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ON (r, p)-CAPACITIES FOR MARKOV PROCESSES

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1. Introduction

For a general Markovian semi-group $\{P_t; t \ge 0\}$ on a measure space, we consider the image $F_{r,p}$ of L_p -space of the r-th order Γ -transformation of P_t . Then $F_{r,p}$ gives rise to a set function $C_{r,p}$ satisfying certain properties of capacity (M. Fukushima and H. Kaneko [6]). When P_t is a symmetric operator on L_2 -space, the capacity $C_{1,2}$ coincides with the capacity related to the Dirichlet space associated with P_t , and consequently, the set of zero $C_{1,2}$ -capacity can be identified with the polar set of the Hunt process corresponding to P_t , if the latter ever exists ([5]). But as r or p becomes greater, the set of (r, p)capacities zero become finer. For instance, when P_t is the heat kernels on \mathbb{R}^n , the Γ -transformations of P_t are equal to the so-called Bessel kernels. Therefore, in that case, $C_{r,p}$ coincides with the Bessel capacity $B_{r;p}$ presented in [11], for which there exists no non-empty sets of zero capacity whenever rp > n ([11]).

The purpose of this paper is to examine whether some basic theorems related to the Markovian semi-group $\{P_t; t \ge 0\}$ can be refined, so that one may take the sets of $C_{r,p}$ -capacity zero for various r and p as exceptional sets in the statement of the theorems. Assuming the analyticity of P_t , we shall show that two refinements (Theorem 1 in §2 and Theorem 3 in §4) of this kind are indeed possible. The first one is for ergodic theorem due to G.C. Rota [13], E.M. Stein [16] (which concerned m-a.e. statements) and due to M. Fukushima [4] (which concerned $C_{1,2}$ -q.e. statement). The second is for the construction of a Hunt process which has been established by M. Fukushima [5] and M. Silverstein [14] in the case that (r, p)=(1, 2) and P_t is symmetric and by S.C. Menendez [10] in a non-symmetric case. In §3, a refinement in the construction of a transition function will be presented.

In this connection, we mention the work of Y. Le Jan [8] who started with a general Markovian semi-group on an L_{∞} -space and constructed a Hunt process with exceptional set being related to a certain family of supermedian functions. While the above mentioned papers and ours start with a Markovian semi-group acting on an L_2 -space or L_p -space, D. Feyel and A. de La Pradelle [3] started with the one acting on a Banach space of functions which are already refined in relation to a capacity. Further we mention a related work of N.G. Meyers [11] who formulated a non-linear potential theory based on a class of kernels with lower semi-continuous density functions.

In this paper, we always assume that the space of potentials $F_{r,p}$ is regular in the sense that $F_{r,p}$ contains sufficiently many continuous functions. For instance, when the semi-group is generated by a strongly elliptic partial differential operator of second order with smooth coefficients, then $F_{r,p}$ coincides with $W_p^r(\mathbb{R}^n)$ (see example at the end of this paper). But in general, it is rather hard to check the regularity of the space $F_{r,p}$ for $(r, p) \neq (1, 2)$.

Finally, as an application of a theory of (r, p)-capacities to other kinds of problems, we like to mention the works by A. B. Cruzerio [2] and A. Nagel, W. Rudin and J.H. Shapiro [12] concerning the boundary limit theorems and by P. Malliavin [9] and M. Takeda [17] concerning infinite dimensional analysis.

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2. Some limit theorems of semi-groups

Let X be a separable metric space and m be a positive σ -finite measure with the support X. Through the paper, let us consider a strongly continuous contractive semi-group $(P_t)_{t\geq 0}$ on $L_p(X; m)$ (1 , which is Markovian;

$$0 \leq f \leq 1$$
 m-a.e. $\Rightarrow 0 \leq P_t f \leq 1$ *m*-a.e.

We also require that it is analytic in t>0 as a bounded operator valued function of t.

