter condition in* (1), *that is,* $\boldsymbol{eA}$ *is symmetric, and there is an* $n\times n$-*matrix* $\boldsymbol{C}=(c_{i,j};\ 1\leqq i,j\leqq n)$ *with* $(\boldsymbol{eA})\cdot\boldsymbol{C}\cdot(\boldsymbol{eA})=\boldsymbol{eA}$. *Then,* $\langle[\boldsymbol{A}]\rangle$ *is associative if and only if an* n^4-*matrix* $(\sum_{s,t=1}^{n} a_{h,i,s} c_{s,t} a_{t,j,k};\ 1\leqq h,i,j,k\leqq n)$ *is cyclically symmetric.*

Proof. Let $\boldsymbol{A}=(a_{i,j,k};\ 1\leqq i,j,k\leqq n)$ be a cyclically symmetric n^3-matrix, and $\boldsymbol{e}=(e_1, e_2, \cdots, e_n)$ an element in R^n. (1): Put $B(\boldsymbol{xA}, \boldsymbol{yA})=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{eA})$ for $\boldsymbol{xA}, \boldsymbol{yA}\in R^n\boldsymbol{A}$. The bilinear form B is symmetric if and only if $n\times n$-matrix $\boldsymbol{eA}=(\sum_{k=1}^{n} e_k a_{i,j,k};\ 1\leqq i,j\leqq n)$ is symmetric. Suppose that $\boldsymbol{eA}$ is symmetric. By Theorem 1, $(R^n\boldsymbol{A}, B)$ is nondegenerate and R-projective if and only if Ψ: $(R^n\boldsymbol{A}, B)\rightarrow\langle[\boldsymbol{eA}]\rangle$; $\boldsymbol{vA}\leadsto\boldsymbol{x}\,(\boldsymbol{eA})\ (=\boldsymbol{x}_{(1)}\,(\boldsymbol{e}_{(3)}\boldsymbol{A}))$ is an isomorphism and $\boldsymbol{eA}$ is a Von Neumann regular $n\times n$-matrix. Hence, we get that $\boldsymbol{eA}$ is a regular element of $\langle[\boldsymbol{A}]\rangle$ and $\langle[\boldsymbol{A}]\rangle\ (=(R^n\boldsymbol{A}, \theta_A)$ is R-projective, if and only if $\boldsymbol{e}$ is regular with respect to $\boldsymbol{A}$ and $\boldsymbol{eA}$ is a symmetric and Von Neumann regular $n\times n$-matrix. (2): Suppose that $\boldsymbol{eA}$ is a regular element of $\langle[\boldsymbol{A}]\rangle$ and there is an $n\times n$-matrix $\boldsymbol{C}=(c_{i,j})$ with $(\boldsymbol{eA})\cdot\boldsymbol{C}\cdot(\boldsymbol{eA})=\boldsymbol{eA}$. A multiplication $*$ on $R^n\boldsymbol{A}$ is defined by $\theta_A(\boldsymbol{xA}*\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{eA})=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA})$. Since $\langle[\boldsymbol{A}]\rangle$ is associative if and only if $\theta_A(\boldsymbol{xA}*\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{wA})=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}*\boldsymbol{zA}, \boldsymbol{wA})$ holds for every $\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{wA}\in R^n\boldsymbol{A}$, it is sufficient to show that $\theta_A(\boldsymbol{xA}*\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{wA})=\sum_{s,t,i,j,k=1}^{n} x_i y_j z_k w_h a_{k,h,s} c_{s,t} a_{t,i,j}$ and $\theta_A(\boldsymbol{xA}, \boldsymbol{yA}*\boldsymbol{zA}, \boldsymbol{wA})=\sum_{s,t,i,j,k=1}^{n} x_i y_j z_k w_h a_{h,i,s} c_{s,t} a_{t,j,k}$ hold for avery $\boldsymbol{x}=(x_1, \cdots, x_n)$, $\boldsymbol{y}=(y_1, \cdots, y_n)$, $\boldsymbol{z}=(z_1, \cdots, z_n)$ and $\boldsymbol{w}=(w_1, \cdots, w_n)\in R^n$. We put $\boldsymbol{xA}*\boldsymbol{yA}=\boldsymbol{uA}$ and $\boldsymbol{yA}*\boldsymbol{zA}=\boldsymbol{vA}$ for $\boldsymbol{u}=(u_1, \cdots, u_n)$ and $\boldsymbol{v}=(v_1, \cdots, v_n)\in R^n$. First we shall show the following identity:
(♯); $\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA})\ (=\sum_{i,j,k=1}^{n} x_i y_j z_k a_{i,j,k})=$
$\sum_{i,j,k,s,t,m=1}^{n} x_i y_j z_k a_{i,j,s} c_{s,t} e_m a_{m,t,k}$ for any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\in R^n$.
Using identities $\sum_{i,j,k=1}^{n} x_i y_j z_k a_{i,j,k}(=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA})=\theta_A(\boldsymbol{uA}, \boldsymbol{zA}, \boldsymbol{eA}))=$
$\sum_{j,t,m=1}^{n} u_j z_t e_m a_{j,t,m}$ and $\sum_{m=1}^{n} e_m a_{m,j,t}(=\boldsymbol{eA}=(\boldsymbol{eA})\cdot\boldsymbol{C}\cdot(\boldsymbol{eA}))=$
$\sum_{i,p,q,m=1}^{n} e_i a_{i,j,p} c_{p,q} e_m a_{m,q,t}$, we have $\sum_{1,j=1}^{n} x_i y_j a_{i,j,k}=\sum_{j,m=1}^{n} u_j e_m a_{j,k,m}=$
$\sum_{i,j,p,q,m=1}^{n} u_j e_i a_{i,j,p} c_{p,q} e_m a_{m,q,k}=\sum_{p,q,m=1}^{n} (\sum_{i,j=1}^{n} u_j e_i a_{i,j,p})\, c_{p,q} e_m a_{m,q,k}=$
$\sum_{p,q,m=1}^{n} (\sum_{i,j=1}^{n} x_i y_j a_{i,j,p})\, c_{p,q} e_m a_{m,q,k}=\sum_{i,j,p,q,m=1}^{n} x_i y_j a_{i,j,p} c_{p,q} e_m a_{,q,k}$ for $k=1, 2, \cdots, n$. Hence, we get $\sum_{i,j=1}^{n} x_i y_j a_{i,j,k}=\sum_{i,j,p,q,m=1}^{n} x_i y_j a_{i,j,p} c_{p,q} e_m a_{m,q,k}$; $k=1, 2, \cdots, n$, and the identity (♯). Using (♯), we get $\theta_A(\boldsymbol{xA}*\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{wA})=$
$\theta_A(\boldsymbol{uA}, \boldsymbol{zA}, \boldsymbol{wA})=\theta_A(\boldsymbol{zA}, \boldsymbol{wA}, \boldsymbol{uA})=\sum_{k,g,i,s,t,m=1}^{n} z_k w_h u_i a_{k,h,s} c_{s,t} e_m a_{m,t,j}$.
Since $\theta_A(\boldsymbol{zA}, \boldsymbol{uA}, \boldsymbol{eA})\ (=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA}))=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA})$ means

