Linear independence of special values of formal Laurent series

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Part I

Introduction

Chapter 1

History

We denote an algebraic closure of $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. The properties of special values of formal power series with coefficients $\overline{\mathbb{Q}}$, such as irrationality, transcendence or more generally linear relations, algebraic relations among them, have been studied by many mathematicians. One of the pioneering work on this theme is about the special values of exponential function as follows:

THEOREM 1.0.1. (*Hermite,* 1873) *The Napier's constant e is transcendental number.*

In 1882, Lindemann generalized Theorem 1.0.1 to the transcendency of special values of exponential function at non-zero algebraic numbers. In 1885, after the work of Lindemann, Weierstrass obtained the following result so called Lindemann-Weierstrass Theorem:

THEOREM 1.0.2. (*Lindemann-Weierstrass Theorem*)
Let m be a natural number and $\alpha_1, \ldots, \alpha_m$ algebraic numbers which are linearly independent over rational number field. Then we have the following equality:

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_m}) = m.$$

The next development of this subject was given by Siegel in 1929 (cf. [74]). The main contributions of his studies are as follows:

(a) He defined 2 classes of formal power series, E-functions and G-functions, which are still the main research objects even today.

(b) He gave an algebraic independence result for special values of E-functions at algebraic numbers that is a generalization of Theorem 1.0.2.

Note that the notion of E-function is a generalization of exponential function and that of G-function is a generalization of algebraic functions. Siegel obtained the result (b) from the view point of approximation of special values of E-functions by algebraic numbers (cf. Lemma 5.1.1). Although Siegel could not obtain any results of algebraic relations among special values of G-functions, he conjectured that we should be able to obtain some algebraic relations among them by the same method to prove the result of (b), namely the approximation of special values of G-functions by algebraic numbers. After the work of Siegel as above, many mathematicians studied the algebraic relations among special values of E-functions and G-functions at algebraic numbers. The studies for special values of E-functions had been developed early on stage, but the conjecture of Siegel mentioned as above for special values of G-functions was unsolved for a long time. Afterwards, the conjecture of Siegel was solved (see Theorem 1.2.20), but the algebraic relations among special values of G-functions still have many mysteries even nowadays.

In this thesis, we study the algebraic relations, especially linear relations, of special values of formal power series which relate to G-functions. In the following section, we overview algebraic relations among special values of E-functions and G-functions. Since many studies for G-functions are based on that of E-functions, we firstly explain previous studies of E-functions.

1.1 Algebraic relations among special values of E-functions

In this section, we introduce the results of algebraic relations among special values of E-functions. E-function is a generalization of exponential function which are defined as follows:

Definition 1.1.1. (E-functions) Let $g(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k!} \in \overline{\mathbb{Q}}[[z]]$. We call $g(z)$ an E-function if $g(z)$ satisfies the following conditions.

- (E_1) There exist positive numbers γ_1, C_1 satisfying $\overline{|a_k|} \leq \gamma_1 C_1^k$ for all $k \in \mathbb{Z}_{\geq 0}$,
- (E_2) There exist positive numbers γ_2, C_2 satisfying $\mathrm{den}(a_i)_{0\leq i\leq k} \leq \gamma_2 C_2^k$ for all $k \in \mathbb{Z}_{\geq 0}$,
- (E_3) There exists a non-zero differential operator $\Lambda \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ satisfying $\Lambda g = 0$.

Throughout this Section, we use the following notations:

Let m be a natural number and $f_1, \ldots, f_m$ be formal power series with coefficients $\overline{\mathbb{Q}}$. We assume that $f_1, \ldots, f_m$ satisfy the following differential equation:

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \tag{1.1}$$

where $A \in M_m(\overline{\mathbb{Q}}(z))$. We denote $T(z) \in \overline{\mathbb{Q}}[z]$ be the denominator of the entries of A.

As mention above (cf. (b)), in 1929, Siegel proved a generalization of Lindemann-Weierstrass Theorem for E-functions $f_1, \ldots, f_m$ satisfying the relation (1.1) under the assumption of the "normality" of $\Lambda := \frac{d}{dz} - A$. The method of proof of Siegel was achieved by using the method of approximation of numbers (cf. Lemma 5.1.1). Afterwards, Shidlovsky removed the normality condition on the differential operator Λ and obtain the following theorem so called Siegel-Shidlovsky Theorem:

Theorem 1.1.2. (*Siegel-Shidlovsky Theorem,* 1956) ([72, Chapter 4])
Let $f_1, \ldots, f_m$ be E-functions satisfying (1.1). *Let $\alpha \in \overline{\mathbb{Q}}$ satisfying $\alpha T(\alpha) \neq 0$. Then we have*

$$\mathrm{tr.deg}_{\mathbb{Q}(z)} \mathbb{Q}(z)(f_1, \ldots, f_m) = \mathrm{tr.deg}_{\mathbb{Q}} \mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha)).$$

Using the method of proof of Theorem 1.1.2 by Shidlovsky, Nesterenko showed that the algebraic relations of special values of E-functions $f_1, \ldots, f_m$ satisfying the relation (1.1) at almost of all algebraic numbers come from that of $f_1, \ldots, f_m$. The precise statement as follows:

Theorem 1.1.3. (*Nesterenko-Shidlovsky,* 1996) ([22, Theorem 1.2])
Let $f_1, \ldots, f_m$ be E-functions satisfying (1.1). Then, there exists a finite set S such that for all $\xi \in \overline{\mathbb{Q}} \setminus S$ the following holds.

For any homogeneous polynomial relation

$$P(f_1(\xi), \ldots, f_m(\xi)) = 0 \textit{ with } P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m],$$

there exists $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_m]$ which is homogeneous polynomial with respect to $X_1, \ldots, X_m$ such that $Q(z, f_1(z), \ldots, f_m(z)) = 0$ in $\overline{\mathbb{Q}}[z]$ and $P(X_1, \ldots, X_m) = Q(\xi, X_1, \ldots, X_m)$.

In 2000, André studied the properties of differential operators which annihilate E-functions, so called E-operators, in [6] and obtained an alternative proof of Siegle-Shidlovsky theorem in [7]. Using the results of André in [6], Beukers gives a refinement of Theorem 1.1.3 as follows:

THEOREM 1.1.4. (*Beukers,* 2006) [22, Theorem 1.3]
Theorem 1.1.3 *holds for any* $\xi \in \overline{\mathbb{Q}} \setminus \{\alpha \in \overline{\mathbb{Q}} | \ \alpha T(\alpha) = 0\}$.

1.2 Algebraic relations among special values of G-functions

In this section, we review the properties of G-functions, differential operators which annihilate some G-functions and special values of them. The definition of G-function was given by Siegel in [74] as follows:

DEFINITION 1.2.1. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathbb{Q}}[[z]]$. We call $g(z)$ a G-function if $g(z)$ satisfies the following conditions.

- (G_1) There exist positive numbers γ_1, C_1 satisfying $\overline{|a_k|} \leq \gamma_1 C_1^k$ for all $k \in \mathbb{Z}_{\geq 0}$,
- (G_2) There exist positive numbers γ_2, C_2 satisfying $\mathrm{den}(a_i)_{0 \leq i \leq k} \leq \gamma_2 C_2^k$ for all $k \in \mathbb{Z}_{\geq 0}$,
- (G_3) There exists a non-zero differential operator $\Lambda \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$ satisfying $\Lambda g = 0$.

We denote the set of G-functions by $\mathbb{G}$.

EXAMPLE 1.2.2. We give some example of G-functions.
1. Let $g(z)$ is an element of $\mathbb{Q}[[z]]$. If $g(z)$ is algebraic over $\mathbb{Q}(z)$, the formal power series $g(z)$ is a G-function. (Eisenstein's Theorem)
2. Let a, b, c are rational number satisfying $c \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then the hypergeometric function

$$ {}_2F_1(a, b, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b_k)}{(c)_k k!} z^k $$

is a G-function where $(x)_k := \frac{\Gamma(x+k)}{\Gamma(x)}$ for $x \in \mathbb{R}$.
3. Let m be a natural number and

$$ \mathrm{Li}_m(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^m} $$

the m-th polylogarithm function. Then the formal power series $\mathrm{Li}_m(z)$ is a G-function.
4. Let

$$ \Phi(s, x, z) := \sum_{k=0}^{\infty} \frac{z^k}{(x+k)^s} $$

be the Lerch function. Then for a natural number m and a rational number $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, the formal power series $\Phi(m, a, z)$ is a G-function.

1.2.1 G-operators

The properties of the differential operators that annihilate G-functions are studied by Bombieri, Chudnovsky and André et al. We introduce some of the results of them. In this subsection, we use the following notations.

Let K be an algebraic number field. We denote the set of finite place of K by $\mathcal{P}_f(K)$ and Gauss norm on $K_v(z)$ by

$$|\cdot|_{v,\mathrm{gauss}} : K_v(z) \longrightarrow \mathbb{R}_{\geq 0},$$

for $v \in \mathcal{P}_f(K)$. For a natural number m and $A \in M_m(K(z))$, we define a family of matrices $\{A_k\}_{k\in\mathbb{Z}_{\geq 1}} \subset M_m(K(z))$ inductively as follows:

$$A_1 := A \text{ and } A_{k+1} := A_k A + \frac{d}{dz}A_k. \tag{1.2}$$

Put $\Lambda := \dfrac{d}{dz} - A$. We define the following invariant of Λ by

$$\sigma(\Lambda) := \limsup_{k\to\infty} \frac{1}{k} \sum_{v\in\mathcal{P}_f(K)} \sup_{m\leq k} \left|\frac{A_m}{m!}\right|_{v,\mathrm{gauss}}.$$

Firstly, we introduce the notion of G-operator.

Definition 1.2.3. (G-operator)
For $A \in M_m(K(z))$, we put $\Lambda := \dfrac{d}{dz} - A$. We say that Λ is a G-operator if $\sigma(\Lambda) < \infty$.

Remark 1.2.4. We use the same notation as in Definition 1.2.3. We give some equivalent relations of G-operator. We prepare some notations. Let $T(z) \in K[z]$ be the common denominator of the entries of $A \in M_m(K(z))$. Then, from the definition of $T(z)$ and A_k, we have $T^k \frac{A_k}{k!} \in M_m(K[z])$ and so we can define

$$D_k(\Lambda) := \mathrm{den}\left(T^0\frac{A_0}{0!}, \dots, T^k\frac{A_k}{k!}\right) \text{ for all } k \in \mathbb{Z}_{\geq 0},$$

where A_0 is the（dentity matrix in $M_m(K(z))$.

For an element $\mathbf{P} := (P_1(z), \dots, P_m(z)) \in K[z]^m$, we define subsets $\{\tilde{\mathbf{P}}_k(\Lambda)\}_{k\in\mathbb{Z}_{\geq 0}}$ and $\{\mathbf{P}_k(\Lambda)\}_{k\in\mathbb{Z}_{\geq 0}}$ of $K(z)^m$ and $K[z]^m$ respectively as follows:

$$\begin{aligned} &\tilde{\mathbf{P}}_0(\Lambda) := \mathbf{P},\ \tilde{\mathbf{P}}_{k+1}(\Lambda) := \frac{d}{dz}\tilde{\mathbf{P}}_k(\Lambda) + \tilde{\mathbf{P}}_k(\Lambda)A, \\ &\mathbf{P}_k := T^k(z)\tilde{\mathbf{P}}_k(\Lambda). \end{aligned}$$

Lemma 1.2.5. *Let* $\Lambda = \frac{d}{dz} - A \in M_n(K(z)[\frac{d}{dz}])$. *Then the following statements are equivalent.*
(1) Λ *is a* G*-operator.*
(2) *There exists* $C > 0$ *satisfying* $D_k(\Lambda) \leq C^k$ *for all* $k \in \mathbb{Z}_{\geq 1}$.
(3) *There exists a set* $\{d_k\}_{k\in\mathbb{Z}_{\geq 1}} \subset \mathbb{N}$ *satisfying the following properties* (i) *and* (ii)*:*

(i) *There exists* $C > 0$ *satisfying* $d_k \leq C^k$ *for all* $k \in \mathbb{Z}_{\geq 1}$.

(ii) *For any elements* $\mathbf{P} := (P_1(z), \dots, P_m(z)) \in \mathcal{O}_K[z]^m$, *we have*

$$d_k \frac{\mathbf{P}_n(\Lambda)}{n!} \in \mathcal{O}_K[z]^m \text{ for all } k \in \mathbb{Z}_{\geq 1} \text{ and } 1 \leq n \leq k.$$

Note that the notion of G-operator was defined by Galochkin to obtain a Diophantine property of special values of G-functions by the method of proof of that for special values of E-functions by Siegel and Shidlovsky (cf. Theorem 1.2.20).

Some sufficient conditions that $\Lambda = \frac{d}{dz} - A \in M_m(K(z)\left[\frac{d}{dz}\right])$ becomes a G-operator was obtained by Chudnovsky as follows:

THEOREM 1.2.6. (*Chudnovsky,* 1985) [30]
For $A \in M_m(K(z))$, we put $\Lambda := \dfrac{d}{dz} - A$. Suppose there exists ${}^t(g_1, \ldots, g_m) \in \mathbb{G}^m \cap \ker(\Lambda)$ satisfying $g_1, \ldots, g_m$ are linearly independent over $K(z)$. Then the operator Λ is a G-operator.

THEOREM 1.2.7. (*Chudnovsky,* 1985) [34]
Let K be an algebraic number field and $g \in K[[z]]$ be a G-function. Let $L \in K[z, \frac{d}{dz}]$ be the minimal differential operator up to $K(z)^*$ which annihilates g. Then the operator L is a G-operator.

1.2.2 Periods and special values of G-functions

In this subsection, we explain the conjectural relation between the special values of G-functions at algebraic numbers and periods which were proposed by Kontsevich and Zagier. The definition of periods is as follows:

DEFINITION 1.2.8. ([60, p. 3])
Let α be a real number. We call α a real period if it is represented as an absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients. Let α be a complex number. We call α a period if both real numbers $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ are real periods. We denote the set of periods by $\mathcal{P}_{\mathrm{KZ}}$. Note that the set $\mathcal{P}_{\mathrm{KZ}}$ becomes a commutative ring by using Fubini's Theorem.

Our motivation is to understand the algebraic relations, such as irrationality, transcendence, linear independence and algebraic independence, among some given periods. The algebraic relations among periods were predicted by Grothendieck in some special case in [46, note 10] and formulated by Lang in [61, p. 43]. Nowadays, this conjecture is called "Grothendick period conjecture" and stated as follows:

CONJECTURE 1.2.9. (*Grothendieck period conjecture*)
Let X be a Nori motive over an algebraic number field K (see [50, Definition 9.1.3]). We denote the Tannakian category generated by X whose fiber functor is given by relative singular cohomology by

$$\mathcal{C}_X := (\langle X\rangle^{\otimes}, H^* : \langle X\rangle^{\otimes} \longrightarrow \mathrm{Vec}_{\mathbb{Q}}).$$

We also denote the Tannakian fundamental group of $\mathcal{C}_X$ by G_X. Note that the fundamental group G_X is an algebraic group over $\mathbb{Q}$. We denote the base change of G_X to K by $G_{X,K}$ and the torsor of $G_{X,K}$ with respect to the fiber functor $H^* \otimes_{\mathbb{Q}} K$ and H^*_{dR} by $\mathbb{P}_X$ where the functor

$$H^*_{\mathrm{dR}} : \langle X\rangle^{\otimes} \longrightarrow \mathrm{Vec}_K$$

is given by relative algebraic de Rham cohomology. Then we have

$$\mathbb{P}_X \text{ is connected.} \tag{1.3}$$

$$\mathrm{tr.deg}_K K(\text{periods of } X) = \dim_K G_X. \tag{1.4}$$

where $K(\text{periods of } X)$ is the field generated by the entries of the matrix of the comparison isomorphism $\mathrm{comp}_X : H^*(X)\otimes_{\mathbb{Q}}\mathbb{C} \cong H^*_{\mathrm{dR}}(X)\otimes_K\mathbb{C}$ for a given basis of $H^*(X)$ and $H^*_{\mathrm{dR}}(X)$ over $\mathbb{Q}$ and K respectively.

Remark 1.2.10. We can define the ring of periods of Nori motives (see [50, Definition 11.5.1.2 p. 256]) and denote it by $\mathcal{P}_{\mathrm{Nori}}$. The relation between $\mathcal{P}_{\mathrm{KZ}}$ and $\mathcal{P}_{\mathrm{Nori}}$ is known as follows:

Theorem 1.2.11. [44] (*cf.* [50, Theorem 12.2.1. p. 263])
We have the following equality:

$$\mathcal{P}_{\mathrm{KZ}}\left[\frac{1}{2\pi i}\right] = \mathcal{P}_{\mathrm{Nori}}. \tag{1.5}$$

We explain the relation between the special values of G-functions and periods. We prepare some notations. We denote the map of the radius of convergence of formal power series with coefficients $\mathbb{C}$ by

$$r_\infty : \mathbb{C}[[z]] \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\},\ \sum_{k=0}^{\infty} a_k z^k \mapsto \left(\limsup_{k\to\infty}|a_k|^{\frac{1}{k}}\right)^{-1}.$$

For an complex number α and a non-negative real number r, we denote

$$D_\infty(\alpha, r^-) := \{z \in \mathbb{C} |\ |z-\alpha| < r\}.$$

For an element $\sigma \in \mathrm{Hom}(\overline{\mathbb{Q}}, \mathbb{C})$, we denote the natural extension of σ to $\overline{\mathbb{Q}}[[z]]$ by

$$\sigma : \overline{\mathbb{Q}}[[z]] \longrightarrow \mathbb{C}[[z]],\ g := \sum_{k=0}^{\infty} a_k z^k \mapsto \sigma(g) = \sum_{k=0}^{\infty} \sigma(a_k) z^k.$$

Definition 1.2.12. We denote the set of complex numbers represented by the special values of G-functions at algebraic numbers as follows:

$$\mathcal{G} := \{\alpha \in \mathbb{C} |\ \exists g \in \mathbb{G},\ \exists \sigma \in \mathrm{Hom}(\overline{\mathbb{Q}}, \mathbb{C}),\ \exists \beta \in D_\infty(0, r_\infty(\sigma(g))^-) \text{ s.t. } \alpha = \sigma(g)(\beta)\}.$$

Remark 1.2.13. We denote the set of all values of multivalued analytic continuations of G-functions by $\mathcal{G}^{\mathrm{a.c.}}$. Then we have $\mathcal{G} = \mathcal{G}^{\mathrm{a.c.}}$ and $\zeta(k) \in \mathcal{G}$ for all $k \in \mathbb{Z}_{\geq 2}$, where $\zeta(s)$ is the Riemann zeta function.

The relation between $\mathcal{G}$ and $\mathcal{P}_{\mathrm{KZ}}\left[\frac{1}{2\pi i}\right]$ is expected in the following manner:

Conjecture 1.2.14. *We have the following equality:*

$$\mathcal{G} = \mathcal{P}_{\mathrm{KZ}}\left[\tfrac{1}{2\pi i}\right]. \tag{1.6}$$

Especially, from Theorem 1.2.11*, we have*

$$\mathcal{G} = \mathcal{P}_{\mathrm{Nori}}. \tag{1.7}$$

From Conjecture 1.2.14, studying the algebraic relations among special values of G-functions is useful to understand the algebraic relations among periods.

At the last of this subsection, we explain the reason for supporting Conjecture 1.2.14. Firstly, we define the notion of geometric differential operator as follows:

Definition 1.2.15. (Geometric differential equation) ([3, p. 39])
Let K be an algebraic number field and Λ be an element of $K[z, \frac{d}{dz}]$. We say Λ is geometric differential equation if Λ can be represented by

$$\Lambda = \Lambda_1 \cdots \Lambda_n$$

for some Picard-Fuchs differential equations $\Lambda_i \in K[z, \frac{d}{dz}]$ for $1 \leq i \leq n$.

André proved that solutions of geometric differential equations are G-functions. The precise statement as follows:

THEOREM 1.2.16. [3, p. 110]
Let K be an algebraic number field. Let $\Lambda \in K[z, \frac{d}{dz}]$ be a geometric differential equation and $g \in K[[z]]$ satisfies $\Lambda g = 0$. Then we have $r_v(g_v) = 1$ for almost all place v of K and g is a G-function, where g_v is the image of canonical embedding of $K[[z]] \hookrightarrow K_v[[z]]$ of g and $r_v(g_v)$ is the radius of convergence of g_v with respect to the valuation v in K_v.

Note that the set of geometric differential operator is contained in the set of scalar G-operator from Theorem 1.2.16.

CONJECTURE 1.2.17. *The set of geometric differential operator is equal to the set of scalar G-operator.*

From Conjecture 1.2.17, we have $\mathcal{G} \subseteq \mathcal{P}_{\mathrm{Nori}}$.

REMARK 1.2.18. We remark that Conjecture 1.2.17 is a consequence of much stronger conjecture, so called Bombieri-Dwork conjecture, as follows:

CONJECTURE 1.2.19. (*Bombieri-Dwork conjecture*)
Let K be an algebraic number field and $\Lambda \in K[z, \frac{d}{dz}]$. Suppose p-curvatures of Λ are nilpotent for p running over a set of prime numbers of density 1. Then Λ should be a geometric differential equation. □

1.2.3 Algebraic relations of special values of G-functions

One of the earliest results on Diophantine problem of special values of G-functions is the following result by Galochkin.

THEOREM 1.2.20. (*Galochkin, 1974*) [45]
Let $H \in \mathbb{R}$ and $q, d, m \in \mathbb{N}$. Let K be an algebraic number field and $A \in M_m(K(z))$. Put $\Lambda := \frac{d}{dz} - A$. We assume that Λ is a G-operator. Let $\mathbf{g} := {}^t(g_1(z), \dots, g_m(z)) \in \mathbb{G}^m \cap \ker(\Lambda)$ which are not related to one another by any non-trivial algebraic equation of degree at most d with coefficients $\mathbb{C}(z)$. Then, for a non-zero polynomial $Q(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ whose degree is at most d and coefficients have absolute value at most H, there exist positive constants $A := A(K, \mathbf{g}, d), \lambda = \lambda(K, \mathbf{g}, d)$ and $\mu = \mu(K, \mathbf{g}, d, q)$ satisfying

$$\left| Q\left(g_1\left(\frac{1}{q}\right), \dots, g_m\left(\frac{1}{q}\right)\right) \right| > q^{-\lambda} H^{mu}, \tag{1.8}$$

for integer $q \in \mathbb{Z}_{>A}$. Especially, the dimension of the $\mathbb{Q}$-vector space $\sum_{i=1}^{m} \mathbb{Q} g_i(\frac{1}{q})$ is m.

Theorem 1.2.20 is the first result which supports the conjecture of Siegel on Diophantine property of special values of G-functions. From Theorem 1.2.6 and Lemma 1.2.5, if $\mathbf{g} := {}^t(g_1(z), \dots, g_m(z)) \in \mathbb{G}^m \cap \ker(\Lambda)$ in Theorem 1.2.20 are linearly independent over $K(z)$, we obtain a Diophantine property of special values of $\mathbf{g}$ at enough small rational numbers. A generalization and a p-adic analogue of Theorem 1.2.20 were obtained by Väänänen in 1980 (cf. [79]).

Theorem 1.2.20 and the results of Väänänen are proved by constructing some rational approximation of special values of G-functions. They construct a rational approximation of special values of G-functions by the method of Padé approximation. The Padé approximation of G-functions that are given in the results of Galochkin and Väänänen are constructed by using "Siegel's Lemma" in [74] (cf. Lemma 3.1.3). The "Siegel's Lemma" guarantees only the existence of a Padé approximation which can be used for the proof of the inequality of type (1.8). In this thesis, we will construct an explicit Padé approximation of given G-functions which has an inequality of type (1.8) with better range of algebraic numbers than that of Galochkin and Väänänen.

Remark 1.2.21. The best known result on algebraic independence of special values of G-functions is as follows:

Theorem 1.2.22. (*André,* 1996) [5] (cf. [31])
Let v be a place of $\overline{\mathbb{Q}}$ and $\xi \in D_v(0, |16|_v^-) \cap \overline{\mathbb{Q}}^*$. Then, for the v-adic evaluation of ${}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; \xi)$ and ${}_2F_1(-\frac{1}{2}, -\frac{1}{2}, 1; \xi)$, we have the following equality:

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}\left({}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \xi\right), {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}, 1; \xi\right)\right) = 2.$$

Note that no results are known for algebraic independence among more than two special values of G-function.

In this thesis, we study the dimension of the vector space spanned by the special values of the formal Laurent series which relate to G-functions. In the next section, we explain the detail contents of this thesis.

Chapter 2

Contents of this thesis

2.1 (Type A)$_*$-estimate

In this thesis, we formulate a type of estimate of the dimension of the vector space spanned by the special values of some formal Laurent series over algebraic number fields, so called **(Type A)**$_*$-estimate. Before introduce the **(Type A)**$_*$-estimate, we prepare some notations.

We denote the field of complex numbers by $\mathbb{C}_\infty$ and the absolute value of $\mathbb{C}_\infty$ by $|\cdot|_\infty$. For a prime number p, we denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p$ by $\mathbb{C}_p$ and the normalized p-adic absolute value on $\mathbb{C}_p$ by $|\cdot|_p$. We fix embeddings

$$\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p, \text{ and } \iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_\infty.$$

We also denote the natural extension of ι_p (resp. ι_∞) to the formal power series and the formal Laurent series by

$$\iota_p : \overline{\mathbb{Q}}[[z]] \longrightarrow \mathbb{C}_p[[z]],\ \iota_p : \overline{\mathbb{Q}}\left[\left[\frac{1}{z}\right]\right] \longrightarrow \mathbb{C}_p\left[\left[\frac{1}{z}\right]\right].$$

Let m be a natural number, $* \in \{p, \infty\}$ and $\xi \in \mathbb{P}^1(\overline{\mathbb{Q}})$. For a positive real number r, we denote $\{z \in \mathbb{C}_* \mid |z-\xi|_* < r\}$ by $D_*(\xi, r^-)$. Let L be an algebraic number field. We denote the set of all algebraic number fields containing L by $\mathcal{A}_L$.

Let $\mathbf{f} := (f_1(z), \ldots, f_m(z)) \in \overline{\mathbb{Q}}[[z-\xi]]^m$. We assume that $\mathbf{f}_* := (f_{1,:}(z), \ldots, f_{m,*}(z)) \in \mathbb{C}_*[[z-\xi]]^m$ converges on $D_*(\xi, r^-)$ for $r > 0$. For $K \in \mathcal{A}_L$ and $\beta \in D_*(\xi, r^-) \cap \overline{\mathbb{Q}}$, we denote the vector space spanned by $f_{1,*}(\beta), \ldots, f_{m,*}(\beta)$ over K by $V_K(\mathbf{f}_*, \beta)$. The main purpose of this thesis is estimate the lower bound of $\dim_K V_K(\mathbf{f}_*, \beta)$ for both $* = p, \infty$. The **(Type A)**$_*$-estimate is an estimate of $\dim_K V_K(\mathbf{f}_*, \beta)$ which is formulated as follows:

Definition 2.1.1. (**(Type A)**$_*$-estimate with $(W_*, F^{(*)})$)

Let $\xi \in \overline{\mathbb{Q}}$, $\mathbf{f} := (f_1, \ldots, f_m) \in \overline{\mathbb{Q}}[[z-\xi]]^m$. We assume that $\mathbf{f}_* := (f_{1,*}(z), \ldots, f_{m,*}(z)) \in \mathbb{C}_*[[z-\xi]]^m$ converges on $D_*(\xi, r^-)$ for $r > 0$. We say that $\mathbf{f}$ has the **(Type A)**$_*$-estimate with $(W_*, F^{(*)})$ if there exists a subset

$$W_* \subset (D_*(\xi, r^-) \cap \overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}}$$

and a non-trivial function

$$F^{(*)} : W_* \longrightarrow \mathbb{R}_{>0}$$

We say that the function $F^{(*)} : W_* \longrightarrow \mathbb{R}_{>0}$ is non-trivial if there exists at least one element $(K, \beta) \in W_*$ satisfying $F^{(*)}(K, \beta) > 1$.

satisfying

$$\dim_K V_K(\mathbf{f}_*, \beta) \geq F^{(*)}(\beta, K) \text{ for all } (\beta, K) \in W_*.$$

In the following, we only treat in the case of $\xi = \infty$. In this thesis, we give a sufficient condition that $\mathbf{f} := (f_1(z), \ldots, f_m(z)) \in \overline{\mathbb{Q}}[[\frac{1}{z}]]^m$ have the **(Type A)**$_*$-estimate for $* \in \{p, \infty\}$ (cf. Theorem 5.1.4 and Theorem 5.2.8 for $* = \infty$ and $* = p$ respectively) and give some examples of $\mathbf{f} := (f_1(z), \ldots, f_m(z))$ which relates to G-function and satisfy the sufficient condition to satisfy the **(Type A)**$_*$-estimate. We state our main theorems in the next subsection.

2.2 Main Theorems

Our main results give some results on **(Type A)**$_*$-estimate for some formal Laurent series $\mathbf{f}$ which relate G-function by giving the explicit Padé approximation of $\mathbf{f}$. These results are based on that of [49]. Throughout this subsection, we use the following notations:

$$\begin{aligned} &r: \ \text{a natural number},\\ &s_1, \ldots, s_r: \ \text{natural numbers},\\ &a_1, \ldots, a_r: \ \text{rational numbers satisfying } 0 < a_1 < \cdots < a_r \leq 1. \end{aligned}$$

The first result is about the Lerch function.

Definition 2.2.1. We define the Lerch function as follows:

$$\Phi : \{s \in \mathbb{C} |\ \mathrm{Re}(s) > 1\} \times \{x \in \mathbb{R} |\ x > 0\} \times \{z \in \mathbb{C} |\ |z| \leq 1\} \longrightarrow \mathbb{C}, \quad (s, x, z) \mapsto \sum_{n=0}^{\infty} \frac{z^{-n-1}}{(n+x)^s}.$$

We put

$$\begin{aligned} A &:= \mathrm{l.c.m.}_{1 \leq i \leq r}\{\mathrm{den}(a_i)\},\\ M &:= \mathrm{l.c.m.}\{\mathrm{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\ i \neq i'}\\ S &:= \max_{1 \leq i \leq r} s_i,\\ s &:= \sum_{i=1}^{r} s_i. \end{aligned}$$

Then, the first result is as follows:

Main Theorem 1 (Theorem 6.1.2) [49, Theorem 1.1]

Under the notation as above, we denote the set $\{(\beta, K) \in D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} |\ \beta \in K\}$ by W_∞ and define the following four functions:

$$\begin{aligned} &f : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto S \left(\log A + \sum_{\substack{q:\text{prime} \\ q | A}} \frac{\log q}{q-1} \right) + S(M + A) + \log \mathrm{den}(\beta),\\ &g : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1) \log 2),\\ &h : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto s \log |\beta|,\\ &F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f(\tau\beta) + g(\tau\beta))}. \end{aligned}$$

Then, we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K\Phi(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi(v_r, a_r, \beta) \right) \geq F^{(\infty)}(\beta, K),$$

for all $(\beta, K) \in W_\infty$. $\square$

REMARK 2.2.2. In [54, Theorem 0.2], the author gave a criterion of linear independence of special values of the Lerch function over the rational number field. In [48, Theorem 2.1], N. Hirata, M. Ito and Y. Washio gave a criterion of linear independence of special values of the polylogarithm functions over an algebraic number field. Theorem 6.1.2 is a generalization of both [54, Theorem 0.2] and [48, Theorem 2.1].

We define the p-adic Lerch function as follows:

DEFINITION 2.2.3. We define the p-adic Lerch function Φ_p by

$$\Phi_p : \mathbb{N} \times (\mathbb{C}_p \setminus \mathbb{Z}_{\leq 0}) \times D_p \longrightarrow \mathbb{C}_p, \quad (s, x, z) \mapsto \Phi_p(s, x, z) := \sum_{m=0}^{\infty} \frac{z^{-m-1}}{(m+x)^s}.$$

The second result is a p-adic analogue of Main Theorem 1.

Main Theorem 2 (Theorem 7.1.1) [49, Theorem 1.3]

Under the notation as above, we denote the set $\{(\beta, K) \in D_p(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} | \ \beta \in K\}$ by W_p and define the following four functions:

$$f^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto s \left(\log A + \sum_{\substack{q:\text{prime} \\ q|A}} \frac{\log q}{q-1} \right) + S(M+A) + \log \operatorname{den}(\beta),$$

$$g^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1)\log 2),$$

$$h^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } \beta \mapsto s \log |\beta|_p,$$

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \ \text{ by } (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s \log |\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}.$$

Then, we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K\Phi_p(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi_p(v_r, a_r, \beta) \right) \geq F^{(p)}(\beta, K),$$

for all $(\beta, K) \in W_p$. $\square$

The third result is about **(Type A)**$_p$-estimate for certain p-adic functions.