Let us recall some notations formulated in [6]. The Markovian contractive operator V_r (r>0) is defined by

(1)
$$V_r = \Gamma(r/2)^{-1} \int_0^\infty s^{r/2-1} e^{-s} P_s ds \; .$$

We let $||u||_{r,p} = ||f||_{L_p}$ for $u = V_r f$, $f \in L_p$, then the space $F_{r,p} = V_r(L_p)$ with the norm $|| \quad ||_{r,p}$ is a Banach space. We define a set of function $C_{r,p}$ by

$$C_{r,p}(A) = \inf \{ ||u||_{r,p}^{p}; u \in F_{r,p} \text{ satisfies } u \ge 1$$

m-a.e. on some open set which contains $A \}$.

" $C_{r,p}$ -quasi-everywhere" or briefly " $C_{r,p}$ -q.e." means that the statement holds except on a $C_{r,p}$ (capacity) zero set. A function u is called $C_{r,p}$ -quasi-continuous if for any $\varepsilon > 0$ there exists an open set G such that $C_{r,p}(G) < \varepsilon$ and the function is continuous on X-G. A sequence of functions u_n is said to be $C_{r,p}$ -quasi-uniformly convergent to a function u if for any $\varepsilon > 0$ there exists an open set G such that $C_{r,p}(G) < \varepsilon$ and the sequence of functions u_n converges to u uniformly on X-G.

We make the following assumption:

(2)
$$F_{r,p} \cap C(X)$$
 is dense in the Banach space $F_{r,p}$.

We can show the following ([6]):

(a) $C_{r,p}$ is an outer capacity and stable under the increasing limits of sets.

(b) $C_{r,p}$ is non-decreasing in r.

(c) A function u is $C_{r,p}$ -quasi-continuous and $u \ge 0$ m-a.e. $\Rightarrow u \ge 0$ $C_{r,p}$ -q.e.

(d) $u \in F_{r,p} \Rightarrow$ a $C_{r,p}$ -quasi-continuous modification \tilde{u} of u exists, and it enjoys

(3)
$$C_{r,p}(|\tilde{u}| > \lambda) \leq \lambda^{-p} ||u||_{r,p}^{p}, \lambda > 0$$

(e) The convergence of $C_{r,p}$ -quasi-continuous functions in $F_{r,p}$ implies $C_{r,p}$ -quasi-uniform convergence of some subsequence to a $C_{r,p}$ -quasi-continuous function.

We know that the semi-group restores some potential theoretic feature. Let r>0 and 1 be fixed.

Lemma 1. For each $f \in L_p$, we can take a function $P_t f(x)$ of $x \in X$ and t > 0 which has the following properties.

(i) For each t>0, $P_t f(x)$ is a $C_{r,p}$ -quasi-continuous version of $P_t f(x)$, moreover for any $\varepsilon > 0$ there exists an open set G independent of t such that $C_{r,p}(G) < \varepsilon$ and the functions $\{P_t f(x)\}_{t>0}$ are continuous on X-G.

(ii) For $C_{r,p}$ -quasi-everywhere $x \in X$, the function $P_t f(x)$ is analytic in t.

(iii) For each $t_0 \ge 0$, there exist positive constants C and ε such that

(4)
$$C_{r,p}(\sup_{|t-t_0|<\varepsilon}|P_tf(x)|>\lambda) \leq C\lambda^{-p}||f||_{L_p}^b, \quad f \in L_p, \lambda > 0.$$

Proof. Take a natural number n > r/2. Since V_r has a semi-group property in r, we have

$$P_t f = (V_1)^{2n} (I - A)^n P_t f = V_r V_{2n-r} ((I - d/dt)^n P_t) f,$$

where A is the generator of the semi-group $(P_t)_{t\geq 0}$ and d/dt stands for the derivative in the operator topology. Hence, $P_t f$ is an element of $F_{r,p}$. Consider an operator valued function $S_t = V_{2n-r}((I-d/dt)^n P_t)$. Then analyticity of S_t in t admits the Taylor expansion around $t=t_0$:

$$S_t = \sum_{n=0}^{\infty} B_n (t-t_0)^n, \quad |t-t_0| < \varepsilon,$$

where the B_n 's are bounded operators in L_p such that $\sum_{n=0}^{\infty} ||B_n|| \varepsilon^n < \infty$.