$\Sigma_{i,m=1}^n u_i e_m a_{m,t,i}$ $(=\Sigma_{i,m=1}^n u_i e_m a_{t,i,m})=\Sigma_{i,j=1}^n x_i y_j a_{i,j,t}$ for $t=1, 2, \cdots, n$, we get $\Sigma_{k,h,i,s,t,m=1}^n z_k w_h u_i a_{k,h,s} c_{s,t} e_m a_{m,t,i}=\Sigma_{k,h,i,j,s,t=1}^n z_k w_h a_{k,h,s} c_{s,t} (u_i e_m a_{m,t,i})=$ $\Sigma_{k,h,i,j,s,t=1}^n z_k w_h a_{k,h,s} c_{s,t} (x_i y_j a_{i,j,t})=\Sigma_{k,h,i,j,s,t=1}^n x_i y_j z_k w_h a_{k,h,s} c_{s,t} a_{i,j,t}=$ $\Sigma_{h,i,j,k,s,t=1}^n x_i y_j z_k w_h a_{k,h,s} c_{s,t} a_{t,i,j}$, So we have $\theta_A(\boldsymbol{xA}*\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{wA})=$ $\Sigma_{h,i,j,k,s,t=1}^n x_i y_j z_k w_h a_{k,h,s} c_{s,t} a_{t,i,j}$. Similarly, since $\theta_A(\boldsymbol{vA}, \boldsymbol{xA}, \boldsymbol{eA})$ $(=\theta_A(\boldsymbol{yA}, \boldsymbol{zA}, \boldsymbol{xA}))=\theta_A(\boldsymbol{xA}, \boldsymbol{yA}, \boldsymbol{zA})$ means $\Sigma_{j,m=1}^n v_j e_m a_{m,t,j}(=\Sigma_{j,m=1}^n v_j e_m a_{t,j,m})$ $=\Sigma_{j,k=1}^n y_j z_k a_{t,j,k}$ for $t=1, 2, \cdots, n$, we get $\theta_A(\boldsymbol{xA}, \boldsymbol{yA}*\boldsymbol{zA}, \boldsymbol{wA})=$ $\theta_A(\boldsymbol{wA}, \boldsymbol{xA}, \boldsymbol{vA})=\Sigma_{h,i,j,s,t,m=1}^n w_h x_i v_j a_{h,i,s} c_{s,t} e_m a_{m,t,j}=$ $\Sigma_{h,i,j,s,t,m=1}^n w_h x_i a_{h,i,s} c_{s,t} (v_j e_m a_{m,t,j})=\Sigma_{h,i,j,k,s,t=1}^n w_h x_i a_{h,i,s} c_{s,t} (y_j z_h a_{t,j,k})=$ $\Sigma_{h,i,j,k,s,t=1}^n x_i y_j z_k w_h a_{h,i,s} c_{s,t} a_{t,j,k}$, using (♯). Thus, the proof finished.

