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ON (r, p)-CAPACITIES FOR MARKOV PROCESSES

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1. Introduction

For a general Markovian semi-group $\{P_t;\ t\geqq 0\}$ on a measure space, we consider the image $F_{r,p}$ of L_p-space of the r-th order Γ-transformation of P_t. Then $F_{r,p}$ gives rise to a set function $C_{r,p}$ satisfying certain properties of capacity (M. Fukushima and H. Kaneko [6]). When P_t is a symmetric operator on L_2-space, the capacity $C_{1,2}$ coincides with the capacity related to the Dirichlet space associated with P_t, and consequently, the set of zero $C_{1,2}$-capacity can be identified with the polar set of the Hunt process corresponding to P_t, if the latter ever exists ([5]). But as r or p becomes greater, the set of (r, p)-capacities zero become finer. For instance, when P_t is the heat kernels on R^n, the Γ-transformations of P_t are equal to the so-called Bessel kernels. Therefore, in that case, $C_{r,p}$ coincides with the Bessel capacity $B_{r;p}$ presented in [11], for which there exists no non-empty sets of zero capacity whenever $rp>n$ ([11]).

The purpose of this paper is to examine whether some basic theorems related to the Markovian semi-group $\{P_t;\ t\geqq 0\}$ can be refined, so that one may take the sets of $C_{r,p}$-capacity zero for various r and p as exceptional sets in the statement of the theorems. Assuming the analyticity of P_t, we shall show that two refinements (Theorem 1 in §2 and Theorem 3 in §4) of this kind are indeed possible. The first one is for ergodic theorem due to G.C. Rota [13], E.M. Stein [16] (which concerned m-a.e. statements) and due to M. Fukushima [4] (which concerned $C_{1,2}$-q.e. statement). The second is for the construction of a Hunt process which has been established by M. Fukushima [5] and M. Silverstein [14] in the case that $(r, p)=(1, 2)$ and P_t is symmetric and by S.C. Menendez [10] in a non-symmetric case. In §3, a refinement in the construction of a transition function will be presented.

In this connection, we mention the work of Y. Le Jan [8] who started with a general Markovian semi-group on an L_∞-space and constructed a Hunt process with exceptional set being related to a certain family of supermedian functions. While the above mentioned papers and ours start with a Markovian semi-group acting on an L_2-space or L_p-space, D. Feyel and A. de La Pradelle [3] started with the one acting on a Banach space of functions which are already refined in relation to a capacity. Further we mention a related work of N.G.

Meyers [11] who formulated a non-linear potential theory based on a class of kernels with lower semi-continuous density functions.

In this paper, we always assume that the space of potentials $F_{r,p}$ is regular in the sense that $F_{r,p}$ contains sufficiently many continuous functions. For instance, when the semi-group is generated by a strongly elliptic partial differential operator of second order with smooth coefficients, then $F_{r,p}$ coincides with $W_p^r(R^n)$ (see example at the end of this paper). But in general, it is rather hard to check the regularity of the space $F_{r,p}$ for $(r, p)\neq(1, 2)$.

Finally, as an application of a theory of (r, p)-capacities to other kinds of problems, we like to mention the works by A. B. Cruzerio [2] and A. Nagel, W. Rudin and J.H. Shapiro [12] concerning the boundary limit theorems and by P. Malliavin [9] and M. Takeda [17] concerning infinite dimensional analysis.

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2. Some limit theorems of semi-groups

Let X be a separable metric space and m be a positive σ-finite measure with the support X. Through the paper, let us consider a strongly continuous contractive semi-group $(P_t)_{t\geqq 0}$ on $L_p(X; m)$ $(1<p<\infty)$, which is Markovian;

$$0\leqq f\leqq 1 \quad m\text{-a.e.} \Rightarrow 0\leqq P_t f\leqq 1 \quad m\text{-a.e.}$$

We also require that it is analytic in $t>0$ as a bounded operator valued function of t.

