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## ON $(r, p)$ -CAPACITIES FOR MARKOV PROCESSES

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### 1. Introduction

For a general Markovian semi-group  $\{P_t; t \geq 0\}$  on a measure space, we consider the image  $F_{r,p}$  of  $L_p$ -space of the  $r$ -th order  $\Gamma$ -transformation of  $P_t$ . Then  $F_{r,p}$  gives rise to a set function  $C_{r,p}$  satisfying certain properties of capacity (M. Fukushima and H. Kaneko [6]). When  $P_t$  is a symmetric operator on  $L_2$ -space, the capacity  $C_{1,2}$  coincides with the capacity related to the Dirichlet space associated with  $P_t$ , and consequently, the set of zero  $C_{1,2}$ -capacity can be identified with the polar set of the Hunt process corresponding to  $P_t$ , if the latter ever exists ([5]). But as  $r$  or  $p$  becomes greater, the set of  $(r, p)$ -capacities zero become finer. For instance, when  $P_t$  is the heat kernels on  $R^n$ , the  $\Gamma$ -transformations of  $P_t$  are equal to the so-called Bessel kernels. Therefore, in that case,  $C_{r,p}$  coincides with the Bessel capacity  $B_{r,p}$  presented in [11], for which there exists no non-empty sets of zero capacity whenever  $rp > n$  ([11]).

The purpose of this paper is to examine whether some basic theorems related to the Markovian semi-group  $\{P_t; t \geq 0\}$  can be refined, so that one may take the sets of  $C_{r,p}$ -capacity zero for various  $r$  and  $p$  as exceptional sets in the statement of the theorems. Assuming the analyticity of  $P_t$ , we shall show that two refinements (Theorem 1 in §2 and Theorem 3 in §4) of this kind are indeed possible. The first one is for ergodic theorem due to G.C. Rota [13], E.M. Stein [16] (which concerned  $m$ -a.e. statements) and due to M. Fukushima [4] (which concerned  $C_{1,2}$ -q.e. statement). The second is for the construction of a Hunt process which has been established by M. Fukushima [5] and M. Silverstein [14] in the case that  $(r, p) = (1, 2)$  and  $P_t$  is symmetric and by S.C. Menendez [10] in a non-symmetric case. In §3, a refinement in the construction of a transition function will be presented.

In this connection, we mention the work of Y. Le Jan [8] who started with a general Markovian semi-group on an  $L_\infty$ -space and constructed a Hunt process with exceptional set being related to a certain family of supermedian functions. While the above mentioned papers and ours start with a Markovian semi-group acting on an  $L_2$ -space or  $L_p$ -space, D. Feyel and A. de La Pradelle [3] started with the one acting on a Banach space of functions which are already refined in relation to a capacity. Further we mention a related work of N.G.

Meyers [11] who formulated a non-linear potential theory based on a class of kernels with lower semi-continuous density functions.

In this paper, we always assume that the space of potentials  $F_{r,p}$  is regular in the sense that  $F_{r,p}$  contains sufficiently many continuous functions. For instance, when the semi-group is generated by a strongly elliptic partial differential operator of second order with smooth coefficients, then  $F_{r,p}$  coincides with  $W_p^r(\mathbb{R}^n)$  (see example at the end of this paper). But in general, it is rather hard to check the regularity of the space  $F_{r,p}$  for  $(r, p) \neq (1, 2)$ .

Finally, as an application of a theory of  $(r, p)$ -capacities to other kinds of problems, we like to mention the works by A. B. Cruzeiro [2] and A. Nagel, W. Rudin and J.H. Shapiro [12] concerning the boundary limit theorems and by P. Malliavin [9] and M. Takeda [17] concerning infinite dimensional analysis.

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**2. Some limit theorems of semi-groups**

Let  $X$  be a separable metric space and  $m$  be a positive  $\sigma$ -finite measure with the support  $X$ . Through the paper, let us consider a strongly continuous contractive semi-group  $(P_t)_{t \geq 0}$  on  $L_p(X; m)$  ( $1 < p < \infty$ ), which is Markovian;

$$0 \leq f \leq 1 \quad m\text{-a.e.} \Rightarrow 0 \leq P_t f \leq 1 \quad m\text{-a.e.}$$

We also require that it is analytic in  $t > 0$  as a bounded operator valued function of  $t$ .