H. KANEKO

Take any $f \in L_p$ and quasi-continuous versions $\widetilde{V_r B_n f}$, $n=0, 1, 2, \cdots$. Then by (e), $\sum_{n=0}^{\infty} |\widetilde{V_r B_n f}|(x) \varepsilon^n$ converges except on some Borel set N with $C_{r,p}(N)=0$. Therefore, if we set, for $|t-t_0| < \varepsilon$

$$P_t f(x) = \begin{cases} \widetilde{\sum_{n=0}^{\infty} V_r B_n f(x) (t-t_0)^n}, & \text{if } x \in X - N, \\ 0, & \text{, otherwise,} \end{cases}$$

and patch the functions in t, then we have $P_t f(x)$ which enjoys properties (i) and (ii). (iii) is clear from (3) and

$$\sup_{t-t_0|<\mathfrak{e}}|P_tf(x)|\leq \widetilde{V_r(\sum_{n=0}^\infty |B_nf|\mathcal{E}^n)}(x). \quad \text{q.e.d.}$$

In the remainder of this section, we only consider a strongly continuous contraction semi-group $(P_t)_{t\geq 0}$ which is determined by Markovian symmetric operator $(P_t)_{t\geq 0}$ on $L_2(X; m)$. E. M. Stein ([16]) shows that $(P_t)_{t\geq 0}$ then becomes an analytic semi-group on L_p for each p>1. We introduce for $f \in L_p(X; m)$ the maximal function Mf by

$$Mf(x) = \sup_{t>0} |P_t f(x)|$$
,

where P_{t} is the function in Lemma 1. Then we have the L_{p} -estimate ([16]):

(5)
$$||Mf||_{L_p}^p \leq C_p ||f||_{L_p}^p, \quad f \in L_p$$

for some positive constant C_p .

1

Lemma 2. For each $\lambda > 0$, $u \in F_{r,p}$, we have

$$C_{r,p}(Mu > \lambda) \leq C_p \lambda^{-p} ||u||_{r,p}^p$$

Proof. For $f \in L_p(X; m)$ and $u = V_r f$, we have

$$|P_tV_rf| = |V_rP_tf| \leq V_rMf$$
 m-a.e.

and consequently

$$Mu \leq \widetilde{V_rMf} \quad C_{r,p}$$
-q.e.

Hence, by (5)

$$C_{r,p}(Mu > \lambda) \leq C_{r,p}(\widetilde{V_rMf} > \lambda) \leq \lambda^{-p} ||Mf||_{L_p}^b$$

$$\leq C_p \lambda^{-p} ||f||_{L_p}^b = C_p \lambda^{-p} ||u||_{r,p}^b, \quad u \in F_{r,p}. \quad \text{q.e.d.}$$

Theorem 1. Assume that $(P_t)_{t\geq 0}$ is determined by a Markovian symmetric operator on L_2 .

(i) For any $u \in F_{r,p}$, the limit $\lim_{t \to 0} \hat{P}_{t}u(x)$ exists $C_{r,p}$ -q.e. which is $C_{r,p}$ -quasicontinuous version of u, where r > 0, p > 1. (ii) The limit $\lim_{t \to \infty} P_t f(x) = h(x)$ exists $C_{r,2}$ -q.e., for any $f \in L_2(X; m)$. h satisfies

$$\dot{P}_t \dot{h}(x) = h(x), t > 0, C_{r,2}$$
-a.e.