DEFINITION 2.2.4. We define the p-adic function Ξ_p as follows:

$$\Xi_p : \mathbb{N} \times \mathbb{Z}_p \times (\mathbb{C}_p \setminus D_p(1, 1^-) \cup \{1\}) \times D_p \longrightarrow \mathbb{C}_p$$

$$(s, x_1, x_2, z) \mapsto \Xi_p(s, x_1, x_2, z) := \frac{\epsilon(x_2)}{s-1} \frac{1}{z^{s-1}} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!} B_{m+1}(x_1, x_2)(s)_m \frac{1}{z^{s+m}},$$

where

$$\epsilon(x_2) = \begin{cases} 0 & \text{if } x_2 \neq 1 \\ 1 & \text{if } x_2 = 1, \end{cases} \quad (s)_m = \begin{cases} s(s+1)\cdots(s+m-1) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

and $B_k(x_1, x_2)$ are defined by the following generating function:

$$\frac{te^{x_1 t}}{x_2 e^t - 1} = \sum_{k=0}^{\infty} B_k(x_1, x_2) \frac{t^k}{k!}.$$

Note that the function Ξ_p is related to the p-adic Hurwitz zeta function and its special values are related to special values of the p-adic Riemann zeta function. We also mention that the function Ξ_p is obtained by the formal Mellin transfrom (see Definition 10.1.1) of G-function (see Chapter 11).

Put

$$\begin{aligned} s &:= \sum_{i=1}^{r} s_i, \\ B(b) &:= \text{l.c.m.}\{\text{den}(b + a_i)\}_{1 \leq i \leq r} \text{ for } b \in D_p(\mathbb{Q}), \\ M &:= \text{l.c.m.}\{\text{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\ i \neq i'}, \\ S &:= \max_{1 \leq i \leq r} \{s_i\}, \\ T &:= \min_{1 \leq i \leq r} \{s_i\}. \end{aligned}$$

Our third result is as follows:

Main Theorem 3 (Theorem 12.1.2) [49, Theorem 1.5]

Let $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$. We assume that α satisfies $|\alpha - 1|_p \geq 1$. Under the above notations, we denote W_p be the set $D_p(\mathbb{Q}) \times \mathcal{A}_{\mathbb{Q}}$ and define the following four functions:

$$\begin{aligned} f^{(p)} &: D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto S + M(s + r - T - 1) + \sum_{\substack{q:\text{prime} \\ q \mid B(b)}} \frac{\log q}{q-1} + \log \text{den}(\alpha), \\ g^{(p)} &: D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto s \log 2, \\ h^{(p)} &: D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \sum_{\substack{q:\text{prime} \\ q \mid B(b)}} \frac{\log q}{q-1} - \frac{\log p}{p-1} + \log \text{den}(\alpha) - \log \max\{1, |\alpha|_p\}, \\ F^{(p)} &: W_p \longrightarrow \mathbb{R}_{\geq 0} \text{ by } (b, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(b) + T \log |b|_p)}{[K : \mathbb{Q}](f^{(p)}(b) + g^{(p)}(b))}. \end{aligned}$$

Then, we obtain the following estimates:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1+1} K\Xi_p(v_1, a_1, \alpha, b) + \cdots + \sum_{v_r=1}^{s_r+1} K\Xi_p(v_r, a_r, \alpha, b) \right) \geq F^{(p)}(b, K),$$

for all $(b, K) \in W_p$. □

2.3 Outline of this thesis

In Part II, we prepare some tools which will be used in the rest of this thesis. In Chapter 2, we explain the Padé approximation of formal Laurent series. In Chapter 3, we explain the asymptotic expansion of functions. In Chapter 4, we explain a sufficient condition that $\mathbf{f}$ has the **(Type A)**$_*$-estimate for $* \in \{p, \infty\}$.

In Part Ⅲ , we prove Main Theorem 2 and 3. We give an explicit Padé approximation of the Lerch function and prove Main Theorem 2 (resp. Main Theorem 3) in Chapter 5 (resp. Chapter 6).

In Part Ⅳ, we give some examples of formal Laurent series represented by the image of formal Mellin transform of G-functions which satisfy the sufficient condition to satisfy the $\textbf{(Type A)}_p$-estimate. In Chapter 8, we explain the motivation of the study of this part. In Chapter 9, we prepare some p-adic analysis to obtain some integral representations of the special values of Kubota-Leoplodt p-adic L-functions. In Chapter 10, we introduce the formal Mellin transform and give some basic properties of it. In Chapter 11, we give some power series representation of special values of the Kubota-Leopoldt p-adic L-functions at positive integers. In Chapter 12, we prove Main Theorem 3.

Notations

We denote the field of complex numbers by $\mathbb{C} = \mathbb{C}_\infty$ and the absolute value of $\mathbb{C}_\infty$ by $|\cdot| = |\cdot|_\infty$. For a prime number p, we denote the completion of an algebraic closure of $\overline{\mathbb{Q}}_p$ by $\mathbb{C}_p$. We also denote the ring of integers of $\mathbb{C}_p$ by $\mathcal{O}_{\mathbb{C}_p}$. We denote the normalized valuation of $\mathbb{C}_p$ by $|\cdot|_p$ with $|p|_p = p^{-1}$. We denote the order function of $\mathbb{C}_p^*$ by

$$\mathrm{ord} : \mathbb{C}_p^* \longrightarrow \mathbb{Q},\ x \mapsto -\ln_p(|x|_p).$$

We fix an algebraic closure of $\overline{\mathbb{Q}}$ over the rational number field and denote the set of algebraic integer in $\overline{\mathbb{Q}}$ by $\overline{\mathbb{Z}}$. We regard all the algebraic number fields as subfields of $\overline{\mathbb{Q}}$. We fix embeddings

$$\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p.$$

We regard $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$ (resp. $\mathbb{C}_p$) by the fixed embedding ι_∞ (resp. ι_p).

We define the map of the radius of convergence of formal power series as follows:

$$r_p : \mathbb{C}_p[[z]] \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\},\ f(z) = \sum_{n=0}^{\infty} a_n z^n \mapsto \left(\limsup_n |a_n|_p^{\frac{1}{n}}\right)^{-1},$$

$$r_\infty : \mathbb{C}[[z]] \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\},\ f(z) = \sum_{n=0}^{\infty} a_n z^n \mapsto \left(\limsup_n |a_n|^{\frac{1}{n}}\right)^{-1}.$$

For $a \in \mathbb{C}_p$ (resp. $a \in \mathbb{C}$) and $r > 0$, we define subsets $D_p(a, r)$ and $D_p(a, r^-)$ (resp. $D_\infty(a, r^-)$ and $D_\infty(a, r)$) of $\mathbb{C}_p$ (resp. $\mathbb{C}$) as follows:

$$D_p(a, r) = \{x \in \mathbb{C}_p |\ |x - a|_p \leq r\},\ (\text{resp. } D_\infty(a, r) = \{x \in \mathbb{C} |\ |x - a| \leq r\}),$$
$$D_p(a, r^-) = \{x \in \mathbb{C}_p |\ |x - a|_p < r\}\ (\text{resp. } D_\infty(a, r^-) = \{x \in \mathbb{C} |\ |x - a| < r\}).$$

We also define subsets $D_p(\infty, r^-)$ (resp. $D_\infty(\infty, r^-)$) of $\mathbb{P}^1(\mathbb{C}_p)$ (resp. $\mathbb{P}^1(\mathbb{C})$) for $r > 0$ as follows:

$$D_p(\infty, r^-) = \{x \in \mathbb{C}_p |\ |x|_p > r\} \cup \{\infty\},\ (\text{resp. } D_\infty(\infty, r^-) = \{x \in \mathbb{C} |\ |x| > r\} \cup \{\infty\}).$$

For an algebraic extension K of $\mathbb{Q}$, we use the following notations:

$$D_p := D_p(\infty, 1^-), \quad D_\infty := D_\infty(\infty, 1^-),$$
$$D_p(K) := D_p \cap K,\ D_\infty(K) := D_\infty \cap K.$$

We denote the ring of integers of K by $\mathcal{O}_K$ and put

$$I_K^{(\infty)} := \mathrm{Hom}(K, \mathbb{C}),\ I_K^{(p)} := \mathrm{Hom}(K, \mathbb{C}_p).$$

We denote $I_{\mathbb{Q}}^{(p)}$ (resp for short. $I_{\mathbb{Q}}^{(\infty)}$) by $I^{(p)}$ (resp. $I^{(\infty)}$). We denote the maximal ideal of $\mathcal{O}_K$ determined by the fixed embedding ι_p by $\mathfrak{p}_K$ and the closure of K in $\mathbb{C}_p$ by $K_{\mathfrak{p}_K}$. We denote the set of all algebraic number fields containing K by $\mathcal{A}_K$.

We define the denominator function as follows:

$$\mathrm{den} : \overline{\mathbb{Q}} \longrightarrow \mathbb{N},\ \gamma \mapsto \min\{n \in \mathbb{N} |\ n\gamma \text{ is an algebraic integer}\}.$$

For a natural number n, we denote the least common multiple of $1, 2, \cdots, n$ by d_n. For natural numbers b and n, we put

$$\mu_n(b) := \prod_{\substack{q:\mathrm{prime} \\ q|b}} q^{[n/(q-1)]}.$$

For a field K and a formal Laurent series $f(z) \in K[[\frac{1}{z}]]$, when $f(z)$ satisfies $f(z) \in \left(\frac{1}{z^n}\right)$ for $n \in \mathbb{Z}_{\geq 1}$, we denote $f(z) = O(z^{-n})$

Part II

Preliminaries

Chapter 3

Padé approximations

Throughout this chapter, K denotes a field of characteristic 0. In this chapter, we recall the definition and the existence of a Padé approximation of formal Laurent series.

3.1 Padé approximation at $z = \infty$

Proposition 3.1.1. *Let $s \in \mathbb{N}$, $\mathbf{n} := (n_1, \ldots, n_s) \in \mathbb{N}^s$ and $\mathbf{f} := (f_1(z), \cdots, f_s(z)) \in \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right]^s$. We put $N := \sum_{v=0}^{s}(n_v + 1)$. For a natural number M satisfying $M \leq N$, there exist $(s+1)$-polynomials $\mathcal{P} := (P_0(z), P_1(z), \cdots, P_s(z)) \in K[z]^{s+1} \setminus \{\mathbf{0}\}$ satisfying the following conditions:*

1. *The degree of each polynomial $P_v(z)$ satisfies*

$$\deg P_v \leq n_v \text{ for all } 1 \leq v \leq s.$$

2. *We put $f_0(z) := 1$. the formal power series $\sum_{v=0}^{s} P_v(z) f_v(z)$ satisfies*

$$\sum_{v=0}^{s} P_v(z) f_v(z) \in O(z^{-M}).$$

Proof. We denote $f_v(z) = \sum_{k=0}^{\infty} f_{v,k} \frac{1}{z^{k+1}}$. For a polynomial $P_v(z) := \sum_{j=0}^{n_v} p_{v,j} z^j$, we have the following equalities

$$P_v(z) f_v(z) = \sum_{l=1}^{n_v} \left(\sum_{j=0}^{n_v} f_{v,j-l} \right) z^{l-1} + \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n_v} f_{v,k+j} p_{v,j} \right) \frac{1}{z^{k+1}}, \tag{3.1}$$

for $1 \leq v \leq s$. We put

$$P_0(z) := -\sum_{v=1}^{s} \sum_{l=1}^{n_v} \left(\sum_{j=0}^{n_v} f_{v,j-l} \right) z^{l-1}.$$

By (3.1), the element $(P_0(z), P_1(z), \ldots, P_s(z)) \in K[z]^{s+1}$ satisfies the conditions 1 and 2 in Proposition 3.1.1 if and only if the coefficients of $P_v(z) = \sum_{j=0}^{n_v} p_{v,j} z^j$ satisfy the following linear relations

$$\sum_{v=1}^{s} \left(\sum_{j=0}^{n_v} f_{v,k+j} p_{v,j} \right) = 0 \text{ for all } 0 \leq k \leq M-2. \tag{3.2}$$

By the argument of elementary linear algebra, there exists a set of polynomial $\{P_v(z) = \sum_{j=0}^{n_v} p_{v,j} z^j\}_{1\le v\le s} \neq \{0\}$ satisfying the linear relations (3.2) (cf. Remark 3.1.2 2). This completes the proof of Proposition 3.1.1. $\square$

When we have a family of polynomials $\mathcal{P} := (P_0(z), P_1(z), \cdots, P_s(z))$ satisfying the conditions 1 and 2 in Proposition 3.1.1 in the case of M, we call it is a $\mathbf{n}$-th Padé approximation of $\mathbf{f}$ of degree M. Especially, in the case of $M = N$, we call $\mathcal{P}$ a $\mathbf{n}$-th Padé approximation of $\mathbf{f}$.

Remark 3.1.2. Let $s \in \mathbb{N}$, $\mathbf{n} := (n_1, \ldots, n_s) \in \mathbb{N}^s$ and $\mathbf{f} := (f_1(z), \cdots, f_s(z)) \in \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right]^s$. Put $N := \sum_{v=1}^{s}(n_v + 1)$. Let M be a natural number satisfying $M \le N$.

1. For a $\mathbf{n}$-th Padé approximation of $\mathbf{f}$ of degree M, $\mathcal{P} = (P_0(z), P_1(z), \ldots, P_s(z))$, $P_0(z)$ is equal to the polynomial part of $-\sum_{v=1}^{m} P_v(z) f_v(z)$. For this reason, $P_0(z)$ is uniquely determined by $\mathcal{P}$.

2. For an integer $r \in \mathbb{Z}_{\ge 2}$, we define a matrix $A_r(\mathbf{n}, \mathbf{f}) \in M_{N,r-1}(K)$ by

$$A_r(\mathbf{n},\mathbf{f}) := \begin{pmatrix} f_{1,0} & \cdots & f_{1,n_1} & \cdots & f_{m,0} & \cdots & f_{m,n_m} \\ f_{1,1} & \cdots & f_{1,n_1+1} & \cdots & f_{m,1} & \cdots & f_{m,n_m+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ f_{1,r-2} & \cdots & f_{m,r+n_1-2} & \cdots & f_{m,r-2} & \cdots & f_{m,r+n_m-2} \end{pmatrix}.$$

Then by (3.2), we have the following bijection of sets for $M \le N$:

$$\begin{aligned} &\ker(\times A_M(\mathbf{n},\mathbf{f}) : K^N \longrightarrow K^{M-1}) \setminus \{\mathbf{0}\} \longrightarrow \{\mathcal{P} |\ \mathbf{n}\text{-th Padé approximation of } \mathbf{f} \text{ of degree } M\} \\ &\quad {}^t(p_{1,0}, \ldots, p_{1,n_1}, \ldots, p_{s,0}, \ldots, p_{s,n_s}) \mapsto \mathcal{P} := (P_0(z), P_1(z), \ldots, P_s(z)), \end{aligned}$$

where

$$P_v(z) := \sum_{j=0}^{n_v} p_{v,j} z^j \text{ for } 1 \le v \le s,\ P_0(z) = -\sum_{v=1}^{s}\sum_{l=1}^{n_v}\left(\sum_{j=l}^{n_v} f_{v,j-l} p_{v,j}\right) z^{l-1}.$$

The following lemma is one of the most important lemmas which guarantee the existence of "good" Padé approximation of formal Laurent series and formal power series.

Lemma 3.1.3. (*Siegel*) [*74*]

Let $K/\mathbb{Q}$ be an algebraic extension and s, t natural numbers satisfying $s > t$. We assume that there exist t-linear forms with coefficients $\mathcal{O}_K$

$$L_l(X_1, \ldots, X_s) := a_{1,l}X_1 + \cdots + a_{s,l}X_s, \quad 1 \le l \le t \tag{3.3}$$

satisfying the following condition:

There exists a positive integer A satisfying $|\overline{a_{i,j}}| < A$ for all $1 \le i \le s,\ 1 \le j \le t$.

Then, there exist a non-trivial solution of (3.3), $(x_1, \ldots, x_s) \in \mathcal{O}_K^s$, *and a constant $c > 0$ depending only on K satisfying*

$$\max_{1\le i\le s}\{|x_i|\} < c(csA)^{\frac{t}{s-t}}.$$

$\square$

Note that Lemma 3.1.3 is called "Siegel's Lemma" .

Let $n \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{N}$ and $f_0(z) := 1, f_1(z), \cdots, f_s(z) \in \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right]$. For any $0 \leq w \leq s$, we put

$$n_{v,w} = \begin{cases} n & \text{if } 0 \leq v \leq w, \\ n-1 & \text{if } w+1 \leq v \leq s. \end{cases}$$

Using Proposition 3.1.1, there exists a set $\{P^{(n)}_{v,w}(z)\}_{0 \leq v \leq s}$ of Padé approximation of $(1, f_1(z), \cdots f_s(z))$ of degree $(n_{v,w})_{1 \leq v \leq s}$. We denote the determinant of the following $(s+1) \times (s+1)$ matrix by

$$\Delta^{(n)}(z) := \det \begin{pmatrix} P^{(n)}_{0,0}(z) & P^{(n)}_{1,0}(z) & \dots & P^{(n)}_{s,0}(z) \\ P^{(n)}_{0,1}(z) & P^{(n)}_{1,1}(z) & \dots & P^{(n)}_{s,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ P^{(n)}_{0,s}(z) & P^{(n)}_{1,s}(z) & \dots & P^{(n)}_{s,s}(z) \end{pmatrix} \in K[z]. \tag{3.4}$$

We put

$$\begin{aligned} \sigma_{v,w} &:= \sum_{v=1}^{s} ns + w, \\ P^{(n)}_{v,w}(z) &:= \sum_{j=0}^{n} a^{(n)}_{v,w,j} z^j \ \textit{for all } 1 \leq v \leq s \textit{ and } 0 \leq w \leq s, \\ \mathcal{R}^{(n)}_w(z) &:= \sum_{v=0}^{s} P^{(n)}_{v,w}(z) f_v(z) \in \frac{c^{(n)}_w}{z^{\sigma_{n,w}}} + O\left(z^{-\sigma_{n,w}}\right). \end{aligned}$$

Under the above notations, we have the following lemma.

Lemma 3.1.4. *The determinant $\Delta^{(n)}(z)$ satisfies the following equality:*

$$\Delta^{(n)}(z) = c^{(n)}_0 \prod_{v=1}^{s} a^{(n)}_{v,v,n}.$$

Especially, $\Delta^{(n)}(z)$ is an element of K and if all the numbers $c^{(n)}_0, a^{(n)}_{v,v,n}$ for $1 \leq v \leq s$ are not zero, we have $\Delta^{(n)}(z) \in K^*$.

Proof. By adding the v-th column of the matrix (3.4) multiplied by $f_v(z)$ to the first column of the matrix (3.4) for all $1 \leq v \leq s$, we obtain the following equality: Since we have

$$\Delta^{(n)}(z) = \det \begin{pmatrix} \mathcal{R}^{(n)}_0(z) & P^{(n)}_{1,0}(z) & \dots & P^{(n)}_{s,0}(z) \\ \mathcal{R}^{(n)}_1(z) & P^{(n)}_{1,1}(z) & \dots & P^{(n)}_{s,1}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{R}^{(n)}_s(z) & P^{(n)}_{1,s}(z) & \dots & P^{(n)}_{s,s}(z) \end{pmatrix} \tag{3.5}$$

We denote the (t,u)-th cofactor of the matrix in (3.5) by $\Delta^{(n)}_{t,u}(z)$. Then, we obtain

$$\Delta^{(n)}(z) = \sum_{t=0}^{s} \mathcal{R}^{(n)}_t(z) \Delta^{(n)}_{t,1}(z). \tag{3.6}$$

By the definition of $\mathcal{R}_w^{(n)}(z), P_{v,w}^{(n)}(z)$ and the equality (3.6), we obtain the following relations.

$$\begin{aligned}
&\Delta^{(n)}(z) \in \mathcal{R}_0^{(n)}(z)\prod_{v=1}^{s} P_{v,v}^{(n)}(z) + O\left(\frac{1}{z}\right),\\
&\mathcal{R}_0^{(n)}(z)\prod_{v=1}^{s} P_{v,v}^{(n)}(z) \in c_0^{(n)}\prod_{v=1}^{m} a_{v,v,n}^{(n)} + O\left(\frac{1}{z}\right),\\
&\Delta^{(n)}(z) \in K[z].
\end{aligned}$$

By using the above relations, we obtain

$$\Delta^{(n)}(z) = c_0^{(n)}\prod_{v=1}^{s} a_{v,v,n}^{(n)}.$$

This completes the proof of Lemma 3.4 . $\square$

Chapter 4

Asymptotic expansions

In this chapter, we introduce the asymptotic expansion of functions and some related properties.

Definition 4.0.5. Let A be a real number and $\{\phi_k(z) : \mathbb{R}_{>A} \longrightarrow \mathbb{C}\}_{k\in\mathbb{Z}_{\geq 0}}$ a sequence of functions. If $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$ satisfies

$$\phi_{k+1}(z) = o(\phi_k(z)) \ \ (z \to \infty)$$

for all $k \in \mathbb{Z}_{\geq 0}$, then we call $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$ is an asymptotic sequence at $z = \infty$. Let $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$ be an asymptotic sequence at $z = \infty$ and $f : \mathbb{R}_{>A} \longrightarrow \mathbb{C}$ be a function. We define $f(z)$ has an asymptotic expansion at $z = \infty$ with respect to $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$ over K if there exists a sequence $\{a_k\}_{k\in\mathbb{Z}_{\geq 0}} \subset K$ satisfying the following condition for all $N \in \mathbb{Z}_{\geq 0}$,

$$f(z) = \sum_{k=0}^{N} a_k \phi_k(z) + o\left(\phi_N(z)\right) \ \ (z \to \infty). \tag{4.1}$$

If $f(z)$ has an asymptotic expansion at $z = \infty$ with respect to $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$, we denote

$$f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z) \ (z \to \infty). \tag{4.2}$$

Remark 4.0.6. The coefficient a_k ($k \in \mathbb{Z}_{\geq 0}$) in (4.1) is uniquely determined by $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}}$. In fact, recursively, a_k is determined as follows:

$$a_N = \begin{cases} \lim_{z\to\infty} \dfrac{f(z)}{\phi_0(z)} & \text{if } N = 0 \\ \lim_{z\to\infty} \dfrac{1}{\phi_N(z)} \left(f(z) - \displaystyle\sum_{k=0}^{N-1} a_k \phi_k(z) \right) & \text{if } N > 0. \end{cases} \tag{4.3}$$

In the remaining part, we only deal with the case for $\{\phi_k(z)\}_{k\in\mathbb{Z}_{\geq 0}} = \{z^{-k}\}_{k\in\mathbb{Z}_{\geq 0}}$ or $\{\frac{k!}{z(z+1)\cdots(z+k)}\}_{k\in\mathbb{Z}_{\geq 0}}$. For a real number A, we define the sets of functions that are related to the asymptotic expansion with respect to $\{z^{-k}\}_{k\in\mathbb{Z}_{\geq 0}}$ and $\{\frac{k!}{z(z+1)\cdots(z+k)}\}_{k\in\mathbb{Z}_{\geq 0}}$ as follows:

$$M_K^A := \{f : \mathbb{R}_{>A} \to \mathbb{C} | f \text{ has asymptotic expansion with respect to } \{z^{-k}\} \ \text{ over } K \text{ at } z = \infty\},$$

$$\tilde{M}_K^A := \left\{ f : \mathbb{R}_{>A} \to \mathbb{C} | \, f \text{ has asymptotic expansion with respect to } \{\frac{k!}{z\cdots(z+k)}\} \ \text{ over } K \text{ at } z = \infty \right\}.$$

In the case of $K = \mathbb{C}$, we denote $M_{\mathbb{C}}^A$ by M^A.

Regarding Remark 4.0.6, we introduce the following K-homomorphisms:

$$\pi_A : M_K^A \longrightarrow K\left[\left[\frac{1}{z}\right]\right], \qquad f(z) \mapsto \sum_{k=0}^{\infty} a_k(f)\frac{1}{z^k}, \tag{4.4}$$

$$\tilde{\pi}_A : \tilde{M}_K^A \longrightarrow \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right], \qquad f(z) \mapsto \sum_{k=0}^{\infty} b_k(f)\frac{k!}{z(z+1)\cdots(z+k)}, \tag{4.5}$$

where $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$ (resp. $\{b_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$) is the sequence which is determined in (4.3) for $f \in M_K^A$ (resp. $f \in \tilde{M}_K^A$) with respect to $\{z^{-k}\}_{k\in\mathbb{Z}_{\geq 0}}$ $\left(\text{resp. } \{\frac{k!}{z(z+1)\cdots(z+k)}\}_{k\in\mathbb{Z}_{\geq 0}}\right)$. Note that the K-homomorphism π is also a K-algebra homomorphism. Hereafter, we denote $\pi_A(f)$ by $\hat{f}(z)$ for $f \in M_K^A$.

Proposition 4.0.7. *Let A be a real number. Then we have*

$$\tilde{M}_K^A \subset M_K^A \quad \textit{and} \quad \tilde{M}_K^A = \{f \in M_K^A | \hat{f}(z) \in \tfrac{1}{z}K[[\tfrac{1}{z}]]\}.$$

Proof. Put

$$\frac{k!}{z(z+1)\cdots(z+k)} = \sum_{m=0}^{\infty} c_{k,m} z^{-(m+k+1)}$$

Note that $c_{k,0} = 1$ holds for all $k \in \mathbb{Z}_{\geq 0}$. Firstly, we prove $\tilde{M}_K^A \subset M_K^A$. Take $f \in \tilde{M}_K^A$, then there exists a sequence $\{b_k(f)\}_{k\in\mathbb{Z}_{\geq 0}} \subset K$ which satisfies

$$f(z) = \sum_{k=0}^{N} b_k(f)\frac{k!}{z(z+1)\cdots(z+k)} + o\left(\begin{bmatrix} N \\ z \end{bmatrix}\right) \ (z \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0}.$$

By means of $\{b_k(f)\}_{k\in\mathbb{Z}_{\geq 0}} \subset K$, we obtain a sequence $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}} \subset K$ which satisfies the following condition:

$$f(z) = \sum_{k=0}^{N} a_k(f) z^{-k} + o\left(z^{-N}\right) \ (z \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0}, \tag{4.6}$$

where

$$a_N(f) = \begin{cases} 0 & \text{if } N = 0 \\ \sum_{0\leq i,j\leq N-1, i+j=N-1} b_i(f)c_{i,j} & \text{if } N > 0. \end{cases}$$

The equality (4.6) means that $f \in M_K^A$, *i.e.* $\tilde{M}_K^A \subset M_K^A$. For $f \in \tilde{M}_K^A$, since $a_0(f) = 0$, we have $\hat{f}(z) \in \frac{1}{z}K[[\frac{1}{z}]]$, which implies $\tilde{M}_K^A \subset \{f \in M_K^A | \hat{f}(z) \in \frac{1}{z}K[[\frac{1}{z}]]\}$.

Finally, we show $\{f \in M_K^A | \hat{f}(z) \in \frac{1}{z}K[[\frac{1}{z}]]\} \subset \tilde{M}_K^A$. Take $f \in \{f \in M_K^A | \hat{f}(z) \in \frac{1}{z}K[[\frac{1}{z}]]\}$.

There exists a sequence $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}} \subset K$ satisfying the condition

$$f(z) = \sum_{k=0}^{N} a_k(f) z^{-k} + o\left(z^{-N}\right) \ (z \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0}.$$

We inductively define a sequence of K, $(b_k(f))_{k\in\mathbb{Z}_{\geq 0}}$, as follows:

$$b_0(f) = a_1(f), \quad b_{N+1}(f) = a_{N+2}(f) - \sum_{0\leq i,j\leq N+1, i+j=N+1, i\neq N+1} b_i(f)c_{i,j} \text{ for } N \geq 0.$$

Then we see that $(b_k(f))_{k\in\mathbb{Z}_{\geq 0}}$ satisfies:

$$f(z)=\sum_{k=0}^{N} b_k(f)\frac{k!}{z(z+1)\cdots(z+k)}+o\left(\frac{N!}{z(z+1)\cdots(z+N)}\right)\ (z\to\infty) \text{ for all } N\in\mathbb{Z}_{\geq 0}. \tag{4.7}$$

From the equalities (4.7), we have $\{f\in M_K^A|\hat{f}(z)\in\frac{1}{z}K[[\frac{1}{z}]]\}\subset\tilde{M}_K^A$. This completes the proof of Proposition 4.0.7. $\square$

Remark 4.0.8. Let A and A' be real numbers. If we assume $A<A'$, there are natural ring homomorphisms

$$\phi_{A,A'}:M_K^A\longrightarrow M_K^{A'},\ \ \tilde{\phi}_{A,A'}:\tilde{M}_K^A\longrightarrow\tilde{M}_K^{A'}.$$

The sets $\{M_K^A,\phi_{A,A'}\}_{A\in\mathbb{R}}$ and $\{\tilde{M}_K^A,\tilde{\phi}_{A,A'}\}_{A\in\mathbb{R}}$ become direct systems of rings and denote the rings of direct limit of the above direct systems by

$$M_K:=\varinjlim_{A\in\mathbb{R}} M_K^A,\ \ \tilde{M}_K:=\varinjlim_{A\in\mathbb{R}}\tilde{M}_K^A.$$

Note that, since there are equalities (4.3), the coefficients $\{a_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$ (resp. $\{b_k(f)\}_{k\in\mathbb{Z}_{\geq 0}}$) do not depend on the choice of a representative of $[f]\in M_K$ (resp. $[f]\in\tilde{M}_K$). We can also define an K-algebra homomorphisms:

$$\pi:M_K\longrightarrow K\left[\left[\frac{1}{z}\right]\right],\ \ [f]\mapsto\sum_{k=0}^{\infty}a_k(f)\frac{1}{z^k},$$
$$\tilde{\pi}:\tilde{M}_K\longrightarrow\frac{1}{z}K\left[\left[\frac{1}{z}\right]\right],\ \ [f]\mapsto\sum_{k=0}^{\infty}b_k(f)\frac{k!}{z(z+1)\cdots(z+k)}.$$

By the definition of $\tilde{\pi}$ and π, we have the following commutative diagram for all $A\in\mathbb{R}$,

$$\begin{array}{ccccc}
\tilde{M}_K^A & \xrightarrow{\tilde{i}_A} & \tilde{M}_K & \xrightarrow{\tilde{\pi}} & \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right] \\
{\scriptstyle\iota_A}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle i}\downarrow \\
M_K^A & \xrightarrow[i_A]{} & M_K & \xrightarrow[\pi]{} & K\left[\left[\frac{1}{z}\right]\right],
\end{array}$$

where all the morphisms in the above diagram except for π and $\tilde{\pi}$ are canonical homomorphisms. Note that the commutativity of the above diagram is obtained by Proposition 4.0.7.

If $K\subset\overline{\mathbb{Q}}$, there is a natural homomorphism $\iota_p\circ\iota_\infty^{-1}:K[[\frac{1}{z}]]\longrightarrow\iota_p\circ\iota_\infty^{-1}(K)[[\frac{1}{z}]]$. In this situation, for $f\in M_K^A$, we denote $\iota_p\circ\iota_\infty^{-1}(\hat{f}(z))$ by $\hat{f}_p(z)\in\iota_p\circ\iota_\infty^{-1}(K)[[\frac{1}{z}]]$.

Chapter 5

A criterion of linear independence of special values of formal Laurent series

5.1 Complex case

Let $\mathbf{f}_\infty := \{f_{1,\infty}(z), \cdots, f_{s,\infty}(z)\}$ be a finite set of analytic functions defined on D_∞. For an algebraic number field K and an element $\beta \in D_\infty(K)$, we denote the K-vector space spanned by $f_{1,\infty}(\beta), \cdots, f_{s,\infty}(\beta)$ by $V_K(\mathbf{f}_\infty, \beta)$. In this section, we give a sufficient condition for (K, β) to satisfy $\dim_K V_K(\mathbf{f}_\infty, \beta) = s+1$ for $\mathbf{f}_\infty := \{f_{1,\infty}(z), \cdots, f_{s,\infty}(z)\}$ with some assumptions (see Assumption 5.1.2 and Assumption 5.1.3).

5.1.1 Lower bound of the dimension of vector space spanned by complex numbers

In this subsection, we recall a result of Marcovecchio (cf. [63]) on a lower bound of the dimension of the vector space spanned by some complex numbers over an algebraic number field, which is based on a result of Siegel's article [77]. Before stating the results, we prepare some notations. Let K be an algebraic number field. For $\tau \in I_K^{(\infty)}$, we denote the completion of K with respect to$\nu_\tau := |\cdot| \circ \tau$ by K_{ν_τ} and the degree of $K_{\nu_\tau}/\mathbb{Q}_{\nu_\tau}$ by η_τ. We put

$$\delta_K := [K : \mathbb{Q}]/[K_\infty : \mathbb{R}]. \tag{5.1}$$

For a natural number s and a vector $\mathbf{x} = (x_1, \cdots, x_s) \in K^s$, we define

$$h_0(\mathbf{x}) = \frac{1}{[K : \mathbb{Q}]} \sum_{\substack{\tau \in I_K^{(\infty)} \\ \tau \neq id_K, F \circ id_K}} \log |\tau \mathbf{x}|,$$

where F is the complex conjugate and $|\tau \mathbf{x}| = \max_{1 \le v \le s}\{|\tau x_v|\}$. From now on throughout the chapter, we fix a natural number $s \in \mathbb{N}$.

