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ON \((r, \rho)-CAPACITIES\) FOR MARKOV PROCESSES

HIROSHI KANEKO

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1. Introduction

For a general Markovian semi-group \(\{P_t; t \geq 0\}\) on a measure space, we consider the image \(F_{r, \rho}\) of \(L_\rho\)-space of the \(r\)-th order \(\Gamma\)-transformation of \(P_t\). Then \(F_{r, \rho}\) gives rise to a set function \(C_{r, \rho}\) satisfying certain properties of capacity (M. Fukushima and H. Kaneko [6]). When \(P_t\) is a symmetric operator on \(L_2\)-space, the capacity \(C_{1,2}\) coincides with the capacity related to the Dirichlet space associated with \(P_t\), and consequently, the set of zero \(C_{1,2}\)-capacity can be identified with the polar set of the Hunt process corresponding to \(P_t\), if the latter ever exists ([5]). But as \(r\) or \(\rho\) becomes greater, the set of \((r, \rho)\)-capacities zero become finer. For instance, when \(P_t\) is the heat kernels on \(\mathbb{R}^n\), the \(\Gamma\)-transformations of \(P_t\) are equal to the so-called Bessel kernels. Therefore, in that case, \(C_{r, \rho}\) coincides with the Bessel capacity \(B_{r, \rho}\) presented in [11], for which there exists no non-empty sets of zero capacity whenever \(rp > n\) ([11]).

The purpose of this paper is to examine whether some basic theorems related to the Markovian semi-group \(\{P_t; t \geq 0\}\) can be refined, so that one may take the sets of \(C_{r, \rho}\)-capacity zero for various \(r\) and \(\rho\) as exceptional sets in the statement of the theorems. Assuming the analyticity of \(P_t\), we shall show that two refinements (Theorem 1 in §2 and Theorem 3 in §4) of this kind are indeed possible. The first one is for ergodic theorem due to G.C. Rota [13], E.M. Stein [16] (which concerned \(m\)-a.e. statements) and due to M. Fukushima [4] (which concerned \(C_{1,2}\)-q.e. statement). The second is for the construction of a Hunt process which has been established by M. Fukushima [5] and M. Silverstein [14] in the case that \((r, \rho) = (1, 2)\) and \(P_t\) is symmetric and by S.C. Menendez [10] in a non-symmetric case. In §3, a refinement in the construction of a transition function will be presented.

In this connection, we mention the work of Y. Le Jan [8] who started with a general Markovian semi-group on an \(L_\infty\)-space and constructed a Hunt process with exceptional set being related to a certain family of supermedian functions. While the above mentioned papers and ours start with a Markovian semi-group acting on an \(L_2\)-space or \(L_\rho\)-space, D. Feyel and A. de La Pradelle [3] started with the one acting on a Banach space of functions which are already refined in relation to a capacity. Further we mention a related work of N.G.

In this paper, we always assume that the space of potentials $F_{r,p}$ is regular in the sense that $F_{r,p}$ contains sufficiently many continuous functions. For instance, when the semi-group is generated by a strongly elliptic partial differential operator of second order with smooth coefficients, then $F_{r,p}$ coincides with $W^r_p(R^n)$ (see example at the end of this paper). But in general, it is rather hard to check the regularity of the space $F_{r,p}$ for $(r, p) \neq (1, 2)$.

Finally, as an application of a theory of $(r, p)$-capacities to other kinds of problems, we like to mention the works by A. B. Cruzerio [2] and A. Nagel, W. Rudin and J.H. Shapiro [12] concerning the boundary limit theorems and by P. Malliavin [9] and M. Takeda [17] concerning infinite dimensional analysis.

The author wishes to thank Professor M. Fukushima for his valuable suggestions and encouragement.

2. Some limit theorems of semi-groups

Let $X$ be a separable metric space and $m$ be a positive $\sigma$-finite measure with the support $X$. Through the paper, let us consider a strongly continuous contractive semi-group $(P_t)_{t \geq 0}$ on $L_p(X; m)$ $(1 < p < \infty)$, which is Markovian;

\[
0 \leq f \leq 1 \quad m\text{-a.e.} \Rightarrow 0 \leq P_t f \leq 1 \quad m\text{-a.e.}
\]

We also require that it is analytic in $t > 0$ as a bounded operator valued function of $t$.