Proof. (i) If we set

$$R(u) = \lim_{n \to \infty} \sup_{0 < t, t' < 1/n} | \hat{P}_t u(x) - \hat{P}_{t'} u(x) | ,$$

then R(u)=0 $C_{r,p}$ -q.e., for any $u \in F_{r,p}$. For the last lemma combining with the inequality

$$R(u) = R(u - P_h u) \leq 2M(u - P_h u) \quad C_{r,p} - q.e.$$

shows that

$$C_{r,p}(R(u) > \lambda) \leq C_p(2\lambda)^{-p} ||u - P_h u||_{r,p}^p$$

which tends to zero as h tends to zero for any $\lambda > 0$. By (e), the pointwise limit $\lim_{t\to 0} \dot{P_t u}(x)$ must be a $C_{r,p}$ -quasi-continuous version of u.

(ii) As in [4], we easily obtain the existence of the $C_{r,2}$ -q.e. limit $h = \lim_{t \to \infty} P_t f$. *h* is $C_{r,2}$ -quasi-continuous. Recalling the analyticity of $\stackrel{\frown}{P_th}$, we have

$$P_t h(x) = h(x), t > 0, C_{r,2}$$
-q.e. q.e.d.

3. Construction of a transition function

In this section, we suppose that X is separable complete metric space, X is covered by some countable family of closed sets with finite *m*-measure and the support of m is X. Given a strongly continuous Markovian semigroup $(P_t)_{t \ge 0}$ on $L_p(X; m)$ (1 satisfying the analyticity in <math>t > 0 and the regularity condition (2), we have constructed a regularized version \dot{P}_{t} , $f \in L_p$ in Lemma 1. We can further construct a transition function as follows.

Theorem 2. There exists a family of kernels $\{p_t(x, E); t > 0, x \in X, E \in X\}$ \mathcal{B} , where \mathcal{B} stands for the set of all Borel subsets of X, which satisfies the following conditions:

(i)
$$p_t(x, X) \leq 1$$
, $t > 0$.
(ii) $\int_X p_t(x, dy) p_s(y, E) = p_{t+s}(x, E)$, $t, s > 0$.

H. KANEKO

(iii) For each $f \in L_p$ and r > 0, there exists a Borel set N such that $C_{r,p}(N) = 0$ and

$$p_t f(x) = P_t f(x)$$

for every t > 0 and $x \in X - N$.

Proof. We only give the proof in the case that $m(X) < \infty$ but the proof is similar to the σ -finite case. Let us embed the space X homeomorphically onto a Borel subset Y of $[0, 1]^N$. Take a countable dense subset C_1 of $C([0, 1]^N)$. Denoting by B_b the set of all bounded Borel functions of $[0, 1]^N$ and by \tilde{f} the restriction to Y of $f \in B_b$, then $B_b \subset L_p([0, 1]^N)$. By virtue of Lemma 1, we get

$$\overbrace{P_{i}(a\tilde{f}+b\tilde{g})}^{\frown}(x) = aP_{i}\widetilde{f}(x) + bP_{i}\widetilde{g}(x) \quad C_{r,p}\text{-q.e.}$$

for $f, g \in B_{b}, a, b \in R$,
 $f_{n}, f \in B_{b}, f_{n} \uparrow f \Rightarrow P_{i}\widetilde{f}_{n}(x) \uparrow P_{i}\widetilde{f}(x) \quad C_{r,p}\text{-q.e.}$

Further we find the set $N \subset X$ with $C_{r,p}(N) = 0$ such that $P_t \tilde{f}(x)$ is analytic function of t > 0 for $f \in C_1$, $x \in X - N$.

By similar way of the proof of Proposition (4.1) in R.K. Getoor [7], we obtain the kernel $q_t(x, E)$ such that

$$q_t f(x) = P_t \tilde{f}(x), \quad x \in X - N, f \in C_1, t \in Q^+,$$

where Q^+ is the set of all positive rational numbers. Since $[0, 1]^N$ is compact, the dual space of $C([0, 1]^N)$ is weakly complete and $q_i(x, \cdot)$ $(t \in Q^+)$ has a continuous extension to the half real line. Denote by $p_i(x, \cdot)$, $t \in (0, \infty)$ the restriction of $q_i(x, \cdot)$ to X, then we have

$$p_t \tilde{f}(x) = P_t \tilde{f}(x)$$
 for any $t > 0, x \in X - N, f \in C_1$.

