ON \((r, p)\)-CAPACITIES FOR MARKOV PROCESSES

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1. Introduction

For a general Markovian semi-group \(\{P_t; t \geq 0\}\) on a measure space, we consider the image \(F_{r,p}\) of \(L_p\)-space of the \(r\)-th order \(\Gamma\)-transformation of \(P_t\). Then \(F_{r,p}\) gives rise to a set function \(C_{r,p}\) satisfying certain properties of capacity (M. Fukushima and H. Kaneko [6]). When \(P_t\) is a symmetric operator on \(L_2\)-space, the capacity \(C_{1,2}\) coincides with the capacity related to the Dirichlet space associated with \(P_t\), and consequently, the set of zero \(C_{1,2}\)-capacity can be identified with the polar set of the Hunt process corresponding to \(P_t\), if the latter ever exists ([5]). But as \(r\) or \(p\) becomes greater, the set of \((r, p)\)-capacities zero become finer. For instance, when \(P_t\) is the heat kernels on \(\mathbb{R}^n\), the \(\Gamma\)-transformations of \(P_t\) are equal to the so-called Bessel kernels. Therefore, in that case, \(C_{r,p}\) coincides with the Bessel capacity \(B_{r, p}\) presented in [11], for which there exists no non-empty sets of zero capacity whenever \(rp > n\) ([11]).

The purpose of this paper is to examine whether some basic theorems related to the Markovian semi-group \(\{P_t; t \geq 0\}\) can be refined, so that one may take the sets of \(C_{r,p}\)-capacity zero for various \(r\) and \(p\) as exceptional sets in the statement of the theorems. Assuming the analyticity of \(P_t\), we shall show that two refinements (Theorem 1 in §2 and Theorem 3 in §4) of this kind are indeed possible. The first one is for ergodic theorem due to G.C. Rota [13], E.M. Stein [16] (which concerned \(m\)-a.e. statements) and due to M. Fukushima [4] (which concerned \(C_{1,2}\)-q.e. statement). The second is for the construction of a Hunt process which has been established by M. Fukushima [5] and M. Silverstein [14] in the case that \((r, p) = (1, 2)\) and \(P_t\) is symmetric and by S.C. Menendez [10] in a non-symmetric case. In §3, a refinement in the construction of a transition function will be presented.

In this connection, we mention the work of Y. Le Jan [8] who started with a general Markovian semi-group on an \(L_\infty\)-space and constructed a Hunt process with exceptional set being related to a certain family of supermedian functions. While the above mentioned papers and ours start with a Markovian semi-group acting on an \(L_2\)-space or \(L_p\)-space, D. Feyel and A. de La Pradelle [3] started with the one acting on a Banach space of functions which are already refined in relation to a capacity. Further we mention a related work of N.G.

In this paper, we always assume that the space of potentials \( F_{r,p} \) is regular in the sense that \( F_{r,p} \) contains sufficiently many continuous functions. For instance, when the semi-group is generated by a strongly elliptic partial differential operator of second order with smooth coefficients, then \( F_{r,p} \) coincides with \( W^r_p(\mathbb{R}^n) \) (see example at the end of this paper). But in general, it is rather hard to check the regularity of the space \( F_{r,p} \) for \((r, p) \neq (1, 2)\).

Finally, as an application of a theory of \((r, p)\)-capacities to other kinds of problems, we like to mention the works by A. B. Cruzerio [2] and A. Nagel, W. Rudin and J.H. Shapiro [12] concerning the boundary limit theorems and by P. Malliavin [9] and M. Takeda [17] concerning infinite dimensional analysis.

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2. Some limit theorems of semi-groups

Let \( X \) be a separable metric space and \( m \) be a positive \( \sigma \)-finite measure with the support \( X \). Through the paper, let us consider a strongly continuous contractive semi-group \((P_t)_{t \geq 0}\) on \( L_p(X; m) \) \((1 < p < \infty)\), which is Markovian;

\[
0 \leq f \leq 1 \quad m\text{-a.e.} \quad \Rightarrow \quad 0 \leq P_t f \leq 1 \quad m\text{-a.e.}
\]

We also require that it is analytic in \( t > 0 \) as a bounded operator valued function of \( t \).