Let us recall some notations formulated in [6]. The Markovian contractive operator $V_r$ $(r > 0)$ is defined by

\[
V_r = \Gamma(r/2) \int_0^{\infty} e^{s^2 - 1} e^{-s t} P_s ds.
\]

We let $\|u\|_{r,p} = \|f\|_{L_p}$ for $u = V_r f, f \in L_p$, then the space $F_{r,p} = V_r(L_p)$ with the norm $\| \|_{r,p}$ is a Banach space. We define a set of function $C_{r,p}$ by

\[
C_{r,p}(A) = \inf \{ \|u\|_{r,p}^r; u \in F_{r,p}, \text{ satisfies } u \geq 1 \text{ m-a.e. on some open set which contains } A \}.
\]

"$C_{r,p}$-quasi-everywhere" or briefly "$C_{r,p}$-q.e." means that the statement holds except on a $C_{r,p}$ (capacity) zero set. A function $u$ is called $C_{r,p}$-quasi-continuous if for any $\varepsilon > 0$ there exists an open set $G$ such that $C_{r,p}(G) < \varepsilon$ and the function is continuous on $X - G$. A sequence of functions $u_n$ is said to be $C_{r,p}$-quasi-uniformly convergent to a function $u$ if for any $\varepsilon > 0$ there exists an open set $G$ such that $C_{r,p}(G) < \varepsilon$ and the sequence of functions $u_n$ converges to $u$ uniformly on $X - G$.

We make the following assumption:
(2) \( F_{r,p} \cap C(X) \) is dense in the Banach space \( F_{r,p} \).

We can show the following ([6]):

(a) \( C_{r,p} \) is an outer capacity and stable under the increasing limits of sets.

(b) \( C_{r,p} \) is non-decreasing in \( r \).

(c) A function \( u \) is \( C_{r,p} \)-quasi-continuous and \( u \geq 0 \) m.a.e. \( \Rightarrow u \geq 0 \) \( C_{r,p} \)-q.e.

(d) \( u \in F_{r,p} \) \( \Rightarrow \) a \( C_{r,p} \)-quasi-continuous modification \( \bar{u} \) of \( u \) exists, and it enjoys

\[
C_{r,p}(|\bar{u}| > \lambda) \leq \lambda^{-p} ||u||_{r,p}, \lambda > 0.
\]

(e) The convergence of \( C_{r,p} \)-quasi-continuous functions in \( F_{r,p} \) implies \( C_{r,p} \)-quasi-uniform convergence of some subsequence to a \( C_{r,p} \)-quasi-continuous function.

We know that the semi-group restores some potential theoretic feature. Let \( r > 0 \) and \( 1 < p < \infty \) be fixed.

**Lemma 1.** For each \( f \in L_p \), we can take a function \( \hat{P}_t f(x) \) of \( x \in X \) and \( t > 0 \) which has the following properties.

(i) For each \( t > 0 \), \( \hat{P}_t f(x) \) is a \( C_{r,p} \)-quasi-continuous version of \( P_t f(x) \), moreover for any \( \varepsilon > 0 \) there exists an open set \( G \) independent of \( t \) such that \( C_{r,p}(G) < \varepsilon \) and the functions \( \{\hat{P}_t f(x)\}_{t > 0} \) are continuous on \( X - G \).

(ii) For \( C_{r,p} \)-quasi-everywhere \( x \in X \), the function \( \hat{P}_t f(x) \) is analytic in \( t \).

(iii) For each \( t_0 \geq 0 \), there exist positive constants \( C \) and \( \varepsilon \) such that

\[
C_{r,p}(\sup_{|t-t_0| < \varepsilon} |\hat{P}_t f(x)| > \lambda) \leq C \lambda^{-p} ||f||_{r,p}, \quad f \in L_p, \quad \lambda > 0.
\]