Lemma 5.1.1. [63, Proposition 4.1] Let K be an algebraic number field. Let $\theta_1, \cdots, \theta_s \in \mathbb{C}$. Suppose that, for all $n \in \mathbb{N}$ there exist $(s+1)$-linear forms

$$L_w^{(n)}(X_0, \cdots, X_s) = \sum_{v=0}^{s} A_{v,w}^{(n)} X_v \quad \text{for all } 0 \le w \le s,$$

with coefficients $\{A_{v,w}^{(n)}\}_{1\leq v,w\leq s}\subset \mathcal{O}_K$ satisfying

$$\det((A_{v,w}^{(n)})_{0\leq v,w\leq s})\neq 0,\ \text{for infinitely many} n\in\mathbb{N}.$$

Assume also that there exist positive real numbers ρ, c, c' satisfied $c, c', \rho+c' > 0$ and the following conditions for all $0\leq v\leq s$:

$$\limsup_n \frac{\log ||L_w^{(n)}||}{n}\leq c,$$
$$\limsup_n \frac{h_0(\mathbf{L}_w^{(n)})}{n}\leq c^{'},$$
$$\limsup_n \frac{\log |L_w^{(n)}(\boldsymbol{\theta})|}{n}\leq -\rho.$$

where $||L_w^{(n)}||=\max_{0\leq v\leq s}\{|A_{v,w}|\}$, $L_w^{(n)}(\boldsymbol{\theta}):=L_w^{(n)}(1,\theta_1,\cdots,\theta_s)$ and $\mathbf{L}_w^{(n)}=(A_{0,w}^{(n)},\cdots,A_{s,w}^{(n)})$. Then we have

$$\dim_K(K+K\theta_1+\cdots+K\theta_s)\geq \frac{c+\rho}{c+\delta_K c^{'}}.$$

Using Lemma 5.1.1, we axiomatize the estimation of a lower bound of the dimension of the vector space spanned by the special values of certain functions.

5.1.2 Lower bound of the dimension of the vector space spanned by the special values of formal Laurnet series

Let m be a natural number and

$$f_v(\mathbf{x},z)=f_v(x_1,\cdots,x_m,z):\left(\prod_{1\leq i\leq m} U_i\right)\times D_\infty\longrightarrow\mathbb{C},$$

be $(m+1)$-variable functions for $1\leq v\leq s$ where U_i are some non-empty subsets of $\mathbb{C}$. We fix a set

$$\{(\alpha_{1,v},\cdots,\alpha_{m,v})\}_{1\leq v\leq s}\subset \prod_{1\leq i\leq m}\left(U_i\cap\overline{\mathbb{Q}}\right).$$

We put $f_v(\alpha_{1,v},\cdots,\alpha_{m,v},z)=f_v(\boldsymbol{\alpha}_v,z)$ for $1\leq v\leq s$, $f_0(\boldsymbol{\alpha}_0,z):=1$ and $\mathbb{Q}(\{\alpha_{1,v},\cdots,\alpha_{m,v}\}_{1\leq v\leq s})$ by $\mathbb{Q}(\boldsymbol{\alpha})$. We consider a family of polynomials

$$\{A_{v,w}^{(n)}(z)\}_{0\leq v,w\leq s,n\in\mathbb{N}}\subset\mathbb{Q}(\boldsymbol{\alpha})[z],$$

to approximate $\{f_v(\boldsymbol{\alpha}_v,z)\}_{0\leq v\leq s}$. Let us introduce a family of functions on D_∞, $\{\mathcal{R}_w^{(n)}(z)\}_{n\in\mathbb{N}}$ defined by

$$\mathcal{R}_w^{(n)}(z)=\sum_{v=0}^{s} A_{v,w}^{(n)}(z)f_v(\boldsymbol{\alpha}_v,z),$$

for all $0\leq w\leq s$. We assume the following assumptions on $\{A_{v,w}^{(n)}(z)\}_{0\leq v,w\leq s,n\in\mathbb{N}}$:

Assumption 5.1.2. Suppose that there exists a non-empty subset $V_\infty\subset D(\overline{\mathbb{Q}})$ satisfying the following assumptions:

There exists an integer l such that we have: (5.2)

$\mathcal{R}_w^{(n)}(z) = o(z^{-ns+w+l})$ $(z \to \infty)$ for all $n \in \mathbb{N}$ and $0 \le w \le s$.

Put $\Delta_n(x) = \det((A_{v,w}^{(n)}(z))_{0\le v,w\le s}) \in \mathbb{Q}(\boldsymbol{\alpha})[z]$. We have : (5.3)

$\Delta_n(z)$ satisfies $\Delta_n(\beta) \neq 0$ for all $\beta \in V_\infty$ and infinitely many $n \in \mathbb{N}$,

There exists a family of functions $\{D_n : V_\infty \longrightarrow \overline{\mathbb{Z}} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying (5.4)

$D_n(\beta) \in \mathcal{O}(\boldsymbol{\alpha}, \beta)$ and $D_n(\beta)A_{v,w}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}$ for all $\beta \in V_\infty,\ 0 \le v, w \le s$ and $n \in \mathbb{N}$.

Assumption 5.1.3. We use the notations as above. Suppose that there exist some constants $c_1, c_2, c_3 > 0$ such that the following assumptions hold for all sufficiently large n.

There exists a function $f : V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau D_n(\beta)| \le n^{c_1+o(1)}e^{nf(\tau\beta)}$ (5.5)

for all $\beta \in V_\infty$ and $\tau \in I_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}^{(\infty)}$.

There exists a function $g : V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau A_{v,w}^{(n)}(\beta)| \le n^{c_2+o(1)}e^{ng(\tau\beta)}$ (5.6)

for all $\beta \in V_\infty$ and $\tau \in I_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}^{(\infty)}$.

There exists a family of functions $h : V_\infty \longrightarrow \mathbb{R}_{>0}$ satisfying $|\mathcal{R}_w^{(n)}(\beta)| \le n^{c_3+o(1)}e^{-nh(\beta)}$ (5.7)

for all $\beta \in V_\infty$.

Under Assumptions 5.1.2 and 5.1.3, we obtain the following estimate of the dimension of the vector space spanned by the special values of $\mathbf{f}_\infty = \{f_v(\boldsymbol{\alpha}_v, z)\}_{0\le v\le s}$.

Theorem 5.1.4. *Let m be a natural number and*

$$f_v(\mathbf{x}, z) = f_v(x_1, \cdots, x_m, z) : \left(\prod_{1\le i\le m} U_i\right) \times D_\infty \longrightarrow \mathbb{C},$$

be $(m+1)$-variable functions for $1 \le v \le s$ where U_i are some non-empty subsets of $\mathbb{C}$. We fix a set

$$\{(\alpha_{1,v}, \cdots, \alpha_{m,v})\}_{1\le v\le s} \subset \prod_{1\le i\le m} \left(U_i \cap \overline{\mathbb{Q}}\right).$$

We assume Assumptions 5.1.2 and 5.1.3 for $(f_v(\boldsymbol{\alpha}_v, z))_{0\le v\le s}$. We denote the complex conjugation by F and also assume that the functions f and g in (7) and (8) respectively satisfy the following relations:

$$f(\tau\beta) = f(F \circ \tau\beta), \qquad (5.8)$$
$$g(\tau\beta) = g(F \circ \tau\beta),$$

for all $\beta \in V_\infty$ and $\tau \in I_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}^{(\infty)}$. We denote the subset $\{(\beta, K)|\ \beta \in K\}$ of $V_\infty \times \mathcal{A}_{\mathbb{Q}}$ by W_∞. We define the function

$$F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau\in I_{\mathbb{Q}(\beta)}^{(\infty)}} (f(\tau\beta) + g(\tau\beta))}.$$

Then we obtain the following inequality:

$$\dim_K (K + Kf_1(\boldsymbol{\alpha}_1, \beta) + \cdots + Kf_s(\boldsymbol{\alpha}_s, \beta)) \ge F^{(\infty)}(\beta, K),$$

for all $(\beta, K) \in W_\infty$.

Proof. We fix $(\beta, K) \in W_\infty$ and put

$$\begin{aligned}
A_{v,w}^{(n)} &:= D_n(\beta) A_{v,w}^{(n)}(\beta) \quad \text{for all} \quad 0 \leq v, w \leq s, \\
L_w^{(n)}(X_0, \cdots, X_s) &:= \sum_{v=0}^{s} A_{v,w}^{(n)} X_q \quad \text{for all} \quad 0 \leq w \leq s, \\
\mathbf{L}_v^{(n)} &:= (A_{0,w}^{(n)}, A_{1,w}^{(n)}, \cdots, A_{s,w}^{(n)}) \quad \text{for all} \quad 0 \leq w \leq s, \\
\Delta^{(n)} &:= \det \begin{pmatrix} A_{0,0}^{(n)} & A_{0,1}^{(n)} & \dots & A_{0,s}^{(n)} \\ A_{1,0}^{(n)} & A_{1,1}^{(n)} & \dots & A_{1,s}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s,0}^{(n)} & A_{s,1}^{(n)} & \dots & A_{s,s}^{(n)} \end{pmatrix},
\end{aligned}$$

for each $n \in \mathbb{N}$. Using the hypothesis (5.3), we get

$$\Delta^{(n)} \neq 0. \tag{5.9}$$

Using the hypothesis (5.4), we get

$$\{L_w^{(n)}(X_0, \cdots, X_s)\}_{0 \leq w \leq s} \subset \mathcal{O}_K[X_0, \cdots, X_s]. \tag{5.10}$$

Using the hypothesis (5.5) and (5.6) in Assumption 5.1.3, we obtain:

$$\limsup_n \frac{\log |\tau L_w^{(n)}|}{n} \leq f(\tau\beta) + g(\tau\beta) \quad \text{for all} \quad 1 \leq w \leq s+1, \tag{5.11}$$

for any $\tau \in I_K^{(\infty)}$. Using the inequality (5.11), we also obtain:

$$\limsup_n \frac{h_0(\mathbf{L}_v^{(n)})}{n} \leq \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{\tau \in I_K^{(\infty)} \\ \tau \neq id_K, F \circ id_K}} (f(\tau\beta) + g(\tau\beta)) \quad \text{for all} \quad 1 \leq v \leq s+1. \tag{5.12}$$

Using the hypothesis (5.7), we obtain:

$$\limsup_n \frac{\log |\tau L_w^{(n)}(\boldsymbol{\theta})|}{n} \leq -(h(\beta) - f(\beta)) \quad \text{for all} \quad 0 \leq w \leq s. \tag{5.13}$$

Since (5.9) and (5.10) are satisfied, we can use Lemma 5.1.1 for (5.11), (5.12) and (5.13). Then we have the following inequality:

$$\dim_K (K + Kf_1(\boldsymbol{\alpha}_1, \beta) + \cdots + Kf_s(\boldsymbol{\alpha}_s, \beta)) \geq \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K_\infty : \mathbb{R}](f(\beta) + g(\beta)) + \sum_{\substack{\tau \in I_K^{(\infty)} \\ \tau \neq id_K, F \circ id_K}} (f(\tau\beta) + g(\tau\beta))}.$$

Since we have the relation (5.8), we have the following equality:

$$\frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K_\infty : \mathbb{R}](f(\beta) + g(\beta)) + \sum_{\substack{\tau \in I_K^{(\infty)} \\ \tau \neq id_K, F \circ id_K}} (f(\tau\beta) + g(\tau\beta))} = \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}^{(\infty)}} (f(\tau\beta) + g(\tau\beta))}.$$

This completes the proof of Theorem 5.1.4. $\square$

Using Theorem 5.1.4, we obtain the following criterion of linear independence of special values of $\{f_v(\boldsymbol{\alpha}_v, z)\}_{0 \leq v \leq s}$.

Corollary 5.1.5. *Under the same assumption of Theorem* 5.1.4*, we obtain the following sufficient condition for linear independence of special values of* $\{f_v(\boldsymbol{\alpha}_v, z)\}_{0\le v\le s}$.

Suppose $(\beta, K) \in W_\infty$ *satisfies*

$$s[K:\mathbb{Q}(\beta)] \sum_{\tau\in I^{(\infty)}_{\mathbb{Q}(\beta)}} (f(\tau\beta)+g(\tau\beta)) < [K_\infty:\mathbb{R}](g(\beta)+h(\beta)).$$

Then we obtain:

$$\dim_K (K + Kf_1(\boldsymbol{\alpha}_1,\beta) + \cdots + Kf_s(\boldsymbol{\alpha}_s,\beta)) = s+1.$$

5.2 p-adic case

Let $\mathbf{f}_p := \{f_{1,p}(z), \cdots, f_{s,p}(z)\}$ be a finite set of p-adic analytic functions defined on D_p. For an algebraic number field K and an element $\beta \in D_p(K)$, we denote the K-vector space spanned by $f_{1,p}(\beta), \cdots, f_{s,p}(\beta)$ by $V_K(\mathbf{f}_p, \beta)$. In this section, for $\mathbf{f}_p := \{f_{1,p}(z), \cdots, f_{s,p}(z)\}$ with some assumptions (see Assumption 5.2.4 and Assumption 5.2.5), we give a sufficient condition for (K, β) to satisfy $\dim_K V_K(\mathbf{f}_p, \beta) = s+1$. This is a p-adic analogue of Section 4.1.

5.2.1 Lower bound of the dimension of vector space spanned by p-adic numbers

In this subsection, we recall an estimate of a lower bound of the dimension of the vector space over a number field spanned by p-adic numbers. The method is based on that of Siegel (cf. [77]) and a p-adic analog of Lemma 5.1.1. The following lemma was proved by Pierre Bel in [17].

Lemma 5.2.1. [17, lemma 4.1] Let K be an algebraic number field. Let $\theta_1, \cdots, \theta_s \in \mathbb{C}_p$. Suppose that there exist $(s+1)$-linear forms

$$L^{(n)}_w(X_0, \cdots, X_s) = \sum_{v=0}^{s} A^{(n)}_{v,w} X_v \quad \text{for all } 0 \le w \le s,$$

with coefficients $\{A^{(n)}_{v,w}\}_{0\le v,w\le s} \subset \mathcal{O}_K$ satisfying

$$\det((A^{(n)}_{v,w})_{0\le v,w\le s}) \neq 0 \text{ for infinitely many} n \in \mathbb{N}.$$

Suppose there exist positive real numbers $\{c_\tau\}_{\tau\in I^{(\infty)}_K}$ and ρ satisfying the following conditions:

$$\max_{0\le v,w\le s} |\tau(A^{(n)}_{v,w})| \le e^{n(c_\tau + o(1))} \text{ for each } \tau \in I^{(\infty)}_K \text{ and } n \in \mathbb{N},$$
$$\max_{0\le w\le s} |L^{(n)}_w(\boldsymbol{\theta})|_p \le e^{n(o(1)-\rho)} \text{ for each } n \in \mathbb{N}$$

where $L^{(n)}_w(\boldsymbol{\theta}) = L^{(n)}_w(1, \theta_1, \cdots, \theta_s)$. Then we have

$$\dim_K (K + K\theta_1 + \cdots + K\theta_s) \ge \frac{[K_p:\mathbb{Q}_p]\rho}{\sum_{\tau\in I^{(\infty)}_K} c_\tau}. \tag{5.14}$$

Remark 5.2.2. Under the assumption of Lemma 5.2.1, we also assume that $\{A^{(n)}_{v,w}\}_{0\le v,w\le s} \subset \mathbb{Z}$ and there exists $c > 0$ satisfying

$$\limsup_n \frac{|\tau L^{(n)}_w|}{n} \le c \text{ for all } \tau \in I^{(\infty)}_K \text{ and } 0 \le w \le s.$$

Then for an algebraic number field K satisfying $[K_p : \mathbb{Q}_p] = [K : \mathbb{Q}]$ (i.e. p is completely decomposable in K), the inequality (5.14) becomes

$$\dim_K (K + K\theta_1 + \cdots + K\theta_s) \geq \frac{\rho}{c}. \tag{5.15}$$

We remark that the right hand side of (5.15) does not depend on K satisfying $[K_p : \mathbb{Q}_p] = [K : \mathbb{Q}]$.

5.2.2 Lower bound of the dimension of the vector space spanned by the special values of for formal Laurnet series

Let m be a natural number, A a real number and

$$f_v(\mathbf{x}, z) = f_v(x_1, \cdots, x_m, z) : \left(\prod_{1\leq i\leq m} U_i\right) \times \mathbb{R}_{>A} \longrightarrow \mathbb{C},$$

be $(m+1)$-variable functions for all $1 \leq v \leq s$ where U_i are some non-empty subsets of $\mathbb{C}$. We fix a set

$$\{(\alpha_{1,v}, \cdots, \alpha_{m,v})\}_{1\leq v\leq s} \subset \prod_{1\leq i\leq m} \left(U_i \cap \overline{\mathbb{Q}}\right).$$

We put $f_v(\alpha_{1,v}, \cdots, \alpha_{m,v}, z) = f_v(\boldsymbol{\alpha}_v, z)$ for $1 \leq v \leq s$, $f_0(\boldsymbol{\alpha}_v, z) := 1$ and $\mathbb{Q}(\{\alpha_{1,v}, \cdots, \alpha_{m,v}\}_{1\leq v\leq s})$ by $\mathbb{Q}(\boldsymbol{\alpha})$. We assume

$$\{f_v(\boldsymbol{\alpha}_v, z)\}_{1\leq v\leq s} \subset M^A_{\mathbb{Q}(\boldsymbol{\alpha})}.$$

Firstly, we assume the following important assumption:

Assumption 5.2.3. We assume $\hat{f}_{v,p}(\boldsymbol{\alpha}_v, z) \in \frac{1}{z}\mathbb{Q}(\boldsymbol{\alpha})[[\frac{1}{z}]]$ can be regarded as the functions on D_p for all $1 \leq v \leq s$. Namely, $\hat{f}_{v,p}(\boldsymbol{\alpha}_v, \beta)$ converges for any $\beta \in D_p$ and $1 \leq v \leq s$.

From now on we put $f_0(\boldsymbol{\alpha}_v, z) := 1$ and give a lower bound of the dimension of the vector space spanned by the special values of $\mathbf{f}_p := \{\hat{f}_{v,p}(\boldsymbol{\alpha}_v, z)\}_{0\leq v\leq s}$ over algebraic number fields under some assumptions.

As in the case of Subsection 5.2, we consider a family of polynomials

$$\{A^{(n)}_{v,w}(z)\}_{0\leq v,w\leq s, n\in\mathbb{N}} \subset \mathbb{Q}(\boldsymbol{\alpha})[z],$$

to approximate $\{f_v(\boldsymbol{\alpha}_v, z)\}_{0\leq v\leq s}$. Let us introduce a family of functions of $M^A_{\mathbb{Q}(\boldsymbol{\alpha})}$, $\{\mathcal{R}^{(n)}_w(z)\}_{n\in\mathbb{N}}$ defined by

$$\mathcal{R}^{(n)}_w(z) = \sum_{v=0}^{s} A^{(n)}_{v,w}(z) f_v(\boldsymbol{\alpha}_v, z) \in M^A_{\mathbb{Q}(\boldsymbol{\alpha})},$$

for all $0 \leq w \leq s$. We assume the following assumptions on $\{A^{(n)}_{v,w}(z)\}_{0\leq v,w\leq s, n\in\mathbb{N}}$:

We assume that the following assumptions on $\{A^{(n)}_{v,w}(z)\}_{0\leq v,w\leq s, n\in\mathbb{N}}$:

Assumption 5.2.4.

Suppose there exists a non-empty subset $V_p \subset D_p(\overline{\mathbb{Q}})$ and the following assumptions.

There exists an integer l satisfying the following condition: (5.16)

$\mathcal{R}^{(n)}_w(z) = o(z^{-ns+w+l})$ $(z \to \infty)$ for all $n \in \mathbb{N}$ and $0 \leq w \leq s$.

We denote $\Delta_n(z) = \det((A^{(n)}_{v,w}(z))_{0\leq v,w\leq s}) \in \mathbb{Q}(\boldsymbol{\alpha})[z]$. (5.17)

$\Delta_n(z)$ satisfies $\Delta_n(\beta) \neq 0$ for all $\beta \in V_\infty$ and infinitely many $n \in \mathbb{N}$.

There exists a family of functions $\{D_n : V_p \longrightarrow \overline{\mathbb{Z}} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying (5.18)

$D_n(\beta) \in \mathcal{O}(\alpha, \beta)$ and $D_n(\beta)A^{(n)}_{v,w}(\beta) \in \mathcal{O}_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}$ for all $\beta \in V_p$, $0 \leq v, w \leq s$ and $n \in \mathbb{N}$.

ASSUMPTION 5.2.5. We use the notations as above. Suppose that there exist some constants $c_1, c_2, c_3 > 0$ such that the following assumptions hold for all sufficiently large n.

There exists function $f^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau D_n(\beta)| \le n^{c_1+o(1)} e^{nf(\tau\beta)}$ (5.19)
for all $\beta \in V_p$ and $\tau \in I^{(\infty)}_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}$.

There exists a function $g^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying $|\tau A^{(n)}_{v,w}(\beta)| \le n^{c_2+o(1)} e^{ng(\tau\beta)}$ (5.20)
for all $\beta \in V_p$ and $\tau \in I^{(\infty)}_{\mathbb{Q}(\boldsymbol{\alpha},\beta)}$.

There exists functions $\{E_n : V_p \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying (5.21)
$|E_n(\beta)\hat{\mathcal{R}}^{(n)}_w(\beta)|_p \le n^{c_3+o(1)}|\beta|_p^{-ns}$ for all $\beta \in V_p$.

There exists a function $h^{(p)} : V_p \longrightarrow \mathbb{R}_{>0}$ satisfying (5.22)
$|D_n(\beta)/E_n(\beta)|_p \le n^{c_4+o(1)} e^{-nh^{(p)}(\beta)}$ for all $\beta \in V_p$.

REMARK 5.2.6. We use the notations as above. Without assumption (5.22), we have the following estimation by using Proposition 12.2.1:

$$|D_n(\beta)\hat{\mathcal{R}}^{(n)}_w(\beta)|_p \le n^{c_1+o(1)}|\beta|_p^{-ns},$$

for all $\beta \in V_p$, $0 \le w \le s$ and $n \in \mathbb{N}$. But the assumption (5.22) is important to improve the estimate.

REMARK 5.2.7. We denote $\hat{f}_{v,p}(\boldsymbol{\alpha}_v, z) = \sum_{m=0}^{\infty} c^{(\boldsymbol{\alpha}_v)}_{v,m} \frac{1}{z^{m+1}}$ and assume that there exists $c > 0$ satisfying $|c^{(\boldsymbol{\alpha}_v)}_{v,m}|_p < m^{c+o(1)}$ $(m \to \infty)$ for all $0 \le v \le s$. We also assume that there exists a family of functions $\{E_n : V_p \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\}\}_{n\in\mathbb{N}}$ satisfying

$$E_n(\beta)A^{(n)}_{v,w}(z) \in \mathcal{O}_{\mathbb{C}_p}[z] \text{ for all } \beta \in V_p.$$

Then there exists $c_3 > 0$ satisfying

$$|E_n(\beta)\hat{\mathcal{R}}^{(n)}_w(\beta)|_p \le n^{c_3}|\beta|_p^{-ns} \text{ for all } \beta \in V_p.$$

Under the assumption 5.2.3, 5.2.4 and 5.2.5, we obtain the following **(Type A)$_p$**-estimate of lower bound of the dimension of the vector space spanned by the special values of $\{\hat{f}_{v,p}(\mathbf{x}, z)\}_{1\le v\le s}$.

THEOREM 5.2.8. *Let m be a natural number and*

$$f_v(\mathbf{x}, z) = f_v(x_1, \cdots, x_m, z) : \left(\prod_{1\le i\le m} U_i\right) \times \mathbb{R}_{>A} \longrightarrow \mathbb{C},$$

be $(m+1)$-variables functions for $1 \le v \le s$ where U_i are some non-empty subsets in $\mathbb{C}$. We fix a set

$$\{(\alpha_{1,v}, \cdots, \alpha_{m,v})\}_{1\le v\le s} \subset \prod_{1\le i\le m} \left(U_i \cap \overline{\mathbb{Q}}\right).$$

We put $f_0(\boldsymbol{\alpha}_0, z) := 1$. We assume Assumptions 5.2.3, 5.2.4 and 5.2.5 for $\{f_v(\boldsymbol{\alpha}_v, z)\}_{1\le v\le s}$. We denote the subset $\{(\beta, K)|\ \beta \in K\}$ of $V_p \times \mathcal{A}_{\mathbb{Q}(\boldsymbol{\alpha})}$ by W_p. We define a function

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\ge 0}, \quad (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h(\beta) + s\log|\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau\in I^{(\infty)}_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}.$$

Then we obtain the following inequality:

$$\dim_K \left(K + K\hat{f}_{1,p}(\boldsymbol{\alpha}_1, \beta) + \cdots + K\hat{f}_{s,p}(\boldsymbol{\alpha}_s, \beta)\right) \ge F^{(p)}(\beta, K)$$

for all $(\beta, K) \in W_p$. □

Since we can prove Theorem 5.2.8 by the same argument as Theorem 5.1.4, we skip the proof of it. Theorem 5.2.8 is a p-adic analog of Theorem 5.1.4.

Corollary 5.2.9. *Under the same assumptions of Theorem* 5.2.8*, we obtain the following sufficient condition for linear independence of special values of* $\{\hat{f}_{v,p}(\boldsymbol{\alpha}_v, z)\}_{0\le v\le s}$.

Suppose $(\beta, K) \in W_p$ *satisfies*

$$s[K_p : \mathbb{Q}_p](h(\beta) + s\log|\beta|_p) < [K : \mathbb{Q}(\beta)] \sum_{\tau\in I^{(\infty)}_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta)).$$

Then we obtain:

$$\dim_K \left(K + K\hat{f}_{1,p}(\boldsymbol{\alpha}_1, \beta) + \cdots + K\hat{f}_{s,p}(\boldsymbol{\alpha}_s, \beta)\right) = s + 1.$$

Part III

Linear independence of special values of the Lerch function

In this chapter, we give some examples of **(Type A)**$_*$-estimate for $* \in \{p, \infty\}$. More precisely, by constructing a Padé approximation of the Lerch function, we give estimates of a lower bound of the dimension of the vector space spanned by the special values of the Lerch function in both archimedian and p-adic setting. These results are given in [49] (cf. [54]).

Chapter 6

Archimedian case

6.1 Statement of Theorem 6.1.2

In this section, we give an example of **(Type A)**$_\infty$-estimate of a lower bound of the dimension of the vector space spanned by the special values of the Lerch function in archimedian case. Firstly, we recall the definition of the Lerch function.

Definition 6.1.1. We define the Lerch function as follows:

$$\Phi : \{s \in \mathbb{C} |\ \mathrm{Re}(s) > 1\} \times \{x \in \mathbb{R} |\ x > 0\} \times \{z \in \mathbb{C} |\ |z| \leq 1\} \longrightarrow \mathbb{C}, \quad (s, x, z) \mapsto \sum_{n=0}^{\infty} \frac{z^{-n-1}}{(n+x)^s}.$$

Our result is as follows:

Theorem 6.1.2. ([49]) (*cf.* [54]) *Let r be a natural number, $s_1, \dots, s_r$ natural numbers and $a_1, \dots, a_r$ rational numbers. We assume that $a_1, \dots, a_r$ satisfy $0 < a_1 < \cdots < a_r \leq 1$. We put*

$$\begin{aligned} A &:= \mathrm{l.c.m.}_{1 \leq i \leq r} \{\mathrm{den}(a_i)\}, \\ M &:= \mathrm{l.c.m.} \{\mathrm{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r,\ i \neq i'} \\ S &:= \max_{1 \leq i \leq r} s_i, \\ s &:= \sum_{i=1}^{r} s_i. \end{aligned}$$

We denote the set $\{(\beta, K) \in D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} |\ \beta \in K\}$ by W_∞ and define the following four functions:

$$\begin{aligned} &f : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto S\left(\log A + \sum_{\substack{q:\mathrm{prime} \\ q|A}} \frac{\log q}{q-1}\right) + S(M + A) + \log \mathrm{den}(\beta), \\ &g : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1) \log 2), \\ &h : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto s \log |\beta|, \\ &F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f(\tau\beta) + g(\tau\beta))}. \end{aligned}$$

Then we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K\Phi(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi(v_r, a_r, \beta) \right) \geq F^{(\infty)}(\beta, K),$$

for all $(\beta, K) \in W_\infty$.

6.2 A Padé approximation of the Lerch function

Let r be a natural number. From here to the last section, we fix the following numbers:

$$
\begin{aligned}
&s_1, \dots, s_r : \text{natural numbers},\\
&s := \sum_{i=1}^{r} s_i,\\
&a_1, \dots, a_r : \text{rational numbers satisfying } 0 < a_1 < \cdots < a_r \le 1.
\end{aligned}
$$

In this section, we give a Padé approximation of the Lerch function which is a generalization of that in [54]. For a positive integer n and an r-tuple of non-negative integers $\mathbf{w} = (w_1, \dots, w_r)$ with $0 \le w_i \le s_i$ for all $1 \le i \le r$, we put

$$
\begin{aligned}
Q_{\mathbf{w}}^{(n)}(u) &:= \frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i=1}^{r}\left[(u+a_i)_n^{s_i}(u+n+a_i)^{w_i}\right]},\\
\mathcal{R}_{\mathbf{w}}^{(n)}(z) &:= \sum_{m=0}^{\infty} Q_{\mathbf{w}}^{(n)}(m) z^{-m-1},
\end{aligned}
$$

where $w = \displaystyle\sum_{i=1}^{r} w_i$ and $\sigma_{n,w} = ns+w$. We define a family of rational numbers $\{b_{i,j,v_i,\mathbf{w}}^{(n)}\}_{1 \le i \le r, 1 \le v_i \le s_i, 0 \le j \le n}$ by

$$
Q_{\mathbf{w}}^{(n)}(u) = \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i}\sum_{j=0}^{n} \frac{b_{i,j,v_i,\mathbf{w}}^{(n)}}{(u+a_i+j)^{v_i}}\right), \tag{6.1}
$$

and a family of polynomials $\{A_{i,v_i,\mathbf{w}}^{(n)}(z)\}_{1 \le i \le r, 0 \le v_i \le s_i} \subset \mathbb{Q}[z]$ by

$$
A_{i,v_i,\mathbf{w}}^{(n)}(z) = \sum_{j=0}^{n} b_{i,j,v_i,\mathbf{w}}^{(n)} z^j, \tag{6.2}
$$

$$
P_{\mathbf{w}}^{(n)}(z) = \sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{v_i=1}^{s_i}\sum_{l=0}^{j-1} b_{i,j,v_i,\mathbf{w}}^{(n)} \frac{z^{j-1-l}}{(l+a_i)^{v_i}}. \tag{6.3}
$$

By the definition of $A_{i,v_i,\mathbf{w}}^{(n)}(z)$, we obtain

$$
\deg_z A_{i,v_i,\mathbf{w}}^{(n)} \le \begin{cases} n-1 & \text{for } w_i < v_i,\\ n & \text{for } w_i \ge v_i. \end{cases} \tag{6.4}
$$

Remark 6.2.1. For $1 \le i \le r$ and $\mathbf{w} = (w_1, \cdots, w_r) \in \prod_{i=1}^{r}\{0, 1, \cdots, s_i\}$, we have the following equality:

$$
\deg A_{i,w_i,\mathbf{w}}^{(n)}(z) = n,
$$

for every $n \in \mathbb{N}$. In fact, the coefficient of z^n of the polynomial $A_{i,w_i,\mathbf{w}}^{(n)}(z) \in \mathbb{Q}[z]$ is

$$
\begin{aligned}
b_{i,n,w_i,\mathbf{w}}^{(n)} &= \left.\frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i'=1}^{r}\left[\left(\prod_{j=0}^{n-1}(u+a_{i'}+j)^{s_{i'}}\right)(u+a_{i'}+n)^{w_{i'}}\right]}(u+a_i+n)^{w_i}\right|_{u=-a_i-n}\\
&\neq 0.
\end{aligned}
$$

Using the rational functions $\{A^{(n)}_{i,v_i,\mathbf{w}}(z), P^{(n)}_{\mathbf{w}}(z)\}_{1\le i\le r, 0\le v_i\le s_i} \subset \mathbb{Q}[z]$, we obtain the following Padé approximation of the Lerch function in [54, Proposition 2.1].