Let us recall some notations formulated in [6]. The Markovian contractive operator  $V_r$  ( $r > 0$ ) is defined by

$$(1) \quad V_r = \Gamma(r/2)^{-1} \int_0^\infty s^{r/2-1} e^{-s} P_s ds .$$

We let  $\|u\|_{r,p} = \|f\|_{L_p}$  for  $u = V_r f, f \in L_p$ , then the space  $F_{r,p} = V_r(L_p)$  with the norm  $\| \cdot \|_{r,p}$  is a Banach space. We define a set of function  $C_{r,p}$  by

$$C_{r,p}(A) = \inf \{ \|u\|_{r,p}^p; u \in F_{r,p} \text{ satisfies } u \geq 1 \text{ m-a.e. on some open set which contains } A \} .$$

“ $C_{r,p}$ -quasi-everywhere” or briefly “ $C_{r,p}$ -q.e.” means that the statement holds except on a  $C_{r,p}$  (capacity) zero set. A function  $u$  is called  $C_{r,p}$ -quasi-continuous if for any  $\varepsilon > 0$  there exists an open set  $G$  such that  $C_{r,p}(G) < \varepsilon$  and the function is continuous on  $X - G$ . A sequence of functions  $u_n$  is said to be  $C_{r,p}$ -quasi-uniformly convergent to a function  $u$  if for any  $\varepsilon > 0$  there exists an open set  $G$  such that  $C_{r,p}(G) < \varepsilon$  and the sequence of functions  $u_n$  converges to  $u$  uniformly on  $X - G$ .

We make the following assumption:

(2)  $F_{r,p} \cap C(X)$  is dense in the Banach space  $F_{r,p}$ .

We can show the following ([6]):

(a)  $C_{r,p}$  is an outer capacity and stable under the increasing limits of sets.

(b)  $C_{r,p}$  is non-decreasing in  $r$ .

(c) A function  $u$  is  $C_{r,p}$ -quasi-continuous and  $u \geq 0$   $m$ -a.e.  $\Rightarrow u \geq 0$   $C_{r,p}$ -q.e.

(d)  $u \in F_{r,p} \Rightarrow$  a  $C_{r,p}$ -quasi-continuous modification  $\tilde{u}$  of  $u$  exists, and it enjoys

$$(3) \quad C_{r,p}(|\tilde{u}| > \lambda) \leq \lambda^{-p} \|u\|_{r,p}^p, \lambda > 0.$$

(e) The convergence of  $C_{r,p}$ -quasi-continuous functions in  $F_{r,p}$  implies  $C_{r,p}$ -quasi-uniform convergence of some subsequence to a  $C_{r,p}$ -quasi-continuous function.

We know that the semi-group restores some potential theoretic feature. Let  $r > 0$  and  $1 < p < \infty$  be fixed.

**Lemma 1.** For each  $f \in L_p$ , we can take a function  $\widehat{P}_t f(x)$  of  $x \in X$  and  $t > 0$  which has the following properties.

(i) For each  $t > 0$ ,  $\widehat{P}_t f(x)$  is a  $C_{r,p}$ -quasi-continuous version of  $P_t f(x)$ , moreover for any  $\varepsilon > 0$  there exists an open set  $G$  independent of  $t$  such that  $C_{r,p}(G) < \varepsilon$  and the functions  $\{\widehat{P}_t f(x)\}_{t > 0}$  are continuous on  $X - G$ .

(ii) For  $C_{r,p}$ -quasi-everywhere  $x \in X$ , the function  $\widehat{P}_t f(x)$  is analytic in  $t$ .

(iii) For each  $t_0 \geq 0$ , there exist positive constants  $C$  and  $\varepsilon$  such that

$$(4) \quad C_{r,p}(\sup_{|t-t_0| < \varepsilon} |\widehat{P}_t f(x)| > \lambda) \leq C \lambda^{-p} \|f\|_{L_p}^p, \quad f \in L_p, \lambda > 0.$$

Proof. Take a natural number  $n > r/2$ . Since  $V_r$  has a semi-group property in  $r$ , we have

$$P_t f = (V_1)^{2n} (I - A)^n P_t f = V_r V_{2n-r} ((I - d/dt)^n P_t) f,$$

where  $A$  is the generator of the semi-group  $(P_t)_{t \geq 0}$  and  $d/dt$  stands for the derivative in the operator topology. Hence,  $P_t f$  is an element of  $F_{r,p}$ . Consider an operator valued function  $S_t = V_{2n-r} ((I - d/dt)^n P_t)$ . Then analyticity of  $S_t$  in  $t$  admits the Taylor expansion around  $t = t_0$ :

$$S_t = \sum_{n=0}^{\infty} B_n (t - t_0)^n, \quad |t - t_0| < \varepsilon,$$

where the  $B_n$ 's are bounded operators in  $L_p$  such that  $\sum_{n=0}^{\infty} \|B_n\| \varepsilon^n < \infty$ .