Proof. Take a natural number \( n > r/2 \). Since \( V_r \) has a semi-group property in \( r \), we have

\[
P_t f = (V_r)^{2n}(I-A)^n P_t f = V_r V_{2n-r} ((I-d/dt)^n P_t) f,
\]

where \( A \) is the generator of the semi-group \( (P_t)_{t \geq 0} \) and \( d/dt \) stands for the derivative in the operator topology. Hence, \( P_t f \) is an element of \( F_{r,p} \). Consider an operator valued function \( S_t = V_{2n-r} ((I-d/dt)^n P_t) \). Then analyticity of \( S_t \) in \( t \) admits the Taylor expansion around \( t = t_0 \):

\[
S_t = \sum_{n=0}^{\infty} B_n (t-t_0)^n, \quad |t-t_0| < \varepsilon,
\]

where the \( B_n \)'s are bounded operators in \( L_p \) such that \( \sum_{n=0}^{\infty} ||B_n||e^n < \infty \).
Take any \( f \in L_\rho \) and quasi-continuous versions \( \widehat{V_rB_nf} \), \( n = 0, 1, 2, \ldots \). Then by (e), \( \sum_{n=0}^\infty |\widehat{V_rB_nf}(x)| \in \mathbb{E}^n \) converges except on some Borel set \( N \) with \( C_{r\rho}(N) = 0 \). Therefore, if we set, for \( |t-t_0| < \varepsilon \)

\[
P_t f(x) = \begin{cases} 
\sum_{n=0}^\infty \widehat{V_rB_nf}(x) (t-t_0)^n, & \text{if } x \in X-N, \\
0, & \text{otherwise}, 
\end{cases}
\]

and patch the functions in \( t \), then we have \( \widehat{P_t f}(x) \) which enjoys properties (i) and (ii). (iii) is clear from (3) and

\[
\sup_{|t-t_0| < \varepsilon} |P_t f(x)| \leq V_r(\sum_{n=0}^\infty |B_nf| \in \mathbb{E}^n)(x). \quad \text{q.e.d.}
\]

In the remainder of this section, we only consider a strongly continuous contraction semi-group \( (P_t)_{t \geq 0} \) which is determined by Markovian symmetric operator \( (P_t)_{t \geq 0} \) on \( L_2(X; m) \). E. M. Stein ([16]) shows that \( (P_t)_{t \geq 0} \) then becomes an analytic semi-group on \( L_\rho \) for each \( \rho > 1 \). We introduce for \( f \in L_\rho(X; m) \) the maximal function \( Mf \) by

\[
Mf(x) = \sup_{t \geq 0} |\widehat{P_t f}(x)|,
\]

where \( \widehat{P_t f} \) is the function in Lemma 1. Then we have the \( L_\rho \)-estimate ([16]):

\[
\|Mf\|_{L_\rho} \leq C_{r\rho}\|f\|_{L_\rho}, \quad f \in L_\rho
\]

for some positive constant \( C_{r\rho} \).

**Lemma 2.** For each \( \lambda > 0, u \in F_{r\rho} \), we have

\[
C_{r\rho}(Mu > \lambda) \leq C_{r\rho}\lambda^{-\rho}\|u\|_{r,\rho}^\rho
\]

Proof. For \( f \in L_\rho(X; m) \) and \( u = V_rf \), we have

\[
|P_t V_rf| = |V_rP_t f| \leq V_rMf \quad m\text{-a.e.}
\]

and consequently

\[
Mu \leq \widehat{V_rMf} \quad C_{r\rho}\text{-q.e.}
\]

Hence, by (5)

\[
C_{r\rho}(Mu > \lambda) \leq C_{r\rho}(\widehat{V_rMf} > \lambda) \leq \lambda^{-\rho}\|Mf\|_{L_\rho}^\rho
\]

\[
\leq C_{r\rho}\lambda^{-\rho}\|f\|_{L_\rho}^\rho = C_{r\rho}\lambda^{-\rho}\|u\|_{r,\rho}^\rho, \quad u \in F_{r\rho}. \quad \text{q.e.d.}
\]
Theorem 1. Assume that $(P_t)_{t \geq 0}$ is determined by a Markovian symmetric operator on $L_2$.
(i) For any $u \in F_{r,p}$, the limit $\lim_{t \to \infty} \hat{P}_t u(x)$ exists $C_{r,p}$-q.e. which is $C_{r,p}$-quasi-continuous version of $u$, where $r > 0, p > 1$.
(ii) The limit $\lim_{t \to \infty} \hat{P}_t f(x) = h(x)$ exists $C_{r,2}$-q.e., for any $f \in L_2(X; m)$. $h$ satisfies
$$\hat{P}_t h(x) = h(x), \quad t > 0, \quad C_{r,2}$-a.e.$$