Let us recall some notations formulated in [6]. The Markovian contractive operator V_r $(r>0)$ is defined by

$$V_r = \Gamma(r/2)^{-1}\int_0^\infty s^{r/2-1}e^{-s}P_s ds\,. \tag{1}$$

We let $||u||_{r,p} = ||f||_{L_p}$ for $u=V_r f$, $f\in L_p$, then the space $F_{r,p}=V_r(L_p)$ with the norm $||\ \ ||_{r,p}$ is a Banach space. We define a set of function $C_{r,p}$ by

$$C_{r,p}(A) = \inf\ \{||u||_{r,p}^p;\ u\in F_{r,p} \quad \text{satisfies} \quad u\geq 1$$
$$m\text{-a.e. on some open set which contains } A\}\ .$$

"$C_{r,p}$-quasi-everywhere" or briefly "$C_{r,p}$-q.e." means that the statement holds except on a $C_{r,p}$ (capacity) zero set. A function u is called $C_{r,p}$-quasi-continuous if for any $\varepsilon>0$ there exists an open set G such that $C_{r,p}(G)<\varepsilon$ and the function is continuous on $X-G$. A sequence of functions u_n is said to be $C_{r,p}$-quasi-uniformly convergent to a function u if for any $\varepsilon>0$ there exists an open set G such that $C_{r,p}(G)<\varepsilon$ and the sequence of functions u_n converges to u uniformly on $X-G$.

We make the following assumption:

$$(2) \qquad F_{r,p} \cap C(X) \quad \text{is dense in the Banach space } F_{r,p}\,.$$

We can show the following ([6]):

(a) $C_{r,p}$ is an outer capacity and stable under the increasing limits of sets.

(b) $C_{r,p}$ is non-decreasing in r.

(c) A function u is $C_{r,p}$-quasi-continuous and $u \geqq 0$ m-a.e. $\Rightarrow u \geqq 0$ $C_{r,p}$-q.e.

(d) $u \in F_{r,p} \Rightarrow$ a $C_{r,p}$-quasi-continuous modification $\tilde{u}$ of u exists, and it enjoys

$$(3) \qquad C_{r,p}(|\tilde{u}| > \lambda) \leqq \lambda^{-p} \|u\|_{r,p}^{p},\ \lambda > 0\,.$$

(e) The convergence of $C_{r,p}$-quasi-continuous functions in $F_{r,p}$ implies $C_{r,p}$-quasi-uniform convergence of some subsequence to a $C_{r,p}$-quasi-continuous function.

We know that the semi-group restores some potential theoretic feature. Let $r > 0$ and $1 < p < \infty$ be fixed.

Lemma 1. *For each $f \in L_p$, we can take a function $\widehat{P_t f}(x)$ of $x \in X$ and $t > 0$ which has the following properties.*

(i) *For each $t > 0$, $\widehat{P_t f}(x)$ is a $C_{r,p}$-quasi-continuous version of $P_t f(x)$, moreover for any $\varepsilon > 0$ there exists an open set G independent of t such that $C_{r,p}(G) < \varepsilon$ and the functions $\{\widehat{P_t f}(x)\}_{t>0}$ are continuous on $X - G$.*

(ii) *For $C_{r,p}$-quasi-everywhere $x \in X$, the function $\widehat{P_t f}(x)$ is analytic in t.*

(iii) *For each $t_0 \geqq 0$, there exist positive constants C and ε such that*

$$(4) \qquad C_{r,p}\Big(\sup_{|t-t_0|<\varepsilon} |\widehat{P_t f}(x)| > \lambda\Big) \leqq C\lambda^{-p} \|f\|_{L_p}^{p}, \qquad f \in L_p,\ \lambda > 0\,.$$

Proof. Take a natural number $n > r/2$. Since V_r has a semi-group property in r, we have

$$P_t f = (V_1)^{2n}(I-A)^n P_t f = V_r V_{2n-r}((I-d/dt)^n P_t) f\,,$$

where A is the generator of the semi-group $(P_t)_{t \geqq 0}$ and d/dt stands for the derivative in the operator topology. Hence, $P_t f$ is an element of $F_{r,p}$. Consider an operator valued function $S_t = V_{2n-r}((I-d/dt)^n P_t)$. Then analyticity of S_t in t admits the Taylor expansion around $t = t_0$:

$$S_t = \sum_{n=0}^{\infty} B_n (t-t_0)^n, \quad |t-t_0| < \varepsilon\,,$$

where the B_n's are bounded operators in L_p such that $\sum_{n=0}^{\infty} \|B_n\| \varepsilon^n < \infty$.