5. Appendix: Projectivity of R^nA

Let R be, in general, a non commutative ring with identity 1, and U a left R-module. Then, $U^m=\{(u_1, u_2, \cdots, u_m) \mid u_i \in U\}$ and the set $U_{n,m}$ of all $n\times m$-matrices $(u_{i,j})$ with (i,j)-entry $u_{i,j}$ in U become left R-modules. For any $\boldsymbol{H}=(u_{i,j})\in U_{n,m}$, $R^n \boldsymbol{H}=\{\boldsymbol{aH}=(\Sigma_{i=1}^n a_1 u_{i1}, \Sigma_{i=1}^n a_i u_{i2}, \cdots, \Sigma_{i=1}^n a_i u_{im}) \mid \boldsymbol{a}=(a_1, a_2, \cdots, a_n)\in R^n\}$ is a finitely generated R-submodule of U^m. By $E=E({}_RR)$, one denotes an injective hull of left R-module R, and put $U^*=\mathrm{Hom}_R(U, E)$. Every element $\boldsymbol{F}$ in $\mathrm{Hom}_R(U^m, E^n)$ can be regarded as an $m\times n$-matrix $(f_{i,j})$ with (i,j)-entry $f_{i,j}\in U^*$, that is, $\mathrm{Hom}_R(U^m, E^n)=U^*_{m,n}$. For $\boldsymbol{F}=(f_{i,j})\in \mathrm{Hom}_R(U^m, E^n)$ and $\boldsymbol{H}=(u_{i,j})\in U_{n,m}$, $\boldsymbol{HF}$ denotes an $n\times n$-matrix with (i,j)-entry $\Sigma_{k=1}^m f_{k,j}(u_{i,k})$ $(\in E)$. For an $n\times s$-matrix $\boldsymbol{C}$ with entries in R and an $s\times t$-matrix $\boldsymbol{D}$ with entries in U (or R), the ordinary product of matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ will be denoted by $\boldsymbol{C}\cdot\boldsymbol{D}$. Furthermore, by R_n one denotes the ring of $n\times n$-matrices over R.