Proof. (i) If we set
$$R(u) = \lim_{n \to \infty} \sup_{0 < t, t' < 1} |\hat{P}_t u(x) - \hat{P}_{t'} u(x)|,$$
then $R(u) = 0$ $C_{r,p}$-q.e., for any $u \in F_{r,p}$. For the last lemma combining with the inequality
$$R(u) = R(u - P_t u) \leq 2M(u - P_t u) \quad C_{r,p}$-q.e.$$
shows that
$$C_{r,p}(R(u) > \lambda) \leq C_{r}(2\lambda)^{-p} ||u - P_t u||_{r,p}^p,$$
which tends to zero as $h$ tends to zero for any $\lambda > 0$. By (e), the pointwise limit $\lim_{t \to \infty} \hat{P}_t u(x)$ must be a $C_{r,p}$-quasi-continuous version of $u$.
(ii) As in [4], we easily obtain the existence of the $C_{r,2}$-q.e. limit $h = \lim_{t \to \infty} \hat{P}_t f$. $h$ is $C_{r,2}$-quasi-continuous. Recalling the analyticity of $\hat{P}_t h$, we have
$$P_t h(x) = h(x), \quad t > 0, \quad C_{r,2}$-q.e.$$ q.e.d.

3. Construction of a transition function

In this section, we suppose that $X$ is separable complete metric space, $X$ is covered by some countable family of closed sets with finite $m$-measure and the support of $m$ is $X$. Given a strongly continuous Markovian semi-group $(P_t)_{t \geq 0}$ on $L_p(X; m)$ ($1 < p < \infty$) satisfying the analyticity in $t > 0$ and the regularity condition (2), we have constructed a regularized version $\hat{P}_t$, $f \in L_p$ in Lemma 1. We can further construct a transition function as follows.

Theorem 2. There exists a family of kernels $\{p_t(x, E); t > 0, x \in X, E \in \mathcal{B}\}$, where $\mathcal{B}$ stands for the set of all Borel subsets of $X$, which satisfies the following conditions:
(i) $p_t(x, X) \leq 1, \quad t > 0$.
(ii) $\int_X p_t(x, dy) p_s(y, E) = p_{t+s}(x, E), \quad t, s > 0$. 
(iii) For each $f \in L_p$ and $r > 0$, there exists a Borel set $N$ such that $C_{r, p}(N) = 0$ and

$$p_t f(x) = \hat{P}_t f(x)$$

for every $t > 0$ and $x \in X - N$.

Proof. We only give the proof in the case that $m(X) < \infty$ but the proof is similar to the $\sigma$-finite case. Let us embed the space $X$ homeomorphically onto a Borel subset $Y$ of $[0, 1]^n$. Take a countable dense subset $C_1$ of $C([0, 1]^n)$. Denoting by $B_b$ the set of all bounded Borel functions of $[0, 1]^n$ and by $\hat{f}$ the restriction to $Y$ of $f \in B_b$, then $B_b \subset L_p([0, 1]^n)$. By virtue of Lemma 1, we get

$$P_t(af + bg)(x) = a\hat{P}_t f(x) + b\hat{P}_t g(x) \quad C_{r, p} \text{-q.e.}$$

for $f, g \in B_b$, $a, b \in R$,

$$f_n, f \in B_b, f_n \uparrow f \Rightarrow \hat{P}_n f(x) \uparrow \hat{P}_t f(x) \quad C_{r, p} \text{-q.e.}$$

Further we find the set $N \subset X$ with $C_{r, p}(N) = 0$ such that $\hat{P}_t f(x)$ is analytic function of $t > 0$ for $f \in C_1, x \in X - N$.