Take any $f \in L_p$ and quasi-continuous versions $\widetilde{V_r B_n f}$, $n=0, 1, 2, \cdots$. Then by (e), $\sum_{n=0}^{\infty} |\widetilde{V_r B_n f}|(x)\, \varepsilon^n$ converges except on some Borel set N with $C_{r,p}(N)=0$. Therefore, if we set, for $|t-t_0|<\varepsilon$

$$P_t f(x) = \begin{cases} \sum_{n=0}^{\infty} \widetilde{V_r B_n f}(x)\, (t-t_0)^n\,, & \text{if } x \in X-N\,, \\ 0 & \text{, otherwise}\,, \end{cases}$$

and patch the functions in t, then we have $\widehat{P_t f}(x)$ which enjoys properties (i) and (ii). (iii) is clear from (3) and

$$\sup_{|t-t_0|<\varepsilon} |P_t f(x)| \leqq \widetilde{V_r(\sum_{n=0}^{\infty} |B_n f|\, \varepsilon^n)}\,(x)\,. \quad \text{q.e.d.}$$

In the remainder of this section, we only consider a strongly continuous contraction semi-group $(P_t)_{t \geqq 0}$ which is determined by Markovian symmetric operator $(P_t)_{t \geqq 0}$ on $L_2(X; m)$. E. M. Stein ([16]) shows that $(P_t)_{t \geqq 0}$ then becomes an analytic semi-group on L_p for each $p>1$. We introduce for $f \in L_p(X; m)$ the maximal function Mf by

$$Mf(x) = \sup_{t>0} |\widehat{P_t f}(x)|\,,$$

where $\widehat{P_t f}$ is the function in Lemma 1. Then we have the L_p-estimate ([16]):

$$\|Mf\|_{L_p}^p \leqq C_p \|f\|_{L_p}^p\,, \quad f \in L_p \tag{5}$$

for some positive constant C_p.

Lemma 2. *For each $\lambda>0$, $u \in F_{r,p}$, we have*

$$C_{r,p}(Mu>\lambda) \leqq C_p \lambda^{-p} \|u\|_{r,p}^p$$

Proof. For $f \in L_p(X; m)$ and $u=V_r f$, we have

$$|P_t V_r f| = |V_r P_t f| \leqq V_r Mf \quad m\text{-a.e.}$$

and consequently

$$Mu \leqq \widetilde{V_r Mf} \quad C_{r,p}\text{-q.e.}$$

Hence, by (5)

$$\begin{aligned} C_{r,p}(Mu>\lambda) &\leqq C_{r,p}(\widetilde{V_r Mf}>\lambda) \leqq \lambda^{-p} \|Mf\|_{L_p}^p \\ &\leqq C_p \lambda^{-p} \|f\|_{L_p}^p = C_p \lambda^{-p} \|u\|_{r,p}^p\,, \quad u \in F_{r,p}\,. \quad \text{q.e.d.} \end{aligned}$$

Theorem 1. *Assume that $(P_t)_{t\geqq 0}$ is determined by a Markovian symmetric operator on L_2.*

(i) *For any $u\in F_{r,p}$, the limit $\lim_{t\to 0}\widehat{P_t u}(x)$ exists $C_{r,p}$-q.e. which is $C_{r,p}$-quasi-continuous version of u, where $r>0$, $p>1$.*

(ii) *The limit $\lim_{t\to\infty}\widehat{P_t f}(x)=h(x)$ exists $C_{r,2}$-q.e., for any $f\in L_2(X; m)$. h satisfies*

$$\widehat{P_t h}(x) = h(x),\quad t>0\,,\quad C_{r,2}\text{-}a.e.$$

Proof. (i) If we set

$$R(u) = \lim_{n\to\infty}\ \sup_{0<t,t'<1/n}\ |\widehat{P_t u}(x)-\widehat{P_{t'} u}(x)|\,,$$

then $R(u)=0$ $C_{r,p}$-q.e., for any $u\in F_{r,p}$. For the last lemma combining with the inequality

$$R(u) = R(u-P_h u)\leqq 2M(u-P_h u)\quad C_{r,p}\text{-q.e.}$$

shows that

$$C_{r,p}(R(u)>\lambda)\leqq C_p(2\lambda)^{-p}||u-P_h u||^p_{r,p}\,,$$

which tends to zero as h tends to zero for any $\lambda>0$. By (e), the pointwise limit $\lim_{t\to 0}\widehat{P_t u}(x)$ must be a $C_{r,p}$-quasi-continuous version of u.