Lemma 6.2.2. *Under the same notation as above, we have*

$$\mathcal{R}^{(n)}_{\mathbf{w}}(z) = o(z^{-\sigma_{n,w}+1}) \tag{6.5}$$

and the following Padé approximation:

$$\mathcal{R}^{(n)}_{\mathbf{w}}(z) = \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i} A^{(n)}_{i,v_i,\mathbf{w}}(z)\Phi(v_i,a_i,z^{-1})\right) - P^{(n)}_{\mathbf{w}}(z). \tag{6.6}$$

Proof. From the definition of $\mathcal{R}^{(n)}_{\mathbf{w}}(z)$, we have the relation (6.5). The relation (6.6) follows from the following calculation:

$$\begin{aligned}
\mathcal{R}^{(n)}_{\mathbf{w}}(z) &= \sum_{m=0}^{\infty}\sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n}\frac{b^{(n)}_{i,j,v_i,\mathbf{w}}}{(m+a_i+j)^{v_i}}z^{-m-1}\\
&= \sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n} b^{(n)}_{i,j,v_i,\mathbf{w}}\sum_{m=0}^{\infty}\frac{z^{-m-1}}{(m+a_i+j)^{v_i}}\\
&= \sum_{i=1}^{r}\sum_{v_i=0}^{s_i}\sum_{j=0}^{n} b^{(n)}_{i,j,v_i,\mathbf{w}}\left(z^j\Phi(v_i,a_i,z) - \sum_{l=0}^{j-1}\frac{z^{j-l-1}}{(m+a_i)^{v_i}}\right)\\
&= \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s_i} A^{(n)}_{i,v_i,\mathbf{w}}(z)\Phi(v_i,a_i,z)\right) - P^{(n)}_{\mathbf{w}}(z).
\end{aligned}$$

This gives the proof of Lemma 6.2.2. □

6.3 Some estimations

Lemma 6.3.1. *Let β be a non-zero complex number. There exists $c>0$ such that the inequality*

$$\max_{1\le i\le r, 1\le v_i\le s_i}\{|A^{(n)}_{i,v_i,\mathbf{w}}(\beta)|, |P^{(n)}_{\mathbf{w}}(\beta)|\} \le n^c \max\{1,|\beta|^n\}\exp\{n(s\log s + (2s+1)\log 2)\} \tag{6.7}$$

holds for sufficiently large $n\in\mathbb{N}$.

Proof. We fix an enough large natural number k satisfying the following conditions:

$$|a_{i_1}-a_{i_2}| > \frac{2}{k} \text{ and } 1 > |a_{i_1}-a_{i_2}| + \frac{2}{k} \text{ for all } 1\le i_1, i_2\le r, i_1\ne i_2. \tag{6.8}$$

Firstly, we give an upper bound of $\{b^{(n)}_{i,j,v_i,\mathbf{w}}\}_{1\le j\le n, 1\le v_i\le s_i}$ for fixed i. Using the definition of $b^{(n)}_{i,j,v_i,\mathbf{w}}$ given by (6.1), we get

$$b^{(n)}_{i,j,v_i,\mathbf{w}} = \frac{1}{2\pi\sqrt{-1}}\int_{|u+j+a_i|=\frac{1}{k}} Q^{(n)}_{\mathbf{w}}(u)(u+a_i+j)^{v_i-1}du. \tag{6.9}$$

From the equality (6.9) and the definition of $Q^{(n)}_{\mathbf{w}}(u)$, we obtain

$$\begin{aligned}
|b^{(n)}_{i,j,v_i,\mathbf{w}}| &\le k^{-v_i}\sup\nolimits_{|u+a_i+j|=\frac{1}{k}}|Q^{(n)}_{\mathbf{w}}(u)| \qquad (6.10)\\
&\le k^{-v_i}\sup\nolimits_{|u+j+a_i|=\frac{1}{k}}\left|\frac{(u+\sigma_{n,w}-2)_{\sigma_{n,w}-1}}{\prod_{i'=1}^{r}(u+a_{i'})_n^{s_{i'}}(u+n+a_{i'})^{w_{i'}}}\right|.
\end{aligned}$$

We give an upper bound of $\left|\dfrac{(u+\sigma_{n,w}-2)_{\sigma_{n,w}-1}}{\prod_{i'=1}^{r}(u+a_{i'})_n^{s_{i'}}(u+n+a_{i'})^{w_{i'}}}\right|$. We have the following inequalities for $u \in \{u \in \mathbb{C} |\ |u+j+a_i| = \frac{1}{k}\}$:

$$\begin{aligned}|(u-\sigma_{n,w}+2)_{\sigma_{n,w}-1}| &= |(u+j+a_i-j-a_i)\cdots(u+j+a_i-\sigma_{n,w}-j-a_i+2)| \\ &\leq (j+1)\cdots(\sigma_{n,w}+j+3).\end{aligned} \tag{6.11}$$

Estimating a lower bound of $|(u+a_{i'})_n|$ and $|u+n+a_{i'}|$ for $u \in \{u \in \mathbb{C} |\ |u+j+a_i| = \frac{1}{k}\}$, we give a lower bound of $|u+j+a_i+(a_{i'}-a_i)+(l-j)|$ for $1 \leq i' \leq r$, $0 \leq l \leq n$:

(In the case of $i' = i$)

$$|u+j+a_i+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, j+1, \\ j-l-1 & \text{if } j-1 > l, \\ l-j-1 & \text{if } l > j+1. \end{cases} \tag{6.12}$$

(In the case of $i' > i$)

$$|u+j+a_i+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, \\ j-l-1 & \text{if } j-1 > l, \\ l-j & \text{if } l > j. \end{cases} \tag{6.13}$$

(In the case of $i > i'$)

$$|u+j+a_i+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j, j+1, \\ j-l & \text{if } j > l, \\ l-j-1 & \text{if } l > j+1. \end{cases} \tag{6.14}$$

From the inequalities (6.12), (6.13) and (6.14), we have the following estimation for all $1 \leq i' \leq r$:

$$\begin{aligned}|(u+a_{i'})_n| &= \prod_{l=0}^{n-1}|u+l+a_{i'}| \\ &= \prod_{l=0}^{n-1}|u+j+a_i+(a_{i'}-a_i)+(l-j)| \\ &\geq \frac{(n-j)!j!}{k^3n^3}.\end{aligned} \tag{6.15}$$

We also have the inequality:

$$|u+n+a_{i'}| = |u+j+a_i+(a_{i'}-a_i)+(n-j)| \geq \frac{1}{k}. \tag{6.16}$$

From the inequalities $(6.10), (6.11), (6.15)$ and (6.16), we obtain

$$\begin{aligned}&k^{-v_i}\sup_{|u+j+a_i|=\frac{1}{k}}\left|\frac{(u-\sigma_{n,w}+2)_{\sigma_{n,w}-1}}{\prod_{i'=1}^{r}(u+a_{i'})_n{}^{s_{i'}}(u+n+a_{i'})^{w_{i'}}}\right| \\ &\leq n^{c_1}\frac{(j+1)\cdots(\sigma_{n,w}+j+3)}{((n-j)!j!)^s} \\ &= n^{c_1}\frac{(j+\sigma_{n,w}+3)!}{j!(n!)^s}\binom{n}{j}^s,\end{aligned} \tag{6.17}$$

where c_1 is a positive constant. By using the inequality $\binom{n}{j} \leq 2^n$ for the inequality (6.17), we get

$$\begin{aligned}\frac{(j+\sigma_{n,w}+3)!}{j!(n!)^s}\binom{n}{j}^s &= \frac{(\sigma_{n,w}+3)!}{(n!)^s}\binom{j+\sigma_{n,w}+3}{j}\binom{n}{j}^s \\ &\leq n^{c_2}\frac{(\sigma_{n,w}+3)!}{(n!)^s}2^{2ns+n},\end{aligned} \tag{6.18}$$

for some positive constant c_2. By using the Stirling formula for the inequality (6.18), we obtain

$$\begin{aligned}\frac{(\sigma_{n,w}+3)!}{(n!)^s}2^{2ns+n} &\leq n^{c_3}\frac{(\sigma_{n,w}+3)^{(\sigma_{n,w}+3)}e^{-(\sigma_{n,w}+3)}}{n^{ns}e^{-ns}}2^{2ns+n} \\ &\leq n^{c_4}\exp\{n(s\log s+(2s+1)\log 2)\},\end{aligned} \tag{6.19}$$

where c_3, c_4 are some positive constants. From the inequality (6.19), we conclude that

$$|b^{(n)}_{i,j,v_i,\mathbf{w}}| \leq n^{c_5}\exp\{n(s\log s+(2s+1)\log 2)\}, \tag{6.20}$$

for some positive constant c_5. From the definition (6.2), (6.3) and inequality (6.20), we obtain the desired estimation:

$$\max_{1\leq i\leq r, 1\leq v_i\leq s_i}\{|A^{(n)}_{i,v_i,\mathbf{w}}(\beta)|, |P^{(n)}_{\mathbf{w}}(\beta)|\} \leq \max\{1,|\beta|^n\}n^c\exp\{n(s\log s+(2s+1)\log 2)\},$$

for some constant $c>0$. This completes the proof of Lemma 6.3.1. □

Lemma 6.3.2. *Let β be an element of D_∞. There exists a positive real number C satisfying*

$$|\mathcal{R}^{(n)}_{\mathbf{w}}(\beta)| \leq C|\beta|^{-ns}, \tag{6.21}$$

for all $n \in \mathbb{N}$.

Proof. Let m be a positive integer. Since there is a trivial inequality $|Q^{(n)}_{\mathbf{w}}(m)| \leq 1$ for $m \geq \sigma_{n,w}-1$, we have the following estimation:

$$\begin{aligned}|\mathcal{R}^{(n)}_{\mathbf{w}}(\beta)| &\leq \sum_{m=\sigma_{n,w}-1}^{\infty}|Q^{(n)}_{\mathbf{w}}(m)||\beta|^{-m-1} \\ &\leq |\beta|^{-\sigma_{n,w}}\sum_{m=0}^{\infty}|\beta|^{-m}.\end{aligned}$$

Since $|\beta|>1$, the sum $\sum_{m=0}^{\infty}|\beta|^{-m}$ converges, we obtain the desired estimate. □

For a non-zero algebraic number β, we construct an integer $D_n(\beta)=D_n$ which satisfies $D_nA^{(n)}_{i,v_i,\mathbf{w}}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$ and $D_nP^{(n)}_{\mathbf{w}}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$. Before stating next lemma, we prepare some notations:

$$\begin{aligned}&A := \text{l.c.m.}_{1\leq i\leq r}\{\text{den}(a_i)\}, \\ &b_i := a_i\text{den}(a_i) \text{ for } 1\leq i\leq r, \\ &b := \max_{1\leq i\leq r}\{b_i\}, \\ &M := \text{l.c.m.}\{\text{den}(a_{i'}-a_i)\}_{1\leq i,i'\leq r,\ i\neq i'}, \\ &e_{i',i} := M(a_{i'}-a_i) \text{ for all } 1\leq i,i'\leq r, \\ &e := \max_{1\leq i,i'\leq r}\{e_{i',i}\}, \\ &S := \max_{1\leq i\leq r}\{s_i\}.\end{aligned}$$

We give the following lemma.

Lemma 6.3.3. *We use the same notation as above. Let β be a non-zero algebraic number. Then we have the following relations:*

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i-v_i} \mathrm{den}(\beta)^n A_{i,v_i,\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)},$$
$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{S} \mathrm{den}(\beta)^n d_{b+(n-1)A}^{S} P_{\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)},$$

for all $1 \le i \le r, 1 \le v_i \le s_i$.

Proof. We construct an integer which divide the denominator of $b_{i,j,v_i,\mathbf{w}}^{(n)}$ for $1 \le i \le r, 1 \le v_i \le s_i$. According to the equation (6.1), we get

$$b_{i,j,v_i,\mathbf{w}}^{(n)} = \begin{cases} \frac{1}{(s_i-v_i)!}\left(\frac{d}{du}\right)^{s_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j} & \text{for } \ 0 \le j \le n-1, 1 \le v_i \le s_i, \\ \frac{1}{(w_i-v_i)!}\left(\frac{d}{du}\right)^{w_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+n)^{w_i}|_{u=-a_i-n} & \text{for } \ j = n, 1 \le v_i \le w_i, \\ 0 & \text{for } \ j=n, v_i > w_i. \end{cases} \tag{6.22}$$

First we calculate $\frac{1}{(s_i-v_i)!}\left(\frac{d}{du}\right)^{s_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j}$ for $0 \le j \le n-1, 1 \le v_i \le s_i$. We put

$$R_{i,\mathbf{w}}^{(n)}(u) = (u-\sigma_{n,w}+2)(u-\sigma_{n,w}+3)\cdots(u-\sigma_{n,w}+s_i),$$
$$Q_{i,c,n,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-1))}{(u+a_i)(u+a_i+1)\cdots(u+a_i+j-1)(u+a_i+j+1)\cdots(u+a_i+n)} \text{ for } c \ge 0,$$
$$Q_{i,c,n-1,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-2))}{(u+a_i)(u+a_i+1)\cdots(u+a_i+j-1)(u+a_i+j+1)\cdots(u+a_i+n-1)} \text{ for } c \ge 0,$$
$$S_{i',c,n,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-n))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n)} \text{ for } c \ge 0 \text{ and } i' \ne i,$$
$$S_{i',c,n-1,\mathbf{w}}(u) = \frac{(u-c)(u-c-1)\cdots(u-c-(n-2))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n-1)} \text{ for } c \ge 0 \text{ and } i' \ne i.$$

From the equality

$$Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i} = \frac{u(u-1)\cdots(u-\sigma_{n,w}+2)}{\prod_{i'=1,i'\ne i}^{r}\left(\prod_{j=0}^{n-1}(u+a_{i'}+j)^{s_{i'}}(u+a_{i'}+n)^{w_{i'}}\right)\times\left(\prod_{j'=0,j'\ne j}^{n-1}(u+a_i+j')^{s_i}(u+a_i+n)^{w_i}\right)},$$

we can express $Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}$ for $1 \le i \le r,\ 0 \le j \le n$ as follows:

$$Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i} := R_{i,\mathbf{w}}^{(n)}(u)\prod_{m_i=1}^{s_i} Q_{i,c_{m_i},\mathbf{w}}^{(n)}(u) \times \prod_{i'=1,i'\ne i}^{r}\left(\prod_{m_{i'}=1}^{s_{i'}} S_{i',c_{m_{i'}},\mathbf{w}}^{(n)}(u)\right),$$

where $Q_{i,c_{m_i},\mathbf{w}}^{(n)}(u)$ stands for either $Q_{i,c,n,\mathbf{w}}(u)$ or $Q_{i,c,n-1,\mathbf{w}}(u)$ and $S_{i',c_m,\mathbf{w}}^{(n)}(u)$ stands for either $S_{i',c,n,\mathbf{w}}(u)$ or $S_{i',c,n-1,\mathbf{w}}(u)$. Hence, we get

$$\left(\frac{d}{du}\right)^{s_i-v_i} Q_{\mathbf{w}}^{(n)}(u)(u+a_i+j)^{s_i}|_{u=-a_i-j}$$
$$= \sum_{l_0+l_1+\cdots+l_s=s_i-v_i}\frac{(s_i-v_i)!}{l_0!\cdots l_s!}\times\left(\frac{d}{du}\right)^{l_0} R_{i,\mathbf{w}}{}^{(n)}(u)\times\prod_{m_i=1}^{s_i}\left(\frac{d}{du}\right)^{l_{m_i}} Q_{i,c_{m_i},\mathbf{w}}^{(n)}(u)$$
$$\times\prod_{i'=1,i'\ne i}^{r}\left(\prod_{m_{i'}=1}^{s_{i'}}\left(\frac{d}{du}\right)^{l_{m_{i'}}} S_{i',c_{m_{i'}},\mathbf{w}}^{(n)}(u)\right)|_{u=-a_i-j}.$$

The same argument of the proof of [54, Lemma 3.3] p. 184, we have

$$\mu_n(\mathrm{den}(a_i))\mathrm{den}(a_i)^n d_n^l \left(\frac{d}{du}\right)^l Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)\Big|_{u=-a_i-j} \in \mathbb{Z} \text{ for } 0 \le l \le s_i - v_i, \tag{6.23}$$

where $Q^{(n)}_{i,c_{m_i},\mathbf{w}}(u)$ stands for either $Q_{i,c,n,\mathbf{w}}(u)$ or $Q_{i,c,n-1,\mathbf{w}}(u)$. We can express

$$\begin{aligned} S_{i',c,n,\mathbf{w}}(u) &= \frac{(u-c)(u-c-1)\cdots(u-c-n))}{(u+a_{i'})(u+a_{i'}+1)\cdots(u+a_{i'}+n)} \\ &= 1 + \frac{B_{i',0,\mathbf{w}}}{(u+a_{i'})} + \frac{B_{i',1,\mathbf{w}}}{(u+a_{i'}+1)} + \cdots + \frac{B_{i',n,\mathbf{w}}}{(u+a_{i'}+n)}, \end{aligned}$$

where

$$B_{i',l,\mathbf{w}} = (-1)^{n+l+1}\frac{(a_{i'}+l+c)\cdots(a_{i'}+l+c+n)}{l!(n-l)!}.$$

Substituting $b_{i'}/\mathrm{den}(a_{i'})$ for $a_{i'}$, we get

$$B_{i',l,\mathbf{w}} = \frac{(-1)^{n+l+1}\mathrm{den}(a_{i'})^{-n-1}\prod_{k=0}^n(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{l!(n-l)!}$$

Since

$$\frac{\prod_{k=0}^n(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{l!(n-l)!} = \frac{\prod_{k=0}^n(b_{i'}+\mathrm{den}(a_{i'})(c+l+k))}{(n+1)!}\frac{n!(n+1)}{l!(n-l)!},$$

we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))B_{i',l,\mathbf{w}} \in \mathbb{Z} \qquad \text{for } n \in \mathbb{N}. \tag{6.24}$$

Using the relation (6.24), and the equation

$$\begin{aligned} \left(\frac{d}{du}\right)^l S_{i',c,n,\mathbf{w}}(u)\Big|_{u=-a_i-j} &= \frac{d}{du}^l 1 + (-1)^l l!\Bigg[\frac{B_{i',0,\mathbf{w}}}{(a_{i'}-a_i-j)^{l+1}} \\ &\quad + \frac{B_{i',1,\mathbf{w}}}{(a_{i'}-a_i+1-j)^{l+1}} + \cdots + \frac{B_{i',n,\mathbf{w}}}{(a_{i'}-a_i+n-j)^{l+1}}\Bigg], \end{aligned} \tag{6.25}$$

for $0 \le l \le s_i - v_i$. and the definition of $e_{i',i}$, we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))d^{l+1}_{e_{i',i}+Mn}\left(\frac{d}{du}\right)^l S_{i',c,n,\mathbf{w}}(u)\Big|_{u=-a_i-j} \in \mathbb{Z}, \tag{6.26}$$

for $0 \le l \le s_i - v_i$. Similarly, we obtain

$$\mathrm{den}(a_{i'})^{n+1}\mu_n(\mathrm{den}(a_{i'}))d^{l+1}_{e_{i',i}+Mn}\left(\frac{d}{du}\right)^l S_{i',c,n-1,\mathbf{w}}(u)\Big|_{u=-a_i-j} \in \mathbb{Z},$$

for $0 \le l \le s_i - v_i$. Thus we obtain

$$S!\mu_n(A)^s A^{s(n+1)} d^{s_i-v_i}_{e+Mn} b^{(n)}_{i,j,v_i,\mathbf{w}} \in \mathbb{Z}. \tag{6.27}$$

We conclude that $S!\mu_n(A)^s A^{s(n+1)} d^{s_i-v_i}_{e+Mn}\mathrm{den}(\beta)^n A^{(n)}_{i,v_i,\mathbf{w}}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}$. By the definition of $P^{(n)}_{\mathbf{w}}(z)$ (cf. (6.3)), we get the following equalities

$$\begin{aligned} P^{(n)}_{\mathbf{w}}(\beta) &= \sum_{i=1}^r\sum_{j=0}^n\sum_{v_i=1}^{s_i} b^{(n)}_{i,j,v_i,\mathbf{w}}\left(\frac{\beta^{j-1}}{a_i^{v_i}} + \cdots + \frac{1}{(j-1+a_i)^{v_i}}\right) \\ &= \sum_{i=1}^r\sum_{j=0}^n\sum_{v_i=1}^{s_i} b^{(n)}_{i,j,v_i,\mathbf{w}}\left(\frac{\mathrm{den}(a_i)^{v_i}\beta^{j-1}}{b_i^{v_i}} + \cdots + \frac{\mathrm{den}(a_i)^{v_i}}{(\mathrm{den}(a_i)(j-1)+b_i)^{v_i}}\right). \end{aligned}$$

Since we have $S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{s_i - v_i} b_{i,j,v_i,\mathbf{w}}^{(n)} \in \mathbb{Z}$, then we conclude that

$$S!\mu_n(A)^s A^{s(n+1)} d_{e+Mn}^{S} \mathrm{den}(\beta)^n d_{b+(n-1)A}^{S} P_{\mathbf{w}}^{(n)}(\beta) \in \mathcal{O}_{\mathbb{Q}(\beta)}.$$

This completes the proof of Lemma 6.3.3. □

We denote $(0,\dots,0) \in \prod_{i=1}^{r}\{0,\dots,s_i\}$ by $\mathbf{0}$. Hereafter, we fix a subset

$$\{\mathbf{w}_{i,j} := (w_{i,j}^{(1)},\dots,w_{i,j}^{(r)})\}_{1\le i\le r, 1\le j\le s_i} \subset \prod_{i=1}^{r}\{0,\dots,s_i\}$$

satisfying

$$w_{i,j}^{(k)} = \begin{cases} 0 & \text{if } k \neq i, \\ j & \text{if } k = i. \end{cases}$$

We put $\Delta^{(n)}(z)$ by the determinant of the following $(s+1)\times(s+1)$ matrix

$$\begin{pmatrix}
-P_{\mathbf{0}}^{(n)}(z) & A_{1,1,\mathbf{0}}^{(n)}(z) & \dots & A_{1,s_1,\mathbf{0}}^{(n)}(z) & \cdots & A_{r,1,\mathbf{0}}^{(n)}(z) & \dots & A_{r,s_r,\mathbf{0}}^{(n)}(z) \\
-P_{\mathbf{w}_{1,1}}^{(n)}(z) & A_{1,1,\mathbf{w}_{1,1}}^{(n)}(z) & \dots & A_{1,s_1,\mathbf{w}_{1,1}}^{(n)}(z) & \cdots & A_{r,1,\mathbf{w}_{1,1}}^{(n)}(z) & \dots & A_{r,s_r,\mathbf{w}_{1,1}}^{(n)}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P_{\mathbf{w}_{1,s_1}}^{(n)}(z) & A_{1,1,\mathbf{w}_{1,s_1}}^{(n)}(z) & \dots & A_{1,s_1,\mathbf{w}_{1,s_1}}^{(n)}(z) & \cdots & A_{r,1,\mathbf{w}_{1,s_1}}^{(n)}(z) & \dots & A_{r,s_r,\mathbf{w}_{1,s_1}}^{(n)}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P_{\mathbf{w}_{r,1}}^{(n)}(z) & A_{1,1,\mathbf{w}_{r,1}}^{(n)}(z) & \dots & A_{1,s_1,\mathbf{w}_{r,1}}^{(n)}(z) & \cdots & A_{r,1,\mathbf{w}_{r,1}}^{(n)}(z) & \dots & A_{r,s_r,\mathbf{w}_{r,1}}^{(n)}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-P_{\mathbf{w}_{r,s_r}}^{(n)}(z) & A_{1,1,\mathbf{w}_{r,s_r}}^{(n)}(z) & \dots & A_{1,s_1,\mathbf{w}_{r,s_r}}^{(n)}(z) & \cdots & A_{r,1,\mathbf{w}_{r,s_r}}^{(n)}(z) & \dots & A_{r,s_r,\mathbf{w}_{r,s_r}}^{(n)}(z)
\end{pmatrix} \tag{6.28}$$

for every $n \in \mathbb{N}$. Then we have the following lemma.

Lemma 6.3.4. *Let $\Delta^{(n)}(z)$ be as above. Then $\Delta^{(n)}(z)$ are non-zero rational numbers for all $n \in \mathbb{N}$.*

Proof. For $\mathbf{w} \in \prod_{i=1}^{r}\{0,\dots,s_i\}$, we define $\mathcal{R}_{\mathbf{w}}^{(n)}(z)$, $c_{0,\mathbf{w}}^{(n)}$ and $b^{(n)}$ by

$$\mathcal{R}_{\mathbf{w}}^{(n)}(z) := \sum_{i=1}^{r}\left(\sum_{v_i=1}^{s} A_{i,v_i,\mathbf{w}}^{(n)}(z)\Phi(v_i,a_i,z)\right) - P_{\mathbf{w}}^{(n)}(z) \in \frac{c_{0,\mathbf{w}}^{(n)}}{z^{\sigma_{n,w}}} + O(z^{-\sigma_{n,w}}),$$

and define

$$b^{(n)} := \prod_{i=1}^{r}\left(\prod_{j=1}^{s_i} b_{i,n,j,\mathbf{w}_{i,j}}^{(n)}\right).$$

Then, from Lemma 3.1.4, we get

$$\Delta^{(n)}(z) = b^{(n)} c_{0,\mathbf{0}}^{(n)} \in \mathbb{Q}.$$

Using Remark 6.2.1 and by the definition of $c_{0,\mathbf{0}}^{(0)}$, we obtain $b^{(n)} \neq 0$ and $c_{0,\mathbf{0}}^{(n)} \neq 0$. This completes the proof of Lemma 6.3.4. □

6.4 Proof of Theorem 6.1.2

We use the same notations as the previous sections. Fix a set of rational numbers $\{a_1,\dots,a_r\} \subset \mathbb{Q}$ satisfying $0 < a_1 < \cdots < a_r \le 1$. We use Theorem 5.1.4 for

$$f_{(i,v_i)}(x) = \Phi(v_i,x,z) : \{x \in \mathbb{R} \mid x > 0\} \times D_\infty \longrightarrow \mathbb{C}, \quad 1 \le i \le r,\ 1 \le v_i \le s_i,$$

and $\alpha_{(i,v_i)} = a_i$ for $1 \le i \le r$.

Proof of Theorem 6.1.2. We denote the set $\{(\beta, K) \in D_\infty(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} |\ \beta \in K\}$ by W_∞ and fix an element $(\beta, K) \in W_\infty$. We take the polynomials $\{A^{(n)}_{i,v_i,\mathbf{w}}(z)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{P^{(n)}_{\mathbf{w}}(z)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ defined in (6.2) and (6.3). As a result of the equality (6.6), we obtain

$$\mathcal{R}^{(n)}_{\mathbf{w}}(z) = \sum_{i=1}^{r} \sum_{v_i=1}^{s} A^{(n)}_{i,v_i,\mathbf{w}}(z)\Phi(v_i, a_i, z) - P^{(n)}_{\mathbf{w}}(z) = o(z^{-\sigma_{n,w}+1}).$$

The above relation shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(z), -P^{(n)}_{\mathbf{w}}(z)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfy the assumption (5.2) in Assumption 5.1.2. We define the following functions:

$$\begin{aligned}
&D_n : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{e+Mn} \mathrm{den}(\beta)^n d^S_{b+(n-1)A},\\
&f : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\left(\log A + \sum_{\substack{q:\mathrm{prime}\\ q|A}} \frac{\log q}{q-1}\right) + S(M+A) + \log \mathrm{den}(\beta),\\
&g : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto \log\max\{1, |\beta|\} + (s\log s + (2s+1)\log 2),\\
&h : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log|\beta|.
\end{aligned}$$

We note that the functions f and g satisfy the relation (5.8). Using Lemma 6.3.4, we have $\Delta^{(n)}(z) \in \overline{\mathbb{Q}}^*$. This shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(z), P^{(n)}_{\mathbf{w}}(z)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfies the assumption (5.3). Using Lemma 6.3.3, the family of functions $\{D_n : D_\infty(\overline{\mathbb{Q}}) \longrightarrow \mathbb{N}\}_{n\in\mathbb{N}}$ satisfy the assumption (5.4). Using Lemma 6.3.3, the function f satisfies the assumption (5.5). Using Lemma 6.3.1 (6.7), the function g satisfies the assumption (5.6). Using Lemma 6.3.2, the function h satisfies the assumption (5.7). We define the following function:

$$F^{(\infty)} : W_\infty \longrightarrow \mathbb{R}_{\ge 0} \text{ by } (\beta, K) \mapsto \frac{[K_\infty : \mathbb{R}](g(\beta) + h(\beta))}{[K : \mathbb{Q}(\beta)] \sum_{\tau\in I_{\mathbb{Q}(\beta)}} (f(\tau\beta) + g(\tau\beta))}.$$

Using Theorem 5.1.4, we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K\Phi(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi(v_r, a_r, \beta)\right) \ge F^{(\infty)}(\beta, K),$$

for all $(\beta, K) \in W_\infty$. This completes the proof of Theorem 6.1.2. □

Chapter 7

p-adic case

In this section, we give a p-adic analogue of Theorem 6.1.2.

7.1 Statement of Theorem 7.1.1

We recall the definition of the p-adic Lerch function as follows:

$$\Phi_p : \mathbb{N} \times (\mathbb{C}_p \setminus \mathbb{Z}_{\leq 0}) \times D_p \longrightarrow \mathbb{C}_p, \quad (s,x,z) \mapsto \Phi_p(s,x,z) := \sum_{m=0}^{\infty} \frac{z^{-m-1}}{(m+x)^s}.$$

For an algebraic number field K, we denote the completion of K with respect to the p-adic absolute value of K determined by the fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ by K_p. A p-adic analog of Theorem 6.1.2 is as follows:

Theorem 7.1.1. ([49]) *Let r be a natural number, $s_1, \ldots, s_r$ natural numbers and $a_1, \ldots, a_r$ rational numbers. We assume that $a_1, \ldots, a_r$ satisfy $0 < a_1 < \cdots < a_r \leq 1$. We denote the set $\{(\beta, K) \in D_p(\overline{\mathbb{Q}}) \times \mathcal{A}_{\mathbb{Q}} |\ \beta \in K\}$ by W_p and use the same notations as in Theorem 6.1.2. We define the following four functions:*

$$\begin{aligned}
&f^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto s\left(\log A + \sum_{\substack{q:\text{prime}\\ q|A}} \frac{\log q}{q-1}\right) + S(M+A) + \log \mathrm{den}(\beta),\\
&g^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto \log \max\{1, |\beta|\} + (s \log s + (2s+1)\log 2),\\
&h^{(p)} : D_p(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } \beta \mapsto s \log |\beta|_p,\\
&F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \quad \text{by } (\beta, K) \mapsto \frac{[K_p : \mathbb{Q}_p](h^{(p)}(\beta) + s\log|\beta|_p)}{[K : \mathbb{Q}(\beta)] \sum_{\tau \in I_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}.
\end{aligned}$$

Then we obtain the following inequality:

$$\dim_K \left(K + \sum_{v_1=1}^{s_1} K\Phi_p(v_1, a_1, \beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi_p(v_r, a_r, \beta)\right) \geq F^{(p)}(\beta, K),$$

for all $(\beta, K) \in W_p$.

Remark 7.1.2. We denote the s-th p-adic polylogarithm function by

$$\mathrm{Li}_p(s,z) := \Phi_p(s,1,z) = \sum_{m=0}^{\infty} \frac{z^{-m-1}}{(m+1)^s} \quad \text{for } s \in \mathbb{N} \text{ and } z \in D_p.$$

P. Bel gave a **(Type A)**$_p$-estimate of lower bound of the dimension of the K-vector space spanned by the special values of the p-adic polylogarithm functions $\{\mathrm{Li}_p(1,z),\dots,\mathrm{Li}_p(s,z)\}$ in [18, Theorem 3]. Theorem 7.1.1 is a generalization of [18, Theorem 3].

7.2 Proof of Theorem 7.1.1

Fix a set of rational numbers $\{a_1,\dots,a_r\} \subset \mathbb{Q}$ satisfying $0 < a_1 < \cdots < a_r \le 1$. We use Theorem 5.2.8 for

$$f_{(i,v_i)}(x,z) = \Phi(v_i,x,z) : \{x \in \mathbb{R}|\ x>0\} \times \mathbb{R}_{>1} \longrightarrow \mathbb{C}, \quad 1 \le i \le r,\ 1 \le v_i \le s_i,$$

and $\alpha_{(i,v_i)} = a_i$ for $1 \le i \le r$. Note that $\Phi(v_i,a_i,z)$ satisfy the Assumption 5.2.3 for all $1 \le i \le r$ and $1 \le v_i \le s_i$ and $\hat{\Phi}_p(v_i,a_i,z) = \Phi_p(v_i,a_i,z)$.