By similar way of the proof of Proposition (4.1) in R.K. Getoor [7], we obtain the kernel $q_t(x, E)$ such that

$$q_t f(x) = \hat{P}_t f(x), \quad x \in X - N, f \in C_1, t \in Q^+,$$

where $Q^+$ is the set of all positive rational numbers. Since $[0, 1]^n$ is compact, the dual space of $C([0, 1]^n)$ is weakly complete and $q_t(x, \cdot) (t \in Q^+)$ has a continuous extension to the half real line. Denote by $p_t(x, \cdot), t \in (0, \infty)$ the restriction of $q_t(x, \cdot)$ to $X$, then we have

$$p_t \hat{f}(x) = \hat{P}_t f(x) \quad \text{for any } t > 0, x \in X - N, f \in C_1.$$

Hence, we arrive at (iii) by Lemma 1 and a monotone lemma.

On the other hand, there exists a Borel set $Y_1$ with $C_{r, p}(X - N) = 0$ such that for each $x \in Y_1$

$$p_t(p_s \hat{f})(x) = p_{t+s} \hat{f}(x), \quad t, s \in Q^+, f \in C_1$$

and all the functions $p_t \hat{f}(x)$ and $p_t(p_s \hat{f})(x), s \in Q^+$ are continuous in $t > 0$. Now just as the proof of Lemma 6.1.4. in M. Fukushima [5], we can modify $p_t(x, E)$ slightly to get kernels which satisfy not only (i), (iii) but also (ii) of Theorem 2.

q.e.d.

We call the kernels in Theorem 2 a transition function representing the
Once such a transition function is constructed, we get a nice potential kernel by

$$v_r(x, E) = \Gamma(r/2)^{r-1} \int_0^\infty s^{r/2-1} e^{-s} p_s(x, E) ds.$$ 

In fact, we have

**Corollary.** $v_r f(x)$ is a $C, r$-quasi-continuous version of $V, r f$, for every $f$ in $L_p(X; m)$.

**Proof.** It suffices to prove this for bounded functions in $L_p$. If $f \in L_p$ is bounded, we have the pointwise convergence

$$p_t v_r f(x) = v_r p_t f(x) \to v_r f, \quad t \to 0.$$ 

The convergence also takes place in the Banach space $F_{r, p}$ and hence we get the above conclusion. q.e.d.

### 4. Construction of Hunt processes

In this section, we assume that the state space $X$ is a locally compact separable metric space and the measure $m$ is positive Radon with support $X$. Let $(P_t)_{t \geq 0}$ be a Markovian strongly continuous contraction semi-group defined on $L_p(X; m)$ $(1 < p < \infty)$ which is analytic in the sense of §2. When the space $F_{r, p}$ contains continuous functions densely, we saw in §2 and §3 that the semi-group admits some potential theoretic refinements. Our assertion of this section is that under a stronger assumption (6) on $F_{r, p}$ mentioned below we can construct an associated Hunt process starting from $C_r$-quasi-everywhere point of $X$, for $r \geq 2$.

Let $X_\Delta = X \cup \{\Delta\}$ be the one-point compactification of $X$, and extend the transition function of the last section to $X_\Delta$ by

$$p_t(x, E) = \begin{cases} p_t(x, E - \{\Delta\}) + (1 - p_t(x, X)) \delta_\Delta(E), & x \in X \\ \delta_\Delta(E), & x = \Delta \text{ or } t = 0 \\ \end{cases}$$

for Borel subset $E$ of $X$.

By the Kolmogorov extension theorem, there is a Markov process $M = \{\Omega, \mathcal{M}, \mathcal{M}_{0}^\prime, X, P_t\}_{t \in \mathbb{R}}$ with transition probability $(p_t)_{t \in \mathbb{R}^+}$, where $\Omega, \mathcal{M}, \mathcal{M}^\prime_{0}, X^\prime_{0}$ are the following objects:

- $\Omega = X^\prime_{0} \cup \{\emptyset\}$
- $X^\prime_{0}(\omega) = \omega(t), \quad \omega \in \Omega$
- $\mathcal{M} = \sigma[X^\prime_{0}(\omega); t \in \mathbb{R}^+]$
- $\mathcal{M}_{0}^\prime = \sigma[X^\prime_{0}(\omega); s \leq t, s \in \mathbb{R}^+]$.
We let \( r \geq 2 \) and assume that

\[(6) \quad \text{the } F_{r,p} \cap C_{w}(X) \text{ is dense not only in } F_{r,p} \text{ but also in } C_{w}(X), \]

where \( C_{w}(X) = \{ f \in C(X_{\Delta}); f(\Delta) = 0 \} \). Under the assumption, we have a sequence \( \{ t_{j} \}_{j=0}^{\infty} \subseteq \mathbb{Q}^{+} \) decreasing to 0, which satisfies

\[(7) \quad \lim_{j \to \infty} p_{t_j} f(x) = f(x), \text{ for any } f \in C_{w}(X) \text{ and } x \in X - N \]

for some \( N \) with \( C_{r,p}(N) = 0 \), because we can see this for a dense subclass of \( C_{w}(X) \), contained in \( F_{r,p} \), as in the proof of Corollary in §3. An increasing sequence \( \{ F_{k} \}_{k=1}^{\infty} \) of closed sets with \( \lim_{k \to \infty} C_{r,p}(X - F_{k}) = 0 \) is said to be a \( C_{r,p} - \text{nest} \). The condition (6) further implies that each \( u \in F_{r,p} \) admits a \( C_{r,p} - \text{nest} \) \( \{ F_{k} \}_{k=1}^{\infty} \) for which \( u |_{F_{k} \cup \{ \Delta \}} \) are continuous functions vanishing at \( \Delta \), \( k = 1, 2, 3, \ldots \). The totality of such functions is denoted by \( C_{w}(\{ F_{k} \}_{k=1}^{\infty}) \). Here, we shall show a crucial lemma.

**Lemma 3.** Under the assumption (6), we get the followings:

(i) For any decreasing sequence \( \{ O_{n} \}_{n=0}^{\infty} \) of open sets with \( C_{r,p}(O_{n}) \to 0 \), as \( n \to \infty \), we have

\[ P_{x}(\lim_{n \to \infty} \sigma_{0}^{n} = \infty) = 1, \text{ for } C_{r,p} - \text{q.e. } x \in X, \]

where \( \sigma_{0}^{n} = \inf \{ t > 0; X_{t}^{0} \in A \} \).

(ii) If we let \( \Omega_{1} = \{ \omega \in \Omega_{0}; \text{ the sample path } X_{0}^{0}(\omega) \text{ has left- and right-hand limits in } X, \text{ for all } t > 0 \}, \) then

\[ P_{x}(\Omega_{1}) = 1, \text{ for } C_{r,p} - \text{q.e. } x \in X. \]

(iii) If we let \( X_{t}(\omega) = \lim_{s \uparrow t, s \in \mathbb{Q}^{+}} X_{s}^{0}(\omega) \text{ and } \Omega_{2} = \{ \omega \in \Omega_{1}; X_{t}(\omega) = X_{s}^{0}(\omega), t \in \mathbb{Q}^{+} \text{ and } X_{0}^{0}(\omega) = x \}, \) then

\[ P_{x}(\Omega_{2}) = 1, \text{ for } C_{r,p} - \text{q.e. } x \in X. \]

(iv) Let \( \Omega_{3} = \{ \omega \in \Omega_{2}; \text{ if } X_{t}(\omega) \in X \text{ then the trajectory of the sample path up to the time } t \text{ lies in a compact subset of } X \text{ for all } t > 0 \}, \) then

\[ P_{x}(\Omega_{3}) = 1, \text{ for } C_{r,p} - \text{q.e. } x \in X. \]

(v) There exists a Borel set \( Z \subseteq X \) and \( \Gamma_{0} \subseteq \mathcal{M} \) satisfying \( C_{r,p}(X - Z) = 0 \), \( P_{x}(\Gamma_{0}) = 0 \) for all \( x \in Z \) and the inclusion

\[ \{ \omega \in \Omega_{3}; \text{ for some } t \geq 0, \text{ either } X_{t}(\omega) \text{ or } \lim_{s \uparrow t} X_{s}(\omega) \text{ is not in } Z \} \subseteq \Gamma_{0}. \]

(vi) Put \( \mathcal{M}_{t} = \bigcup_{s \leq t, s \in \mathbb{Q}^{+}} \mathcal{M}_{s}. \) Consider the restrictions of \( \mathcal{M}, \mathcal{M}_{t}, X_{t} \) and \( P_{x} \) to
the set $\Omega=\Omega_0-\Gamma_0$ and denote them by the same notations. Then the quintuplet $M_z=\{\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_z\}_{z\in Z}$ becomes a Hunt process on $Z$.