(ii) As in [4], we easily obtain the existence of the $C_{r,2}$-q.e. limit $h=\lim_{t\to\infty}\widehat{P_t f}$. h is $C_{r,2}$-quasi-continuous. Recalling the analyticity of $\widehat{P_t h}$, we have

$$P_t h(x) = h(x),\ t>0\,,\quad C_{r,2}\text{-q.e.}\qquad\qquad \text{q.e.d.}$$

3. Construction of a transition function

In this section, we suppose that X is separable complete metric space, X is covered by some countable family of closed sets with finite m-measure and the support of m is X. Given a strongly continuous Markovian semi-group $(P_t)_{t\geqq 0}$ on $L_p(X; m)$ $(1<p<\infty)$ satisfying the analyticity in $t>0$ and the regularity condition (2), we have constructed a regularized version $\widehat{P_t t}$, $f\in L_p$ in Lemma 1. We can further construct a transition function as follows.

Theorem 2. *There exists a family of kernels $\{p_t(x, E);\ t>0,\ x\in X,\ E\in \mathcal{B}\}$, where $\mathcal{B}$ stands for the set of all Borel subsets of X, which satisfies the following conditions:*

(i) $p_t(x, X)\leqq 1,\quad t>0.$

(ii) $\int_X p_t(x, dy)\,p_s(y, E) = p_{t+s}(x, E),\quad t,\ s>0.$

(iii) *For each $f \in L_p$ and $r>0$, there exists a Borel set N such that $C_{r,p}(N)=0$ and*

$$p_t f(x) = \widehat{P_t f}(x)$$

for every $t>0$ and $x \in X-N$.

Proof. We only give the proof in the case that $m(X)<\infty$ but the proof is similar to the σ-finite case. Let us embed the space X homeomorphically onto a Borel subset Y of $[0, 1]^N$. Take a countable dense subset C_1 of $C([0, 1]^N)$. Denoting by B_b the set of all bounded Borel functions of $[0, 1]^N$ and by $\tilde{f}$ the restriction to Y of $f \in B_b$, then $B_b \subset L_p([0, 1]^N)$. By virtue of Lemma 1, we get

$$\widehat{P_t(a\tilde{f}+b\tilde{g})}\,(x) = a\widehat{P_t\tilde{f}}(x)+b\widehat{P_t\tilde{g}}(x) \quad C_{r,p}\text{-q.e.}$$
$$\text{for} \quad f, g \in B_b,\ a, b \in R\,,$$
$$f_n, f \in B_b, f_n \uparrow f \Rightarrow \widehat{P_t\tilde{f}_n}(x) \uparrow \widehat{P_t\tilde{f}}(x) \quad C_{r,p}\text{-q.e.}$$

Further we find the set $N \subset X$ with $C_{r,p}(N)=0$ such that $\widehat{P_t\tilde{f}}(x)$ is analytic function of $t>0$ for $f \in C_1$, $x \in X-N$.

By similar way of the proof of Proposition (4.1) in R.K. Getoor [7], we obtain the kernel $q_t(x, E)$ such that

$$q_t f(x) = \widehat{P_t\tilde{f}}(x), \quad x \in X-N, f \in C_1, t \in Q^+\,,$$

where Q^+ is the set of all positive rational numbers. Since $[0, 1]^N$ is compact, the dual space of $C([0, 1]^N)$ is weakly complete and $q_t(x, \cdot)$ $(t \in Q^+)$ has a continuous extension to the half real line. Denote by $p_t(x, \cdot)$, $t \in (0, \infty)$ the restriction of $q_t(x, \cdot)$ to X, then we have

$$p_t\tilde{f}(x) = \widehat{P_t\tilde{f}}(x) \quad \text{for any} \quad t>0, x \in X-N, f \in C_1\,.$$

Hence, we arrive at (iii) by Lemma 1 and a monotone lemma.

On the other hand, there exists a Borel set Y_1 with $C_{r,p}(X-N)=0$ such that for each $x \in Y_1$

$$p_t(p_s\tilde{f})\,(x) = p_{t+s}\tilde{f}(x), \quad t, s \in Q^+, f \in C_1$$

and all the functions $p_t\tilde{f}(x)$ and $p_t(p_s\tilde{f})(x)$, $s \in Q^+$ are continuous in $t>0$. Now just as the proof of Lemma 6.1.4. in M. Fukushima [5], we can modify $p_t(x, E)$ slightly to get kernels which satisfy not only (i), (iii) but also (ii) of Theorem 2.

q.e.d.