Proof of Theorem 7.1.1. We define the following functions:

$$\begin{aligned}
&D_n : D_p \longrightarrow \mathbb{N} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{e+Mn} \mathrm{den}(\beta)^n d^S_{b+(n-1)A},\\
&E_n : D_p \longrightarrow \mathbb{Z}\setminus\{0\} \text{ by } \beta \mapsto S!\mu_n(A)^s A^{s(n+1)} d^S_{b+(n-1)A} d^S_{e+Mn},\\
&f^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\left(\log A + \sum_{\substack{q:\text{prime}\\ q|A}} \frac{\log q}{q-1}\right) + S(M+A) + \log \mathrm{den}(\beta),\\
&g^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto \log\max\{1,|\beta|\} + (s\log s + (2s+1)\log 2),\\
&h^{(p)} : D_p \longrightarrow \mathbb{R}_{\ge 0} \text{ by } \beta \mapsto s\log|\beta|_p.
\end{aligned}$$

We put the polynomials $\{A^{(n)}_{i,v_i,\mathbf{w}}(z)\}_{1\le i\le r, 1\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{P^{(n)}_{\mathbf{w}}(z)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ be defined in (6.2) and (6.3). As a result of Lemma 6.3.4, we have $\Delta^{(n)}(z) \in \overline{\mathbb{Q}}^*$. This shows that $\{A^{(n)}_{i,v_i,\mathbf{w}}(z), -P_{\mathbf{w}}(z)\}_{1\le i\le r, 0\le v_i\le s_i, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$ satisfies (5.16) in Assumption 5.2.4. Using Lemma 6.3.3, the family of functions $\{D_n : V_p \longrightarrow \mathbb{N}\}_{n\in\mathbb{N}}$ satisfy (5.18) in Assumption 5.2.4. Using Lemma 6.3.3, $f^{(p)}$ satisfies (5.17) in Assumption 5.2.5. Using Lemma 6.3.1 (6.7), $g^{(p)}$ satisfies (5.20) in Assumption 5.2.5. Using Lemma 6.3.2, $h^{(p)}$ satisfies (5.22) in Assumption 5.2.5. We set $W_p := \{(\beta,K)|\ \beta\in K\}$ and define the following function:

$$F^{(p)} : W_p \longrightarrow \mathbb{R}_{\ge 0} \ \text{ by } (\beta,K) \mapsto \frac{[K_p:\mathbb{Q}_p](h^{(p)}(\beta) + s\log|\beta|_p)}{[K:\mathbb{Q}(\beta)]\sum_{\tau\in I^{(\infty)}_{\mathbb{Q}(\beta)}} (f^{(p)}(\tau\beta) + g^{(p)}(\tau\beta))}.$$

As a result of Theorem 5.2.8, we get:

$$\dim_K\left(K + \textstyle\sum_{v_1=1}^{s_1} K\Phi_p(v_1,a_1,\beta) + \cdots + \sum_{v_r=1}^{s_r} K\Phi_p(v_r,a_r,\beta)\right) \ge F^{(p)}(\beta,K),$$

for all $(\beta,K)\in W_p$. This completes the proof of Theorem 7.1.1. □

Part IV

(Type A)$_p$-estimate for formal Laurent series represented as the image of formal Mellin transform

Chapter 8

Motivation

We define the Riemann zeta function by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{8.1}$$

which is absolutely convergent on $\{s \in \mathbb{C} |\ \mathrm{Re}(s) > 1\}$ and has meromorphic continuation to the whole $\mathbb{C}$-plane with only simple pole at $s = 1$.

Conjecture 1.2.9 applied to the mixed Tate motives implies that we have the following conjecture for the special values of the Riemann zeta function at positive integers.

Conjecture 8.0.1. *Let m be a natural number. We have the following equality:*

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(\{\zeta(2), \zeta(3), \zeta(5), \ldots, \zeta(2m+1)\}) = m + 1. \tag{8.2}$$

Though Conjecture 8.0.1 is far from being solved, we will recall some of the known results related to the conjecture.

Theorem 8.0.2. (*Euler*)
Let m be a natural number. Then we have the following equality:

$$\zeta(2m) = (-1)^{n-1} 2^{2n-1} B_{2n} \frac{\pi^{2n}}{(2n)!}, \tag{8.3}$$

where π is the circular constant.

The proof of Theorem 8.0.2 is obtained by the method of contour integration applied to $\zeta(s)$. Using Theorem 8.0.2 and the fact that π is a transcendental number, we have

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(\{\zeta(2m)\}_{m\in\mathbb{N}}) = \mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}(\pi) = 1.$$

On the other hand, it is less known about the arithmetic properties of the special values of $\zeta(s)$ at positive odd integers. The known results of algebraic properties of special values of $\zeta(s)$ at odd positive integers are only as follows:

Theorem 8.0.3. (*Apéry,* 1979, [11]) *We have $\zeta(3) \notin \mathbb{Q}$.*

Theorem 8.0.4. (*Zudilin,* 2002, [83]) *One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.*

Theorem 8.0.5. (*Rivoal,* 2001, [70, Theorem 1]) *Let m be a natural number. We denote the $\mathbb{Q}$-vector space*

$$V_m := \mathbb{Q} + \sum_{i=1}^{m} \mathbb{Q}\zeta(2i+1).$$

Then, we have

$$\dim_{\mathbb{Q}} V_m \geq \frac{1}{3}\log(2m+1). \tag{8.4}$$

Especially, there are infinitely many irrational numbers in the set $\{\zeta(2m+1)\}_{m\in\mathbb{N}}$.

Theorem 8.0.3, 8.0.5 and 8.0.4 are proved by constructing Padé approximations of the polylogarithm functions and the Lerch function.

We would like to consider the p-adic analogue of Theorem 8.0.3 and 8.0.5. A p-adic counterpart of $\zeta(s)$ is $\zeta_p(s)$, the Kubota-Leopoldt p-adic L-function with trivial character (see Theorem 11.1.1 in Chapter 11). Since we don't have the formula for the special values of $\zeta_p(s)$ at positive even integer correspond to Theorem 8.0.2, the arithmetic properties of the special values of $\zeta_p(s)$ at positive integer are much more mysterious than that of $\zeta(s)$. The known results for the arithmetic properties of special values of $\zeta_p(s)$ at positive integers are the following results.

Theorem 8.0.6. (*Calegari,* 2005, [26]) (*cf. Beukers,* 2008, [23])
Let $p = 2$ or $p = 3$. Then we have $\zeta_p(2), \zeta_p(3) \in \mathbb{Q}_p \setminus \mathbb{Q}$.

In [26], Calegari proved Theorem 8.0.6 by using p-adic modular forms. In [23], Beukers gives the alternative proof of Theorem 8.0.6 by using an explicit Padé approximation of certain formal Laurent series studied by Stieltjes in [77]. In this thesis, from the viewpoint to generalize the results that Beukers obtained in [23], we study a p-adic analogue of Theorem 8.0.5 for $\zeta_p(s)$ at positive integers.

Chapter 9

Some p-adic analysis

In this chapter, we recall some basic notions of p-adic analysis, for example p-adic exponential function and p-adic distribution, and some properties of them that we will use later in this Chapter. In this chapter, we use the following notations:

We denote the Iwasawa p-adic logarithm function by $\log_p : \mathbb{C}_p^* \longrightarrow \mathbb{C}_p$.

For admissible open sets $U = D_p(0, t^-)$ or D_p for $0 < t \leq 1$ of $\mathbb{P}^1(\mathbb{C}_p)$, we denote the ring of rigid analytic functions on U by $\mathcal{O}(U)$. Note that we can explicitly represent the set $\mathcal{O}(U)$ as follows:

$$\mathcal{O}(D_p(0,t^-)) = \{\sum_{n=0}^{\infty} a_n x^n |\ |a_n| r^n \to 0 (n \to \infty) \text{ for any } 0 < r < t\},$$
$$\mathcal{O}(D_p) = \{\sum_{n=0}^{\infty} a_n x^{-n} |\ |a_n| r^{-n} \to 0 (n \to \infty) \text{ for any } 1 < r < \infty\}.$$

9.1 p-adic exponential function

We denote the set $\{p^{a/b} \in \mathbb{R} |\ a/b \in \mathbb{Q}\}$ by $p^{\mathbb{Q}}$. We fix a section $\iota : p^{\mathbb{Q}} \longrightarrow \mathbb{C}_p^*$ and define $\tilde{\cdot}_\iota : \mathbb{C}_p^* \longrightarrow \mathcal{O}_{\mathbb{C}_p}$ by $x \mapsto \tilde{x}_\iota := \frac{x}{\iota(x)}$. Since there exist the following exact sequences

$$1 \longrightarrow 1 + \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \longrightarrow \mathcal{O}_{\mathbb{C}_p}^* \xrightarrow{\text{red}} \overline{\mathbf{F}}_p^* \longrightarrow 1,$$

$$1 \longrightarrow \mathcal{O}_{\mathbb{C}_p}^* \longrightarrow \mathbb{C}_p^* \xrightarrow{|\cdot|_p^{-1}} p^{\mathbb{Q}} \longrightarrow 1,$$

we obtain the following isomorphism depending on ι:

$$\mathbb{C}_p^* \longrightarrow p^{\mathbb{Q}} \times (1 + \mathfrak{m}_{\mathbb{C}_p}) \times \overline{\mathbf{F}}_p^* \quad x \mapsto \iota(x) \langle \tilde{x}_\iota \rangle \omega(\tilde{x}_\iota),$$

where

$$\omega : \overline{\mathbf{F}}_p^* \longrightarrow \mathcal{O}_{\mathbb{C}_p},\ x \mapsto \omega(x),$$

is the Teichmuller character and

$$\langle \cdot \rangle : \mathcal{O}_{\mathbb{C}_p}^* \longrightarrow 1 + \mathfrak{m}_{\mathbb{C}_p},\ x \mapsto \langle x \rangle := \frac{x}{\omega(x)},$$

is the Iwasawa braket function.

Definition 9.1.1. We use the above notation. For $x \in \mathbb{C}_p^*$, we define the following function on $\mathbb{Z}_p$:

$$\langle x \rangle_\iota^s : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \quad s \mapsto \langle x \rangle_\iota^s := \sum_{n=0}^{\infty} \binom{s}{n} (\langle \tilde{x}_\iota \rangle - 1)^n.$$

Hereafter for an element $x \in \mathbb{C}_p^*$, we denote $r(x) := p^{-\frac{1}{p-1}} |\log_p(x)|_p^{-1}$.

Remark 9.1.2. By the definition of $\langle x \rangle_\iota^s$, if $x \in \mathcal{O}_{\mathbb{C}_p}^*$, $\langle x \rangle_\iota^s$ is independent of the choice of ι. If $x \in \mathbb{C}_p^*$ satisfies $|x|_p \in p^{\mathbb{Z}}$, then $\langle x \rangle_\iota^s = \langle x p^{-\mathrm{ord}_p(x)} \rangle^s$. In this case, $\langle x \rangle_\iota^s$ is also independent on the choice of ι. If $x \in \mathbb{C}_p^*$ satisfies the above condition, we denote $\langle x \rangle_\iota^s$ by $\langle x \rangle^s$.

We have the following properties of $\langle x \rangle_\iota^s$.

Lemma 9.1.3. 1. *Let x be an element of $\mathbb{C}_p^*$. Then $\langle x \rangle_\iota^s$ is a continuous function on $\mathbb{Z}_p$.*

2. *Let x, y be elements of $\mathbb{C}_p^*$. Then we have $\langle xy \rangle_\iota^s = \langle x \rangle_\iota^s \langle y \rangle_\iota^s$ for all $s \in \mathbb{Z}_p$.*

3. *Let x be an element of $\mathbb{C}_p^*$. Then $\langle x \rangle_\iota^s = \exp(s \log_p x)$ for $s \in D_p(0, r(x)^-) \cap \mathbb{Z}_p$. Especially, $\langle x \rangle_\iota^s$ is independent of the choice ι for all $s \in D_p(0, r(x)^-) \cap \mathbb{Z}_p$.*

Proof. 1. Using the Mahler's structure theorem of $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$ (cf. [62]), we can show the continuity of $\langle x \rangle_\iota^s$.

2. Let x, y be elements of $\mathbb{C}_p^*$. By the definition of ι, we have $(\widetilde{xy})_\iota = \tilde{x}_\iota \tilde{y}_\iota$. Then from the definition of $\langle \cdot \rangle_\iota$, we may assume $x, y \in \mathcal{O}_{\mathbb{C}_p}^*$. Note that in this case $\langle x \rangle_\iota$ (resp. $\langle y \rangle_\iota$, $\langle xy \rangle_\iota$) does not depend on a pair of embeddings ι. By the definition of $\langle \cdot \rangle$, we have $\langle xy \rangle^s = \langle x \rangle^s \langle y \rangle^s$ for any $s \in \mathbb{Z}_{\geq 0}$. Since $\langle xy \rangle^s$, $\langle x \rangle^s$ and $\langle y \rangle^s$ are continuous functions on $\mathbb{Z}_p$ and $\mathbb{Z}_{\geq 0} \subset \mathbb{Z}_p$ are dence with respect to the p-adic topology on $\mathbb{Z}_p$, we can prove the statement.

3. Firstly we note that there is the following equality in $\mathbb{Z}_p[[z]]$ for $s \in \mathbb{Z}_p$:

$$\sum_{n=0}^{\infty} \binom{s}{n} (z-1)^n = \sum_{n=0}^{\infty} \frac{s^n \left(-\sum_{k=1}^{\infty} \frac{(1-z)^k}{k} \right)^n}{n!} = \exp(s \log z). \tag{9.1}$$

For elements $x \in \mathbb{C}_p^*$ and $s \in D_p(0, r(x)^-) \cap \mathbb{Z}_p$, we can define $\exp(s \log_p x)$. Using the equality (9.1), we have the desire equality. □

9.2 p-adic distribution on $\mathbb{Z}_p$

Definition 9.2.1. 1. We denote the set of all continuous functions on $\mathbb{Z}_p$ by $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p)$. If we endow $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p)$ the sup norm:

$$|\cdot|_{\sup} : \mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0}, \; f \mapsto \sup_{x \in \mathbb{Z}_p} |f(x)|_p,$$

the set $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p)$ becomes a $\mathbb{C}_p$-Banach space. We define the set of all $\mathbb{C}_p$-valued measures on $\mathbb{Z}_p$ by $\mathrm{Hom}_{\mathbb{C}_p}^{cont}(\mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p)$. We denote $\mathrm{Hom}_{\mathbb{C}_p}^{cont}(\mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p)$ by $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$. Define a norm on $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$ as follows:

$$|| \cdot || : \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0}, \; \mu \mapsto \sup_{0 \neq f \in \mathrm{C}(\mathbb{Z}_p, \mathbb{C}_p)} \frac{\mu(f)}{|f|_{\sup}},$$

then $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$ becomes a $\mathbb{C}_p$-Banach algebra.

2. Let $h \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}_p$. we define $f : a + p^h\mathbb{Z}_p \longrightarrow \mathbb{C}_p$ as to be an $\mathbb{C}_p$-valued analytic function when there exists $g \in \mathcal{O}(D_p(a, p^h))$ satisfying $f = g|_{a+p^h\mathbb{Z}_p}$. We denote the set of $\mathbb{C}_p$-valued analytic function on $a + p^h\mathbb{Z}_p$ by $\mathrm{A}(a + p^h\mathbb{Z}_p, \mathbb{C}_p)$. We endow with $\mathrm{A}(a + p^h\mathbb{Z}_p, \mathbb{C}_p)$ the following norm:

$$|\cdot|_{D_p(a,p^{-h})} : \mathrm{A}(a + p^h\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0},\ f = g|_{a+p^h\mathbb{Z}_p} \mapsto [g]_{D_p(a,p^{-h})},$$

where $g \in \mathcal{O}(D_p(a, p^h))$ and $[g]_{D_p(a+p^h\mathbb{Z}_p)} := \sup_{x\in D_p(a,p^{-h})}|g(x)|_p$. Then the set $\mathrm{A}(a+p^h\mathbb{Z}_p, \mathbb{C}_p)$ becomes a $\mathbb{C}_p$-Banach algebra.

We call $f : \mathbb{Z}_p \longrightarrow \mathbb{C}_p$ as to be a $\mathbb{C}_p$-valued locally analytic function of radius p^{-h} when $f|_{a+p^h\mathbb{Z}_p} \in \mathrm{A}(a + p^h\mathbb{Z}_p, \mathbb{C}_p)$ for all $a \in \mathbb{Z}_p$. We denote the set of $\mathbb{C}_p$-valued locally analytic function of radius p^{-h} by $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p)$. We endow $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p)$ with the following norm:

$$|\cdot| : \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0},\ f \mapsto \sup_{a\in\mathbb{Z}_p/p^h\mathbb{Z}_p}[f|_{a+p^h\mathbb{Z}_p}]_{D_p(a,p^{-h})},$$

where a runs through all the representatives of $\mathbb{Z}_p/p^h\mathbb{Z}_p$. Then $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p)$ becomes a $\mathbb{C}_p$-Banach algebra for each $h \in \mathbb{Z}_{\geq 0}$. We define the set of locally analytic function by

$$\mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p) := \varinjlim_h \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p),$$

and the norm on $\mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p)$ by

$$|\cdot| : \mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0},\ f \mapsto |f|_h,$$

where $f \in \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p)$. Then the norm $|\cdot|$ on $\mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p)$ is well-defined and the inclusion morphism $\mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p) \hookrightarrow \mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p)$ is continuous.

We define the set of distribution of $\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p)$ by $\mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p) := \mathrm{Hom}^{cont}_{\mathbb{C}_p}(\mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p)$. We endow $\mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p)$ with the norm as follows:

$$||\cdot||_h : \mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0},\ \mu \mapsto \sup_{0\neq f\in\mathrm{LA}_h(\mathbb{Z}_p,\mathbb{C}_p)}\frac{|\mu(f)|_p}{|f|_h}.$$

Then $\mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p)$ becomes a $\mathbb{C}_p$-Banach algebra. We define the set of $\mathbb{C}_p$-valued distribution on $\mathbb{Z}_p$ by

$$\mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{C}_p}(\mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p), \mathbb{C}_p),$$

and the norm on $\mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$ by

$$||\cdot|| : \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{R}_{\geq 0},\ \mu \mapsto \sup_{0\neq f\in\mathrm{LA}(\mathbb{Z}_p,\mathbb{C}_p)}\frac{|\mu(f)|_p}{|f|_h}.$$

Then there exists a topological isomorphism

$$\mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \varprojlim_h \mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p),\ \mu \mapsto (\mu_h)_h,$$

where μ_h is the composition of the natural embedding $\nu_n : \mathrm{LA}_h(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p)$ and μ, namely $\mu_h = \mu \circ \nu_n$.

Note that the inclusion morphism

$$\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p) \hookrightarrow \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p),$$

is a continuous $\mathbb{C}_p$-algebra homomorphism.

REMARK 9.2.2. For an element $\mu \in \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$ and $f \in \mathrm{Cont}(\mathbb{Z}_p, \mathbb{C}_p)$ (resp. $\mu \in \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$ and $f \in \mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p)$), we denote $\mu(f)$ by $\int_{\mathbb{Z}_p} f(t)d\mu(t)$. We endow $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$ (resp. $\mathcal{D}_h(\mathbb{Z}_p, \mathbb{C}_p)$ and $\mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$) the $\mathbb{C}_p$-algebra structure by the convolution product defined as follows:

$$* : \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p) \times \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p),\ (\mu, \nu) \mapsto \left(\mu * \nu : f \mapsto \int_{\mathbb{Z}_p \times \mathbb{Z}_p} f(t_1 t_2) d(\mu \times \nu)(t_1, t_2) \right),$$

where $\mu \times \nu$ is the product measure of μ and ν.

EXAMPLE 9.2.3. Fix an element $x \in \mathbb{C}_p^*$. Put $h := \min\{h \in \mathbb{N} \mid p^h \mathbb{Z}_p \subset D_p(0, r(x)^-)\}$. Then we have $\langle x \rangle_\iota^s \in \mathrm{A}(p^h \mathbb{Z}_p, \mathbb{C}_p)$. In fact, we have $\exp(s \log x) \in \mathcal{O}(D_p(0, p^{-h}))$ and $\langle x \rangle_\iota^s = \exp(s \log x)|_{p^h \mathbb{Z}_p}$ from Proposition 9.1.3 3.

THEOREM 9.2.4. (*Amice*) [2] *We define the following map:*

$$\mathcal{A} : \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathcal{O}(D_p(0, 1^-)),\ \mu \mapsto \mathcal{A}_\mu(z) = \int_{\mathbb{Z}_p} (1+z)^t d\mu(t).$$

Then $\mathcal{A}$ is a topological $\mathbb{C}_p$-algebra isomorphism. Furthermore, if we restrict $\mathcal{A}$ to $\mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$, we also obtain a homeomorphism and isomorphism of $\mathbb{C}_p$-algebra:

$$\mathcal{A} : \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathbb{C}_p[[z]]^{bd},$$

where $\mathbb{C}_p[[z]]^{bd} = \{\sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}_p[[z]] \mid \{|a_n|_p\}_{n \in \mathbb{Z}_{\geq 0}}$ *is a bounded set*$\}$.

EXAMPLE 9.2.5. We give two examples of p-adic distribution. These examples are important object in the following sections.

1. Let $\frac{\log(1+z)}{z} \in \mathcal{O}(D_p(0, 1^-))$. We denote the distribution $\mathcal{A}^{-1}\left(\frac{\log(1+z)}{z}\right)$ by $\mu_{\mathrm{Haar}} \in \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$. For $a \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 0}$, we put the characteristic function on $a + p^n \mathbb{Z}_p$ by $1_{a + p^n \mathbb{Z}_p} \in \mathrm{LA}(\mathbb{Z}_p, \mathbb{C}_p)$. Then we have

$$\mu_{\mathrm{Haar}}(1_{a + p^n \mathbb{Z}_p}) = p^{-n}.$$

2. Let $x \in \mathbb{C}_p \setminus D_p(1, 1^-)$ and put $F_x(z) := \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} z^k \in \mathbb{C}_p[[z]]^{bd}$. We denote the measure $\mathcal{A}^{-1}(F_x(z))$ by $\mu_x \in \mathrm{Meas}(\mathbb{Z}_p, \mathbb{C}_p)$. Then we have

$$\mu_x(1_{a + p^n \mathbb{Z}_p}) = \frac{x^a}{1 - x^{p^n}}.$$

PROPOSITION 9.2.6. *Let $\mu \in \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$. We define a function*

$$f_\mu : \mathbb{Z}_p \times (D_p \setminus \{\infty\}) \longrightarrow \mathbb{C}_p,\ f_\mu(s, z) := \int_{\mathbb{Z}_p} \langle z + t \rangle_\iota^{-s} d\mu(t).$$

For $z \in D_p \setminus \{\infty\}$ and $m \in \mathbb{Z}_{\geq 0}$, we have

$$f_\mu(m+1, z) = \frac{\langle z \rangle_\iota^{-m-1}}{m!} \sum_{k=0}^{\infty} (k+1) \cdots (k+m) \int_{\mathbb{Z}_p} t^k d\mu(t) \left(\frac{-1}{z} \right)^k.$$

Especially, if $z = \frac{a}{F}$ where a be a integer satisfying $(a, p) = 1$ and F be a natural number divisible by p, we have the following equality:

$$f_\mu(m+1, \tfrac{a}{F}) = \frac{(-1)^{m+1} \langle F \rangle^{m+1} \omega(a)^{m+1}}{F^{m+1} m!} \sum_{k=0}^{\infty} (k+1) \cdots (k+m) \int_{\mathbb{Z}_p} t^k d\mu(t) \left(-\frac{F}{a} \right)^{k+m+1}.$$

Proof. Let $z \in D_p \setminus \{\infty\}$ and $k \in \mathbb{Z}_{\geq 1}$. From Lemma 9.1.3 2, we have $\langle z+t \rangle_\iota^{-k} = \langle z \rangle_\iota^{-k} \langle 1 + \frac{t}{z} \rangle_\iota^{-k}$. Since $z \in D_p \setminus \{\infty\}$, we have $1 + \frac{t}{z} \in 1 + \mathfrak{m}_{\mathbb{C}_p}$ for all $t \in \mathbb{Z}_p$. Then we obtain

$$\begin{aligned}
\langle 1 + \tfrac{t}{z} \rangle_\iota^{-k} &= \sum_{n=0}^{\infty} \binom{-k}{n} \left(\frac{t}{z}\right)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{k(k+1)\cdots(k+n-1)}{n!} \left(\frac{t}{z}\right)^n \\
&= \frac{1}{(k-1)!} \sum_{n=0}^{\infty} (n+1)\cdots(k+n-1) \left(\frac{-t}{z}\right)^n . \qquad (9.2)
\end{aligned}$$

Since the equality (9.2) converges uniformly on $\mathbb{Z}_p$, we obtain the desire equality. $\square$

Chapter 10

Formal Mellin transform

10.1 Some properties of formal Mellin transform

In this section, we introduce formal Mellin transform of formal power series. Throughout this section, we denote K by a field of characteristic 0.

Definition 10.1.1. We define the formal Mellin transform with coefficients K as follows:

$$\mathcal{M}_K : K[[z]] \longrightarrow \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right],\ g(z) \mapsto \mathcal{M}_K(g) := \sum_{k=0}^{\infty} b_k \left(\frac{-1}{z}\right)^{k+1},$$

where $\{b_k\}_{k\in\mathbb{Z}_{\geq 0}}$ are defined by $g(e^z - 1) = \sum_{k=0}^{\infty} \frac{b_k}{k!} z^k$. If $K = \overline{\mathbb{Q}}$, for simplicity, we denote $\mathcal{M}_{\overline{\mathbb{Q}}}$ by $\mathcal{M}$.

We have the following property of formal Mellin transform.

Proposition 10.1.2. *Let $g(z) = \sum_{k=0}^{\infty} a_k z^k \in K[[z]]$. Then we have the following identities:*

$$\mathcal{M}_K(g) = -\sum_{k=0}^{\infty} \frac{(-1)^k a_k k!}{z(z+1)\dots(z+k)}. \tag{10.1}$$

Especially, we have the following identity:

$$\frac{1}{z}K\left[\left[\frac{1}{z}\right]\right] = \left\{\sum_{k=0}^{\infty} \frac{c_k k!}{z(z+1)\dots(z+k)} \mid c_k \in K \text{ for all } k \in \mathbb{Z}_{\geq 0}\right\}.$$

To prove Proposition 10.1.2, we introduce formal Laplace transform.

Definition 10.1.3. We define the following K-linear isomorphism

$$\mathcal{L}_K : K[[z]] \longrightarrow K[[z]],\ \mathcal{L}_K\left(\sum_{k=0}^{\infty} a_k \frac{z^k}{k!}\right) = \sum_{k=0}^{\infty} a_k z^k.$$

We call $\mathcal{L}_K$ the formal Laplace transform.

Endowing $K[[z]]$ with the (z)-adic topology, then we see that the formal Laplace transform is a homeomorphism.

Lemma 10.1.4. *Let $m \in \mathbb{Z}_{\geq 0}$. We have the following identity in $K[[z]]$:*

$$\mathcal{L}_K\left(\frac{(e^z-1)^m}{m!}\right) = \frac{z^m}{(1-z)\cdots(1-mz)}.$$

Proof. For $(n,m) \in (\mathbb{Z}_{\geq 0})^2$, we define the (n,m)-th Stirling number of the second kind $a_{n,m}$ which satisfies the following recurrence relations. The initial values $a_{0,0}, a_{n,0}, a_{0,m}$ satisfy:

$$a_{0,0} = 1 \text{ and } a_{n,0} = a_{0,m} = 0 \text{ for } (n,m) \neq (0,0) \tag{10.2}$$

and the value $a_{n,m}$ satisfies the recurrence relation

$$a_{n,m} = a_{n-1,m-1} + m a_{n-1,m} \text{ for } n,m \in \mathbb{Z}_{\geq 1}. \tag{10.3}$$

For $m \in \mathbb{Z}_{\geq 0}$, we define two sequences of rational numbers $\{b_{k,m}\}_{k \in \mathbb{Z}_{\geq 0}}$ and $\{c_{k,m}\}_{k \in \mathbb{Z}_{\geq 0}}$ which satisfy the following relations respectively.

$$\frac{(e^z-1)^m}{m!} = \sum_{k=m}^{\infty} b_{k,m}\frac{z^k}{k!},\ \frac{z^m}{(1-z)\cdots(1-mz)} = \sum_{k=m}^{\infty} c_{k,m} z^k. \tag{10.4}$$

It is easy to show that $\{b_{k,m}\}_{k \in \mathbb{Z}_{\geq 0}}$ and $\{c_{k,m}\}_{k \in \mathbb{Z}_{\geq 0}}$ satisfy the following recurrence relations (10.2) and (10.3). Namely, $a_{k,m} = b_{k,m} = c_{k,m}$ for all $(k,m) \in \mathbb{Z}_{\geq 0}$. Applying the formal Laplace transform for $\dfrac{(e^z-1)^m}{m!}$, we obtain Lemma 10.1.4. $\square$

Proof of Proposition 10.1.2. We define the following map:

$$\Psi_K : K[[z]] \longrightarrow \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right],\ \sum_{k=0}^{\infty} b_k z^k \mapsto \sum_{k=0}^{\infty} b_k \left(\frac{-1}{z}\right)^{k+1}. \tag{10.5}$$

From the definition of $\mathcal{M}_K, \mathcal{L}_K$ and Ψ_K, we have the equality:

$$\mathcal{M}_K = \Psi_K \circ \mathcal{L}_K \circ (z \mapsto e^z - 1). \tag{10.6}$$

The equality (10.1) is easily proved by using Lemma 10.1.4 and the equality (10.6). Since $\mathcal{M}_K$ is surjective, we obtain the second assertion of Proposition 10.1.2. $\square$

Definition 10.1.5. We denote the isomorphism of K-algebra $\mathcal{M}_K^{\mathrm{ope}} : K[[z]] \longrightarrow K\left[\left[\frac{d}{dz}\right]\right]$ by $\mathcal{M}_K^{\mathrm{ope}}(z) := \exp(\frac{d}{dz}) - 1$. We denote $\exp(\frac{d}{dz})$ by Δ.

We define the action $K\left[\left[\frac{d}{dz}\right]\right] \times \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right] \longrightarrow \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right]$ by

$$\left(\sum_{k=0}^{\infty} a_k \left(\frac{d}{dz}\right)^k, \sum_{k=0}^{\infty} b_k \left(\frac{-1}{z}\right)^{k+1}\right) \mapsto \sum_{k=0}^{\infty}\left[\sum_{l=0}^{k} l!a_l \binom{k}{k-l} b_{k-l}\right]\left(\frac{-1}{z}\right)^{k+1}. \tag{10.7}$$

Note that the action of $\Delta \in K\left[\left[\frac{d}{dz}\right]\right]$ for $f(z) \in \frac{1}{z}K\left[\left[\frac{1}{z}\right]\right]$ can be written explicitly as follows:

$$\Delta(f(z)) = f(z+1).$$

We have the following proposition.

Proposition 10.1.6. *The following diagram is commutative.*

$$\begin{array}{ccc}
K[[z]] \times K[[z]] & \xrightarrow{\mathcal{M}_K^{\mathrm{ope}} \times \mathcal{M}_K} & K\left[\left[\dfrac{d}{dz}\right]\right] \times \frac{1}{z}K\left[\left[\dfrac{1}{z}\right]\right] \\
\downarrow & & \downarrow \\
K[[z]] & \xrightarrow{\mathcal{M}_K} & \dfrac{1}{z}K\left[\left[\dfrac{1}{z}\right]\right]
\end{array},$$

where the first vertical morphism is defined by product and the second one is by (10.7).

PROOF. It is enough to show $\mathcal{M}_K^{\mathrm{ope}}(z)\mathcal{M}_K(g)=\mathcal{M}_K(zg)$ for any $g\in K[[z]]$. Let $g(z)=\sum_{k=0}^{\infty}a_kz^k\in K[[z]]$. Firstly, we calculate $\mathcal{M}_K(zg)$. From Proposition 10.1.2, we have

$$\mathcal{M}_K(zg)=-\sum_{k=1}^{\infty}\frac{(-1)^k a_{k-1}z^k}{z(z+1)\dots(z+k)}. \tag{10.8}$$

On the other hand, for $\mathcal{M}_K^{\mathrm{ope}}(z)\mathcal{M}_K(g)$, we have the following equalities:

$$\begin{aligned}
\mathcal{M}^{\mathrm{ope}}(z)\mathcal{M}(g)&=(\Delta-1)\left[-\sum_{k=0}^{\infty}\frac{(-1)^k a_k k!}{z(z+1)\dots(z+k)}\right]\\
&=-\sum_{k=0}^{\infty}\frac{(-1)^k a_k k!}{(z+1)(z+2)\dots(z+k+1)}+\sum_{k=0}^{\infty}\frac{(-1)^k a_k k!}{z(z+1)\dots(z+k)}\\
&=\sum_{k=0}^{\infty}(-1)^k a_k k!\left[\frac{-1}{(z+1)(z+2)\dots(z+k+1)}+\frac{1}{z(z+1)\dots(z+k)}\right]\\
&=-\sum_{k=1}^{\infty}\frac{(-1)^k a_{k-1}k!}{z(z+1)\dots(z+k)} \qquad (10.9)
\end{aligned}$$

From equalities (10.8) and (10.9), we obtain the proof of Proposition 10.1.6. □

The following lemma is a fundamental property of $\mathcal{M}_K(g)$ for $g\in\overline{\mathbb{Q}}[[z]]$.

LEMMA 10.1.7. *Let j be a non-negative integer. For $g(z)=\sum_{k=0}^{\infty}a_kz^k\in K[[z]]$, we have the following equality:*

$$\begin{aligned}
z(z+1)\dots(z+j-1)\mathcal{M}_K(g)=&-z(z+1)\dots(z+j-1)\sum_{k=0}^{j-1}(-1)^k a_k\frac{k!}{z(z+1)\dots(z+k)}\\
&+(-1)^j\Delta^j\mathcal{M}_K\left(\left(\frac{d}{dz}\right)^j g\right),
\end{aligned}$$

where, in the case of $j=0$, we denote $z(z+1)\dots(z+j-1)=1$ and $\sum_{k=0}^{j-1}(-1)^k a_k\frac{k!}{z(z+1)\dots(z+k)}=0$.