Let us recall some notations formulated in [6]. The Markovian contractive operator \( V_r \) \((r > 0)\) is defined by

\[
V_r = \Gamma(r/2) \int_0^\infty e^{s/2} e^{-s} P_s ds.
\]

We let \( \|u\|_{r,p} = \|f\|_{L_p} \) for \( u = V_r f, f \in L_p \), then the space \( F_{r,p} = V_r(L_p) \) with the norm \( \| \|_{r,p} \) is a Banach space. We define a set of function \( C_{r,p} \) by

\[
C_{r,p}(A) = \inf \{ \|u\|_{r,p}^p; u \in F_{r,p} \text{ satisfies } u \geq 1 \text{ m-a.e. on some open set which contains } A \}.
\]

"\( C_{r,p}\)-quasi-everywhere" or briefly "\( C_{r,p}\)-q.e." means that the statement holds except on a \( C_{r,p} \) (capacity) zero set. A function \( u \) is called \( C_{r,p}\)-quasi-continuous if for any \( \varepsilon > 0 \) there exists an open set \( G \) such that \( C_{r,p}(G) < \varepsilon \) and the function is continuous on \( X - G \). A sequence of functions \( u_n \) is said to be \( C_{r,p}\)-quasi-uniformly convergent to a function \( u \) if for any \( \varepsilon > 0 \) there exists an open set \( G \) such that \( C_{r,p}(G) < \varepsilon \) and the sequence of functions \( u_n \) converges to \( u \) uniformly on \( X - G \).

We make the following assumption:
We can show the following ([6]):

(a) \( C_{r,p} \) is an outer capacity and stable under the increasing limits of sets.

(b) \( C_{r,p} \) is non-decreasing in \( r \).

(c) A function \( u \) is \( C_{r,p} \)-quasi-continuous and \( u \geq 0 \) m.a.e. \( \Rightarrow u \geq 0 \) \( C_{r,p} \)-q.e.

(d) \( u \in F_{r,p} \) \( \Rightarrow \) a \( C_{r,p} \)-quasi-continuous modification \( \bar{u} \) of \( u \) exists, and it enjoys

\[ C_{r,p}(|\bar{u}|^r > \lambda) \leq \lambda^{-r} ||u||_{r,p}^r, \lambda > 0. \]

(e) The convergence of \( C_{r,p} \)-quasi-continuous functions in \( F_{r,p} \) implies \( C_{r,p} \)-quasi-uniform convergence of some subsequence to a \( C_{r,p} \)-quasi-continuous function.

We know that the semi-group restores some potential theoretic feature. Let \( r > 0 \) and \( 1 < p < \infty \) be fixed.

**Lemma 1.** For each \( f \in L_p \), we can take a function \( \hat{P}_t f(x) \) of \( x \in X \) and \( t > 0 \) which has the following properties.

(i) For each \( t > 0 \), \( \hat{P}_t f(x) \) is a \( C_{r,p} \)-quasi-continuous version of \( P_t f(x) \), moreover for any \( \varepsilon > 0 \) there exists an open set \( G \) independent of \( t \) such that \( C_{r,p}(G) < \varepsilon \) and the functions \( \{\hat{P}_t f(x)\}_{t > 0} \) are continuous on \( X - G \).

(ii) For \( C_{r,p} \)-quasi-everywhere \( x \in X \), the function \( \hat{P}_t f(x) \) is analytic in \( t \).

(iii) For each \( t_0 \geq 0 \), there exist positive constants \( C \) and \( \varepsilon \) such that

\[ C_{r,p}(\sup_{|t - t_0| < \varepsilon} |\hat{P}_t f(x)|^r > \lambda) \leq C \lambda^{-r} ||f||_{L_p}^r, \quad f \in L_p, \lambda > 0. \]

**Proof.** Take a natural number \( n > r/2 \). Since \( V_r \) has a semi-group property in \( r \), we have

\[ P_t f = (V_r)^{2n}(I - A)^n P_t f = V_r V_{2n-r}((I - d/dt)^n P_t) f, \]

where \( A \) is the generator of the semi-group \( (P_t)_{t \geq 0} \) and \( d/dt \) stands for the derivative in the operator topology. Hence, \( P_t f \) is an element of \( F_{r,p} \). Consider an operator valued function \( S_t = V_{2n-r}((I - d/dt)^n P_t) \). Then analyticity of \( S_t \) in \( t \) admits the Taylor expansion around \( t = t_0 \):

\[ S_t = \sum_{n=0}^{\infty} B_n (t - t_0)^n, \quad |t - t_0| < \varepsilon, \]

where the \( B_n \)'s are bounded operators in \( L_p \) such that \( \sum_{n=0}^{\infty} ||B_n|| e^n < \infty \).
Take any \( f \in L_p \) and quasi-continuous versions \( \overline{V_r B_n f}, n = 0, 1, 2, \ldots \). Then by (e), \( \sum_{n=0}^{\infty} |V_r B_n f|^\epsilon \) converges except on some Borel set \( N \) with \( C_r \mu(N) = 0 \). Therefore, if we set, for \( |t - t_0| < \epsilon \)

\[
P_t f(x) = \begin{cases} 
\sum_{n=0}^{\infty} V_r B_n f(x) (t-t_0)^n, & \text{if } x \in X - N, \\
0, & \text{otherwise,}
\end{cases}
\]

and patch the functions in \( t \), then we have \( \overline{P_t f}(x) \) which enjoys properties (i) and (ii). (iii) is clear from (3) and

\[
\sup_{|t-t_0|<\epsilon} |P_t f(x)| \leq V_r (\sum_{n=0}^{\infty} |B_n f|^\epsilon) (x). \quad \text{q.e.d.}
\]

In the remainder of this section, we only consider a strongly continuous contraction semi-group \( (P_t)_{t \geq 0} \) which is determined by Markovian symmetric operator \( (P_t)_{t \geq 0} \) on \( L_2(X; \mu) \). E. M. Stein ([16]) shows that \( (P_t)_{t \geq 0} \) then becomes an analytic semi-group on \( L_p \) for each \( p > 1 \). We introduce for \( f \in L_p(X; \mu) \) the maximal function \( Mf \) by

\[
Mf(x) = \sup_{t > 0} |\overline{P_t f}(x)|,
\]

where \( \overline{P_t f} \) is the function in Lemma 1. Then we have the \( L_p \)-estimate ([16]):

\[
|||Mf|||_{L_p} \leq C_p |||f|||_{L_p}, \quad f \in L_p
\]

for some positive constant \( C_p \).

**Lemma 2.** For each \( \lambda > 0 \), \( u \in F_{r, p} \), we have

\[
C_{r, p}(Mu > \lambda) \leq C_{r, p} \lambda^{-p} ||u||_{r, p}^p
\]

Proof. For \( f \in L_p(X; \mu) \) and \( u = V_r f \), we have

\[
|P_t V_r f| = |V_r P_t f| \leq V_r Mf \quad m\text{-a.e.}
\]

and consequently

\[
Mu \leq \overline{V_r Mf} \quad C_{r, p}\text{-q.e.}
\]

Hence, by (5)

\[
C_{r, p}(Mu > \lambda) \leq C_{r, p}(\overline{V_r Mf} > \lambda) \leq \lambda^{-p} ||Mf||_{L_p}^p,
\]

\[
\leq C_{r, p} \lambda^{-p} ||f||_{L_p}^p = C_{r, p} \lambda^{-p} ||u||_{r, p}^p, \quad u \in F_{r, p}. \quad \text{q.e.d.}
\]
**Theorem 1.** Assume that $(P_t)_{t \geq 0}$ is determined by a Markovian symmetric operator on $L^p$.

(i) For any $u \in F_{r,p}$, the limit $\lim_{t \to 0} \hat{P}_t u(x)$ exists $C_{r,p}$-q.e. which is $C_{r,p}$-quasi-continuous version of $u$, where $r > 0, p > 1$.

(ii) The limit $\lim_{t \to 0} \hat{P}_t f(x) = h(x)$ exists $C_{r,p}$-q.e., for any $f \in L_p(X; m)$. $h$ satisfies

$$\hat{P}_t h(x) = h(x), \quad t > 0, \quad C_{r,p}$$

Proof. (i) If we set

$$R(u) = \lim_{n \to \infty} \sup_{0 < t < 1/t_n} |\hat{P}_t u(x) - \hat{P}_t u(x)|,$$

then $R(u) = 0 C_{r,p}$-q.e., for any $u \in F_{r,p}$. For the last lemma combining with the inequality

$$R(u) = R(u - P_h u) \leq 2M(u - P_h u) C_{r,p}$$

shows that

$$C_{r,p}(R(u) > \lambda) \leq C_p (2\lambda)^{-\theta} ||u - P_h u||_{r,p},$$

which tends to zero as $h$ tends to zero for any $\lambda > 0$. By (e), the pointwise limit $\lim_{t \to 0} \hat{P}_t u(x)$ must be a $C_{r,p}$-quasi-continuous version of $u$.

(ii) As in [4], we easily obtain the existence of the $C_{r,p}$-q.e. limit $h = \lim_{t \to 0} \hat{P}_t f$. $h$ is $C_{r,p}$-quasi-continuous. Recalling the analyticity of $\hat{P}_t h$, we have

$$P_t h(x) = h(x), \quad t > 0, \quad C_{r,p}$$

q.e.d.

3. **Construction of a transition function**

In this section, we suppose that $X$ is separable complete metric space, $X$ is covered by some countable family of closed sets with finite $m$-measure and the support of $m$ is $X$. Given a strongly continuous Markovian semigroup $(P_t)_{t \geq 0}$ on $L_p(X; m)$ $(1 < p < \infty)$ satisfying the analyticity in $t > 0$ and the regularity condition (2), we have constructed a regularized version $\hat{P}_t, f \in L_p$ in Lemma 1. We can further construct a transition function as follows.

**Theorem 2.** There exists a family of kernels $\{p_t(x, E); t \geq 0, x \in X, E \in \mathcal{B}\}$, where $\mathcal{B}$ stands for the set of all Borel subsets of $X$, which satisfies the following conditions:

(i) $p_t(x, X) \leq 1, \quad t > 0$.

(ii) $\int_X p_t(x, dy) p_s(y, E) = p_{t+s}(x, E), \quad t, s > 0$. 

(iii) For each \( f \in L^p \) and \( r > 0 \), there exists a Borel set \( N \) such that \( C_{r, p}(N) = 0 \) and
\[
p_t f(x) = \hat{P}_t f(x)
\]
for every \( t > 0 \) and \( x \in X - N \).

Proof. We only give the proof in the case that \( m(X) < \infty \) but the proof is similar to the \( \sigma \)-finite case. Let us embed the space \( X \) homeomorphically onto a Borel subset \( Y \) of \([0, 1]^n\). Take a countable dense subset \( C_1 \) of \( C([0, 1]^n) \). Denoting by \( B_b \) the set of all bounded Borel functions of \([0, 1]^n\) and by \( \tilde{f} \) the restriction to \( Y \) of \( f \in B_b \), then \( B_b \subset L^p([0, 1]^n) \). By virtue of Lemma 1, we get
\[
\hat{P}_t (af + bg)(x) = a \hat{P}_t \tilde{f}(x) + b \hat{P}_t \tilde{g}(x) \quad C_{r, p}\text{-q.e.}
\]
for \( f, g \in B_b, \ a, b \in \mathbb{R} \),
\[
f_n, f \in B_b, f_n \uparrow f \Rightarrow \hat{P}_t \tilde{f}_n(x) \uparrow \hat{P}_t \tilde{f}(x) \quad C_{r, p}\text{-q.e.}
\]
Further we find the set \( N \subset X \) with \( C_{r, p}(N) = 0 \) such that \( \hat{P}_t \tilde{f}(x) \) is analytic function of \( t > 0 \) for \( f \in C_1, \ x \in X - N \).

By similar way of the proof of Proposition (4.1) in R.K. Getoor [7], we obtain the kernel \( q_t(x, E) \) such that
\[
q_t f(x) = \hat{P}_t \tilde{f}(x), \quad x \in X - N, f \in C_1, t \in Q^+,
\]
where \( Q^+ \) is the set of all positive rational numbers. Since \([0, 1]^n\) is compact, the dual space of \( C([0, 1]^n) \) is weakly complete and \( q_t(x, \cdot) (t \in Q^+) \) has a continuous extension to the half real line. Denote by \( \tilde{p}_t(x, \cdot), t \in (0, \infty) \) the restriction of \( q_t(x, \cdot) \) to \( X \), then we have
\[
\tilde{p}_t \tilde{f}(x) = \hat{P}_t \tilde{f}(x) \quad \text{for any} \quad t > 0, \ x \in X - N, f \in C_1.
\]
Hence, we arrive at (iii) by Lemma 1 and a monotone lemma.

On the other hand, there exists a Borel set \( Y_1 \) with \( C_{r, p}(X - N) = 0 \) such that for each \( x \in Y_1 \)
\[
\tilde{p}_t(p_t \tilde{f})(x) = \tilde{p}_{t+s} \tilde{f}(x), \quad t, s \in Q^+, f \in C_1
\]
and all the functions \( p_t \tilde{f}(x) \) and \( p_t(p_t \tilde{f})(x), s \in Q^+ \) are continuous in \( t > 0 \). Now just as the proof of Lemma 6.1.4. in M. Fukushima [5], we can modify \( p_t(x, E) \) slightly to get kernels which satisfy not only (i), (iii) but also (ii) of Theorem 2.

q.e.d.

We call the kernels in Theorem 2 a transition function representing the
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Once such a transition function is constructed, we get a nice potential kernel by

\[ v_r(x, E) = \Gamma(r/2)^{-1} \int_0^\infty s^{r/2 - 1} e^{-s} p_t(x, E) ds. \]

In fact, we have

**Corollary.** \( v_r f(x) \) is a \( C_{r,p} \)-quasi-continuous version of \( V_r f \), for every \( f \) in \( L_p(X; m) \).

**Proof.** It suffices to prove this for bounded functions in \( L_p \). If \( f \in L_p \) is bounded, we have the pointwise convergence

\[ p_t v_r f(x) = v_r p_t f(x) \to v_r f, \quad t \to 0. \]

The convergence also takes place in the Banach space \( F_{r,p} \), and hence we get the above conclusion. q.e.d.

4. Construction of Hunt processes

In this section, we assume that the state space \( X \) is a locally compact separable metric space and the measure \( m \) is positive Radon with support \( X \). Let \( (P_t)_{t \geq 0} \) be a Markovian strongly continuous contraction semi-group defined on \( L_p(X; m) \) \((1 < p < \infty)\) which is analytic in the sense of §2. When the space \( F_{r,p} \) contains continuous functions densely, we saw in §2 and §3 that the semi-group admits some potential theoretic refinements. Our assertion of this section is that under a stronger assumption (6) on \( F_{r,p} \) mentioned below we can construct an associated Hunt process strating from \( C_{r,p} \)-quasi-everywhere point of \( X \), for \( r \geq 2 \).

Let \( X_\Delta = X \cup \{\Delta\} \) be the one-point compactification of \( X \), and extend the transition function of the last section to \( X_\Delta \) by

\[ p_t(x, E) = \begin{cases} p_t(x, E - \{\Delta\}) + (1 - p_t(x, X)) \delta_\Delta(E), & x \in X \\ \delta_\Delta(E), & x = \Delta \text{ or } t = 0, \end{cases} \]

for Borel subset \( E \) of \( X \).

By the Kolmogorov extension theorem, there is a Markov process \( M_0 = \{\Omega_0, \mathcal{M}, \mathcal{M}_0, X^0, P_t\}_{x \in X} \) with transition probability \( (p_t)_{t \in \mathbb{Q}^+} \), where \( \Omega_0, \mathcal{M}, \mathcal{M}_0, X^0 \) are the following objects:

\[ \Omega_0 = X^0_\Delta \cup \{\emptyset\} \]

\[ X^0_\omega(t) = \omega(t), \quad \omega \in \Omega_0 \]

\[ \mathcal{M} = \sigma[X^0_\omega(t); t \in \mathbb{Q}^+] \]

\[ \mathcal{M}_t^0 = \sigma[X^0_\omega(s); s \leq t, s \in \mathbb{Q}^+] \].
We let $r \geq 2$ and assume that
\begin{equation}
(6) \quad \text{the } F_{r,\phi} \cap C_\infty(X) \text{ is dense not only in } F_{r,\phi} \text{ but also in } C_\infty(X),
\end{equation}
where $C_\infty(X) = \{ f \in C(X_\Delta); f(\Delta) = 0 \}$. Under the assumption, we have a sequence $\{t_j\}_{j=0}^\infty \subset \mathbb{Q}^+$ decreasing to 0, which satisfies
\begin{equation}
(7) \quad \lim_{j \to \infty} p_{t_j} f(x) = f(x), \text{ for any } f \in C_\infty(X) \text{ and } x \in X - N
\end{equation}
for some $N$ with $C_{r,\phi}(N) = 0$, because we can see this for a dense subclass of $C_\infty(X)$, contained in $F_{r,\phi}$, as in the proof of Corollary in §3. An increasing sequence $\{F_k\}_{k=1}^\infty$ of closed sets with $\lim_{k \to \infty} C_{r,\phi}(X - F_k) = 0$ is said to be a $C_{r,\phi}$-nest. The condition (6) further implies that each $u \in F_{r,\phi}$ admits a $C_{r,\phi}$-nest $\{F_k\}_{k=1}^\infty$ for which $u|_{F_k \cup \{\Delta\}}$ are continuous functions vanishing at $\Delta$, $k = 1, 2, 3, \ldots$. The totality of such functions is denoted by $C_\infty(F_{r,\phi})$. Here, we shall show a crucial lemma.

**Lemma 3.** Under the assumption (6), we get the followings:

(i) For any decreasing sequence $\{O_n\}_{n=0}^\infty$ of open sets with $C_{r,\phi}(O_n) \to 0$, as $n \to \infty$, we have
\[ P_x(\lim_{n \to \infty} \sigma_0^O_n = \infty) = 1, \text{ for } C_{r,\phi}-q.e. \ x \in X, \]
where $\sigma_0^O = \inf \{t > 0; X_t^O \in A \}$.

(ii) If we let $\Omega_1 = \{\omega \in \Omega_0; \text{ the sample path } X_0^\omega(\omega) \text{ has left- and right-hand limits in } X, \text{ for all } t > 0\}$, then
\[ P_x(\Omega_1) = 1, \text{ for } C_{r,\phi}-q.e. \ x \in X. \]

(iii) If we let $X_0(\omega) = \lim_{t \uparrow, t \in \mathbb{Q}^+} X_t^\omega(\omega)$ and $\Omega_2 = \{\omega \in \Omega_1; X_0(\omega) = X_0^\omega(\omega), t \in \mathbb{Q}^+ \text{ and } X_0^\omega(\omega) = x\}$, then
\[ P_x(\Omega_2) = 1, \text{ for } C_{r,\phi}-q.e. \ x \in X. \]

(iv) Let $\Omega_3 = \{\omega \in \Omega_2; \text{ if } X_t(\omega) \in X \text{ then the trajectory of the sample path up to the time } t \text{ lies in a compact subset of } X \text{ for all } t > 0\}$, then
\[ P_x(\Omega_3) = 1, \text{ for } C_{r,\phi}-q.e. \ x \in X. \]

(v) There exists a Borel set $Z \subset X$ and $\Gamma_0 \subset \mathcal{M}$ satisfying $C_{r,\phi}(X - Z) = 0$, $P_x(\Gamma_0) = 0$ for all $x \in Z$ and the inclusion
\[ \{\omega \in \Omega_3; \text{ for some } t \geq 0, \text{ either } X_t(\omega) \text{ or } \lim_{t \uparrow} X_t(\omega) \text{ is not in } Z\} \subset \Gamma_0. \]

(vi) Put $\mathcal{M}_i = \bigcup_{s \leq t, t \in \mathbb{Q}^+} \mathcal{M}_0^s$. Consider the restrictions of $\mathcal{M}$, $\mathcal{M}_i$, $X_t$ and $P_x$ to
the set \( \Omega = \Omega_0 - \Gamma_0 \) and denote them by the same notations. Then the quintuplet \( \mathcal{M}_z = \{ \Omega, \mathcal{M}_t, \mathcal{M}_s, X_t, P_x \}_{s \in \mathbb{Z}} \) becomes a Hunt process on \( Z \).

Proof. (i) Every open set \( O \) of finite capacity possesses a unique norm-minimizing element \( e_0 \) in the set \( \{ u \in F_{r,p}; u \geq 1 \text{ m.a.e. on } O \} \). As a version of \( e_0 \), take \( e_0 \) a function expressed as \( v_r f \) for some non-negative function \( f \) in \( L_p \). Clearly we have then

\[
e^{-t} p_r e_0(x) \leq e_0(x)
\]

which means that \( \{ Y_t = e^{-t} \hat{e}_0(X_0(t)) \}_{t \in \mathbb{Q}^+} \) is \( \{ \mathcal{M}_t, P_x \} \)-supermartingale for each \( x \in X \).

Applying Doob's optional sampling theorem to \( \{ Y_t, \mathcal{M}_t, P_x \} \) \( x \in X \) and noting that the process \( \{ X_0^t \}_{t \in \mathbb{Q}^+} \) does not hit the set \( \{ x \in X; \hat{e}_0(x) < 1 \} \) with \( P_x \)-a.e. \( x \in X \), we obtain

\[
E_x(\exp(-\sigma^0_0)) \leq e_0 \quad C_{r,s} \text{-q.e.}
\]

The statement (i) follows from this inequality.

(ii) Take a countably dense subset \( C_2 \subset C_0^0(X) \). There exists a nest \( \{ F^*_k \}_{k=1}^\infty \) such that

\[
(8) \quad \text{The convergence (7) holds on } \bigcup_{k=1}^\infty F_k,
\]

\[
(9) \quad \bigcup_{l=1}^{\infty} \bigcup_{r \in \{ r_l^k \}_{k=1}^\infty} v_l(C_2) \subset C^\infty(\{ F^*_k \}_{k=1}^\infty).
\]

Here, we know that the family of functions of the left hand side of (9) separates the point of \( Z_0 = ( \bigcup_{k=1}^\infty F_k ) \cup \{ \Delta \} \). In fact, if we suppose for \( x, y \in Z_0 \)

\[
v_l f(x) = v_l f(y), \quad \text{for any } f \in C_2, l = 1+[-r], 2+[-r], \ldots
\]

then \( p_i f(x) = p_i f(y), t > 0, f \in C_2 \), by the uniqueness of the Laplace transformation. Letting \( t \) tend to 0 along the sequence \( \{ t_j \}_{j=1}^\infty \), we see that \( f(x) = f(y), f \in C_2 \) by (8) and \( x = y \).

Hence, for the event \( \Omega_0^\infty = \{ \omega \in \Omega; \lim_{k \to \infty} \sigma^0_{-F^*_k}(\omega) = \infty \} \), we have that

\[
\Omega_0^\infty - \Omega_1 \subset \{ \omega \in \Omega_0; \text{ for some } k \text{ and some } t < \sigma^0_{-F^*_k} \}
\]

\[
X^0_t(\omega) \text{ does not have the right- or left-hand limit at } t
\]

Since the process \( \{ e^{-t} v_l f(X^0_t(\omega)) \}, \mathcal{M}_t^0, P_x \} \) is a non-negative supermartingale, the \( P_x \) measure of the right hand side is zero. In view of (i), we know that \( P_x(\Omega_0^\infty \cup \Omega_1) = 1, C_{r,s} \text{-q.e.} \) and so is \( \Omega_1 \).

(iii), (iv), (v) and (vi) The proofs can be performed in the same way as in [5; Chapter 6, §2].

\( \text{q.e.d.} \)
We extend the Hunt process of Lemma 3 (vi) to the Hunt process on \( X \) by letting each point of \( X - Z \) be trap.

**Theorem 3.** There exists a Hunt process \( M = \{ \Omega, \mathcal{M}, X_t, P_x \}_{x \in X} \) satisfying that

\[
\text{for each } f \in L_p, E_x(f(X_t)) \text{ is a } C_{r,p}^-\text{-quasi-continuous modification of } P_x f.
\]

If \( M' = \{ \Omega', \mathcal{M}', X'_t, P'_x \}_{x \in X} \) is another Hunt process with property (10), then the induced probability laws of \( X_t \) and \( X'_t \) on the path space \( \Omega = \{ \tilde{\omega}; [0, \infty) \mapsto X, \tilde{\omega}(t) \text{ is right continuous with left limits in } t \} \) coincide for \( C_{r,p}^-\text{-q.e. } x \in X \).

Proof. The existence is already shown. To prove the part of the uniqueness, it suffices to show that for \( M' \) with the property (10)

\[
E_x(f_1(X_{t_1})f_2(X_{t_2}) \cdots f_n(X_{t_n})) = E'_x(f_1(X'_{t_1})f_2(X'_{t_2}) \cdots f_n(X'_{t_n})) \text{,}
\]

where \( f_1, f_2, \ldots, f_n \in C_2, t_1, t_2, \ldots, t_n \in \mathbb{Q}^+ \). But this is clear from (10). q.e.d.

In the symmetric case, we have a criterion for the sample path continuity of the Hunt process \( M \).

Let us consider a strongly continuous semi-group \( (P_t)_{t \geq 0} \) of Markovian symmetric operator on \( L_2 \). As stated in §2, it can be regarded as a strongly contraction analytic semi-group in \( L_p \) \( (1 < p < \infty) \). We assume that the regularity (6) for the associate space \( F_{r,p} \) and \( F_{1,2} \).

**Theorem 4.** The following conditions are equivalent.

(i) The Dirichlet space \( F_{1,2} \) is local in the sense that the pair \( u, v \in F_{1,2} \) with disjoint supports always enjoys the property \( (u, v)_{F_{1,2}} = 0 \).

(ii) \( M \) is a diffusion in the sense

\[
P_x(\omega \in \Omega; \text{the sample path is continuous}) = 1, \ C_{r,p}^-\text{-q.e.}
\]

Proof. Let us set \( q(x) = P_x(\omega \in \Omega; \text{for some } t > 0, \lim_{t \uparrow} X_t(\omega) \neq X_t(\omega)) \). If \( q(x) \) vanishes \( m\text{-a.e.} \), then \( q(x) = 0 \ C_{r,p}^-\text{-q.e.} \). Because the function \( P_x(\omega \in \Omega; \text{for some } t > 1/n, \lim_{t \uparrow} X_t(\omega) \neq X_t(\omega)) = p_{t/n} q(x) \) then vanishing \( C_{r,p}^-\text{-q.e.} \). Since \( M \) can be also regarded as the diffusion as a realization of the \( L_2\)-semi-group, the first statement of Theorem 4 combined with a general theorem related to the Dirichlet space implies that \( q(x) = 0 \ m\text{-a.e.} \). The proof of theorem is completed. q.e.d.

**Example.** Suppose that a uniformly elliptic partial differential operator
\(L = \sum_{i,j=1}^{n} a_{ij}(x) \partial^{2}/\partial x_{i} \partial x_{j} + \sum_{i=1}^{n} b_{i}(x) \partial/\partial x_{i} + c(x)\) possesses bounded smooth coefficients in the sense that \(a_{ij}(x) \in C^{1}_{c}(\mathbb{R}^{n}), 1 \leq i, j \leq n, b_{i}(x) \in C^{1}_{c}(\mathbb{R}^{n}), 1 \leq i \leq n,\) \(\sum_{i,j=1}^{n} a_{ij}(x) \xi^{i} \xi^{j} \geq \delta |\xi|^{2}\) for some \(\delta > 0,\) and that \(c(x)\) is bounded non-positive.

The resolvent \(R_{\lambda}\) on \(L_{2}(\mathbb{R}^{n})\) satisfies \(\|R_{\lambda}\| \leq C/(1 + |\lambda|)\) in the domain \(\{\lambda \in C; \ \text{Re}(\lambda) \geq \alpha\}\) with some positive \(C\) and \(\alpha.\) Owing to a well known theorem of K. Yosida [18; Chapter IX, 10], the corresponding semi-group is analytic in \(L_{2}(\mathbb{R}^{n}).\) Obviously the semi-group \((P_{t})_{t \geq 0}\) is Markovian and contractive. We observe that the dual semi-group has the same properties. By the method of interpolation mentioned in E.M. Stein [16] and above observation, we know that in \(L_{p}(\mathbb{R}^{n})\) \((P_{t})_{t \geq 0}\) is analytic whenever \(1 < p < \infty.\)

The Sobolev space \(W_{p}^{2}(\mathbb{R}^{n})\) as the domain of the closed extension of \(L\) with domain \(C_{c}^{2}(\mathbb{R}^{n})\) coincides with the space of potentials \(F_{2,p}\) with equivalent norms. Since \(W_{p}^{2}(\mathbb{R}^{n})\) satisfies the assumption (6), Theorem 3 gives us the corresponding Hunt process in the \(C_{2,p}\)-refined sense. The Sobolev imbedding theorem assures that "\(C_{2,p}\)-q.e." becomes "everywhere" when \(2p > n.\) Consequently the Hunt process is uniquely associated without exceptional starting point.

References


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