We call the kernels in Theorem 2 a transition function representing the semi-group $(P_t)_{t\geqq 0}$. Once such a transition function is constructed, we get a nice potential kernel by

$$v_r(x, E) = \Gamma(r/2)^{-1}\int_0^\infty s^{r/2-1}\, e^{-s}\, p_s(x, E)ds\,.$$

In fact, we have

Corollary. *$v_r f(x)$ is a $C_{r,p}$-quasi-continuous version of $V_r f$, for every f in $L_p(X; m)$.*

Proof. It suffices to prove this for bounded functions in L_p. If $f\in L_p$ is bounded, we have the pointwise convergence

$$p_t v_r f(x) = v_r p_t f(x) \to v_r f, \quad t\to 0\,.$$

The convergence also takes place in the Banach space $F_{r,p}$, and hence we get the above conclusion. q.e.d.

4. Construction of Hunt processes

In this section, we assume that the state space X is a locally compact separable metric space and the measure m is positive Radon with support X. Let $(P_t)_{t\geqq 0}$ be a Markovian strongly continuous contraction semi-group defined on $L_p(X; m)$ $(1<p<\infty)$ which is analytic in the sense of §2. When the space $F_{r,p}$ contains continuous functions densely, we saw in §2 and §3 that the semi-group admits some potential theoretic refinements. Our assertion of this section is that under a stronger assumption (6) on $F_{r,p}$ mentioned below we can construct an associated Hunt process strating from $C_{r,p}$-quasi-everywhere point of X, for $r\geqq 2$.

Let $X_\Delta = X\cup\{\Delta\}$ be the one-point compactification of X, and extend the transition function of the last section to X_Δ by

$$p_t(x, E) = \begin{cases} p_t(x, E-\{\Delta\})+(1-p_t(x, X))\,\delta_\Delta(E), & x\in X \\ \delta_\Delta(E), \quad x=\Delta \text{ or } t=0\,, \end{cases}$$

for Borel subset E of X.

By the Kolmogorov extention theorem, there is a Markov process $M_0 = \{\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0, P_x\}_{x\in X}$ with transition probability $(p_t)_{t\in Q^+}$, where Ω_0, $\mathcal{M}$, $\mathcal{M}_t^0$, X_t^0 are the following objects:

$$\begin{aligned} &\Omega_0 = X_\Delta^{Q^+\cup\{0\}} \\ &X_t^0(\omega) = \omega(t), \quad \omega\in\Omega_0 \\ &\mathcal{M} = \sigma[X_t^0(\omega);\, t\in Q^+] \\ &\mathcal{M}_t^0 = \sigma[X_s^0(\omega);\, s\leqq t,\, s\in Q^+]\,. \end{aligned}$$

We let $r \geqq 2$ and assume that

(6) the $F_{r,p} \cap C_\infty(X)$ is dense not only in $F_{r,p}$ but also in $C_\infty(X)$,

where $C_\infty(X)=\{f \in C(X_\Delta);\ f(\Delta)=0\}$. Under the assumption, we have a sequence $\{t_j\}_{j=0}^\infty \subset Q^+$ decreasing to 0, which satisfies

(7) $$\lim_{j\to\infty} p_{t_j} f(x) = f(x), \text{ for any } f \in C_\infty(X) \text{ and } x \in X-N$$

for some N with $C_{r,p}(N)=0$, because we can see this for a dense subclass of $C_\infty(X)$, contained in $F_{r,p}$, as in the proof of Corollary in §3. An increasing sequence $\{F_k\}_{k=1}^\infty$ of closed sets with $\lim_{k\to\infty} C_{r,p}(X-F_k)=0$ is said to be a $C_{r,p}$-nest. The condition (6) further implies that each $u \in F_{r,p}$ admits a $C_{r,p}$-nest $\{F_k\}_{k=1}^\infty$ for which $u|_{F_k \cup \{\Delta\}}$ are continuous functions vanishing at Δ, $k=1, 2, 3, \cdots$. The totality of such functions is denoted by $C_\infty(\{F_k\}_{k=1}^\infty)$. Here, we shall show a crucial lemma.

Lemma 3. *Under the assumption* (6), *we get the followings*:

(i) *For any decreasing sequence* $\{O_n\}_{n=0}^\infty$ *of open sets with* $C_{r,p}(O_n)\to 0$, *as* $n\to\infty$, *we have*

$$P_x(\lim_{n\to\infty} \sigma^0_{O_n} = \infty) = 1, \quad \textit{for} \quad C_{r,p}\text{-}q.e.\ x \in X,$$

where $\sigma^0_A = \inf\{t>0;\ X^0_t \in A\}$.