Hence, we arrive at (iii) by Lemma 1 and a monotone lemma.

On the other hand, there exists a Borel set Y_1 with $C_{r,p}(X-N)=0$ such that for each $x \in Y_1$

$$p_t(p_s \tilde{f})(x) = p_{t+s} \tilde{f}(x), \quad t, s \in Q^+, f \in C_1$$

and all the functions $p_t \tilde{f}(x)$ and $p_t(p_s \tilde{f})(x)$, $s \in Q^+$ are continuous in t > 0. Now just as the proof of Lemma 6.1.4. in M. Fukushima [5], we can modify $p_t(x, E)$ slightly to get kernels which satisfy not only (i), (iii) but also (ii) of Theorem 2. q.e.d.

We call the kernels in Theorem 2 a transition function representing the

semi-group $(P_t)_{t\geq 0}$. Once such a transition function is constructed, we get a nice potential kernel by

$$v_r(x, E) = \Gamma(r/2)^{-1} \int_0^\infty s^{r/2-1} e^{-s} p_s(x, E) ds$$
.

In fact, we have

Corollary. $v_r f(x)$ is a $C_{r,p}$ -quasi-continuous version of $V_r f$, for every f in $L_p(X; m)$.

Proof. It suffices to prove this for bounded functions in L_p . If $f \in L_p$ is bounded, we have the pointwise convergence

$$p_t v_r f(x) = v_r p_t f(x) \rightarrow v_r f, \quad t \rightarrow 0.$$

The convergence also takes place in the Banach space $F_{r,p}$, and hence we get the above conclusion. q.e.d.

4. Construction of Hunt processes

In this section, we assume that the state space X is a locally compact separable metric space and the measure *m* is positive Radon with support X. Let $(P_t)_{t\geq 0}$ be a Markovian strongly continuous contraction semi-group defined on $L_p(X; m)$ (1 which is analytic in the sense of §2. When the space $<math>F_{r,p}$ contains continuous functions densely, we saw in §2 and §3 that the semigroup admits some potential theoretic refinements. Our assertion of this section is that under a stronger assumption (6) on $F_{r,p}$ mentioned below we can construct an associated Hunt process strating from $C_{r,p}$ -quasi-everywhere point of X, for $r \geq 2$.

Let $X_{\Delta} = X \cup \{\Delta\}$ be the one-point compactification of X, and extend the transition function of the last section to X_{Δ} by

$$p_t(x, E) = \begin{cases} p_t(x, E - \{\Delta\}) + (1 - p_t(x, X)) \, \delta_{\Delta}(E), & x \in X \\ \delta_{\Delta}(E), & x = \Delta \text{ or } t = 0, \end{cases}$$

for Borel subset E of X.

By the Kolmogorov extention theorem, there is a Markov process $M_0 = \{\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0, P_x\}_{x \in X}$ with transition probability $(p_t)_{t \in Q^+}$, where $\Omega_0, \mathcal{M}, \mathcal{M}_t^0, X_t^0$ are the following objects:

$$egin{aligned} \Omega_{0} &= X^{q^{+} \, \cup \, (0)}_{\Delta} \ X^{0}_{t}(\omega) &= \omega(t), \quad \omega \in \Omega_{0} \ \mathcal{M} &= \sigma[X^{0}_{t}(\omega); \, t \in Q^{+}] \ \mathcal{M}^{0}_{t} &= \sigma[X^{0}_{s}(\omega); \, s \leq t, \, s \in Q^{+}] \,. \end{aligned}$$

We let $r \ge 2$ and assume that

(6) the $F_{r,p} \cap C_{\infty}(X)$ is dense not only in $F_{r,p}$ but also in $C_{\infty}(X)$,

where $C_{\infty}(X) = \{f \in C(X_{\Delta}); f(\Delta) = 0\}$. Under the assumption, we have a sequence $\{t_j\}_{j=0}^{\infty} \subset Q^+$ decreasing to 0, which satisfies

(7)
$$\lim_{j \to \infty} p_{i_j} f(x) = f(x), \text{ for any } f \in C_{\infty}(X) \text{ and } x \in X - N$$

for some N with $C_{r,p}(N)=0$, because we can see this for a dense subclass of $C_{\infty}(X)$, contained in $F_{r,p}$, as in the proof of Corollary in §3. An increasing sequence $\{F_k\}_{k=1}^{\infty}$ of closed sets with $\lim_{k\to\infty} C_{r,p}(X-F_k)=0$ is said to be a $C_{r,p}$ -nest. The condition (6) further implies that each $u \in F_{r,p}$ admits a $C_{r,p}$ -nest $\{F_k\}_{k=1}^{\infty}$ for which $u|_{F_k \cup (\Delta)}$ are continuous functions vanishing at Δ , $k=1, 2, 3, \cdots$. The totality of such functions is denoted by $C_{\infty}(\{F_k\}_{k=1}^{\infty})$. Here, we shall show a crucial lemma.

Lemma 3. Under the assumption (6), we get the followings:

(i) For any decreasing sequence $\{O_n\}_{n=0}^{\infty}$ of open sets with $C_{r,p}(O_n) \rightarrow 0$, as $n \rightarrow \infty$, we have

$$P_x(\lim_{n\to\infty}\sigma^0_{O_n}=\infty)=1, \quad for \quad C_{r,p}-q.e. \ x\in X,$$

where $\sigma_A^0 = \inf \{t > 0; X_t^0 \in A\}.$

(ii) If we let $\Omega_1 = \{\omega \in \Omega_0; \text{ the sample path } X^0_t(\omega) \text{ has left- and right-hand limits in } X, \text{ for all } t>0\}, \text{ then}$

 $P_x(\Omega_1) = 1$, for $C_{r,p}$ -q.e. $x \in X$.

(iii) If we let $X_t(\omega) = \lim_{s \downarrow t, s \in Q^+} X_s^0(\omega)$ and $\Omega_2 = \{\omega \in \Omega_1; X_t(\omega) = X_t^0(\omega), t \in Q^+ \text{ and } X_s^0(\omega) = x\}$, then

$$P_x(\Omega_2) = 1$$
, for $C_{r,p}$ -q.e. $x \in X$.

(iv) Let $\Omega_3 = \{\omega \in \Omega_2; \text{ if } X_t(\omega) \in X \text{ then the trajectory of the sample path up to the time t lies in a compact subset of X for all <math>t > 0\}$, then

$$P_x(\Omega_3) = 1$$
, for $C_{r,p}$ -q.e. $x \in X$.

(v) There exists a Borel set $Z \subset X$ and $\Gamma_0 \subset \mathcal{M}$ satisfying $C_{r,p}(X-Z)=0$, $P_x(\Gamma_0)=0$ for all $x \in Z$ and the inclusion

 $\{\omega \in \Omega_3; \text{ for some } i \ge 0, \text{ either } X_i(\omega) \text{ or } \lim_{s \neq i} X_s(\omega) \text{ is not in } Z\} \subset \Gamma_0.$

(vi) Put $\mathcal{M}_t = \bigcup_{s \leq t, s \in Q^+} \mathcal{M}_s^0$. Consider the restrictions of \mathcal{M} , \mathcal{M}_t , X_t and P_s to

the set $\Omega = \Omega_3 - \Gamma_0$ and denote them by the same notations. Then the quintuplet $M_Z = \{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_s\}_{s \in Z}$ becomes a Hunt process on Z.

Proof. (i) Every open set O of finite capacity possesses a unique normminimizing element e_0 in the set $\{u \in F_{r,p}; u \ge 1 \text{ m-a.e. on } O\}$. As a version of e_0 , take e_0 a function expressed as $v_r f$ for some non-negative function f in L_p . Clearly we have then

$$e^{-t}p_t\hat{e}_0(x) \leq e_0(x)$$

which means that $\{Y_t = e^{-t} \hat{e}_0(X_t^0)\}_{t \in Q^+}$ is $\{\mathcal{M}_t, P_x\}$ -supermartingale for each $x \in X$.