Proof. (i) Every open set $O$ of finite capacity possesses a unique norm-minimizing element $e_O$ in the set $\{u\in F_{r,p}; u\geq 1 \text{ m-a.e. on } O\}$. As a version of $e_O$, take $e_0$ a function expressed as $v_r f$ for some non-negative function $f$ in $L_p$. Clearly we have then

$$e^{-t} p_t e_0(x) \leq e_0(x)$$

which means that $\{Y_t=e^{-t} \hat{e}_0(X_t)\}_{t\geq 0}$ is $\mathcal{M}_t, P_x$-supermartingale for each $x\in X$.

Applying Doob’s optional sampling theorem to $\{Y_t, \mathcal{M}_t, P_x\}$ $x\in X$ and noting that the process $\{X_t\}_{t\geq 0}$ does not hit the set $\{x\in 0; \hat{e}_0(x)<1\}$ with $P_x$-a.e. $x\in X$, we obtain

$$E_x(\exp(-\sigma_0^0)) \leq e_0 \quad C_{r,s^*} \text{q.e.}$$

The statement (i) follows from this inequality.

(ii) Take a countably dense subset $C_2\subset C_0(X)$. There exists a nest $\{F_k\}_{k=1}^\infty$ such that

$$\text{The convergence (7) holds on } \bigcup_{k=1}^\infty F_k,$$

$$\bigcup_{l=1,s[r,2+r],\ldots} v_l(C_2) \subset C_\infty(\{F^\infty_{k=1}\}).$$

Here, we know that the family of functions of the left hand side of (9) separates the point of $Z_0=(\bigcup_{k=1}^\infty F_k)\cup \{\Delta\}$. In fact, if we suppose for $x, y\in Z_0$

$$v_l f(x) = v_l f(y), \quad \text{for any } f\in C_2, l = 1+\lfloor r \rfloor, 2+\lfloor r \rfloor, \ldots$$

then $p_t f(x)=p_t f(y), t>0, f\in C_2$, by the uniqueness of the Laplace transformation. Letting $t$ tend to 0 along the sequence $\{t_j\}_{j=1}^\infty$, we see that $f(x)=f(y), f\in C_2$ by (8) and $x=y$.

Hence, for the event $\Omega_0=\{\omega\in\Omega_0; \lim_{k\to\infty} \sigma_{X,F_k}(\omega)=\infty\}$, we have that

$\Omega_0-\Omega_2=\{\omega\in\Omega_0; \text{ for some } k \text{ and some } t<\sigma^0_{X,F_k}\}$

$X^t(\omega)$ does not have the right- or left-hand limit at $t$.

Since the process $\{e^{-t} v_l f(X_t(\omega)), \mathcal{M}_t, P_x\}$ is a non-negative supermartingale, the $P_x$ measure of the right hand side is zero. In view of (i), we know that $P_x(\Omega_0)=1, C_{r,s^*} \text{q.e.}$ and so is $\Omega_2$. (iii), (iv), (v) and (vi) The proofs can be performed in the same way as in [5; Chapter 6, §2].
We extend the Hunt process of Lemma 3 (vi) to the Hunt process on \( X \) by letting each point of \( X - Z \) be trap.

**Theorem 3.** There exists a Hunt process \( M = \{ \Omega, \mathcal{M}, X, P_x \} \) satisfying that

\[
(10) \quad \text{for each } f \in L_p, E_x(f(X_t)) \text{ is a } C_{r,p} \text{-quasi-continuous modification of } P_t f.
\]

If \( M' = \{ \Omega', \mathcal{M}', X', P'_x \} \) is another Hunt process with property (10), then the induced probability laws of \( X_t \) and \( X'_t \) on the path space \( \Omega = \{ \omega; [0, \infty) \rightarrow X, \omega(t) \text{ is right continuous with left limits in } t \} \) coincide for \( C_{r,p} \)-a.e. \( x \in X \).