(ii) *If we let* $\Omega_1 = \{\omega \in \Omega_0$; *the sample path* $X^0_t(\omega)$ *has left- and right-hand limits in* X, *for all* $t>0\}$, *then*

$$P_x(\Omega_1) = 1, \quad \textit{for} \quad C_{r,p}\text{-}q.e.\ x \in X.$$

(iii) *If we let* $X_t(\omega) = \lim_{s\downarrow t, s\in Q^+} X^0_s(\omega)$ *and* $\Omega_2 = \{\omega \in \Omega_1;\ X_t(\omega)=X^0_t(\omega), t \in Q^+$ *and* $X^0(\omega)=x\}$, *then*

$$P_x(\Omega_2) = 1, \quad \textit{for} \quad C_{r,p}\text{-}q.e.\ x \in X.$$

(iv) *Let* $\Omega_3 = \{\omega \in \Omega_2$; *if* $X_t(\omega) \in X$ *then the trajectory of the sample path up to the time* t *lies in a compact subset of* X *for all* $t>0\}$, *then*

$$P_x(\Omega_3) = 1, \quad \textit{for} \quad C_{r,p}\text{-}q.e.\ x \in X.$$

(v) *There exists a Borel set* $Z \subset X$ *and* $\Gamma_0 \subset \mathcal{M}$ *satisfying* $C_{r,p}(X-Z)=0$, $P_x(\Gamma_0)=0$ *for all* $x \in Z$ *and the inclusion*

$$\{\omega \in \Omega_3;\ \textit{for some } t \geqq 0, \textit{ either } X_t(\omega) \textit{ or } \lim_{s\uparrow t} X_s(\omega) \textit{ is not in } Z\} \subset \Gamma_0.$$

(vi) *Put* $\mathcal{M}_t = \bigcup_{s\leqq t, s\in Q^+} \mathcal{M}^0_s$. *Consider the restrictions of* $\mathcal{M}$, $\mathcal{M}_t$, X_t *and* P_x *to* *the set* $\Omega=\Omega_3-\Gamma_0$ *and denote them by the same notations. Then the quintuplet* $M_Z=\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x\}_{x\in Z}$ *becomes a Hunt process on* Z.

Proof. (i) Every open set O of finite capacity possesses a unique norm-minimizing element e_O in the set $\{u\in F_{r,p};\ u\geqq 1\ m\text{-a.e. on } O\}$. As a version of e_O, take e_O a function expressed as $v_r f$ for some non-negative function f in L_p. Clearly we have then

$$e^{-t}p_t\hat{e}_O(x)\leqq e_O(x)$$

which means that $\{Y_t=e^{-t}\,\hat{e}_O(X_t^0)\}_{t\in Q^+}$ is $\{\mathcal{M}_t, P_x\}$-supermartingale for each $x\in X$.

Applying Doob's optional sampling theorem to $\{Y_t, \mathcal{M}_t, P_x\}$ $x\in X$ and noting that the process $\{X_t^0\}_{t\in Q^+}$ does not hit the set $\{x\in 0;\ \hat{e}_O(x)<1\}$ with P_x-a.e. $x\in X$, we obtain

$$E_x(\exp(-\sigma_O^0))\leqq\hat{e}_O\quad C_{r,p}\text{-q.e.}$$

The statement (i) follows from this inequality.
(ii) Take a countably dense subset $C_2\subset C_0^+(X)$. There exists a nest $\{F_k\}_{k=1}^{\infty}$ such that

(8) $$\text{The convergence (7) holds on } \bigcup_{k=1}^{\infty} F_k\,,$$

(9) $$\bigcup_{l=1+[r],2+[r],\cdots} v_l(C_2)\subset C^{\infty}(\{F_k\}_{k=1}^{\infty})\,.$$

Here, we know that the family of functions of the left hand side of (9) separates the point of $Z_0=(\bigcup_{k=1}^{\infty} F_k)\cup\{\Delta\}$. In fact, if we suppose for $x, y\in Z_0$

$$v_l f(x)=v_l f(y),\quad \text{for any}\quad f\in C_2,\ l=1+[r],\ 2+[r],\ \cdots$$

then $p_t f(x)=p_t f(y)$, $t>0$, $f\in C_2$, by the uniqueness of the Laplace transformation. Letting t tend to 0 along the sequence $\{t_j\}_{j=1}^{\infty}$, we see that $f(x)=f(y)$, $f\in C_2$ by (8) and $x=y$.