PROOF. In the case of $j=0$, the claim is trivially true. In the following, we assume $j\geq 1$. From the definition of $\mathcal{M}_K(g)$, we have the following identities:

$$\begin{aligned}
z(z+1)\dots(z+j-1)\mathcal{M}_K(g)=&-z(z+1)\dots(z+j-1)\sum_{k=0}^{j-1}\frac{(-1)^k a_k k!}{z(z+1)\dots(z+k)}\\
&+\sum_{k=j}^{\infty}\frac{(-1)^k a_k k!}{(z+j)(z+j+1)\dots(z+k)}.
\end{aligned}$$

From the above equalities, it is enough to show that the following equality:

$$(-1)^j\Delta^j\mathcal{M}_K\left(\left(\frac{d}{dz}\right)^j g\right)=\sum_{k=j}^{\infty}\frac{(-1)^k a_k k!}{(z+j)(z+j+1)\dots(z+k)}. \tag{10.10}$$

Using the equality $\left(\frac{d}{dz}\right)^j g(z) = \sum_{k=0}^{\infty}(k+1)\dots(k+j)a_{k+j}z^k$ and Lemma 10.1.2, we obtain the following equalities:

$$
\begin{aligned}
(-1)^j\Delta^j\mathcal{M}_K\left(\left(\frac{d}{dz}\right)^j g\right) &= \Delta^j\left(\sum_{k=0}^{\infty}\frac{(-1)^{k+j}(k+j)!a_{k+j}}{z(z+1)\dots(z+k)}\right)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k+j}(k+j)!a_{k+j}}{(z+j)(z+j+1)\dots(z+k+j)}\\
&= \sum_{k=j}^{\infty}\frac{(-1)^{k}k!a_{k}}{(z+j)(z+j+1)\dots(z+k)}.
\end{aligned}
$$

Thus we obtain equality (10.10). This completes the proof of Lemma 10.1.7. □

10.2 Relation between formal Mellin transform and asymptotic expansion of complex valued functions

For a function $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C}$, we define the Mellin transform of $f(t)$ by $F(z) := \int_0^{\infty} f(t)e^{-zt}dt$. The following lemma gives the asymptotic expansion of functions defined by Mellin transfrom.

Lemma 10.2.1. (*Watson's Lemma*) (*cf.* [82])
Let $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C}$ be a continuous function. We assume that f satisfies the following conditions:

1. *There exists a set $\{\lambda_n\}_{n\in\mathbb{Z}_{\geq 0}} \subset \mathbb{R}_{>0}$ satisfying $\lambda_0 < \lambda_1 < \dots$ and $f(t) \sim \sum_{k=0}^{\infty} a_k t^{\lambda_k - 1}$ $(t \to 0)$,*
2. *There exists a $C > 0$ satisfying $|f(t)| = O(e^{Ct})$ $(t \to \infty)$.*

Then $F(z) := \int_0^{\infty} f(t)e^{-zt}dt$ is converges on $\{z \in \mathbb{C} |\ |z| > C\}$ and there exists $0 < \delta < \frac{\pi}{2}$ satisfying the following property:

$$
F(z) \sim \sum_{k=0}^{\infty} a_k \frac{\Gamma(\lambda_k)}{z^{\lambda_k}} \ \left(|z| \to \infty \ \text{in} \ \arg(z) \leq \frac{\pi}{2} - \delta\right).
$$

Especially, $F(z)$ is an element of M^C.

We obtain the following corollary of Lemma 10.2.1.

Corollary 10.2.2. *Let $g(z) := \sum_{k=0}^{\infty} a_k z^k$ be an element of $\mathbb{C}[[z]]$. Suppose there exists $0 < B \leq 1$ satisfying $|a_k| \leq B^k$ for all $k \in \mathbb{Z}_{\geq 0}$. Then the function $\int_0^{\infty} g(e^{-t}-1)e^{-zt}dt$ converges on $\{z \in \mathbb{C} |\ |z| > B\}$ and there exists a sequence $\{b_k\}_{k\in\mathbb{Z}_{\geq 0}} \subset \mathbb{C}$ satisfying*

$$
\int_0^{\infty} g(e^{-t}-1)e^{-zt}dt \sim \sum_{k=0}^{\infty} b_k \left(\frac{-1}{z}\right)^{k+1} \quad (|z| \to \infty).
$$

Proof. Put $f(t) := g(e^{-t}-1)$. It is enough to verify the condition 1 and 2 in Lemma 10.2.1 for $f(t)$. Firstly, we show the condition 2. for $f(t)$. Since $f(t)$ is analytic in a neighborhood of $t = 0$ in $\mathbb{C}$, there exists a set $\{b_k\}_{k\in\mathbb{Z}_{\geq 0}} \subset \mathbb{C}$ satisfying $f(t) = \sum_{k=0}^{\infty} b_k \frac{(-t)^k}{k!}$ in a neighborhood of $t = 0$. This verifies the condition 1 for $f(t)$. Secondly, we show the condition 2 for $f(t)$. Then we have the following equalities for any $t \in \mathbb{R}_{>0}$:

$$\begin{aligned}
|f(t)| &= |\sum_{k=0}^{\infty} a_k(e^{-t} - 1)| &(10.11)\\
&\leq \sum_{k=0}^{\infty} |a_k||e^{-t} - 1|^k\\
&\leq \sum_{k=0}^{\infty} B^k|e^{-t} - 1|^k\\
&= \frac{1}{1 - B(1 - e^{-t})}\\
&= O(e^t). &(10.12)
\end{aligned}$$

Note that the second equality of the (10.11) is obtained from the assumption $B \leq 1$. This completes the proof of Corollary 10.2.2. $\square$

At the end of this subsection, we give the relation between formal Mellin transform and Mellin transform. Before giving the statement, we prepare some notations. Let B is a positive real number. We assume that B satisfies $B \leq 1$. For $\tau \in I^{(\infty)}$, we denote

$$\mathcal{C}_\tau^B := \{g(z) = \sum_{k=0}^{\infty} a_k z^k \in \overline{\mathbb{Q}}[[z]] \mid |\tau(a_k)| \leq B^k \text{ for all } k \in \mathbb{Z}_{\geq 0}\}.$$

Then, from the argument of the proof of Corollary 10.2.2, we obtain the map

$$\eta_\tau : \mathcal{C}_\tau^B \longrightarrow M^B_{\tau\overline{\mathbb{Q}}}, \quad g(z) \mapsto \tau g(e^{-z} - 1).$$

The relation between formal Mellin transform and Mellin transform is as follows:

Proposition 10.2.3. *The following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{C}_\tau^B & \xrightarrow{\mathcal{M}} & \mathcal{M}(\mathcal{C}_\tau^B) \\
{\scriptstyle \eta_\tau}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\
\eta(\mathcal{C}_\tau^B) & \xrightarrow{\hat{}} & \frac{1}{z}\mathbb{C}\left[\left[\frac{1}{z}\right]\right]
\end{array}.$$

Proof. Proposition 10.2.3 is a consequence of Corollary 10.2.2. $\square$

10.3 Connections between formal Mellin transform and p-adic Stieltjes transform

In this section, we define the p-adic Stieltjes transform and give the relation between formal Mellin transform and p-adic Stieltjes transform. We recall the p-adic Stieltjes transform.

Definition 10.3.1. We define the p-adic Stieltjes transform as follows:

$$\mathcal{S} : \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) \longrightarrow \mathcal{O}(D_p),\ \mathcal{S}(\mu) := -\int_{\mathbb{Z}_p} \frac{1}{z+t} d\mu(t).$$

Remark 10.3.2. We remark that the well-definedness of $\mathcal{S}$ is obtained by the following argument. We define the set of Krasner analytic functions on $\overline{\mathbb{Z}}_p := \mathbb{C}_p \setminus \mathbb{Z}_p$ by

$$H_0(\overline{\mathbb{Z}}_p) := \{\phi : \overline{\mathbb{Z}_p} \to \mathbb{C}_p \mid \phi \text{ satisfies } (1), (2)\}.$$

(1) For $r > 0$, we denote $\overline{D}_{\mathbb{Z}_p}(r) := \{z \in \mathbb{C}_p \mid \min_{x \in \mathbb{Z}_p} |z - x| \geq r\}$. For any $r > 0$, there exists a family of rational functions $\{f_{r,n}\}_{n \in \mathbb{N}}$ whose member have poles only on $\mathbb{Z}_p$ satisfying the limit of $\{f_{r,n}\}_{n \in \mathbb{N}}$ converges to ϕ uniformly on $\overline{D}_{\mathbb{Z}_p}(r)$.
(2) $\lim_{|z|_p \to \infty} \phi(z) = 0$.

From Theorem of Vishik [81] (cf. [58, p. 138]), the following morphism gives the topological isomorphism

$$\mathcal{S}' : \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) \xrightarrow{\cong} H_0(\overline{\mathbb{Z}}_p),\ \mu \mapsto -\int_{\mathbb{Z}_p} \frac{1}{z+t} d\mu(t).$$

The Stieltjes transform $\mathcal{S}$ defined in Definition 10.3.1 is the composition of $\mathcal{S}'$ and the restriction map $H_0(\overline{\mathbb{Z}}_p) \hookrightarrow \mathcal{O}(D_p)$.

We have the following principal relation between the formal Mellin transform and the p-adic Stieltjes transform.

Proposition 10.3.3. *Let τ be an element of $I^{(p)}$. We denote the set $\{g(z) \in \overline{\mathbb{Q}}[[z]] \mid r_\tau(g) \geq 1\}$ by $\mathcal{B}_{p,\tau}$. Then we have the following commutative diagram:*

$$\begin{array}{ccc}
\mathcal{B}_{p,\tau} & \xrightarrow{\mathcal{M}} & \mathcal{M}(\mathcal{B}_{p,\tau}) \\
{\scriptstyle \varphi_\tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
\mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p) & \xrightarrow[\mathcal{S}]{} & \mathcal{O}(D_p)
\end{array}$$

where φ_τ is defined by $g(z) \mapsto \mathcal{A}^{-1}(\tau(g))$.

Proof. Let $g(z) \in \mathcal{B}_{p,\tau}$. We denote $\varphi_\tau(g)$ by $\mu_{\tau g}$ and $g(e^z - 1) = \sum_{k=0}^{\infty} b_k \frac{z^k}{k!}$. Then we have the following identity (cf. [1]):

$$\tau(b_k) = \int_{\mathbb{Z}_p} t^k d\mu_{\tau g}(t) \text{ for all } k \in \mathbb{Z}_{\geq 0}. \tag{10.13}$$

On the other hand, we have the following equality for $z \in D_p$:

$$\mathcal{S}(\mu) = -\int_{\mathbb{Z}_p} \frac{1}{z+t} d\mu(t) = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p} t^k \mu_{\tau g}(t) \left(\frac{-1}{z}\right)^{k+1}. \tag{10.14}$$

From equalities (10.13) and (10.14), we obtain the proof of Proposition 10.3.3. □

Remark 10.3.4. From Proposition 10.3.3, we can define $\tau(\mathcal{M}(g))(\alpha)$ for $g(z) \in \mathcal{B}_{p,\tau}$ and $\alpha \in D_p$.

Definition 10.3.5. Let $\mu \in \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$. We define the following formal Laurent series for positive integer m:

$$R^{(\mu)}(z) = R_0^{(\mu)}(z) := \mathcal{S}(\mu)(z),$$
$$R_m^{(\mu)}(z) := \frac{d}{dz^m} R^{(\mu)}(z).$$

From Proposition 9.2.6 2., we have the following equality for natural number a satisfying $(a,p)=1$, non-negative integer m and natural number F which is divisible by p:

$$f_\mu\left(m+1,\frac{a}{F}\right)=\frac{(-1)^{m+1}\langle F\rangle^{m+1}\omega(a)^{m+1}}{F^{m+1}m!}R_m^{(\mu)}\left(\frac{a}{F}\right). \tag{10.15}$$

For $\mu\in\mathcal{D}(\mathbb{Z}_p,\mathbb{C}_p)$, we suppose that there exists $g\in\mathcal{B}_{p,\tau}$ satisfying $\varphi_\tau(g)=\mu$. From Proposition 10.3.3 and the equality (10.15), we obtain

$$\begin{aligned} f_\mu\left(m+1,\frac{a}{F}\right)&=\frac{(-1)^{m+1}\langle F\rangle^{m+1}\omega(a)^{m+1}}{F^{m+1}m!}\tau\left(\frac{d}{dz}\right)^m\mathcal{M}(g)(z)|_{z=\frac{a}{F}}\\ &=\frac{(-1)^{m+1}\langle F\rangle^{m+1}\omega(a)^{m+1}}{F^{m+1}m!}\tau\mathcal{M}(\log(1+z)^m g)(z)|_{z=\frac{a}{F}}. \end{aligned} \tag{10.16}$$

Note that the equality (10.16) is obtained from Proposition 10.1.6.

10.4 Relation between formal Mellin transform and Padé approximation of formal Laurent series

Let m be a natural number and K a field of characteristic 0. We fix m formal power series

$$g_1(z)=\sum_{k=0}^\infty a_{1,k}z^k,\dots,g_m(z)=\sum_{k=0}^\infty a_{m,k}z^k\in K[[z]]$$

and denote

$$f_0:=1,f_1:=\mathcal{M}_K(g_1),\dots,f_m:=\mathcal{M}_K(g_m)\in\frac{1}{z}K\left[\left[\frac{1}{z}\right]\right].$$

We have the following theorem about the Padé approximation of $\mathbf{f}:=(f_1,\dots,f_m)$.

Theorem 10.4.1. *Let $(n_1,\cdots,n_m)\in\mathbb{N}^m$. Let $\{p_{v,j}^{(\mathbf{n})}\}_{1\le v\le m,0\le j\le n_v}$ be a subset of K. The following are equivalent.*

(i) *For all natural number k satisfying $0\le k\le\sum_{v=1}^m(n_v+1)-2$, we have the following linear relations between $\{p_{v,j}^{(\mathbf{n})}\}_{1\le v\le m,o\le j\le n_v}$.*

$$\sum_{v=1}^m\sum_{j=0}^{n_v}\left((-1)^j\sum_{l=0}^k\binom{j}{l}\binom{k+j-l}{j}a_{v,k+j-l}\right)p_{v,j}^{(\mathbf{n})}=0, \tag{10.17}$$

where we denote $\binom{j}{l}=0$ if l,j satisfy $l>j$.

(ii) *We put $P_v^{(\mathbf{n})}(z)=\sum_{j=0}^{n_v}p_{v,j}^{(\mathbf{n})}\frac{z(z+1)\dots(z+j-1)}{j!}$ for any $1\le v\le m$. Then, there exists a unique polynomial $P_0^{(\mathbf{n})}(z)\in K[z]$ which satisfies that*

$$(P_0^{(\mathbf{n})}(z),P_1^{(\mathbf{n})}(z),\dots,P_m^{(\mathbf{n})}(z))\in K[z]^{m+1}, \tag{10.18}$$

is a $\mathbf{n}$-th Padé approximation of $\mathbf{f}$.

Furthermore, for a Padé approximation given in (10.18)*, we have the following relations:*

$$P_0^{(\mathbf{n})}(z)=-\sum_{v=1}^m\sum_{j=1}^{n_v}\left(p_{v,j}^{(\mathbf{n})}\frac{z(z+1)\dots(z+j-1)}{j!}\right)\left(\sum_{k=0}^{j-1}\frac{(-1)^k a_{v,k}k!}{z(z+1)\dots(z+k)}\right), \tag{10.19}$$

$$\mathcal{R}^{(\mathbf{n})}(z)=\sum_{k=\sum_{v=1}^{m}(n_v+1)-1}\left(\sum_{v=1}^{m}\sum_{j=0}^{n_v}(-1)^j p^{(\mathbf{n})}_{v,j}\sum_{l=0}^{k}\binom{j}{l}\binom{k+j-l}{j}a_{v,k+j-l}\right)\frac{(-1)^{k+1}k!}{z(z+1)\dots(z+k)}. \tag{10.20}$$

Proof. Firstly, we prove that the condition (ii) implies the condition (i). From Lemma 10.1.7, we obtain the following equality for $1\le v\le m$:

$$\begin{aligned}P^{(\mathbf{n})}_v(z)f_v(z)&=\left(\sum_{j=0}^{n_v}\frac{p^{(\mathbf{n})}_{v,j}}{j!}z(z+1)\dots(z+j-1)\right)\mathcal{M}_K(g_v)\\&=\sum_{j=1}^{n_v}p^{(\mathbf{n})}_{v,j}\frac{z(z+1)\dots(z+j-1)}{j!}\sum_{k=0}^{j-1}\frac{(-1)^k a_{v,k}k!}{z(z+1)\dots(z+k)}\\&\quad+\sum_{j=0}^{n_v}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}\Delta^j\mathcal{M}_K\left(\left(\frac{d}{dz}\right)^j g\right).\end{aligned} \tag{10.21}$$

From Proposition 10.1.6, we obtain the following equalities:

$$\begin{aligned}\sum_{j=0}^{n_v}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}\Delta^j\mathcal{M}_K\left(\left(\frac{d}{dz}\right)^j g_v\right)&=\sum_{j=0}^{n_v}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}\left(\mathcal{M}_K^{\mathrm{ope}}\times\mathcal{M}_K\right)\left((z+1)^j\left(\frac{d}{dz}\right)^j g_v\right)\\&=\mathcal{M}_K\left(\sum_{j=0}^{n_v}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}(z+1)^j\left(\frac{d}{dz}\right)^j g_v\right).\end{aligned} \tag{10.22}$$

By the relation (3) in Lemma 6.2.2, we obtain:

$$\mathcal{R}^{(\mathbf{n})}(z)=\mathcal{M}_K\left(\sum_{v=1}^{m}\sum_{j=0}^{n_v}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}(z+1)^j\left(\frac{d}{dz}\right)^j g_v\right)\in\left(\frac{1}{z}\right)^{\sum_{v=1}^{m}(n_v+1)}. \tag{10.23}$$

On the other hand, we have the following equalities:

$$\begin{aligned}(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}(z+1)^j\left(\frac{d}{dz}\right)^j g_v&=(-1)^j\frac{p^{(\mathbf{n})}_{v,j}}{j!}\left(\sum_{k=0}^{j}\binom{j}{k}z^k\right)\left(\sum_{k=0}^{\infty}(k+1)\dots(k+j)a_{v,k+j}z^k\right)\\&=(-1)^j p^{(\mathbf{n})}_{v,j}\left(\sum_{k=0}^{j}\binom{j}{k}z^k\right)\left(\sum_{k=0}^{\infty}\binom{k+j}{j}a_{v,k+j}z^k\right)\\&=(-1)^j p^{(\mathbf{n})}_{v,j}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\binom{j}{l}\binom{k+j-l}{j}a_{v,k+j-l}\right)z^k.\end{aligned} \tag{10.24}$$

For the third equality of (10.24), we mean $\binom{k+j-l}{j}=0$ in the case of $j<l$. By the relations (10.23), and the identity (10.24), we get the following relation:

$$\sum_{k=0}^{\infty}\sum_{v=1}^{m}\sum_{j=0}^{n_v}(-1)^j p^{(\mathbf{n})}_{v,j}\left(\sum_{l=0}^{k}\binom{j}{l}\binom{k+j-l}{j}a_{v,k+j-l}\right)z^k\in(z)^{\sum_{v=1}^{m}(n_v+1)-1}. \tag{10.25}$$

The relation (10.25) shows that $\{p^{(\mathbf{n})}_{v,j}\}_{1\le v\le m,0\le j\le n_v}$ satisfy the relation (10.19). Since the relation (10.25) is equivalent to the relation (10.23), we have that the condition (i) implies the condition (ii).

We prove the relation (10.19). From equality (10.21) and Remark 3.1.2, we obtain the equality:

$$P^{(\mathbf{n})}_0(z)=-\sum_{v=1}^{m}\left(\sum_{j=1}^{n_v}p^{(\mathbf{n})}_{v,j}\frac{z(z+1)\dots(z+j-1)}{j!}\right)\left(\sum_{k=0}^{j-1}\frac{(-1)^k a_{v,k}k!}{z(z+1)\dots(z+k)}\right).$$

Secondly, we prove the relations (10.20). We have the equality

$$\mathcal{R}^{(\mathbf{n})}(z) = \mathcal{M}_K \left(\sum_{k=0}^{\infty} \sum_{v=1}^{m} \sum_{j=0}^{n_v} (-1)^j p_{v,j}^{(\mathbf{n})} \left(\sum_{l=0}^{k} \binom{j}{l} \binom{k+j-l}{j} a_{v,k+j-l} \right) z^k \right). \tag{10.26}$$

Using Proposition 10.1.2 and the relation (10.25) for the equality (10.26), we obtain the equality (10.20). This completes the proof of Theorem 10.4.1. □

Chapter 11

Power series representations of special values of the Kubota-Leopoldt p-adic L-functions

In this chapter, we recall two types of power series representations of special values of Kubota-Leopoldt p-adic L-function at positive integers.

11.1 Some constructions of the Kubota-Leopoldt p-adic L-functions

In this section, we recall two different types of construction of Kubota-Leopoldt p-adic L-functions. Let f be a natural number and χ a Dirichlet character of conductor f.

We denote Dirichlet L-function with character χ and Hurwitz zeta function as follows:

$$L(-,\chi) : \{s \in \mathbb{C} | \ \mathrm{Re}(s) > 1\} \longrightarrow \mathbb{C}, \quad s \mapsto \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

$$\zeta_{\mathrm{H}} : \{s \in \mathbb{C} | \ \mathrm{Re}(s) > 1\} \times \{x \in \mathbb{R} | \ x > 0\} \longrightarrow \mathbb{C}, \quad (s,x) \mapsto \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

If χ is the trivial character, then $L(s,\chi)$ is equal to Riemann zeta function $\zeta(s)$ defined in (8.1).

Firstly, we recall the existence of Kubota-Leopoldt p-adic L-functions.

Theorem 11.1.1. (*Kubota-Leopoldt*) *Let χ be a Dirichlet character of conductor f. Then there exists a unique p-adic continuous function $L_p(-,\chi) : \mathbb{Z}_p \setminus \{1\} \longrightarrow \mathbb{Q}_p$ satisfying*

$$L_p(1-k,\chi) = (1 - \chi\omega^{-k}(p)p^{k-1})L(1-k,\chi\omega^{-k}) \text{ for all } k \in \mathbb{Z}_{\geq 0}.$$

If χ is the trivial character, we denote $L_p(s,\chi)$ by $\zeta_p(s)$ and call it the p-adic Riemann zeta function. Note that if the character χ in Theorem 11.1.1 is odd, we have $L_p(s,\chi) = 0$.

Let c be a natural number. We assume that c satisfies $c \geq 2$ and $(f,c) = 1$. We denote the set

$\{\xi \in \overline{\mathbb{Q}} |\ \xi^c = 1\}$ by μ_c. Then we have the following equalities for $s \in \{s \in \mathbb{C} |\ \mathrm{Re}(s) > 1\}$:

$$L(s,\chi) = f^{-s} \sum_{a=1}^{f-1} \chi(a) \zeta\left(s, \frac{a}{f}\right) \tag{11.1}$$

$$= \frac{F^{-s}}{(\chi(c)c^{1-s}-1)} \sum_{\xi \in \mu_c}{}' \sum_{a=1}^{F-1} \chi(a)\xi^a \Phi\left(s, \xi^F, \frac{a}{F}\right), \tag{11.2}$$

where $\sum_{\xi\in\mu_c}{}'$ means ξ runs through $\mu_c \setminus \{1\}$ and F is a natural number which is divided by f.

Using the equalities (11.1) and (11.2), we obtain two types of construction of Kubota-Leopoldt p-adic L-functions. More precisely, we construct some p-adic continuous functions which interpolate special values of $\zeta_{\mathrm{H}}(s,x)$ and $\Phi(s,x,z)$ for $s \in \mathbb{Z}_{\leq 0}$ and some $x, z \in \overline{\mathbb{Q}}$ and construct Kubota-Leopoldt p-adic L-functions by using equalities (11.1) and (11.2).

We define the numbers $\{B_n(x)\}_{n\in\mathbb{Z}_{\geq 0}}$ and $\{B_{n,z}(x)\}_{n\in\mathbb{Z}_{\geq 0}}$ for some $x, z \in \overline{\mathbb{Q}}$ as follows:

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

$$\frac{e^{xt}}{1-ze^t} = \sum_{n=0}^{\infty} B_{n,z}(x)\frac{t^n}{n!}.$$

Lemma 11.1.2. (*cf.* [12]) *Let k be a non-negative integer x a positive real number and z a complex number satisfying $z \neq 1$ and $|z| \leq 1$. Then we have the following equalities:*
(1) $\zeta_{\mathrm{H}}(-k,x) = -\dfrac{B_{k+1}(x)}{k+1}$.
(2) $\Phi(-k,z,x) = B_{k,x}(z)$.

From Lemma 11.1.2 and Example 9.2.5, we have the following relations for $x \in D_p \cap \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$, $z \in (\mathbb{C}_p \setminus D_p(1,1^-)) \cap \{z \in \overline{\mathbb{Q}} |\ |z| \geq 1\}$ and $k \in \mathbb{Z}_{\geq 0}$:

$$\frac{-1}{k+1}\int_{\mathbb{Z}_p} (x+t)^{k+1} d\mu_{\mathrm{Haar}}(t) = -\frac{B_{k+1}(x)}{k+1} = \zeta_{\mathrm{H}}(-k,x), \tag{11.3}$$

$$\int_{\mathbb{Z}_p} (z+t)^k d\mu_x(t) = B_{k,x}(z) = \Phi(-k,z,x). \tag{11.4}$$

Using (11.3) and (11.4), we define the p-adic Hurwitz zeta function and the p-adic Lerch function of second type as follows:

Definition 11.1.3. We define the following functions:

$$\zeta_{\mathrm{H},p} : \mathbb{Z}_p \times D_p \longrightarrow \mathbb{C}_p,\ \zeta_{\mathrm{H},p}(s,z) := \frac{1}{s-1}\int_{\mathbb{Z}_p} \langle z+t \rangle_\iota^{1-s} d\mu_{\mathrm{Haar}}(t), \tag{11.5}$$

$$\Psi_p : \mathbb{Z}_p \times D_p \times \big(\mathbb{C}_p \setminus D_p(1,1^-)\big) \longrightarrow \mathbb{C}_p,\ \Psi_p(s,z,x) := \int_{\mathbb{Z}_p} \langle z+t \rangle_\iota^{-s} d\mu_x(t). \tag{11.6}$$

We call $\zeta_{\mathrm{H},p}$ the p-adic Hurwitz zeta function and Ψ_p the p-adic Lerch function of second type. Using the same notations as in Definition 10.3.5, we have the following equalities:

$$\zeta_{\mathrm{H},p}(s,z) = \frac{1}{s-1} f_{\mu_{Haar}}(s-1,z), \tag{11.7}$$

$$\Psi_p(s,z,x) = f_{\mu_x}(s,z). \tag{11.8}$$

Using the function $\zeta_{\mathrm{H},p}$ and Ψ_p defined in Definition 11.1.3, we have following equalities:

PROPOSITION 11.1.4. *Let χ be a Dirichlet character of conductor f. We put $F := \begin{cases} 4f & \text{if } p=2, \\ pf & \text{if } p>2. \end{cases}$ Then we have the following equalities:*

1.
$$L_p(s,\chi) = \frac{\langle f\rangle^{1-s}}{F} \sum_{a=1,(a,p)=1}^{F} \chi(a)\zeta_{\mathrm{H},p}\left(s, \frac{a}{F}\right).$$

2. *Let $c \in \mathbb{Z}_{\geq 2}$. We assume that c satisfies $c \equiv 1 \mod p$. Then we have*
$$L_p(s,\chi) = \frac{\langle f\rangle^{1-s}}{(\chi(c)c^{1-s}-1)} \sideset{}{'}\sum_{\xi\in\mu_c} \sum_{a=1,(a,p)=1}^{F} \chi\omega^{-1}(a)\xi^a \Psi_p\left(s,,\xi^F, \frac{a}{F}\right).$$

Especially, we obtain the equalities for p-adic Riemann zeta functions.

$1'$.
$$\zeta_p(s) = \begin{cases} \frac{1}{4}\left(\zeta_{\mathrm{H},2}\left(s,\frac{1}{4}\right) + \zeta_{\mathrm{H},2}\left(s,\frac{3}{4}\right)\right) & \text{if } p=2, \\ \frac{1}{p}\sum_{a=1}^{p-1}\zeta_{\mathrm{H},p}\left(s,\frac{a}{p}\right) & \text{if } p>2. \end{cases}$$

$2'$. *We have the following equalities:*
$$\zeta_p(s) = \begin{cases} \frac{1}{3^{1-s}-1}\sideset{}{'}\sum_{\xi\in\mu_3}\left(\xi\Psi_2\left(s,\xi,\frac{1}{4}\right) - \Psi_2\left(s,\xi,\frac{3}{4}\right)\right) & \text{if } p=2, \\ \frac{1}{(p+1)^{1-s}-1}\sideset{}{'}\sum_{\xi\in\mu_{p+1}}\sum_{a=1}^{p-1}\omega^{-1}(a)\xi^a\Psi_p\left(s,\xi^p,\frac{a}{p}\right) & \text{if } p>2. \end{cases} \qquad \square$$

It is enough to prove the equalities 1 and 2. Since the equality 1 is proved in [69, Theorem 6.2] and the equality 2 can be proved by the same way as that of 1, we omit the proofs of them.

11.2 Power series representations of special values of the Kubota-Leopoldt p-adic L-functions at positive integers

We use the same notations as in the previous section. Using equalities (11.7), (11.8), (10.15) and Proposition 11.1.4, we give some power series representation of special values of $L_p(s,\chi)$ at positive integers as follows:

PROPOSITION 11.2.1. *Let χ be a Dirichlet character of conductor f and m a non-negative integer. We assume that m satisfies $m \geq 1$ if χ is trivial. Then we have the following power series representation of $L_p(m+1,\chi)$.*

1.
$$L_p(m+1,\chi) = \frac{(-1)^m\langle f\rangle^{-m}}{F^{m+1}m!} \sum_{a=1,(a,p)=1}^{F} \chi(a)\omega(a)^m R_{m-1}^{(\mu_{Haar})}\left(\frac{a}{F}\right).$$

2. *Let $c \in \mathbb{Z}_{\geq 2}$ satisfying $c \equiv 1 \mod p$. Then we have*
$$L_p(m+1,\chi) = \frac{(-1)^{m+1}\langle f\rangle^{-m}}{F^{m+1}m!(\chi(c)c^{-m}-1)} \sideset{}{'}\sum_{\xi\in\mu_c} \sum_{a=1,(a,p)=1}^{F} \chi(a)\xi^a\omega(a)^m R_m^{(\mu_{\xi^F})}\left(\frac{a}{F}\right).$$

REMARK 11.2.2. Using Proposition 11.2.1, we obtain the following relation for natural number m:

$$L_p(m+1,\chi) \in \sum_{a=1,(a,p)=1}^{F} \mathbb{Q}(\omega,\chi) R_{m-1}^{(\mu_{Haar})}\left(\frac{a}{F}\right), \tag{11.9}$$

$$L_p(m+1,\chi) \in \sum_{\xi\in\mu_c}{}' \sum_{a=1,(a,p)=1}^{F} \mathbb{Q}(\omega,\chi,\xi_c) R_{m}^{(\mu_{\xi F})}\left(\frac{a}{F}\right), \tag{11.10}$$

where ξ_c is the primitive c-th root of unity. From the above relation (11.9), we study the linear independence of $\{R_i^{(\mu)}(\frac{a}{F})\}_{0\le i\le m, 1\le a\le F-1,(a,p)=1}$ for $m \in \mathbb{Z}_{\ge 0}$, $F \in \mathbb{N}$ which is divisible by p and $\mu \in \mathcal{D}(\mathbb{Z}_p, \mathbb{C}_p)$ over an algebraic number field.

In the case of p-adic Riemann zeta function, we have more concise power series representation of special values of it at positive integers. We prepare some notations.

DEFINITION 11.2.3. (*cf.* [23]) Let $m \in \mathbb{Z}_{\ge 1}$. We define the following formal Laurent series:

$$\begin{aligned}
R_0(z) &:= \sum_{k=0}^{\infty} B_k \left(-\frac{1}{z}\right)^{n+1} \\
R_m(z) &:= \frac{d^m}{d^m z} R_0(z) = \sum_{k=0}^{\infty} (k+1)\cdots(k+m) B_k \left(-\frac{1}{z}\right)^{k+m+1} \\
\Theta_0(z) &:= \sum_{k=0}^{\infty} (2^{k+1}-2) B_k \left(-\frac{1}{z}\right)^{n+1} \\
\Theta_m(z) &:= \frac{d^m}{d^m z} \Theta_0(z) = \sum_{k=0}^{\infty} (k+1)\cdots(k+m)(2^{k+1}-2) B_k \left(-\frac{1}{z}\right)^{k+m+1}.
\end{aligned}$$

Note that $R_m(z)$ is equal to $R_m^{(\mu_{\mathrm{Haar}})}(z)$ in Definition 10.3.5. For each $m \in \mathbb{Z}_{\ge 0}$, the functions $R_m(z)$ and $\Theta_m(z)$ have several functional equations. That is as follows:

PROPOSITION 11.2.4. (*cf.* [23, Propotition 4.4, Corollary 4.5]) *Let $m \in \mathbb{Z}_{\ge 0}$ and $l \in \mathbb{Z}_{\ge 1}$. We have the following identities in $\mathbb{Q}[[\frac{1}{z}]]$:*

(i) $R_m(z+1) - R_m(z) = \dfrac{(-1)^m (m+1)!}{z^{m+2}}$,

(ii) $R_m(z) = (-1)^{m+1} R_m(1-z)$,

(iii) $\displaystyle\sum_{a=0}^{l-1} R_m\left(z + \frac{a}{l}\right) = l^{m+2} R_m(lz)$,

(iv) $2^{m+1}\Theta_m(z) = R_m\left(\dfrac{z}{2}\right) - R_m\left(\dfrac{z+1}{2}\right)$.