Take any  $f \in L_p$  and quasi-continuous versions  $\widetilde{V_r B_n f}$ ,  $n=0, 1, 2, \dots$ . Then by (e),  $\sum_{n=0}^{\infty} |\widetilde{V_r B_n f}|(x) \varepsilon^n$  converges except on some Borel set  $N$  with  $C_{r,p}(N)=0$ . Therefore, if we set, for  $|t-t_0| < \varepsilon$

$$P_t f(x) = \begin{cases} \sum_{n=0}^{\infty} \widetilde{V_r B_n f}(x) (t-t_0)^n, & \text{if } x \in X-N, \\ 0, & \text{otherwise,} \end{cases}$$

and patch the functions in  $t$ , then we have  $\widehat{P_t f}(x)$  which enjoys properties (i) and (ii). (iii) is clear from (3) and

$$\sup_{|t-t_0| < \varepsilon} |P_t f(x)| \leq \widetilde{V_r(\sum_{n=0}^{\infty} |B_n f| \varepsilon^n)}(x). \quad \text{q.e.d.}$$

In the remainder of this section, we only consider a strongly continuous contraction semi-group  $(P_t)_{t \geq 0}$  which is determined by Markovian symmetric operator  $(P_t)_{t \geq 0}$  on  $L_2(X; m)$ . E. M. Stein ([16]) shows that  $(P_t)_{t \geq 0}$  then becomes an analytic semi-group on  $L_p$  for each  $p > 1$ . We introduce for  $f \in L_p(X; m)$  the maximal function  $Mf$  by

$$Mf(x) = \sup_{t > 0} |\widehat{P_t f}(x)|,$$

where  $\widehat{P_t f}$  is the function in Lemma 1. Then we have the  $L_p$ -estimate ([16]):

$$(5) \quad \|Mf\|_{L_p}^p \leq C_p \|f\|_{L_p}^p, \quad f \in L_p$$

for some positive constant  $C_p$ .

**Lemma 2.** For each  $\lambda > 0$ ,  $u \in F_{r,p}$ , we have

$$C_{r,p}(Mu > \lambda) \leq C_p \lambda^{-p} \|u\|_{r,p}^p$$

Proof. For  $f \in L_p(X; m)$  and  $u = V_r f$ , we have

$$|P_t V_r f| = |V_r P_t f| \leq V_r Mf \quad m\text{-a.e.}$$

and consequently

$$Mu \leq \widetilde{V_r Mf} \quad C_{r,p}\text{-q.e.}$$

Hence, by (5)

$$\begin{aligned} C_{r,p}(Mu > \lambda) &\leq C_{r,p}(\widetilde{V_r Mf} > \lambda) \leq \lambda^{-p} \|Mf\|_{L_p}^p \\ &\leq C_p \lambda^{-p} \|f\|_{L_p}^p = C_p \lambda^{-p} \|u\|_{r,p}^p, \quad u \in F_{r,p}. \quad \text{q.e.d.} \end{aligned}$$

**Theorem 1.** Assume that  $(P_t)_{t \geq 0}$  is determined by a Markovian symmetric operator on  $L_2$ .

(i) For any  $u \in F_{r,p}$ , the limit  $\lim_{t \rightarrow 0} \widehat{P}_t u(x)$  exists  $C_{r,p}$ -q.e. which is  $C_{r,p}$ -quasi-continuous version of  $u$ , where  $r > 0, p > 1$ .

(ii) The limit  $\lim_{t \rightarrow \infty} \widehat{P}_t f(x) = h(x)$  exists  $C_{r,2}$ -q.e., for any  $f \in L_2(X; m)$ .  $h$  satisfies

$$\widehat{P}_t h(x) = h(x), \quad t > 0, \quad C_{r,2}\text{-a.e.}$$

Proof. (i) If we set

$$R(u) = \lim_{n \rightarrow \infty} \sup_{0 < t, t' < 1/n} |\widehat{P}_t u(x) - \widehat{P}_{t'} u(x)|,$$

then  $R(u) = 0$   $C_{r,p}$ -q.e., for any  $u \in F_{r,p}$ . For the last lemma combining with the inequality

$$R(u) = R(u - P_h u) \leq 2M(u - P_h u) \quad C_{r,p}\text{-q.e.}$$

shows that

$$C_{r,p}(R(u) > \lambda) \leq C_p (2\lambda)^{-p} \|u - P_h u\|_{r,p}^p,$$

which tends to zero as  $h$  tends to zero for any  $\lambda > 0$ . By (e), the pointwise limit  $\lim_{t \rightarrow 0} \widehat{P}_t u(x)$  must be a  $C_{r,p}$ -quasi-continuous version of  $u$ .

(ii) As in [4], we easily obtain the existence of the  $C_{r,2}$ -q.e. limit  $h = \lim_{t \rightarrow \infty} \widehat{P}_t f$ .  $h$  is  $C_{r,2}$ -quasi-continuous. Recalling the analyticity of  $\widehat{P}_t h$ , we have

$$P_t h(x) = h(x), \quad t > 0, \quad C_{r,2}\text{-q.e.} \qquad \text{q.e.d.}$$

### 3. Construction of a transition function

In this section, we suppose that  $X$  is separable complete metric space,  $X$  is covered by some countable family of closed sets with finite  $m$ -measure and the support of  $m$  is  $X$ . Given a strongly continuous Markovian semi-group  $(P_t)_{t \geq 0}$  on  $L_p(X; m)$  ( $1 < p < \infty$ ) satisfying the analyticity in  $t > 0$  and the regularity condition (2), we have constructed a regularized version  $\widehat{P}_t$ ,  $f \in L_p$  in Lemma 1. We can further construct a transition function as follows.

**Theorem 2.** There exists a family of kernels  $\{p_t(x, E); t > 0, x \in X, E \in \mathcal{B}\}$ , where  $\mathcal{B}$  stands for the set of all Borel subsets of  $X$ , which satisfies the following conditions:

(i)  $p_t(x, X) \leq 1, \quad t > 0$ .

(ii)  $\int_X p_t(x, dy) p_s(y, E) = p_{t+s}(x, E), \quad t, s > 0$ .

(iii) For each  $f \in L_p$  and  $r > 0$ , there exists a Borel set  $N$  such that  $C_{r,p}(N) = 0$  and

$$p_t f(x) = \widehat{P}_t f(x)$$

for every  $t > 0$  and  $x \in X - N$ .

Proof. We only give the proof in the case that  $m(X) < \infty$  but the proof is similar to the  $\sigma$ -finite case. Let us embed the space  $X$  homeomorphically onto a Borel subset  $Y$  of  $[0, 1]^N$ . Take a countable dense subset  $C_1$  of  $C([0, 1]^N)$ . Denoting by  $B_b$  the set of all bounded Borel functions of  $[0, 1]^N$  and by  $\tilde{f}$  the restriction to  $Y$  of  $f \in B_b$ , then  $B_b \subset L_p([0, 1]^N)$ . By virtue of Lemma 1, we get

$$\widehat{P}_t(a\tilde{f} + b\tilde{g})(x) = a\widehat{P}_t\tilde{f}(x) + b\widehat{P}_t\tilde{g}(x) \quad C_{r,p}\text{-q.e.}$$

for  $f, g \in B_b, a, b \in R,$

$$f_n, f \in B_b, f_n \uparrow f \Rightarrow \widehat{P}_t\tilde{f}_n(x) \uparrow \widehat{P}_t\tilde{f}(x) \quad C_{r,p}\text{-q.e.}$$

Further we find the set  $N \subset X$  with  $C_{r,p}(N) = 0$  such that  $\widehat{P}_t\tilde{f}(x)$  is analytic function of  $t > 0$  for  $f \in C_1, x \in X - N$ .