Applying Doob's optional sampling theorem to $\{Y_t, \mathcal{M}_t, P_x\} x \in X$ and noting that the process $\{X_t^0\}_{t \in Q^+}$ does not hit the set $\{x \in 0; \hat{e}_o(x) < 1\}$ with P_x -a.e. $x \in X$, we obtain

$$E_{\mathbf{x}}(\exp(-\sigma_0^0)) \leq \hat{e}_0 \quad C_{r,p}$$
-q.e.

The statement (i) follows from this inequality.

(ii) Take a countably dense subset $C_2 \subset C_0^+(X)$. There exists a nest $\{F_k\}_{k=1}^{\infty}$ such that

(8) The convergence (7) holds on
$$\bigcup_{k=1}^{\infty} F_k$$
,

(9)
$$\bigcup_{l=1+\lceil r \rceil, 2+\lceil r \rceil, \cdots} v_l(C_2) \subset C^{\infty}(\{F_k\}_{k=1}^{\infty}).$$

Here, we know that the family of functions of the left hand side of (9) separates the point of $Z_0 = (\bigcup_{k=1}^{\infty} F_k) \cup \{\Delta\}$. In fact, if we suppose for $x, y \in Z_0$

$$v_l f(x) = v_l f(y)$$
, for any $f \in C_2$, $l = 1 + [r], 2 + [r], \cdots$

then $p_t f(x) = p_t f(y)$, t > 0, $f \in C_2$, by the uniqueness of the Laplace transformation. Letting t tend to 0 along the sequence $\{t_j\}_{j=1}^{\infty}$, we see that f(x)=f(y), $f \in C_2$ by (8) and x=y.

Hence, for the event $\Omega_{00} = \{\omega \in \Omega_0; \lim_{k \to \infty} \sigma^0_{X-F_k}(\omega) = \infty\}$, we have that

$$\begin{split} \Omega_{00} - \Omega_1 \subset \{ \omega \in \Omega_0; \text{ for some } k \text{ and some } t < \sigma_{X-F_k}^0 \\ X_s^0(\omega) \text{ does not have the right- or left-hand limit at } t \} . \end{split}$$

Since the process $\{e^{-s}v_{l}f(X_{s}^{0}(\omega)), \mathcal{M}_{s}^{0}, P_{x}\}\$ is a non-negative supermartingale, the P_{x} measure of the right hand side is zero. In view of (*i*), we know that $P_{x}(\Omega_{00})=1, C_{r,p}$ -q.e. and so is Ω_{1} .

(iii), (iv), (v) and (vi) The proofs can be performed in the same way as in [5; Chapter 6, §2]. q.e.d.

We extend the Hunt process of Lemma 3 (vi) to the Hunt process on X by letting each point of X-Z be trap.

Theorem 3. There exists a Hunt process $M = \{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x\}_{x \in X_\Delta}$ satisfying that

(10) for each $f \in L_p$, $E_x(f(X_t))$ is a $C_{r,p}$ -quasi-continuous modification of $P_t f$.

If $M' = \{\Omega', \mathcal{M}', \mathcal{M}'_t, X'_t, P'_x\}_{x \in X_{\Delta}}$ is another Hunt process with property (10), then the induced probability laws of X_t and X'_t on the path space $\tilde{\Omega} = \{\tilde{\omega}; [0, \infty) \mapsto X, \tilde{\omega}(t) \text{ is right continuous with left limits in } t \}$ coincide for $C_{r,p}$ -q.e. $x \in X$.