PROOF. Here we only give proofs of (i) and (iii).

(i) We have the following equalities:

$$\begin{aligned}
R_m(z+1) - R_m(z) &= \Delta(R_m(z)) \\
&= \mathcal{M}^{\mathrm{ope}} \times \mathcal{M}\left(z, \frac{\log(1+z)^{m+1}}{z}\right) \\
&= \mathcal{M}\left(z \frac{\log(1+z)^{m+1}}{z}\right) \\
&= \frac{(-1)^m (m+1)!}{z^{m+2}}.
\end{aligned}$$

Note that the third equality is obtained by Proposition 10.1.6. This completes the proof of (i).

(iii) It is enough to prove in the case of $m = 0$. Let l be a natural number. The equality (iii) comes from the following trivial equality

$$\sum_{a=0}^{l-1} r^l = \frac{r^l - 1}{r - 1}. \tag{11.11}$$

Substituting $r = e^z$ to the equality (11.11) and multiplying $e^{lz} - 1$ to the both sides of equality (11.11), we obtain

$$\sum_{a=0}^{l-1} \frac{e^{az}}{e^{lz} - 1} = \frac{1}{e^z - 1}. \tag{11.12}$$

Multiplying by lz to both sides of the equation (11.12) and change the variable z to $\frac{z}{l}$, we get

$$\sum_{a=0}^{l-1} e^{\frac{a}{l}z} \frac{z}{e^z - 1} = l \frac{\frac{z}{l}}{e^{\frac{z}{l}} - 1}. \tag{11.13}$$

Acting formal Laplace transform $\mathcal{L}_{\mathbb{Q}}$ (see Definition 10.1.3) to both side of the equality (11.13) and use the following equality:

$$\mathcal{L}_{\mathbb{Q}}(e^{az} f(z)) = \frac{1}{1 - az} \mathcal{L}_{\mathbb{Q}}(f) \left(\frac{z}{1 - az} \right) \quad \text{for } a \in \mathbb{Q}^*$$

then we obtain

$$\sum_{a=0}^{l-1} \frac{1}{1 - \frac{a}{l} z} F\left(\frac{z}{1 - \frac{a}{l} z} \right) = l F\left(\frac{z}{l} \right) \tag{11.14}$$

where $F(z) := \mathcal{L}_{\mathbb{Q}}\left(\frac{z}{e^z - 1} \right)$. Acting $\Psi_{\mathbb{Q}}$ (see Definition (10.5)) to both side of the equality (11.14), we obtain the equality (iii). This completes the proof of (iii). $\square$

Remark 11.2.5. The statement of Proposition 11.2.4 (i), (ii) and (iii) in the case of $m = 0, 1$ and (iv) in the case of $l = 2$ were proved by Beukers in [23]. Since our proof of (ii), (iv) in Proposition 11.2.4 are the same as that of [23], we omit them. In this thesis, we give an alternative proof of Proposition 11.2.4 (i), (iii) by using formal Mellin transfrom.

By using Proposition 11.9 1. and 11.2.4, we express the special values of $\zeta_p(s)$ at positive integers as follows.

Proposition 11.2.6. (cf. [23, Propositon 5.1]) *Let $m \in \mathbb{Z}_{\geq 1}$. The following identities hold.*

(i) *When $p = 2$, we have*

$$\zeta_2(m+1) = \begin{cases} \dfrac{-1}{2^{m+2} m!} \Theta_{m-1}\left(\dfrac{1}{2}\right) & \text{if } m \text{ is odd} \\[2ex] \dfrac{1}{2^{m+1} m!} R_{m-1}\left(\dfrac{1}{4}\right) & \text{if } m \text{ is even.} \end{cases}$$

(ii) *When p is odd, we have*

$$\zeta_p(m+1) = \frac{(-1)^m 2}{p^{m+1} m!} \sum_{a=1}^{\frac{p-1}{2}} \omega(a)^m R_{m-1}\left(\frac{a}{p}\right).$$

Especially, for $p = 3$, we obtain:

$$\zeta_3(m+1) = \frac{(-1)^m 2}{3^{m+1} m!} R_{m-1}\left(\frac{1}{3}\right) \in \mathbb{Q}^* R_{m-1}\left(\frac{1}{3}\right). \tag{11.15}$$

PROOF. Let p be a prime number. From the definition of $\zeta_{\mathrm{H},p}(m+1, \frac{a}{p})$ and using the equality (10.15), we have

$$\zeta_{\mathrm{H},p}\left(m+1, \frac{a}{F}\right) = \frac{(-1)^m \omega(a)^m}{F^m m!} R_{m-1}\left(\frac{a}{F}\right). \tag{11.16}$$

where F is a power of p. From Proposition 11.1.4 1., we calculate $\zeta_p(m+1)$ in each case in Proposition 11.2.6.
(i) Proof in the case of $p = 2$:
Suppose m is odd. From the identity (11.16), we have the following identity:

$$\zeta_2(m+1) = -\frac{1}{4^{m+1} m!}\left(R_{m-1}\left(\frac{1}{4}\right) - R_{m-1}\left(\frac{3}{4}\right)\right). \tag{11.17}$$

We use the equality (iii) which is

$$2^{m+1}\Theta_m(z) = R_m\left(\frac{z}{2}\right) - R_m\left(\frac{z+1}{2}\right)$$

in Proposition 11.2.4 for (11.17), we obtain:

$$\zeta_2(m+1) = -\frac{1}{2^{m+2} m!}\Theta_{m-1}\left(\frac{1}{2}\right).$$

Suppose m is even. From the definition of $\zeta_2(s)$, we have:

$$\zeta_2(m+1) = \frac{1}{4^{m+1} m!}\left(R_{m-1}\left(\frac{1}{4}\right) + R_{m-1}\left(\frac{3}{4}\right)\right). \tag{11.18}$$

Substituting $z = \frac{1}{4}$ in the equality (i):

$$R_m(z+1) - R_m(z) = \frac{(-1)^m (m+1)!}{z^{m+2}},$$

in Proposition 11.2.4, we obtain

$$R_{m-1}(\tfrac{1}{4}) = \frac{1}{2}\left(R_{m-1}\left(\frac{1}{4}\right) + R_{m-1}\left(\frac{3}{4}\right)\right). \tag{11.19}$$

By the identities (11.18) and (11.19) above, we have

$$\zeta_2(m+1) = \frac{1}{2^{m+1} m!} R_{m-1}\left(\frac{1}{4}\right).$$

This proves (i).
(ii) Proof in the case of $p > 2$:
By using the identity (11.16), we have

$$\zeta_p(m+1) = \frac{(-1)^m}{p^{m+1} m!}\sum_{a=1}^{p} \omega(a)^m R_{m-1}\left(\frac{a}{p}\right).$$

From the equality (i):

$$R_m(z+1) - R_m(z) = \frac{(-1)^m (m+1)!}{z^{m+2}},$$

in Proposition 11.2.4, we obtain

$$\begin{aligned}\zeta_p(m+1) =& \frac{(-1)^m}{p^{m+1}m!}\sum_{a=1}^{\frac{p-1}{2}}(\omega(a)^m + (-1)^m\omega(p-a)^m)R_{m-1}\left(\frac{a}{p}\right)\\ =& \frac{(-1)^m 2}{p^{m+1}m!}\sum_{a=1}^{\frac{p-1}{2}}\omega(a)^m R_{m-1}\left(\frac{a}{p}\right).\end{aligned}$$

This proves (ii). □

REMARK 11.2.7. Let m be a natural number. By Proposition 11.2.6, the following are equivalent.
$(a)_2$ We assume m is even (resp. odd). The p-adic number $\zeta_2(m+1)$ is irrational.
$(b)_2$ The p-adic number $R_{m-1}(\frac{1}{4})$ (resp. $\Theta_{m-1}(\frac{1}{2})$) is irrational.
Similarly, in the case of $p=3$, the following are equivalent.
$(a)_3$ The p-adic number $\zeta_3(m+1)$ is irrational.
$(b)_3$ The p-adic number $R_{m-1}(\frac{1}{3})$ is irrational.

Beukers proved Proposition 11.2.6 in the case of $m=1,2$ and also proved the irrationality of $\zeta_3(2)$ and $\zeta_2(2)$ by proving $(b)_2$ and $(b)_3$ in [23].

Next, we give some functional equations for $R^{(\mu_\xi)}(z)$ for $\xi \in \mathbb{C}_p \setminus D_p(1,1^-)$.

LEMMA 11.2.8. *Let $m \in \mathbb{Z}_{\geq 0}$, $l \in \mathbb{Z}_{\geq 1}$ and $\xi \in \mathbb{C}_p \setminus D_p(1,1^-)$. We have the following identities in $\mathbb{Q}(\xi)[[\frac{1}{z}]]$:*
(i) $R_m^{(\mu_\xi)}(z+1) - R_m^{(\mu_\xi)}(z) = \dfrac{(-1)^m m!}{\xi z^{m+1}}$,
(ii) $R_m^{(\mu_\xi)}(1-z) = \dfrac{(-1)^m}{\xi} R_m^{(\mu_{\xi^{-1}})}(z)$.

Since Lemma 11.2.8 is proved by the same argument of that of Lemma 11.2.4, we omit it. Using Proposition 11.2.1 2. and Lemma 11.2.8, we obtain the following power series representation of special values of $\zeta_p(s)$ at positive integers.

PROPOSITION 11.2.9. *Let $m \in \mathbb{N}$. Then we have the following equalities:*

$$\zeta_p(m+1) = \begin{cases} \dfrac{2(-1)^{m+1}}{3^{m+1}m!(3^{-m}-1)}{\sum_{\xi\in\mu_3}}' \xi R^{(\mu_{\xi^{-1}})}\left(\dfrac{1}{4}\right) & \textit{if } p=2,\\ \dfrac{2(-1)^{m+1}}{p^{m+1}m!(c^{-m}-1)}{\sum_{\xi\in\mu_{p+1}}}' \sum_{a=1}^{\frac{p-1}{2}} \xi^a\omega(a)^m R^{(\mu_{\xi^{-1}})}\left(\dfrac{a}{p}\right) & \textit{if } p>2.\end{cases}$$

PROOF. We use the equality of Proposition 11.2.1 2. for $\chi = \mathbf{1}$ and $c = p+1$. Then we obtain

$$\zeta_p(m+1) = \begin{cases} \dfrac{(-1)^{m+1}}{3^{m+1}m!(3^{-m}-1)}{\sum_{\xi\in\mu_3}}' \sum_{a=1,(a,2)=1}^{4} \xi^a\omega(a)^m R_m^{(\mu_{\xi^{-1}})}\left(\dfrac{a}{4}\right) & \text{if } p=2,\\ \dfrac{(-1)^{m+1}}{p^{m+1}m!((p+1)^{-m}-1)}{\sum_{\xi\in\mu_{p+1}}}' \sum_{a=1}^{p-1} \xi^a\omega(a)^m R_m^{(\mu_{\xi^{-1}})}\left(\dfrac{a}{p}\right) & \text{if } p>2.\end{cases}$$

Since the equalities in the case of $p>2$ are proved by the same method of that in the case of $p=2$ (cf. the prove of Proposition 11.2.6 (ii)), we only prove the case of $p=2$. Let ξ_3 be a primitive 3-th root of unity. Then, we get

$$\sum_{\xi\in\mu_3}{}' \sum_{a=1,(a,p)=1}^{4} \xi^a \omega(a)^m R_m^{(\mu_{\xi^{-1}})}\left(\frac{a}{4}\right)$$
$$= \xi_3 R_m^{(\mu_{\xi_3^{-1}})}\left(\frac{1}{4}\right) + (-1)^m R_m^{(\mu_{\xi_3^{-1}})}\left(\frac{3}{4}\right) + \xi_3^{-1} R_m^{(\mu_{\xi_3})}\left(\frac{1}{4}\right) + (-1)^m R_m^{(\mu_{\xi_3})}\left(\frac{3}{4}\right). \tag{11.20}$$

From the equality (ii) in Lemma 11.2.8, we obtain

$$R_m^{(\mu_{\xi_3^{-1}})}\left(\frac{3}{4}\right) = R_m^{(\mu_{\xi_3^{-1}})}\left(1-\frac{1}{4}\right) = \frac{(-1)^m}{\xi_3} R_m^{(\mu_{\xi_3})}\left(\frac{1}{4}\right). \tag{11.21}$$

From the equality (11.20) and (11.21), we obtain

$$\sum_{\xi\in\mu_3}{}' \sum_{a=1,(a,p)=1}^{4} \xi^a \omega(a)^m R_m^{(\mu_{\xi^{-1}})}\left(\frac{a}{4}\right) = 2\left(\xi_3 R_m^{(\mu_{\xi_3^{-1}})}\left(\frac{1}{4}\right) + \xi_3^{-1} R_m^{(\mu_{\xi_3})}\left(\frac{1}{4}\right)\right).$$

This completes the proof of Proposition 11.2.9. □

Chapter 12

(Type A)$_p$-estimate of formal Laurent series related to the p-adic Hurwitz zeta function

12.1 Statement of Theorem 12.1.2

In this chapter, we give some examples of **(Type A)$_p$**-estimate of formal Laurent series represented by the image of formal Mellin transform. More precisely, we give a lower bound of the dimension of the vector space spanned by the special values of the following p-adic function:

$$\Xi_p : \mathbb{N} \times \mathbb{Z}_p \times (\mathbb{C}_p \setminus D_p(1,1^-) \cup \{1\}) \times D_p \longrightarrow \mathbb{C}_p$$

$$(s, x_1, x_2, z) \mapsto \Xi_p(s, x_1, x_2, z) := \frac{\epsilon(x_2)}{s-1}\frac{1}{z^{s-1}} + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!} B_{m+1}(x_1, x_2)(s)_m \frac{1}{z^{s+m}},$$

where

$$\epsilon(x_2) = \begin{cases} 0 & \text{if } x_2 \neq 1 \\ 1 & \text{if } x_2 = 1, \end{cases} \quad (s)_m = \begin{cases} s(s+1)\cdots(s+m-1) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

and $B_k(x_1, x_2)$ are defined by the following generating function:

$$\frac{te^{x_1 t}}{x_2 e^t - 1} = \sum_{k=0}^{\infty} B_k(x_1, x_2)\frac{t^k}{k!}.$$

REMARK 12.1.1. 1. Let $x_1 \in \mathbb{Z}_p$, $x_2 \in \mathbb{C}_p \setminus D_p(1, 1^-)$ and s be a positive integer. We assume that s satisfies $s \geq 2$ if $x_2 = 1$. Since we have the following equalities:

$$\mathcal{M}\left(\frac{(-1)^{s-1}}{(s-1)!}\frac{\log(1+z)^{s-1}(1+z)^{x_1}}{x_2(1+z)-1}\right) = \frac{\epsilon(x_2)}{s-1}\frac{1}{z^{s-1}} + \sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{(m+1)!}B_{m+1}(x_1,x_2)(s)_m\frac{1}{z^{s+m}}, \qquad (12.1)$$

$$r_p\left(\frac{\log(1+z)^{s-1}(1+z)^{x_1}}{x_2(1+z)-1}\right) = 1.$$

the function $\Xi_p(s, x_1, x_2, z)$ converges on D_p.

2. Using the equality (12.1), we obtain the following equalities:

$$\begin{aligned}
\Xi_p(s,0,1,z) &= \frac{(-1)^{s-1}}{(s-1)!}R_{s-2}(z),\\
\Xi_p(s,1,1,z) &= \frac{(-1)^{s-1}}{(s-1)!}R_{s-2}(z) - \frac{1}{z^s} = \frac{(-1)^{s-1}}{s-1}R_{s-2}(z+1),
\end{aligned} \tag{12.2}$$

for $s \in \mathbb{Z}_{\geq 2}$. Note that the second equality of (12.2) is obtained by Lemma 11.2.4 (i).

$$\begin{aligned}
\Xi_p(s,0,\xi,z) &= \frac{(-1)^{s}}{(s-1)!}R^{(\mu_\xi)}_{s-1}(z),\\
\Xi_p(s,1,\xi,z) &= \frac{(-1)^{s}}{\xi(s-1)!}R^{(\mu_\xi)}_{s-1}(z) - \frac{1}{\xi z^s} = \frac{(-1)^{s}}{(s-1)!}R^{(\mu_\xi)}_{s-1}(z+1),
\end{aligned} \tag{12.3}$$

for $s \in \mathbb{Z}_{\geq 1}$ and $\xi \in \mathbb{C}_p \setminus D_p(1,1^-)$. Note that the second equality of (12.3) is obtained by Lemma 11.2.8 (i).

3. Let $\alpha_1 \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p$ and $\alpha_2 \in \overline{\mathbb{Q}}$. We assume that α_2 satisfies $|\alpha_2| = 1$ and $|\alpha_2 - 1|_p \geq 1$. Then we have the following relations:

$$\begin{aligned}
&\Phi(s, z+\alpha_1, \alpha_2) \in M^0_{\mathbb{Q}(\alpha_1,\alpha_2)},\\
&\hat{\Phi}_p(s, z+\alpha_1, \alpha_2) = \Xi_p(s,\alpha_1,\alpha_2,z).
\end{aligned} \tag{12.4}$$

Remark that the equality (12.4) was proved by Katsurada in [52, Theorem 1]. □

We give the following estimate of a lower bound of the dimension of the vector space spanned by the special values of $\Xi_p(s,x_1,x_2,z)$:

Theorem 12.1.2. *We use the same notations as before. Let r be a natural number, $s_1,\ldots,s_r$ natural numbers, $a_1,\ldots,a_r \in \mathbb{Q}\cap\mathbb{Z}_p$ and $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\}$. We assume that $a_1,\ldots,a_r$ and α satisfy $0 < a_1 < \cdots < a_r \leq 1$ and $|\alpha - 1|_p \geq 1$ respectively. Let W_p be the set $D_p(\mathbb{Q}) \times \mathcal{A}_{\mathbb{Q}}$. We put the following numbers:*

$$\begin{aligned}
&s := \sum_{i=1}^{r} s_i,\\
&B(b) := \mathrm{l.c.m.}\{\mathrm{den}(b+a_i)\}_{1\leq i\leq r} \text{ for } b \in D_p(\mathbb{Q}),\\
&M := \mathrm{l.c.m.}\{\mathrm{den}(a_{i'}-a_i)\}_{1\leq i,i'\leq r,\ i\neq i'},\\
&S := \max_{1\leq i\leq r}\{s_i\},\\
&T := \min_{1\leq i\leq r}\{s_i\},
\end{aligned}$$

and define the following four functions:

$$\begin{aligned}
&f^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto S + M(s+r-T-1) + \sum_{\substack{q:\text{prime}\\ q\mid B(b)}} \frac{\log q}{q-1} + \log\mathrm{den}(\alpha),\\
&g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \log\max\{1,|\alpha|\} + s\log 2,\\
&h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \sum_{\substack{q:\text{prime}\\ q\mid B(b)}} \frac{\log q}{q-1} - \frac{\log p}{p-1} + \log\mathrm{den}(\alpha) - \log\max\{1,|\alpha|_p\},\\
&F^{(p)} : W_p \longrightarrow \mathbb{R}_{\geq 0} \text{ by } (b,K) \mapsto \frac{[K_p:\mathbb{Q}_p](h^{(p)}(b) + T\log|b|_p)}{[K:\mathbb{Q}](f^{(p)}(b)+g^{(p)}(b))}.
\end{aligned}$$

Then we obtain the following estimate:

$$\dim_K\left(K+\sum_{v_1=1}^{s_1+1}K\Xi_p(v_1,a_1,\alpha,b)+\cdots+\sum_{v_r=1}^{s_r+1}K\Xi_p(v_r,a_r,\alpha,b)\right)\geq F^{(p)}(b,K),$$

for all $(b,K)\in W_p$.

REMARK 12.1.3. When $r=1$, P. Bel in [17, Theorem 3.1] also gave a **(Type A)**$_p$-estimate of the dimension of the vector space spanned by the special values of $\{\Xi_p(2,a,1,b),\ldots,\Xi_p(s+1,a,1,b)\}$ for $s\in\mathbb{N}$ and $a\in\mathbb{Q}_{>0}\cap\mathbb{Z}_p$:

$$\dim_K\left(K+\sum_{v=2}^{s+1}K\Xi_p(v,a,1,b)\right)\geq F_1^{(p)}(b,K)\text{ for all }(b,K)\in W_p,$$

where $F_1^{(p)}(b,K)$ is defined by the same way in Theorem 12.1.2 for $\alpha=1$. In Theorem 12.1.2, r is general but we exclude the case $\alpha=1$. Thus, Theorem 12.1.2 is not regarded as a complete generalization of [17, Theorem 3.1]. (see Remark 12.4.8 for the reason why we exclude $\alpha=1$ in Theorem 12.1.2.)

12.2 A construction of Padé approximation of formal Laurent series

In this section, we recall a method of construction of Padé approximation of formal Laurent series obtained by Rivoal (cf. [71, Proposition 4]).

PROPOSITION 12.2.1. (*cf.* [71, Proposition 4]) *Let K be a subfield of $\mathbb{C}$ and l,s be natural numbers. Let $f_0(z):=1,f_1(z),\cdots,f_s(z)\in M_K^A$. Suppose there exist family of polynomials $\{P_v^{(n)}(z)\}_{0\leq v\leq s}\subset K[z]$ which satisfy the following condition:*

The function $\mathcal{R}(z)=\sum_{v=0}^{s}P_v(z)f_v(z)$ satisfies

$$\mathcal{R}(z)=o(z^{-l+1})\ \ (z\to\infty).$$

Then we have $\mathrm{deg}P_0\leq\max_{1\leq v\leq s}\mathrm{deg}P_v,\ \mathcal{R}(z)\in M_K^A$ *and* $\hat{\mathcal{R}}(z)\in K[[\frac{1}{z}]]$ *satisfies*

$$\hat{\mathcal{R}}(z)=\sum_{v=0}^{s}P_v(z)\hat{f}_v(z)=O(z^{-l}).$$

PROOF. Put $q_v=\mathrm{deg}P_v(z),\ q=\max_{1\leq v\leq s}q_v$ and $P_v(z)=\sum_{j=0}^{q_v}b_{v,j}z^j$ for $1\leq v\leq s$. From the definition of M_K^A, there exists a subset of K $\{a_k(f_v)\}_{k\in\mathbb{Z}_{\geq0}}$ satisfying the following condition:

$$f_v(z)=\sum_{k=0}^{N}a_k(f_v)z^{-k}+o(z^{-N})\ (z\to\infty)\text{ for all }N\in\mathbb{Z}_{\geq0}. \tag{12.5}$$

We use the equalities (12.5) for any $N>q$, we get the following equalities:

$$\sum_{v=1}^{s}P_v(z)f_v(z)=Q(z)+\sum_{v=1}^{s}\left[\sum_{j=0}^{q_v}\sum_{\substack{0\leq k\leq N\\-N+q\leq j-k<0}}b_{v,j}a_k(f_v)z^{j-k}\right]+o(z^{-N+q}) \tag{12.6}$$

where $Q(z)$ is a polynomial with coefficients K which satisfies $\deg Q \leq q$. Using the equality (12.6) in the case of $N = l + q - 1$ and the assumption

$$\mathcal{R}(z) = \sum_{v=0}^{s} P_v(z) f_v(z) = o(z^{-l+1}),$$

we obtain

$$P_0(z) = -Q(z) \text{ and } \mathcal{R}(z) = \sum_{k=0}^{N} a_k(\mathcal{R}) z^{-k} + o(z^{-N}) \ (x \to \infty) \text{ for all } N \in \mathbb{Z}_{\geq 0},$$

where

$$a_N(\mathcal{R}) = \begin{cases} 0 & \text{if } N < l, \\ \sum_{v=1}^{s} \sum_{j=0}^{n} b_{v,j} a_{N+j}(f_v) & \text{if } N \geq l. \end{cases}$$

This shows that $\deg P_0 \leq \max_{1 \leq v \leq s} \deg P_v, \mathcal{R}(z) \in M_K^A$ and $\hat{\mathcal{R}}(z) \in (\frac{1}{z})^l$. This completes the proof of Proposition 12.2.1. $\square$

12.3 A Padé approximation of the Lerch function

To prove Theorem 12.1.2, we give a Padé approximation of the Lerch function that is different from the one given in Chapter 7. Let r be a natural number. From here to the last section, we fix r natural numbers $s_1, \dots, s_r$, and r rational numbers $a_1, \dots, a_r$ satisfying $0 < a_1 < \cdots < a_r \leq 1$ and put the following numbers:

$$\begin{aligned}
s &:= \sum_{i=1}^{r} s_i, \\
A &:= \text{l.c.m.}\{\text{den}(a_i)\}_{1 \leq i \leq r}, \\
M &:= \text{l.c.m.}\{\text{den}(a_{i'} - a_i)\}_{1 \leq i, i' \leq r, \ i \neq i'}, \\
e_{i',i} &:= M(a_{i'} - a_i) \text{ for all } 1 \leq i, i' \leq r, \\
e &:= \max_{1 \leq i, i' \leq r} \{|e_{i',i}|\}, \\
S &:= \max_{1 \leq i \leq r} \{s_i\}, \\
T &:= \min_{1 \leq i \leq r} \{s_i\}.
\end{aligned}$$

In this section, we give a Padé approximation of the Lerch function $\{\Phi(v_i, z + a_i, x_1)\}_{1 \leq i \leq r, 1 \leq v_i \leq s_i + 1}$ with variable z.

For a positive integer n and $\mathbf{w} := (w_1, \dots, w_r) \in \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$, we put

$$\begin{aligned}
H_{\mathbf{w}}^{(n)}(u, z) &:= (n!)^{s+r-1} \frac{u(u+1)\cdots(u+n)}{\prod_{i=1}^{r} \left[(u + z + a_i)_n^{s_i+1} (u + z + a_i + n)^{w_i}\right]}, \\
\mathcal{H}_{\mathbf{w}}^{(n)}(x_1, z) &:= \sum_{m=0}^{\infty} H_{\mathbf{w}}^{(n)}(z, m) {x_1}^{-m-1}.
\end{aligned}$$

We define a family of rational functions $\{d_{i,j,v_i,\mathbf{w}}^{(n)}(z)\}_{1 \leq i \leq r, 0 \leq j \leq n, 1 \leq v_i \leq s_i + 1}$ by

$$H_{\mathbf{w}}^{(n)}(u, z) = \sum_{i=1}^{r} \sum_{v_i=1}^{s_i+1} \sum_{j=0}^{n} \frac{d_{i,j,v_i,\mathbf{w}}^{(n)}(z)}{(u + z + a_i + j)^{v_i}}, \tag{12.7}$$

and a family of polynomials $\{A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z), Q^{(n)}_{\mathbf{w}}(x_1,z)\}_{1\le i\le r, 1\le v_i\le s_i+1} \subset \mathbb{Q}(z)[x_1]$ by

$$A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z) = \sum_{j=0}^{n} d^{(n)}_{i,j,v_i,\mathbf{w}}(z) x_1^j, \tag{12.8}$$

$$Q^{(n)}_{\mathbf{w}}(x_1,z) = \sum_{i=1}^{r}\sum_{j=1}^{n}\sum_{v_i=1}^{s_i+1}\sum_{l=0}^{j-1} d^{(n)}_{i,j,v_i,\mathbf{w}}(z)\frac{x_1^{j-1-l}}{(l+a_i)^{v_i}}. \tag{12.9}$$

In the following, we denote $\mathbf{w}$ as an element of $\prod_{i=1}^{r}\{0,\dots,s_i+1\}$.

Remark 12.3.1.

1. By the same argument of the proof of [71, Theorem 1], we can prove that

$$A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z) \in \mathbb{Q}[x_1,z] \text{ and } Q^{(n)}_{\mathbf{w}}(x_1,z) \in \mathbb{Q}[x_1,z].$$

2. By the same argument as is given in Remark 6.2.1, we can obtain the following:

$$\deg_{x_1} A^{(n)}_{i,w_i,\mathbf{w}}(x_1,z) = n \text{ for all } n\in\mathbb{N} \text{ with } \mathbf{w} \text{ satisfying } w_i \ge 1.$$

3. Since we have the following equality:

$$d^{(n)}_{i,j,v_i,\mathbf{w}}(z) =$$
$$\begin{cases} \dfrac{(-1)^{s_i-v_i+1}}{(s_i-v_i+1)!}\left(\dfrac{d}{du}\right)^{s_i-v_i+1} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+j)^{s_i+1}|_{u=j} & \text{for } \ 0\le j\le n-1, 1\le v_i\le s_i+1, \\ \dfrac{(-1)^{w_i-v_i}}{(w_i-v_i)!}\left(\dfrac{d}{du}\right)^{w_i-v_i} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+n)^{w_i}|_{u=n} & \text{for } \ j=n, 1\le v_i\le w_i, \\ 0 & \text{for } \ j=n, v_i>w_i, \end{cases}$$

we have

$$\deg_z A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z) = n+1 \text{ for all } 1\le i\le r,\ 1\le v_i\le s_i+1 \text{ and } \mathbf{w}. \tag{12.10}$$

We denote $A^{(n)}_{i,v_i,\mathbf{w}}(1,z)$ by $A^{(n)}_{i,v_i,\mathbf{w}}(z)$ and $Q^{(n)}_{\mathbf{w}}(1,z)$ by $Q^{(n)}_{\mathbf{w}}(z)$. From the definition of $\mathcal{H}^{(n)}_{\mathbf{w}}(\alpha,z)$ and the same argument of the proof of Lemma 6.2.2 and that of [71, Corollary 2], we get the following proposition.

Proposition 12.3.2. (cf. [71, Corollary 2], [17, Corollary 5.2]) *Let $\alpha\in\overline{\mathbb{Q}}$. We assume that α satisfies $|\alpha|\ge 1$. Under the notation as above, we put $w=\sum_{i=1}^{r} w_i$. Then we obtain*

$$\mathcal{H}^{(n)}_{\mathbf{w}}(\alpha,z) = o(z^{-(ns+w+n(r-1)-3)}) \quad (z\to\infty),$$

and the following Padé approximation of the Lerch function:

$$\mathcal{H}^{(n)}_{\mathbf{w}}(\alpha,z) = \sum_{i=1}^{r}\sum_{v_i=1}^{s_i+1} A^{(n)}_{i,v_i,\mathbf{w}}(\alpha,z)\Phi(v_i,z+a_i,\alpha) - Q^{(n)}_{\mathbf{w}}(\alpha,z), \tag{12.11}$$

$$\mathcal{H}^{(n)}_{\mathbf{w}}(1,z) = \sum_{i=1}^{r}\sum_{v_i=1}^{s_i+1} A^{(n)}_{i,v_i,\mathbf{w}}(z)\Phi(v_i,z+a_i,1) - Q^{(n)}_{\mathbf{w}}(z). \tag{12.12}$$

12.4 Some estimations

We denote $(0,\dots,0) \in \prod_{i=1}^{r}\{0,\dots,s_i+1\}$ by $\mathbf{0}$. We fix a subset

$$\{\mathbf{w}_{i,j} := (w_{i,j}^{(1)},\dots,w_{i,j}^{(r)})\}_{1\le i\le r, 1\le j\le s_i+1} \subset \prod_{i=1}^{r}\{0,\dots,s_i+1\}$$

satisfying

$$w_{i,j}^{(k)} = \begin{cases} 0 & \text{if } k \neq i, \\ j & \text{if } k = i. \end{cases}$$

We denote the determinant of $(s+r+1)\times(s+r+1)$ matrix

$$\begin{pmatrix}
-Q_{\mathbf{0}}^{(n)}(x_1,z) & A_{1,1,\mathbf{0}}^{(n)} & \cdots & A_{1,s_1+1,\mathbf{0}}^{(n)} & \cdots & A_{r,1,\mathbf{0}}^{(n)} & \cdots & A_{r,s_r+1,\mathbf{0}}^{(n)} \\
-Q_{\mathbf{w}_{1,1}}^{(n)}(x_1,z) & A_{1,1,\mathbf{w}_{1,1}}^{(n)} & \cdots & A_{1,s_1+1,\mathbf{w}_{1,1}}^{(n)} & \cdots & A_{r,1,\mathbf{w}_{1,1}}^{(n)} & \cdots & A_{r,s_r+1,\mathbf{w}_{1,1}}^{(n)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q_{\mathbf{w}_{1,s_1+1}}^{(n)}(x_1,z) & A_{1,1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & A_{1,s_1+1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & A_{r,1,\mathbf{w}_{1,s_1+1}}^{(n)} & \cdots & A_{r,s_r+1,\mathbf{w}_{1,s_1+1}}^{(n)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q_{\mathbf{w}_{r,1}}^{(n)}(x_1,z) & A_{1,1,\mathbf{w}_{r,1}}^{(n)} & \cdots & A_{1,s_1+1,\mathbf{w}_{r,1}}^{(n)} & \cdots & A_{r,1,\mathbf{w}_{r,1}}^{(n)} & \cdots & A_{r,s_r,\mathbf{w}_{r,1}}^{(n)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q_{\mathbf{w}_{r,s_r+1}}^{(n)}(x_1,z) & A_{1,1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & A_{1,s_1+1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & A_{r,1,\mathbf{w}_{r,s_r+1}}^{(n)} & \cdots & A_{r,s_r+1,\mathbf{w}_{r,s_r+1}}^{(n)}
\end{pmatrix}$$

by $\Delta^{(n)}(x_1,z)$ for $n \in \mathbb{N}$ where we denote $A_{i,j,\mathbf{w}}^{(n)}(x_1,z)$ by $A_{i,j,\mathbf{w}}^{(n)}$.