By similar way of the proof of Proposition (4.1) in R.K. Gettoor [7], we obtain the kernel  $q_t(x, E)$  such that

$$q_t f(x) = \widehat{P}_t\tilde{f}(x), \quad x \in X - N, f \in C_1, t \in Q^+,$$

where  $Q^+$  is the set of all positive rational numbers. Since  $[0, 1]^N$  is compact, the dual space of  $C([0, 1]^N)$  is weakly complete and  $q_t(x, \cdot)$  ( $t \in Q^+$ ) has a continuous extension to the half real line. Denote by  $p_t(x, \cdot)$ ,  $t \in (0, \infty)$  the restriction of  $q_t(x, \cdot)$  to  $X$ , then we have

$$p_t\tilde{f}(x) = \widehat{P}_t\tilde{f}(x) \quad \text{for any } t > 0, x \in X - N, f \in C_1.$$

Hence, we arrive at (iii) by Lemma 1 and a monotone lemma.

On the other hand, there exists a Borel set  $Y_1$  with  $C_{r,p}(X - N) = 0$  such that for each  $x \in Y_1$

$$p_t(p_s\tilde{f})(x) = p_{t+s}\tilde{f}(x), \quad t, s \in Q^+, f \in C_1$$

and all the functions  $p_t\tilde{f}(x)$  and  $p_t(p_s\tilde{f})(x)$ ,  $s \in Q^+$  are continuous in  $t > 0$ . Now just as the proof of Lemma 6.1.4. in M. Fukushima [5], we can modify  $p_t(x, E)$  slightly to get kernels which satisfy not only (i), (iii) but also (ii) of Theorem 2. q.e.d.

We call the kernels in Theorem 2 a transition function representing the

semi-group  $(P_i)_{i \geq 0}$ . Once such a transition function is constructed, we get a nice potential kernel by

$$v_r(x, E) = \Gamma(r/2)^{-1} \int_0^\infty s^{r/2-1} e^{-s} p_s(x, E) ds .$$

In fact, we have

**Corollary.**  $v_r f(x)$  is a  $C_{r,p}$ -quasi-continuous version of  $V_r f$ , for every  $f$  in  $L_p(X; m)$ .

Proof. It suffices to prove this for bounded functions in  $L_p$ . If  $f \in L_p$  is bounded, we have the pointwise convergence

$$p_t v_r f(x) = v_r p_t f(x) \rightarrow v_r f, \quad t \rightarrow 0 .$$

The convergence also takes place in the Banach space  $F_{r,p}$ , and hence we get the above conclusion. q.e.d.

#### 4. Construction of Hunt processes

In this section, we assume that the state space  $X$  is a locally compact separable metric space and the measure  $m$  is positive Radon with support  $X$ . Let  $(P_i)_{i \geq 0}$  be a Markovian strongly continuous contraction semi-group defined on  $L_p(X; m)$  ( $1 < p < \infty$ ) which is analytic in the sense of §2. When the space  $F_{r,p}$  contains continuous functions densely, we saw in §2 and §3 that the semi-group admits some potential theoretic refinements. Our assertion of this section is that under a stronger assumption (6) on  $F_{r,p}$  mentioned below we can construct an associated Hunt process starting from  $C_{r,p}$ -quasi-everywhere point of  $X$ , for  $r \geq 2$ .

Let  $X_\Delta = X \cup \{\Delta\}$  be the one-point compactification of  $X$ , and extend the transition function of the last section to  $X_\Delta$  by

$$p_i(x, E) = \begin{cases} p_i(x, E - \{\Delta\}) + (1 - p_i(x, X)) \delta_\Delta(E), & x \in X \\ \delta_\Delta(E), & x = \Delta \text{ or } t = 0, \end{cases}$$

for Borel subset  $E$  of  $X$ .

By the Kolmogorov extension theorem, there is a Markov process  $M_0 = \{\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0, P_x\}_{x \in X}$  with transition probability  $(p_t)_{t \in \mathbb{Q}^+}$ , where  $\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0$  are the following objects:

$$\begin{aligned} \Omega_0 &= X_\Delta^{\mathbb{Q}^+ \cup \{0\}} \\ X_t^0(\omega) &= \omega(t), \quad \omega \in \Omega_0 \\ \mathcal{M} &= \sigma[X_t^0(\omega); t \in \mathbb{Q}^+] \\ \mathcal{M}_t^0 &= \sigma[X_s^0(\omega); s \leq t, s \in \mathbb{Q}^+] . \end{aligned}$$