Hence, fcr the event $\Omega_{00}=\{\omega\in\Omega_0;\ \lim_{k\to\infty}\sigma^0_{X-F_k}(\omega)=\infty\}$, we have that

$$\Omega_{00}-\Omega_1\subset\{\omega\in\Omega_0;\ \text{for some } k \text{ and some } t<\sigma^0_{X-F_k}$$
$$X_s^0(\omega) \text{ does not have the right- or left-hand limit at } t\}\,.$$

Since the process $\{e^{-s}v_l f(X_s^0(\omega)), \mathcal{M}_s^0, P_x\}$ is a non-negative supermartingale, the P_x measure of the right hand side is zero. In view of (i), we know that $P_x(\Omega_{00})=1$, $C_{r,p}$-q.e. and so is Ω_1.
(iii), (iv), (v) and (vi) The proofs can be performed in the same way as in [5; Chapter 6, §2]. q.e.d.

We extend the Hunt process of Lemma 3 (vi) to the Hunt process on X by letting each point of $X-Z$ be trap.

Theorem 3. *There exists a Hunt process* $M=\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x\}_{x\in X_\Delta}$ *satisfying that*

(10) *for each* $f\in L_p$, $E_x(f(X_t))$ *is a* $C_{r,p}$*-quasi-continuous modification of* $P_t f$.

If $M'=\{\Omega', \mathcal{M}', \mathcal{M}'_t, X'_t, P'_x\}_{x\in X_\Delta}$ *is another Hunt process with property* (10), *then the induced probability laws of* X_t *and* X'_t *on the path space* $\tilde{\Omega}=\{\tilde{\omega}; [0, \infty)\mapsto X, \tilde{\omega}(t)$ *is right continuous with left limits in* $t\}$ *coincide for* $C_{r,p}$*-q.e.* $x\in X$.

Proof. The existence is already shown. To prove the part of the uniqueness, it suffices to show that for M' with the property (10)

$$\begin{aligned} &E_x(f_1(X_{t_1})f_2(X_{t_2})\cdots f_n(X_{t_n})) \\ &\quad = E'_x(f_1(X'_{t_1})f_2(X'_{t_2})\cdots f_n(X'_{t_n}))\,, \\ &\quad C_{r,p}\text{-q.e.}\,, \end{aligned}$$

where $f_1, f_2, \cdots, f_n\in C_2$, $t_1, t_2, \cdots, t_n\in Q^+$. But this is clear from (10). q.e.d.

In the symmetric case, we have a criterion for the sample path continuity of the Hunt process M.

Let us consider a strongly continuous semi-group $(P_t)_{t\geq 0}$ of Markovian symmetric operator on L_2. As stated in §2, it can be regarded as a strongly contraction analytic semi-group in L_p $(1<p<\infty)$. We assume that the regularity (6) for the associate space $F_{r,p}$ and $F_{1,2}$.

Theorem 4. *The following conditions are equivalent.*

(i) *The Dirichlet space* $F_{1,2}$ *is local in the sense that the pair* $u, v\in F_{1,2}$ *with disjoint supports always enjoys the property* $(u, v)_{F_{1,2}}=0$.

(ii) *M is a diffusion in the sense*

$$P_x(\omega\in\Omega;\ \textit{the sample path is continuous}) = 1, \quad C_{r,p}\textit{-q.e.}$$

Proof. Let us set $q(x)=P_x(\omega\in\Omega;$ for some $t>0$, $\lim_{s\uparrow t} X_s(\omega)\neq X_t(\omega))$. If $q(x)$ vanishes m-a.e., then $q(x)=0$ $C_{r,p}$-q.e. Because the function $P_x(\omega\in\Omega$: for some $t>1/n$, $\lim_{s\uparrow t} X_s(\omega)\neq X_t(\omega))=p_{1/n}q(x)$ then vanishing $C_{r,p}$-q.e. Since M can be also regarded as the diffusion as a realization of the L_2-semi-group, the first statement of Theorem 4 combined with a general theorem related to the Dirichlet space implies that $q(x)=0$ m-a.e. The proof of theorem is completed. q.e.d.

EXAMPLE. Suppose that a uniformly elliptic partial differential operator

$L=\sum_{i,j=1}^{n} a_{ij}(x)\, \partial^2/\partial x_i\, \partial x_j + \sum_{i=1}^{n} b_i(x)\, \partial/\partial x_i + c(x)$ possesses bounded smooth coefficients in the sense that $a_{ij}(x)\in C_b^2(R^n)$, $1\leqq i,\ j\leqq n$, $b_i(x)\in C_b^1(R^n)$, $1\leqq i\leqq n$, $\sum_{i,j=1}^{n} a_{ij}(x)\, \xi^i\, \xi^j \geqq \delta\,|\,\xi\,|^2$ for some $\delta>0$, and that $c(x)$ is bounded non-positive.