Proposition A. *Let $\boldsymbol{H}\in U_{n,m}$. $R^n\boldsymbol{H}$ is R-projective if and only if there exists an $\boldsymbol{F}\in U^*_{m,n}$ such that $\boldsymbol{HF}\in R_n$ and $(\boldsymbol{HF})\cdot\boldsymbol{H}=\boldsymbol{H}$.*

Proof. Suppose $R^n\boldsymbol{H}$ is projective over R. An epimorphism $h: R^n\to R^n\boldsymbol{H}$; $\boldsymbol{a}\leadsto\boldsymbol{aH}$ is split, that is, there is an R-homomorphism $g: R^n \boldsymbol{H}\to R^n$ with $h\cdot g=\mathrm{I}$. Since E^n is injective over R, an R-homomorphism $\iota\cdot g: R^n \boldsymbol{H}\to R^n\hookrightarrow E^n$ is extended to an R-homomorphism $f: U^m\to E^n$. Then, there exist $f_{i,j}\in U^*(=\mathrm{Hom}_R(U, E))$; $i=1, \cdots, m, j=1, \cdots, n$, such that, for any $(u_1, u_2, \cdots, u_m)\in U^m$, $f(u_1, \cdots, u_m)=(\Sigma_{i=1}^m f_{i,1}(u_i), \Sigma_{i=1}^m f_{i,2}(u_i), \cdots, \Sigma_{i=1}^m f_{i,n}(u_i))$ holds. $\boldsymbol{F}=(f_{i,j})$ is in $U^*_{m,n}$. It is easy to see that $f(R^n\boldsymbol{H})=g(R^n\boldsymbol{H})$ and $g(R^n\boldsymbol{H})\subseteq R^n$ mean $\boldsymbol{HF}\in R_n$. From the fact that $f|_{R^n\boldsymbol{H}}=\iota\cdot g$ and $h\cdot g=\mathrm{I}$, it follows that $h\cdot f|_{R^n\boldsymbol{H}}=\mathrm{I}$, and $h\cdot f|_{R^n\boldsymbol{H}}=\mathrm{I}$ means $(\boldsymbol{HF})\cdot\boldsymbol{H}=\boldsymbol{H}$. Because, the i-th row of $(\boldsymbol{HF})\cdot\boldsymbol{H}$ is

$$
\begin{aligned}
&(\Sigma_{j,k=1}^{m,n} f_{j,k}(u_{i,j})\, u_{k,1}, \Sigma_{j,k=1}^{m,n} f_{j,k}(u_{i,j})\, u_{k,2}, \cdots, \Sigma_{j,k=1}^{m,n} f_{j,k}(u_{i,j})\, u_{k,m})=\\
&h(\Sigma_{j=1}^m f_{j,1}(u_{i,j}), \Sigma_{j=1}^m f_{j,2}(u_{i,j}), \cdots, \Sigma_{j=1}^m f_{j,n}(u_{i,j}))=\\
&h\cdot f(u_{i,1}, u_{i,2}, \cdots, u_{i,m})=(u_{i,1}, u_{i,2}, \cdots, u_{i,m})
\end{aligned}
$$

which is i-th row of $\boldsymbol{H}$. Conversely, suppose that there is an $\boldsymbol{F}\in U^*_{m,n}$ such that

$HE \in R_n$ and $(HF) \cdot H = H$. Then, the epimorphism h: $R^n \to R^n H$ is split, that is, there is an R-homomorphism f':$R^n H \to R^n$; $aH \rightsquigarrow (aH)F$ with $h \cdot f' = \mathrm{I}$. Because, $h \cdot f'(aH) = (a(FH)) \cdot H = a((FH) \cdot H) = aH$ for every $aH \in R^n H$, since $(aH)F = a(FH)$ for $a \in R^n$. Hence $R^n H$ is projective over R. Thus, the proof finished.

Especially, if $U = R$, one can regard $R^*_{m,n} (= \mathrm{Hom}_R(R^m, E^n))$ as $E_{m,n}$ by a natural isomorphism $\mathrm{Hom}_R(R^m, E^n) \to E_{m,n}$; $(f_{i,j}) \rightsquigarrow (f_{i,j}(1))$. Then, for $H \in R_{n,m}$ and $F \in E_{m,n}$, the product HF coincides with the ordinary product of matrices H and F.

Corollary A. *Let A be an $n \times m$-matrix over R. Then, $R^n A$ is projective over R if and only if there exists an $F \in E_{m,n}$ such that $A \cdot F \in R_n$ and $(A \cdot F) \cdot A = A$.*

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