We let  $r \geq 2$  and assume that

(6) the  $F_{r,p} \cap C_\infty(X)$  is dense not only in  $F_{r,p}$ , but also in  $C_\infty(X)$ ,

where  $C_\infty(X) = \{f \in C(X_\Delta); f(\Delta) = 0\}$ . Under the assumption, we have a sequence  $\{t_j\}_{j=0}^\infty \subset \mathbb{Q}^+$  decreasing to 0, which satisfies

(7)  $\lim_{j \rightarrow \infty} p_{t_j} f(x) = f(x)$ , for any  $f \in C_\infty(X)$  and  $x \in X - N$

for some  $N$  with  $C_{r,p}(N) = 0$ , because we can see this for a dense subclass of  $C_\infty(X)$ , contained in  $F_{r,p}$ , as in the proof of Corollary in §3. An increasing sequence  $\{F_k\}_{k=1}^\infty$  of closed sets with  $\lim_{k \rightarrow \infty} C_{r,p}(X - F_k) = 0$  is said to be a  $C_{r,p}$ -nest. The condition (6) further implies that each  $u \in F_{r,p}$  admits a  $C_{r,p}$ -nest  $\{F_k\}_{k=1}^\infty$  for which  $u|_{F_k \cup \{\Delta\}}$  are continuous functions vanishing at  $\Delta$ ,  $k=1, 2, 3, \dots$ . The totality of such functions is denoted by  $C_\infty(\{F_k\}_{k=1}^\infty)$ . Here, we shall show a crucial lemma.

**Lemma 3.** *Under the assumption (6), we get the followings:*

(i) *For any decreasing sequence  $\{O_n\}_{n=0}^\infty$  of open sets with  $C_{r,p}(O_n) \rightarrow 0$ , as  $n \rightarrow \infty$ , we have*

$$P_x(\lim_{n \rightarrow \infty} \sigma_{O_n}^0 = \infty) = 1, \text{ for } C_{r,p}\text{-q.e. } x \in X,$$

where  $\sigma_A^0 = \inf \{t > 0; X_t^0 \in A\}$ .

(ii) *If we let  $\Omega_1 = \{\omega \in \Omega_0; \text{the sample path } X_t^0(\omega) \text{ has left- and right-hand limits in } X, \text{ for all } t > 0\}$ , then*

$$P_x(\Omega_1) = 1, \text{ for } C_{r,p}\text{-q.e. } x \in X.$$

(iii) *If we let  $X_t(\omega) = \lim_{s \uparrow t, s \in \mathbb{Q}^+} X_s^0(\omega)$  and  $\Omega_2 = \{\omega \in \Omega_1; X_t(\omega) = X_t^0(\omega), t \in \mathbb{Q}^+ \text{ and } X^0(\omega) = x\}$ , then*

$$P_x(\Omega_2) = 1, \text{ for } C_{r,p}\text{-q.e. } x \in X.$$

(iv) *Let  $\Omega_3 = \{\omega \in \Omega_2; \text{if } X_t(\omega) \in X \text{ then the trajectory of the sample path up to the time } t \text{ lies in a compact subset of } X \text{ for all } t > 0\}$ , then*

$$P_x(\Omega_3) = 1, \text{ for } C_{r,p}\text{-q.e. } x \in X.$$

(v) *There exists a Borel set  $Z \subset X$  and  $\Gamma_0 \subset \mathcal{M}$  satisfying  $C_{r,p}(X - Z) = 0$ ,  $P_x(\Gamma_0) = 0$  for all  $x \in Z$  and the inclusion*

$$\{\omega \in \Omega_3; \text{for some } t \geq 0, \text{ either } X_t(\omega) \text{ or } \lim_{s \uparrow t} X_s(\omega) \text{ is not in } Z\} \subset \Gamma_0.$$

(vi) *Put  $\mathcal{M}_t = \bigcup_{s \leq t, s \in \mathbb{Q}^+} \mathcal{M}_s^0$ . Consider the restrictions of  $\mathcal{M}$ ,  $\mathcal{M}_t$ ,  $X_t$  and  $P_x$  to*

the set  $\Omega = \Omega_3 - \Gamma_0$  and denote them by the same notations. Then the quintuplet  $M_Z = \{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x\}_{x \in Z}$  becomes a Hunt process on  $Z$ .