The resolvent R_λ on $L_2(R^n)$ satisfies $||R_\lambda||\leqq C/(1+|\lambda|)$ in the domain $\{\lambda\in C;\ \mathrm{Re}(\lambda)\geqq\alpha\}$ with some positive C and α. Owing to a well known theorem of K. Yosida [18; Chapter IX, 10], the corresponding semi-group is analytic in $L_2(R^n)$. Obviously the semi-group $(P_t)_{t\geqq 0}$ is Markovian and contractive. We observe that the dual semi-group has the same properties. By the method of interpolation mentioned in E.M. Stein [16] and above observation, we know that in $L_p(R^n)$ $(P_t)_{t\geqq 0}$ is analytic whenever $1<p<\infty$.

The Sobolev space $W_p^2(R^n)$ as the domain of the closed extension of L with domain $C_0^\infty(R^n)$ coincides with the space of potentials $F_{2,p}$ with equivalent norms. Since $W_p^2(R^n)$ satisfies the assumption (6), Theorem 3 gives us the corresponding Hunt process in the $C_{2,p}$-refined sense. The Sobolev imbedding theorem assures that "$C_{2,p}$-q.e." becomes "everywhere" when $2p>n$. Consequently the Hunt process is uniquely associated without exceptional starting point.

References

[1] D.R. Adams: *Maximal operators and capacity*, Proc. Amer. Math. Soc. **34** (1972), 152–156.

[2] A.B. Cruzerio: *Convergence au bord pour les fonctions harmoniques dans R^d de la classe de Sobolev W_1^q*, C.R. Acad. Sci. Paris Sér. I Math. **294** (1982), 71–74.

[3] D. Feyel and A. de La Pradelle: *Processus associates a une classe d'espaces de Banach fonctionnels,* Z. Wahrsch. Verw. Gebiete **43** (1978), 339–351.

[4] M. Fukushima: *Capacitary maximal inequalities and an ergodic theorem*, Lecture Notes in Math. No. 1021, Springer-Verlag, Berlin, 1982, 130–136.

[5] M. Fukushima: Dirichlet forms and Markov processes, North-Holland and Kodonshia, Tokyo, 1980.

[6] M. Fukushima and H. Kaneko: *On (r, p)-capacities for general Markovian semi-groups*, Pioc. USP-meeting Bielefeld 'Stochastic process and infinite dimensional analysis', 1983, ed. by S. Albeverio, Pitman.

[7] R.K. Getoor: *On the construction of kernels*, Lecture Notes in Math. No. 465, Springer-Verlag, Berlin, 1974/75, 443–463.

[8] Y. Le Jan: *Quasi-continuous functions and Hunt processes,* J. Math. Soc. Japan **35** (1983), 37–42.

[9] P. Malliavin: *Implicit functions in finite corank on the Wiener space*, Proc. Taniguchi Intern. Symp. on Stochastic Analysis, Katata and Kyoto, Kinokuniya, Tokyo, 1982.

[10] S.C. Menendez: *Processus de Markov associé a une forme de Dirichlet non symetrique*, Z. Wahrsch. Verw. Gebiete **33** (1975), 139–154.

[11] N.G. Meyers: *A theory of capacities for potentials of functions in Lebesgue classes*, Math. Scand. **68** (1970), 255–292.

[12] A. Nagel, W. Rudin and J.H. Shapiro: *Tangential boundary behavior of function in Dirichlet-type spaces*, Ann. of Math. **116** (1982), 331–360.

[13] G.C. Rota: *An "Alernieven de Verfahren" for general positive operators*, Bull. Amer. Math. Soc. **68** (1968), 95–102.

[14] M.L. Silverstein: Symmetric Markov processes, Lecture Notes in Math. No. 426, Springer-Verlag, Berlin, 1974.

[15] E.M. Stein: Singular integral and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.

[16] E.M. Stein: Topics in harmonic analysis related to the Littlewood-Paley theory, Princeton Univ. Press, Princeton, 1970.

[17] M. Takeda: *(r, p)-capacity on the Wiener space and properties of Brownian motion*, Z. Wahrsch. Verv. Gebiete **68** (1984), 149–162.

[18] K. Yosida: Functional analysis, Springer-Verlag, Berlin, 1965.

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