Proof. The existence is already shown. To prove the part of the uniqueness, it suffices to show that for M' with the property (10)

$$E_{\mathbf{x}}(f_1(X_{t_1})f_2(X_{t_2})\cdots f_n(X_{t_n})) = E'_{\mathbf{x}}(f_1(X'_{t_1})f_2(X'_{t_2})\cdots f_n(X'_{t_n})), C_{\mathbf{r},\mathbf{p}}\text{-q.e.},$$

where $f_1, f_2, \dots, f_n \in C_2, t_1, t_2, \dots, t_n \in Q^+$. But this is clear from (10). q.e.d.

In the symmetric case, we have a criterion for the sample path continuity of the Hunt process M.

Let us consider a strongly continuous semi-group $(P_t)_{t\geq 0}$ of Markovian symmetric operator on L_2 . As stated in §2, it can be regarded as a strongly contraction analytic semi-group in L_p (1 . We assume that the re $gularity (6) for the associate space <math>F_{r,p}$ and $F_{1,2}$.

Theorem 4. The following conditions are equivalent.

(i) The Dirichlet space $F_{1,2}$ is local in the sense that the pair $u, v \in F_{1,2}$ with disjoint supports always enjoys the property $(u, v)_{F_{1,2}} = 0$.

(ii) M is a diffusion in the sense

 $P_{\mathbf{x}}(\omega \in \Omega; \text{ the sample path is continuous}) = 1, \quad C_{r,p}$ -q.e.

Proof. Let us set $q(x) = P_x(\omega \in \Omega;$ for some t > 0, $\lim_{s \neq t} X_s(\omega) \neq X_t(\omega)$). If q(x) vanishes *m*-a.e., then q(x) = 0 $C_{r,p}$ -q.e. Because the function $P_x(\omega \in \Omega;$ for some t > 1/n, $\lim_{s \neq t} X_s(\omega) \neq X_t(\omega)$) $= p_{1/n}q(x)$ then vanishing $C_{r,p}$ -q.e. Since M can be also regarded as the diffusion as a realization of the L_2 -semi-group, the first statement of Theorem 4 combined with a general theorem related to the Dirichlet space implies that q(x) = 0 *m*-a.e. The proof of theorem is completed. q.e.d.

EXAMPLE. Suppose that a uniformly elliptic partial differential operator

 $L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i} \frac{\partial x_j}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \text{ possesses bounded smooth coef$ $ficients in the sense that } a_{ij}(x) \in C_b^2(\mathbb{R}^n), \ 1 \leq i, \ j \leq n, \ b_i(x) \in C_b^1(\mathbb{R}^n), \ 1 \leq i \leq n, \\ \sum_{i=1}^{n} a_{ij}(x) \xi^i \xi^j \geq \delta |\xi|^2 \text{ for some } \delta > 0, \text{ and that } c(x) \text{ is bounded non-positive.}$

The resolvent R_{λ} on $L_2(\mathbb{R}^n)$ satisfies $||R_{\lambda}|| \leq C/(1+|\lambda|)$ in the domain $\{\lambda \in C; \operatorname{Re}(\lambda) \geq \alpha\}$ with some positive C and α . Owing to a well known theorem of K. Yosida [18; Chapter IX, 10], the corresponding semi-group is analytic in $L_2(\mathbb{R}^n)$. Obviously the semi-group $(P_t)_{t\geq 0}$ is Markovian and contractive. We observe that the dual semi-group has the same properties. By the method of interpolation mentioned in E.M. Stein [16] and above observation, we know that in $L_p(\mathbb{R}^n)$ $(P_t)_{t\geq 0}$ is analytic whenever 1 .

The Sobolev space $W_p^2(\mathbb{R}^n)$ as the domain of the closed extension of L with domain $C_0^{\infty}(\mathbb{R}^n)$ coincides with the space of potentials $F_{2,p}$ with equivalent norms. Since $W_p^2(\mathbb{R}^n)$ satisfies the assumption (6), Theorem 3 gives us the corresponding Hunt process in the $C_{2,p}$ -refined sense. The Sobolev imbedding theorem assures that " $C_{2,p}$ -q.e." becomes "everywhere" when 2p > n. Consequently the Hunt process is uniquely associated without exceptional starting point.

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H. KANEKO

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