Proof. (i) Every open set  $O$  of finite capacity possesses a unique norm-minimizing element  $e_o$  in the set  $\{u \in F_{r,p}; u \geq 1 \text{ m-a.e. on } O\}$ . As a version of  $e_o$ , take  $e_o$  a function expressed as  $v_r f$  for some non-negative function  $f$  in  $L_p$ . Clearly we have then

$$e^{-t} p_t \hat{e}_o(x) \leq e_o(x)$$

which means that  $\{Y_t = e^{-t} \hat{e}_o(X_t)\}_{t \in \mathbb{Q}^+}$  is  $\{\mathcal{M}_t, P_x\}$ -supermartingale for each  $x \in X$ .

Applying Doob's optional sampling theorem to  $\{Y_t, \mathcal{M}_t, P_x\}$   $x \in X$  and noting that the process  $\{X_t^0\}_{t \in \mathbb{Q}^+}$  does not hit the set  $\{x \in 0; \hat{e}_o(x) < 1\}$  with  $P_x$ -a.e.  $x \in X$ , we obtain

$$E_x(\exp(-\sigma_0^0)) \leq \hat{e}_o \quad C_{r,p}\text{-q.e.}$$

The statement (i) follows from this inequality.

(ii) Take a countably dense subset  $C_2 \subset C_0^+(X)$ . There exists a nest  $\{F_k\}_{k=1}^\infty$  such that

$$(8) \quad \text{The convergence (7) holds on } \bigcup_{k=1}^\infty F_k,$$

$$(9) \quad \bigcup_{l=1+[r], 2+[r], \dots} v_l(C_2) \subset C^\infty(\{F_k\}_{k=1}^\infty).$$

Here, we know that the family of functions of the left hand side of (9) separates the point of  $Z_0 = (\bigcup_{k=1}^\infty F_k) \cup \{\Delta\}$ . In fact, if we suppose for  $x, y \in Z_0$

$$v_l f(x) = v_l f(y), \quad \text{for any } f \in C_2, l = 1+[r], 2+[r], \dots$$

then  $p_t f(x) = p_t f(y)$ ,  $t > 0, f \in C_2$ , by the uniqueness of the Laplace transformation. Letting  $t$  tend to 0 along the sequence  $\{t_j\}_{j=1}^\infty$ , we see that  $f(x) = f(y)$ ,  $f \in C_2$  by (8) and  $x = y$ .

Hence, for the event  $\Omega_{00} = \{\omega \in \Omega_0; \lim_{k \rightarrow \infty} \sigma_{X-F_k}^0(\omega) = \infty\}$ , we have that

$$\Omega_{00} - \Omega_1 \subset \{\omega \in \Omega_0; \text{for some } k \text{ and some } t < \sigma_{X-F_k}^0 \\ X_s^0(\omega) \text{ does not have the right- or left-hand limit at } t\}.$$

Since the process  $\{e^{-s} v_l f(X_s^0(\omega)), \mathcal{M}_s, P_x\}$  is a non-negative supermartingale, the  $P_x$  measure of the right hand side is zero. In view of (i), we know that  $P_x(\Omega_{00}) = 1, C_{r,p}$ -q.e. and so is  $\Omega_1$ .

(iii), (iv), (v) and (vi) The proofs can be performed in the same way as in [5; Chapter 6, §2]. q.e.d.

We extend the Hunt process of Lemma 3 (vi) to the Hunt process on  $X$  by letting each point of  $X-Z$  be trap.

**Theorem 3.** *There exists a Hunt process  $M = \{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x\}_{x \in X_\Delta}$  satisfying that*

$$(10) \text{ for each } f \in L_p, E_x(f(X_t)) \text{ is a } C_{r,p}\text{-quasi-continuous modification of } P_t f.$$

If  $M' = \{\Omega', \mathcal{M}', \mathcal{M}'_t, X'_t, P'_x\}_{x \in X_\Delta}$  is another Hunt process with property (10), then the induced probability laws of  $X_t$  and  $X'_t$  on the path space  $\tilde{\Omega} = \{\tilde{\omega}; [0, \infty) \mapsto X, \tilde{\omega}(t) \text{ is right continuous with left limits in } t\}$  coincide for  $C_{r,p}$ -q.e.  $x \in X$ .

Proof. The existence is already shown. To prove the part of the uniqueness, it suffices to show that for  $M'$  with the property (10)

$$\begin{aligned} E_x(f_1(X_{t_1})f_2(X_{t_2}) \cdots f_n(X_{t_n})) \\ = E'_x(f_1(X'_{t_1})f_2(X'_{t_2}) \cdots f_n(X'_{t_n})), \\ C_{r,p}\text{-q.e.}, \end{aligned}$$

where  $f_1, f_2, \dots, f_n \in C_2, t_1, t_2, \dots, t_n \in \mathbb{Q}^+$ . But this is clear from (10). q.e.d.

In the symmetric case, we have a criterion for the sample path continuity of the Hunt process  $M$ .

Let us consider a strongly continuous semi-group  $(P_t)_{t \geq 0}$  of Markovian symmetric operator on  $L_2$ . As stated in §2, it can be regarded as a strongly contraction analytic semi-group in  $L_p$  ( $1 < p < \infty$ ). We assume that the regularity (6) for the associate space  $F_{r,p}$  and  $F_{1,2}$ .

**Theorem 4.** *The following conditions are equivalent.*

- (i) *The Dirichlet space  $F_{1,2}$  is local in the sense that the pair  $u, v \in F_{1,2}$  with disjoint supports always enjoys the property  $(u, v)_{F_{1,2}} = 0$ .*
- (ii)  *$M$  is a diffusion in the sense*

$$P_x(\omega \in \Omega; \text{the sample path is continuous}) = 1, \quad C_{r,p}\text{-q.e.}$$

Proof. Let us set  $q(x) = P_x(\omega \in \Omega; \text{for some } t > 0, \lim_{s \uparrow t} X_s(\omega) \neq X_t(\omega))$ . If  $q(x)$  vanishes  $m$ -a.e., then  $q(x) = 0$   $C_{r,p}$ -q.e. Because the function  $P_x(\omega \in \Omega; \text{for some } t > 1/n, \lim_{s \uparrow t} X_s(\omega) \neq X_t(\omega)) = p_{1/n} q(x)$  then vanishing  $C_{r,p}$ -q.e. Since  $M$  can be also regarded as the diffusion as a realization of the  $L_2$ -semi-group, the first statement of Theorem 4 combined with a general theorem related to the Dirichlet space implies that  $q(x) = 0$   $m$ -a.e. The proof of theorem is completed. q.e.d.

EXAMPLE. Suppose that a uniformly elliptic partial differential operator

$L = \sum_{i,j=1}^n a_{ij}(x) \partial^2/\partial x_i \partial x_j + \sum_{i=1}^n b_i(x) \partial/\partial x_i + c(x)$  possesses bounded smooth coefficients in the sense that  $a_{ij}(x) \in C^2_b(\mathbb{R}^n)$ ,  $1 \leq i, j \leq n$ ,  $b_i(x) \in C^1_b(\mathbb{R}^n)$ ,  $1 \leq i \leq n$ ,  $\sum_{i,j=1}^n a_{ij}(x) \xi^i \xi^j \geq \delta |\xi|^2$  for some  $\delta > 0$ , and that  $c(x)$  is bounded non-positive.

The resolvent  $R_\lambda$  on  $L_2(\mathbb{R}^n)$  satisfies  $\|R_\lambda\| \leq C/(1+|\lambda|)$  in the domain  $\{\lambda \in \mathbb{C}; \operatorname{Re}(\lambda) \geq \alpha\}$  with some positive  $C$  and  $\alpha$ . Owing to a well known theorem of K. Yosida [18; Chapter IX, 10], the corresponding semi-group is analytic in  $L_2(\mathbb{R}^n)$ . Obviously the semi-group  $(P_t)_{t \geq 0}$  is Markovian and contractive. We observe that the dual semi-group has the same properties. By the method of interpolation mentioned in E.M. Stein [16] and above observation, we know that in  $L_p(\mathbb{R}^n)$   $(P_t)_{t \geq 0}$  is analytic whenever  $1 < p < \infty$ .

The Sobolev space  $W^2_p(\mathbb{R}^n)$  as the domain of the closed extension of  $L$  with domain  $C^\infty_0(\mathbb{R}^n)$  coincides with the space of potentials  $F_{2,p}$  with equivalent norms. Since  $W^2_p(\mathbb{R}^n)$  satisfies the assumption (6), Theorem 3 gives us the corresponding Hunt process in the  $C_{2,p}$ -refined sense. The Sobolev imbedding theorem assures that " $C_{2,p}$ -q.e." becomes "everywhere" when  $2p > n$ . Consequently the Hunt process is uniquely associated without exceptional starting point.

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