Under the notation as above, we have the following lemma that corresponds to Assumption (5.18).

Lemma 12.4.1. (cf. [17, Proposition 5.9], [17, Proposition 5.10]) *Let $\Delta^{(n)}(x_1,z)$ be as above. Then $\Delta^{(n)}(\alpha,z)$ has zero only at $z \in \{-a_1,\dots,-a_r\}$ for $\alpha \in \{\alpha \in \overline{\mathbb{Q}} \mid |\alpha| = 1\} \setminus \{1\}$ and $n \in \mathbb{N}$.*

Proof of Lemma 12.4.1 is based on that of [17, Proposition 5.9]. Before proving Lemma 12.4.1, we give some preparation lemmas.

Lemma 12.4.2. *Let n be a natural number. Then $\Delta^{(n)}(x_1,z)$ is divisible by $\prod_{i=1}^{r}(z+a_i)^{s_i+1}$.*

Lemma 12.4.3. *Let n be a natural number and fix $x_1 \in \mathbb{C}$ satisfying $|x_1| > 1$. Then we have the following relation:*

$$\lim_{\mathrm{Re}(z)\to\infty} \frac{\Delta^{(n)}(x_1,z)}{z^{s+r}} < \infty.$$

Especially, we have $\deg_z \Delta^{(n)}(x_1,z) \le s+r$.

Note that from Lemma 12.4.2 and Lemma 12.4.3, there exists a polynomial $Q(x_1) \in \mathbb{Q}[x_1]$ satisfying

$$\Delta^{(n)}(x_1,z) = Q(x_1)\prod_{i=1}^{r}(z+a_i)^{s_i+1}. \tag{12.13}$$

Lemma 12.4.4. *Let n be a natural number. We have $\Delta^{(n)}(x_1,z) \neq 0$ and the following inequality:*

$$\deg_{x_1}\Delta^{(n)}(x_1,z) \le n(s+r)-1.$$

Lemma 12.4.5. *Let n be a natural number. Then $\Delta^{(n)}(x_1,z)$ is divisible by z^{n+1}.*

Lemma 12.4.6. *Let n be a natural number. Then $\Delta^{(n)}(x_1,z)$ is divisible by $(x_1-1)^{(s+r-1)n-2}$.*

Remark 12.4.7. Lemma 12.4.2, 12.4.3, 12.4.4, 12.4.5 and 12.4.6 are generalizations of Lemma 5.11, 5.13, 5.12, 5.15 and 5.16 in [17] respectively. Since Lemma 12.4.2, 12.4.3, 12.4.4, 12.4.5 and 12.4.6 can be proved by the same method of Lemma 5.11, 5.13, 5.12, 5.15 and 5.16 in [17] respectively, we omit the proof of them.

Proof of Lemma 12.4.1. From the equality (11.9), Lemma 12.4.4, Lemma 12.4.5 and Lemma 12.4.6, there exists an element $\delta \in \mathbb{Q}^*$ satisfying

$$\Delta^{(n)}(x_1, z) = \delta z^n (x_1 - 1)^{(s+r-1)n-2} \prod_{i=1}^{r} (z + a_i)^{s_i+1}. \tag{12.14}$$

The equality (12.14) shows Lemma 12.4.1. □

Remark 12.4.8. We explain the reason why we exclude $\alpha = 1$ in Theorem 12.1.2 for $r \geq 2$ (cf. Remark 12.1.3). For a set

$$\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\},$$

satisfying $\mathbf{w}_i \neq \mathbf{w}_j$ for $i \neq j$. We denote the following determinant of $(s+1) \times (s+1)$ matrix

$$\begin{pmatrix}
-Q^{(n)}_{\mathbf{w}_1}(z) & A^{(n)}_{1,2,\mathbf{w}_1}(z) & \dots & A^{(n)}_{1,s_1+1,\mathbf{w}_1}(x) & \cdots & A^{(n)}_{r,2,\mathbf{w}_1}(x) & \dots & A^{(n)}_{r,s_r+1,\mathbf{w}_1}(x) \\
-Q^{(n)}_{\mathbf{w}_2}(z) & A^{(n)}_{1,2,\mathbf{w}_2}(z) & \dots & A^{(n)}_{1,s_1+1,\mathbf{w}_2}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_2}(z) & \dots & A^{(n)}_{r,s_r+1,\mathbf{w}_2}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q^{(n)}_{\mathbf{w}_{s_1+1}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s_1+1}}(z) & \dots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s_1+1}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s_1+1}}(z) & \dots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s_1+1}}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q^{(n)}_{\mathbf{w}_{s-s_r+2}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s-s_r+2}}(z) & \dots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s-s_r+2}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s-s_r+2}}(z) & \dots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s-s_r+2}}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
-Q^{(n)}_{\mathbf{w}_{s+1}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s+1}}(z) & \dots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s+1}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s+1}}(z) & \dots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s+1}}(z)
\end{pmatrix}, \tag{12.15}$$

by $\tilde{\Delta}^{(n)}(z)$ for every $n \in \mathbb{N}$. For $r \geq 2$, we will show the following:

$$\tilde{\Delta}^{(n)}(z) = 0 \text{ for sufficiently large } n \text{ and any } \{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}. \tag{12.16}$$

Then Assumption (5.17) for $\{-Q^{(n)}_{\mathbf{w}_j}(x_1, z)\}_{1 \leq j \leq s+1} \cup \{A^{(n)}_{i,v_i,\mathbf{w}_j}(z)\}_{1 \leq i \leq r, 1 \leq v_i \leq s_i+1, 1 \leq j \leq s+1}$ is no longer satisfied for any $\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$. For this reason, we have to exclude $\alpha = 1$ in Theorem 12.1.2 for $r \geq 2$. We shall prove (12.16). Fix a set $\{\mathbf{w}_1, \dots, \mathbf{w}_{s+1}\} \subset \prod_{i=1}^{r} \{0, \dots, s_i + 1\}$ satisfying $\mathbf{w}_i \neq \mathbf{w}_j$ for $i \neq j$. By the same argument as the proof of Lemma 12.4.2 (cf. [17, Proposition 5.11]), we obtain

$$\prod_{i=1}^{r} (x + a_i)^{s_i} | \tilde{\Delta}^{(n)}(z) \text{ for all } n \in \mathbb{N}.$$

Especially we have

$$\deg \tilde{\Delta}^{(n)}(z) \geq s. \tag{12.17}$$

Next, we show the following:

$$\lim_{\mathrm{Re}(x) \to \infty} \frac{\tilde{\Delta}^{(n)}(z)}{z^s} = 0 \text{ for enough large } n. \tag{12.18}$$

Let i and j be integers. We assume that i and j satisfy $0 \leq i \leq r-1$ and $2 \leq j \leq s_{i+1}+1$. By adding the $\left(j+\sum_{l=1}^{i} s_l\right)$-th column of the matrix (12.15) multiplied by $\Phi(j, x+a_i, 1)$ to the first column of the matrix (12.15), we obtain the matrix (12.19) below. Note that if $i=0$, we mean $\sum_{l=1}^{i} s_l = 0$.

$$\begin{pmatrix}
\mathcal{H}^{(n)}_{\mathbf{w}_1}(z) & A^{(n)}_{1,2,\mathbf{w}_1}(z) & \cdots & A^{(n)}_{1,s_1+1,\mathbf{w}_1}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_1}(z) & \cdots & A^{(n)}_{r,s_r+1,\mathbf{w}_1}(z) \\
\mathcal{H}^{(n)}_{\mathbf{w}_2}(z) & A^{(n)}_{1,2,\mathbf{w}_2}(z) & \cdots & A^{(n)}_{1,s_1+1,\mathbf{w}_2}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_2}(z) & \cdots & A^{(n)}_{r,s_r+1,\mathbf{w}_2}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s_1+1}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s_1+1}}(z) & \cdots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s_1+1}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s_1+1}}(z) & \cdots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s_1+1}}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s-s_r+2}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s-s_r+2}}(z) & \cdots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s-s_r+2}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s-s_r+2}}(z) & \cdots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s-s_r+2}}(z) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\mathcal{H}^{(n)}_{\mathbf{w}_{s+1}}(z) & A^{(n)}_{1,2,\mathbf{w}_{s+1}}(z) & \cdots & A^{(n)}_{1,s_1+1,\mathbf{w}_{s+1}}(z) & \cdots & A^{(n)}_{r,2,\mathbf{w}_{s+1}}(z) & \cdots & A^{(n)}_{r,s_r+1,\mathbf{w}_{s+1}}(z)
\end{pmatrix}. \tag{12.19}$$

Since the determinant of (12.15) is equal to that of (12.19), the determinant of (12.19) coincides with $\tilde{\Delta}^{(n)}(x)$. Thus, it suffices to show that the determinant of (12.19) is zero. Denote the $(1,q)$-th cofactor matrix of the matrix (12.19) by $\tilde{\Delta}^{(n)}_q(z)$. We calculate the cofactor expansion of the matrix (12.19) at the first row, we obtain:

$$\tilde{\Delta}^{(n)}(z) = \sum_{q=1}^{s+1} (-1)^{q+1} \mathcal{H}^{(n)}_{\mathbf{w}_q}(z) \tilde{\Delta}^{(n)}_q(z).$$

From the definition of $\mathcal{H}^{(n)}_{\mathbf{w}_q}(z)$, we have

$$\mathcal{H}^{(n)}_{\mathbf{w}_q}(z)\tilde{\Delta}^{(n)}_q(z) = (n!)^{s+r-1} \sum_{m=0}^{\infty} \frac{(m)_{n+1}\tilde{\Delta}^{(n)}_q(z)}{\prod_{i=1}^{r}\left[(m+x+a_i)_n^{s_i+1}(m+x+a_i+n)^{w_{q,i}}\right]},$$

where $w_{q,i}$ is the i-th factor of $\mathbf{w}_q$. Since we have $\deg\tilde{\Delta}^{(n)}_q(z) \leq s(n+1)$ (see Remark 12.3.1 3 (12.10)),

$$\begin{aligned}
\left|\frac{(m)_{n+1}\tilde{\Delta}^{(n)}_q(z)}{z^s\prod_{i=1}^{r}\left[(m+z+a_i)_n^{s_i+1}(m+z+a_i+n)^{w_{q,i}}\right]}\right| &\leq \left|\frac{\tilde{\Delta}^{(n)}_q(z)}{z^{s(n+1)}} \frac{(m)_{n+1}}{\prod_{i=1}^{r}(m+z+a_i)_n}\right| \\
&\leq \left|\frac{\tilde{\Delta}^{(n)}_q(z)}{z^{s(n+1)}} \frac{(m)_{n+1}}{(m+z)^{n+3}} \frac{1}{z^{n(r-1)-3}}\right|,
\end{aligned}$$

and $r \geq 2$, we obtain $\lim_{\mathrm{Re}(z)\to\infty} \dfrac{\tilde{\Delta}^{(n)}(z)}{z^s} = 0$ for enough large n. By the relation (12.17) and (12.18), we obtain (12.16).

We have the following lemma that corresponds to Assumption (5.18).

Lemma 12.4.9. (cf. [17, Proposition 5.5]) *Let α be a non-zero algebraic number. Then we obtain the following relations:*

$$\begin{aligned}
&d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} p^{[n/(p-1)]} p^{\mathrm{ord}_p(\mathrm{den}(a_i))} \max\{1, |\alpha|_p\}^n A^{(n)}_{i,v_i,\mathbf{w}}(\alpha, z) \in \mathcal{O}_{\mathbb{C}_p}[z], \\
&d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} p^{[n/(p-1)]} p^{\mathrm{ord}_p(\mathrm{den}(a_i))} \max\{1, |\alpha|_p\}^n Q^{(n)}_{\mathbf{w}}(\alpha, z) \in \mathcal{O}_{\mathbb{C}_p}[z].
\end{aligned} \tag{12.20}$$

Let b be a rational number. We assume that b satisfies $b+a_i \neq 0$ for all $1 \leq i \leq r$. Then we also obtain the following relations:

$$\begin{aligned}
&d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} \mu_n(\mathrm{den}(b+a_i))\mathrm{den}(b)\mathrm{den}(\alpha)^n A^{(n)}_{i,v_i,\mathbf{w}}(\alpha, b) \in \mathcal{O}_{\mathbb{Q}(\alpha)}, \\
&d_n^{s_i+1} d_{e+Mn}^{s+r-s_i-1} \mu_n(\mathrm{den}(b+a_i))\mathrm{den}(b)\mathrm{den}(\alpha)^n Q^{(n)}_{\mathbf{w}}(\alpha, b) \in \mathcal{O}_{\mathbb{Q}(\alpha)}.
\end{aligned} \tag{12.21}$$

Proof. From the equality (12.7), we have

$$d^{(n)}_{i,j,v_i,\mathbf{w}}(z) = \tag{12.22}$$
$$\begin{cases} \dfrac{1}{(s_i-v_i+1)!}\left(\dfrac{d}{du}\right)^{s_i-v_i+1} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+j)^{s_i+1}|_{u=j} & \text{for } \ 0\le j\le n-1, 1\le v_i\le s_i+1, \\ \dfrac{1}{(w_i-v_i)!}\left(\dfrac{d}{du}\right)^{w_i-v_i} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+n)^{w_i}|_{u=n} & \text{for } \ j=n, 1\le v_i\le w_i, \\ 0 & \text{for } \ j=n, v_i>w_i. \end{cases}$$

We give natural numbers which are divisible by the denominator of $d^{(n)}_{i,j,v_i,\mathbf{w}}(z)$. Firstly, we calculate

$$\frac{1}{(s_i-v_i+1)!}\left(\frac{d}{du}\right)^{s_i-v_i+1} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+j)^{s_i+1}|_{u=j}$$

for $0\le j\le n-1, 1\le v_i\le s_i+1$. We have the following equality

$$H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+j)^{s_i+1} = \tag{12.23}$$
$$\frac{(n!)^{s+r-1}(-u-z-a_i)(-u-z-a_i+1)\cdots(-u-z-a_i+n)}{\prod_{i'\neq i}\left(\prod_{j=0}^{n-1}(-u+(a_{i'}-a_i)+j)^{s_{i'}+1}(-u+(a_{i'}-a_i)+n)^{w_{i'}}\right)\left(\prod_{j'=0,j'\neq j}^{n-1}(-u+j')^{s_i+1}(-u+n)^{w_i}\right)}$$
$$= \frac{(n!)^{s_i}(-u-z-a_i)\cdots(-u-z-a_i+n)}{\prod_{j'=0,j'\neq j}^{n-1}(-u+j')^{s_i+1}(-u+n)^{w_i}}\prod_{i'\neq i} I_{i',n}(u)^{w_{i'}} I_{i',n-1}(u)^{s_{i'}+1-w'_i},$$

where the functions $I_{i',n}(u)$ and $I_{i',n-1}(u)$ are as follows:

$$I_{i',n}(u) := \frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n)} \ \text{ for } i'\neq i, \tag{12.24}$$
$$I_{i',n-1}(u) := \frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n-1)} \ \text{ for } i'\neq i. \tag{12.25}$$

From the proof of Bel (cf. [17] p. 204), we have the following equality:

$$\frac{(n!)^{s_i}(-u-z-a_i)\cdots(-u-z-a_i+n)}{\prod_{j'=0,j'\neq j}^{n-1}(-u+j')^{s_i+1}(-u+n)^{w_i}} = F(u)G(u)^{s_i}H(u), \tag{12.26}$$

where

$$F(u) = \frac{(-u-z-a_i)_n}{(-u)_{n+1}}(-u+j), \ G(u) = \frac{n!}{(-u)_{n+1}}(-u+j),$$

and

$$H(u) = (n-u)^{s_i+1-w_i}(-u-z-a_i+n).$$

Also we have the equalities:

$$\frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n)} = \sum_{j'=0}^{n}(-1)^{j'}\frac{n!}{j'!(n-j')!}\frac{1}{-u+a'_i-a_i+j'}, \tag{12.27}$$
$$\frac{n!}{(-u+a_{i'}-a_i)\cdots(-u+a_{i'}-a_i+n-1)} = \sum_{j'=0}^{n-1}(-1)^{j'}\frac{n!}{j'!(n-j'-1)!}\frac{1}{-u+a'_i-a_i+j'}. \tag{12.28}$$

For a non-negative integer ν, we denote $\dfrac{1}{\nu!}\left(\dfrac{d}{du}\right)^{\nu}$ by ∂_ν. From the equality (12.23), we obtain

$$\frac{1}{(s_i-v_i+1)!}\left(\frac{d}{du}\right)^{s_i-v_i+1} H^{(n)}_{\mathbf{w}}(z,-u-z-a_i)(-u+j)^{s_i+1}|_{u=j} = \tag{12.29}$$
$$\sum_{\boldsymbol{\nu}}\partial_{\nu_0}(F)|_{u=j}\prod_{k_1=1}^{s_i}\partial_{\nu_{k_1}}(G)|_{u=j}\partial_{\nu_{s_i+1}}(H)|_{u=j}\times\prod_{i'\neq i}\prod_{k_2=1}^{w_{i'}}\partial_{\nu_{k_2}}(I_{i',n})|_{u=j}\prod_{k_3=1}^{s_{i'}+1-w_{i'}}\partial_{\nu_{k_3}}(I_{i',n-1})|_{u=j}.$$

Here the sum $\sum_{\boldsymbol{\nu}}$ stands for all possible summation arising from the Leibniz rule. Note that the "index" of $\partial_{\nu_{s_i+1}}$ is ν_{s_i+1} and it is not $\nu_{s_i}+1$. As for the second case of (12.22), we obtain a similar presentation to (12.29). The argument to deduce the presentation for the second case is the same as (77) to (82) for the first case. Finally, by applying the same argument as the proof of [17, Proposition 5.5] to these representations for (12.22), we conclude (12.20) and (12.21). $\square$

For a rational number b satisfying $b+a_i\neq 0$ for all $1\leq i\leq r$, we denote l.c.m.$\{\mathrm{den}(b+a_i)\}_{1\leq i\leq r}$ by $B(b)$. We define the following functions:

$$D_n: D_p(\mathbb{Q})\longrightarrow \mathbb{Z}\setminus\{0\} \text{ by } b\mapsto d_n^{S+1}d_{e+Mn}^{s+r-T-1}\mu_n(B(b))\mathrm{den}(b)\mathrm{den}(\alpha)^n,$$
$$f^{(p)}: D_p(\mathbb{Q})\longrightarrow \mathbb{R}_{\geq 0} \text{ by } b\mapsto S+M(s+r-T-1)+\sum_{\substack{q:\text{prime}\\ q|B(b)}}\frac{\log q}{q-1}+\log\mathrm{den}(\alpha).$$

From Proposition 12.4.9, $\{D_n\}_{n\in\mathbb{N}}$ satisfies Assumption (5.19) and there exists $c_1>0$ such that

$$|D_n(b)|\leq n^{c_1}e^{nf^{(p)}(b)}. \tag{12.30}$$

The inequality (12.30) corresponds to Assumption (5.19). We have the following estimate that corresponds to Assumption (5.20).

Lemma 12.4.10. *Let β be a complex number. We assume that β satisfies $\beta+a_i\notin\mathbb{Z}_{\leq 0}$ for all $1\leq i\leq r$. If $n\in\mathbb{N}$ is enough large, then there exists $c>0$ which is independent of n and satisfies the following inequality:*

$$\max_{1\leq i\leq r, 1\leq v_i\leq s_i+1}\{|A^{(n)}_{i,v_i,\mathbf{w}}(\alpha,\beta)|,|Q^{(n)}_{\mathbf{w}}(\alpha,\beta)|\}\leq n^c\max\{1,|\alpha|^n\}\exp(ns\log 2). \tag{12.31}$$

Proof. We fix a enough large natural number k satisfying the following conditions:

$$|a_{i_1}-a_{i_2}|>\frac{2}{k} \text{ and } 1>|a_{i_1}-a_{i_2}|+\frac{2}{k} \text{ for all } 1\leq i_1,i_2\leq r, i_1\neq i_2. \tag{12.32}$$

Firstly, we give an upper bound of $\{|d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)|\}_{1\leq j\leq n,1\leq v_i\leq s_i+1}$ for fixed i.

We fix $1\leq i\leq r, 1\leq v_i\leq s_i+1$ and $1\leq j\leq n$. Using the definition of $d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)$ given by (12.7), we get

$$d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)=\frac{1}{2\pi\sqrt{-1}}\int_{|u+j+\beta+a_i|=\frac{1}{k}}H^{(n)}_{\mathbf{w}}(u,z)(u+\beta+a_i+j)^{v_i-1}du. \tag{12.33}$$

From the equality (12.33) and the definition of $Q^{(n)}_{\mathbf{w}}(u)$, we obtain

$$\begin{aligned}|d^{(n)}_{i,j,v_i,\mathbf{w}}(\beta)| &\leq k^{-v_i}\sup_{|u+\beta+a_i+j|=\frac{1}{k}}|H^{(n)}_{\mathbf{w}}(u,z)| \\ &\leq k^{-v_i}\sup_{|u+\beta+a_i+j|=\frac{1}{k}}\left|\frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^r\left[(u+\beta+a_{i'})_n^{s_{i'}+1}(u+\beta+n+a_{i'})^{w_{i'}}\right]}\right|.\end{aligned} \tag{12.34}$$

We give an upper bound of $\left|\frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^r\left[(u+\beta+a_{i'})_n^{s_{i'}+1}(u+\beta+n+a_{i'})^{w_{i'}}\right]}\right|$. We have the following inequalities for $u\in\{u\in\mathbb{C}|\ |u+\beta+a_i+j|=\frac{1}{k}\}$:

$$\begin{aligned}|(u)_{n+1}| &= |(u+\beta+a_i+j-\beta-a_i-j)\cdots(u+\beta+a_i+j+n-\beta-a_i-j)| \\ &\leq (f+j)!(f+n-j)!,\end{aligned} \tag{12.35}$$

where f is a natural number satisfying $\beta + a_i + \frac{1}{k} \leq f$. Estimating a lower bound of $|(u+\beta+a_{i'})_n|$ and $|u+\beta+a_{i'}+n|$ for $u \in \{u \in \mathbb{C} |\ |u+\beta+a_i+j| = \frac{1}{k}\}$, we give a lower bound of $|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)|$ for $1 \leq i' \leq r$, $0 \leq l \leq n$:

(In the case of $i' = i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i-\beta)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, j+1, \\ j-l-1 & \text{if } j-1 > l, \\ l-j-1 & \text{if } l > j+1. \end{cases} \tag{12.36}$$

(In the case of $i' > i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j-1, j, \\ j-l-1 & \text{if } j-1 > l, \\ l-j & \text{if } l > j. \end{cases} \tag{12.37}$$

(In the case of $i' < i$)

$$|u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \geq \begin{cases} \frac{1}{k} & \text{if } l = j, j+1, \\ j-l & \text{if } j > l, \\ l-j-1 & \text{if } l > j+1. \end{cases} \tag{12.38}$$

From the inequalities (12.38), (12.37) and (12.38), we have the following estimation for $1 \leq i' \leq r$:

$$\begin{aligned} |(u+\beta+a_{i'})_n| &= \prod_{l=0}^{n-1} |u+\beta+a_{i'}+l| \\ &= \prod_{l=0}^{n-1} |u+\beta+a_i+j+(a_{i'}-a_i)+(l-j)| \\ &\geq \frac{(n-j)!j!}{k^3n^3}. \end{aligned} \tag{12.39}$$

We also have the inequality:

$$|u+\beta+a_{i'}+n| = |u+\beta+a_i+j+(a_{i'}-a_i)+(n-j)| \geq \frac{1}{k}, \tag{12.40}$$

for $1 \leq i' \leq r$. From the inequalities (12.34), (12.35), (12.39), (12.40), we obtain

$$\begin{aligned} & k^{-v_i} \sup_{|u+\beta+a_i+j|=\frac{1}{k}} \left| \frac{(n!)^{s+r-1}(u)_{n+1}}{\prod_{i'=1}^{r} (u+a_{i'})_n{}^{s_{i'}+1}(u+n+a_{i'})^{w_{i'}}} \right| \\ & \leq n^{c_0} \frac{(n!)^{s+r-1}j!(n-j)!}{((n-j)!j!)^{s+r}} \\ & = n^{c_1} \binom{n}{j}^{s+r-1} \\ & \leq n^{c_2} 2^{sn}. \end{aligned} \tag{12.41}$$

where c_0, c_1, c_2 are positive constants. From the inequality (12.41) and the definition of $A^{(n)}_{i,v_i,\mathbf{w}}(\alpha, \beta)$ and $Q^{(n)}_{\mathbf{w}}(\alpha, \beta)$, we obtain the desired estimate. $\square$

We define the following function on $D_p(\mathbb{Q})$:

$$g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \text{ by } b \mapsto \log\max\{1, |\alpha|\} + s\log 2.$$

Then there exists $c_2 > 0$ such that

$$\max_{1\leq i\leq r, 1\leq v_i\leq s_i+1, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}\{|A^{(n)}_{i,v_i,\mathbf{w}}(\alpha, b)|, |Q^{(n)}_{\mathbf{w}}(\alpha, b)|\} \leq n^{c_2}e^{ng^{(p)}(b)} \text{ for all } b \in D_p(\mathbb{Q}). \tag{12.42}$$

12.5 Proof of Theorem 12.1.2

We use the notations of the previous section. We have the following property of p-adic absolute value of the coefficients of $\Xi_p(v, a, \alpha, z)$ for $a \in \mathbb{Z}_p \cap \mathbb{Q}$ and $\alpha \in \{\alpha \in \overline{\mathbb{Q}}| \ |\alpha| = 1$ satisfying $\alpha = 1$ or $|\alpha - 1|_p \leq 1$.

LEMMA 12.5.1. *Let $a \in \mathbb{Q} \cap \mathbb{Z}_p$. Take $\alpha \in \{\alpha \in \overline{\mathbb{Q}}| \ |\alpha| = 1\}$ satisfying $\alpha = 1$ or $1 \leq |\alpha - 1|_p$. Let v be a natural number. We assume that v satisfies $v \geq 2$ (resp. $v \geq 1$) if $\alpha = 1$ (resp. $1 \leq |\alpha - 1|_p$). Then the set $\{|B_k(a, \alpha)|_p\}_{k\in\mathbb{Z}_{\geq 0}}$ is bound.*

PROOF. Firstly, we assume $\alpha = 1$. By the definition of $B_k(a, 1)$, we have the following equality:

$$B_k(a, 1) = \sum_{i=0}^{k}\binom{k}{i}B_i a^{k-i}.$$

Using Theorem of Clausen-Von Staudt that gives an upper bound of p-adic absolute value of Bernoulli numbers and the assumption for a, we obtain that the set $\{|B_k(a, 1)|_p\}_{k\in\mathbb{Z}_{\geq 0}}$ is a bounded set. Secondly, we assume $1 \leq |\alpha - 1|_p$. Note that, from the definition of $B_k(a, \alpha)$, we have the following equality:

$$\frac{Te^{aT}}{\alpha e^T - 1} = \sum_{k=0}^{\infty} B_k(a, \alpha)\frac{T^k}{k!} = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}B_{i,\alpha}a^{k-i}\right)\frac{T^{k+1}}{k!}$$

where $B_{i,\alpha}$ is defined by the generating function $\dfrac{1}{\alpha e^T - 1} = \displaystyle\sum_{k=0}^{\infty}\frac{B_{k,\alpha}}{k!}T^k$. Since $1 \leq |\alpha - 1|_p$, we have that the set $\{|B_{k,\alpha}|_p\}_{k\in\mathbb{Z}_{\geq 0}}$ is bounded [58, p. 24] . This completes the proof of Lemma 12.5.1. □

We define the following functions:

$$E_n : D_p(\mathbb{Q}) \longrightarrow \mathcal{O}_{\mathbb{C}_p} \setminus \{0\} \text{ by } b \mapsto d_n^{S+1}d_{e+Mn}^{s+r-T-1}p^{[n/(p-1)]}p^{\mathrm{ord}_p(A)}\max\{1, |\alpha|_p\}^n. \tag{12.43}$$

By Lemma 12.5.1, the set of coefficients of $\Xi_p(v_i, a_i, \alpha, z)$ is bounded for all $1 \leq i \leq r, 1 \leq v_i \leq s_i$. Then from Remark 5.2.7 and the relation (12.20), there exists $c_3 > 0$ satisfying

$$|E_n(b)\hat{\mathcal{R}}^{(n)}_{w,p}(b, \alpha)|_p \leq n^{c_3}|b|_p^{-nT} \text{ for some constant } c_3 > 0. \tag{12.44}$$

The inequality (12.44) corresponds to (5.22) in Assumption 5.2.5. We define $h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$h^{(p)}(b) = \sum_{\substack{q:\text{prime}\\ q|A}}\frac{\log q}{q-1} - \frac{\log p}{p-1} + \log\mathrm{den}(\alpha) - \log\max\{1, |\alpha|_p\}.$$

Proof of Theorem 12.1.2. Let $\{a_1, \ldots, a_r\} \subset \mathbb{Q} \cap \mathbb{Z}_p$. We assume that $a_1, \ldots, a_r$ satisfy $0 < a_1 < \cdots < a_r \leq 1$. We use the notations as above and Theorem 5.2.8 for

$$f_{(i,v_i)}(x_1, z) = \Phi(v_i, z + a_1, x_1) : \mathbb{R}_{>0} \longrightarrow \mathbb{C}, \ \ 1 \leq i \leq r, \ 1 \leq v_i \leq s_i + 1.$$

From Section 10.1, we define the following functions:

$$
\begin{aligned}
A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z) &= \sum_{j=0}^{n} d^{(n)}_{i,j,v_i,\mathbf{w}}(z)x_1^j,\\
Q^{(n)}_{\mathbf{w}}(x_1,z) &= \sum_{i=1}^{r}\sum_{j=0}^{n}\sum_{v_i=1}^{s_i+1}\sum_{l=0}^{j-1} d^{(n)}_{i,j,v_i,\mathbf{w}}(z)\frac{x_1^{j-1-l}}{(l+a_i)^{v_i}},\\
\mathcal{H}^{(n)}_{\mathbf{w}}(x_1,z) &:= \sum_{m=0}^{\infty} H^{(n)}_{\mathbf{w}}(z,m){x_1}^{-m-1}.
\end{aligned}
$$

By Lemma 12.5.1, the set $\{f_{(i,v_i)}(x_1,z)\}_{1\le i\le r,\ 1\le v_i\le s_i+1}$ satisfies Assumption 5.2.3. From Section 11.1, we defined the following five functions:

$$
\begin{aligned}
&D_n : D_p(\mathbb{Q}) \longrightarrow \mathbb{N} \text{ by } b \mapsto d_n^{S+1}d_{e+Mn}^{s+r-T-1}\mu_n(B(b))\mathrm{den}(b)\mathrm{den}(\alpha)^n,\\
&E_n : D_p(\mathbb{Q}) \longrightarrow \mathbb{Z}\setminus\{0\} \text{ by } b \mapsto d_n^{S+1}d_{e+Mn}^{s+r-T-1}p^{[n/(p-1)]}p^{\mathrm{ord}_p(A)}\max\{1,|\alpha|_p\}^n,\\
&f^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } b \mapsto S+M(s+r-T-1)+\sum_{\substack{q:\mathrm{prime}\\ q|B(b)}}\frac{\log q}{q-1}+\log\mathrm{den}(\alpha),\\
&g^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } b \mapsto \log\max\{1,|\alpha|\}+s\log 2.\\
&h^{(p)} : D_p(\mathbb{Q}) \longrightarrow \mathbb{R}_{\ge 0} \text{ by } b \mapsto \sum_{\substack{q:\mathrm{prime}\\ q|A}}\frac{\log q}{q-1}-\frac{\log p}{p-1}+\log\mathrm{den}(\alpha)-\log\max\{1,|\alpha|_p\}.
\end{aligned}
$$

From the lemmas in Section 11.1, we can easily check that the functions $\mathcal{R}^{(n)}_{\mathbf{w}}(\alpha,z) := \mathcal{H}^{(n)}_{\mathbf{w}}(\alpha,z)$, $\{A^{(n)}_{i,v_i,\mathbf{w}}(x_1,z)\}_{1\le i\le r, 0\le j\le s_i+1, \mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}} \cup \{-Q^{(n)}_{\mathbf{w}}(x_1,z)\}_{\mathbf{w}\in\{\mathbf{0},\mathbf{w}_{i,j}\}}$, D_n, E_n, $f^{(p)}$, $g^{(p)}$ and $h^{(p)}$ satisfy Assumption 5.2.4 and Assumption 5.2.5. Applying Theorem 5.2.8, we obtain the desired estimate in Theorem 12.1.2. $\square$

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