**Proof.** The existence is already shown. To prove the part of the uniqueness, it suffices to show that for \( M' \) with the property (10)

\[
E_x(f(X_{t_1})f(X_{t_2}) \cdots f(X_{t_n})) = E_x(f(X'_{t_1})f(X'_{t_2}) \cdots f(X'_{t_n})),
\]

where \( f_1, f_2, \cdots, f_n \in C_2, t_1, t_2, \cdots, t_n \in \mathbb{Q}^+ \). But this is clear from (10). q.e.d.

In the symmetric case, we have a criterion for the sample path continuity of the Hunt process \( M \).

Let us consider a strongly continuous semi-group \( (P_t)_{t \geq 0} \) of Markovian symmetric operator on \( L_2 \). As stated in §2, it can be regarded as a strongly contraction analytic semi-group in \( L_p \) \((1 < p < \infty) \). We assume that the regularity (6) for the associate space \( F_{r,p} \) and \( F_{1,2} \).

**Theorem 4.** The following conditions are equivalent.

(i) The Dirichlet space \( F_{1,2} \) is local in the sense that the pair \( u, v \in F_{1,2} \) with disjoint supports always enjoys the property \((u, v)_{F_{1,2}} = 0 \).

(ii) \( M \) is a diffusion in the sense

\[
P_x(\omega \in \Omega; \text{the sample path is continuous}) = 1, \quad C_{r,p}\text{-a.e.}
\]

**Proof.** Let us set \( q(x) = P_x(\omega \in \Omega; \text{ for some } t > 0, \lim_{t \uparrow t} X_t(\omega) = X_t(\omega)) \). If \( q(x) \) vanishes \( m \text{-a.e.} \), then \( q(x) = 0 \) \( C_{r,p} \text{-a.e.} \). Because the function \( P_x(\omega \in \Omega; \text{ for some } t > 1/n, \lim_{t \uparrow t} X_t(\omega) = X_t(\omega)) = p_{\mu,q}(x) \) then vanishing \( C_{r,p} \text{-a.e.} \). Since \( M \) can be also regarded as the diffusion as a realization of the \( L_2 \)-semi-group, the first statement of Theorem 4 combined with a general theorem related to the Dirichlet space implies that \( q(x) = 0 \) \( m \text{-a.e.} \). The proof of theorem is completed.

**Example.** Suppose that a uniformly elliptic partial differential operator
\[ L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \] possesses bounded smooth coefficients in the sense that \( a_{ij}(x) \in C^1_c(\mathbb{R}^n), \) \( 1 \leq i, j \leq n, \) \( b_i(x) \in C^1(\mathbb{R}^n), \) \( 1 \leq i \leq n, \) \( \sum_{i,j=1}^{n} \xi^i \xi^j \geq \delta |\xi|^2 \) for some \( \delta > 0, \) and that \( c(x) \) is bounded non-positive.

The resolvent \( R_\lambda \) on \( L_2(\mathbb{R}^n) \) satisfies \( ||R_\lambda|| \leq C/(1+|\lambda|) \) in the domain \( \{ \lambda \in C; \ Re(\lambda) \geq \alpha \} \) with some positive \( C \) and \( \alpha. \) Owing to a well known theorem of K. Yosida [18; Chapter IX, 10], the corresponding semi-group is analytic in \( L_2(\mathbb{R}^n). \) Obviously the semi-group \((P_t)_{t \geq 0}\) is Markovian and contractive. We observe that the dual semi-group has the same properties. By the method of interpolation mentioned in E.M. Stein [16] and above observation, we know that in \( L_p(\mathbb{R}^n) \) \((P_t)_{t \geq 0}\) is analytic whenever \( 1 < p < \infty. \)

The Sobolev space \( W^2_p(\mathbb{R}^n) \) as the domain of the closed extension of \( L \) with domain \( C_c^\infty(\mathbb{R}^n) \) coincides with the space of potentials \( F_{2,p} \) with equivalent norms. Since \( W^2_p(\mathbb{R}^n) \) satisfies the assumption (6), Theorem 3 gives us the corresponding Hunt process in the \( C_2,p\)-refined sense. The Sobolev embedding theorem assures that "\( C_2,p\)-q.e." becomes "everywhere" when \( 2p > n. \) Consequently the Hunt process is uniquely associated without exceptional starting point.

References


Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan