Twist deformations in affine geometry

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Doctoral thesis

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Contents

1 Introduction

The content of this doctoral thesis is a theory of affine deformations related to the twist deformations in the hyperbolic geometry. The goal of the thesis is to prove the three main theorem in [26]. We also mention the general results of the theorems, which are shown in [27].

In the theory of affine deformations, the cocycles play important roles. Indeed, we regard the cycles as mathematical objects in the two aspects, the Lorentzian geometry and the deformation theory of hyperbolic structures. In the Lorentzian geometry, cocycles correspond to the translation parts of the affine actions, and in the deformation theory of hyperbolic structures, cocycles represent the infinitesimal deformations of the structures. We will define special cocycles, which are called the *affine twist cocycles*, and observe an analogue between the two aspects.

1.1 Affine deformation

Let G be a finitely generated Fuchsian group, which is isomorphic to a free group in $\mathrm{PSL}(2,\mathbb{R})$. An *affine deformation* of G is a homomorphism of G into the isometry group on the Minkowski space-time $\mathbb{R}^3_1$, whose restriction to the the linear part is an identity map through the identification $\mathrm{SO}(2,1)^\circ \cong \mathrm{PSL}(2,\mathbb{R})$. We call the restricted map to the translation part a *cocycle*. With G fixed, a space of affine deformations is canonically identified with the space of the cocycles up to translational conjugacy, which we denote by $\mathrm{H}^1(G,\mathbb{R}^3_1)$. We define the affine twist cocycle $\mathbf{AT}_g$ for $g \in G$ which corresponds to a separating loop on the quotient surface $S = \mathbb{H}/G$.

Suppose that S is homeomorphic to an interior of a compact orientable surface with boundary, which consists of more than three boundary components. Let us take a pants-decomposition of S, and assign each of the dividing curves and the original boundary curves to an real value, respectively. We denote the set of the values by $\mathbf{a} \in \mathbb{R}^{2b-1}$, where b is the number of the boundary components of S. Then there exists a cocycle $\mathbf{u}_0^{\mathbf{a}}$ whose Margulis invariants corresponding to the above closed curves are equal to the given value in $\mathbf{a} \in \mathbb{R}^{2b-1}$ (*cf.* Proposition 11), where the Margulis invariant is a translation length of an affine action of $\mathbb{R}^3_1$. By using the affine twist cocycles and Proposition 11, we will parametrize $\mathrm{H}^1(G, \mathbb{R}^3_1)$. Namely, we will prove the following:

Theorem 1 (Theorem 1.1 in [26]). *Fix a pants-decomposition of a punctured sphere S without cusps. For the values $(\mathbf{a}, \mathbf{t}) \in \mathbb{R}^{2b-1} \times \mathbb{R}^{b-2}$, the map corresponding to the following cocycle gives a canonical linear isomorphism from $\mathbb{R}^{3b-3}$ to $\mathrm{H}^1(G, \mathbb{R}^3_1)$;*

$$\mathbf{u}_0^{\mathbf{a}} + \sum_{k=1}^{b-2} t_k \mathbf{A}\mathbf{T}_{g_k}, \tag{1}$$

where $t_k \in \mathbf{t}$ and g_k is the separating simple closed curve $(k = 1, \ldots, b-2)$ with respect to the pants-decomposition.

This claim is a little stronger than Theorem 1.1 of [26], however it follows from the same argument as that for the proof of Theorem 1 in [26].

On the other hand, the works of Goldman and Margulis [24] implies the parametrization of $\mathrm{H}^1(G, \mathbb{R}^3_1)$, however our parametrization is different from theirs.

The cases that G be the free group of rank two are concretely studied by

V.Charette T.Drumm and Goldman(See [8, 9, 10]). Theorem 1 in this thesis is recognized as a natural extension of their works of an affine deformation theory.

1.2 Infinitesimal deformation

We discuss the deformation of hyperbolic structures along the affine twist cocycles. Since there is a canonical isomorphism between $\mathbb{R}^3_1$ and the Lie algebra $\mathfrak{psl}_2(\mathbb{R})$, Goldman and Margulis applied the cocycles and the Margulis invariant for investigating the infinitesimal deformation of a hyperbolic surface. They regards a cocycle as an infinitesimal deformation of $\sigma \in G$ as follows:

$$\sigma^{\mathbf{u}}(s) := \sigma \exp(s\mathbf{u}(\sigma) + o(s)), \tag{2}$$

where s moves in the small interval containing 0. Through this correspondence, it is shown that the value of the Margulis invariant of σ is equal to half of $\frac{d}{ds}\mathbf{L}(\sigma^{\mathbf{u}}(s))|_{s=0}$, where $\mathbf{L}$ is the displacement length of an element of G in the hyperbolic geometry. Above results are referred to Section 5 and 6 in [24]. We apply the two relations to show the following formula.

Theorem 2 ([26, 27])**.** *Let $S = \mathbb{H}/G$ be a punctured sphere without cusps, and g be any separating simple closed curve in S. Then*

$$\left.\frac{d}{ds}\right|_{s=0} \mathbf{L}(\sigma^{\mathbf{AT}_g}(s)) = 2 \sum_{p \in \sigma \cap g} \cos\theta_p \tag{3}$$

holds, where σ is a simple closed curve in S. Plus, θ_p is the angle at p between the geodesics g and σ.

The angle at p is chosen by the segment of the left side of g along σ. The equation (3) is observed when σ is only a special loop in [26] and when σ is a general closed loop in [27]. The equation (3) is regarded as an analogue of the formula of Wolpert, which shows the correspondence between the twist deformation and the angles by the closed curves(introduced in Section 3 of [28]).

Theorem 3. *The affine twist cocycle* $\mathbf{AT}_g$ *along the separating simple closed curve* g *can be regarded as a cocycle corresponding to the infinitesimal twist deformation along* g*.*

1.3 Case of surfaces with positive genus

When G is a Fuchsian group whose associated surface is not planer, the same conclusions are shown as the extended results of the three theorems (*cf.* [27]). The arguments of the proofs in the higher genus case is almost same as that for the planer case. However, we need some discussion for getting over difficulties rising from the complexity from the handles.

1.4 Organization of this thesis

This thesis is organized as follows:

Some definitions and notations are introduced briefly in Section 2, including topics from a Lorentzian geometry and a hyperbolic geometry. In Section 3 and 4, we state the theories related directly to our main theorems. In Section 3, we define the affine deformation and its related topics. Furthermore, we mention proper affine deformations in Subsection 3.2. In Section 4, we introduce the theory of infinitesimal deformations of hyperbolic surfaces. There are two topics in this section: First is the application of the Lorentzian geometry to the theory of infinitesimal deformations of a surface, and second is the theory of infinitesimal Fenchel-Nielsen twist deformations.

Finally we show the main results in Section 5. An affine twist cocycle is formally defined in Subsection 5.1. In Subsection 5.2, we prove Theorem 1 on the basis of Section 3. In Subsection 5.3, we show the corollary following immediately by Theorem 2, and subsequently prove Theorem 2. With the two topics of Section 4, we obtain Theorem 3 as mentioned above.

2 Basic theory and Notation

In this section, basic topics will be introduced briefly, and notations will be defined.

Lorentzian geometry The Lorentzian space-time $\mathbb{R}^3_1$ of three dimension is a vector space equipped with the Lorentzian inner product $\mathbf{B}$. The Minkowski space-time of dimention three is an affine space with $\mathbb{R}^3_1$ as the base space. By abuse of the notation, we use the same notation $\mathbb{R}^3_1$ as both a vector space and an affine space. A *linear* isometry on $\mathbb{R}^3_1$ is represented by the matrix g of the Lorentzian group $\mathrm{O}(2,1)$, which is denoted by $\mathrm{Isom}(\mathbb{R}^3_1)$. An *affine* isometry on $\mathbb{R}^3_1$ has a form $(g, \mathbf{u})$ for $\mathbf{u} \in \mathbb{R}^3_1$, where $(g, \mathbf{u})$ acts on $\mathbb{R}^3_1$ by

$$(g, \mathbf{u})\mathbf{x} = g\mathbf{x} + \mathbf{u}, \ \mathbf{x} \in \mathbb{R}^3_1. \tag{4}$$

Affine isometries forms the isometry group $\mathrm{O}(2,1) \ltimes \mathbb{R}^3$ on the Minkowski space-time, which is denoted by $\mathrm{AfIsom}(\mathbb{R}^3_1)$. From now on, we will only consider the identity component $\mathbf{G} := \mathrm{SO}(\mathbf{2},\mathbf{1})^\circ$ of $\mathrm{O}(2,1)$.

The hyperbolic plane $\mathbb{H}$ is realized as a subspace of the Lorentzian space-time $\mathbb{R}^3_1$; the following subset consists of two disjoint sheets:

$$\{\mathbf{x} \in \mathbb{R}^3_1 \mid \mathrm{B}(\mathbf{x}, \mathbf{x}) = -1\} \tag{5}$$

We take one sheet from the set (5) and fix it, then we can naturally regard the sheet as a hyperbolic plane $\mathbb{H}$. We call a vector starting the origin in $\mathbb{R}^3_1$ *future-pointing* if a ray defined by the vector intersects the sheet $\mathbb{H}$. The restriction of the action of an isometry of $\mathbf{G}$ on $\mathbb{H}$ corresponds to the action of Möbius transformation of $\mathrm{PSL}(2,\mathbb{R})$, which preserves the hyperbolic metric on $\mathbb{H}$. This gives the identification

$$\mathbf{G} = \mathrm{SO}(2,1)^{\circ} \cong \mathrm{PSL}(2,\mathbb{R}). \tag{6}$$

Lie algebra The Lie algebra $\mathfrak{psl}(2,\mathbb{R})$ is a tangent space at the identity of $\mathrm{PSL}(2,\mathbb{R})$ as a Lie group. The Lie algebra $\mathfrak{psl}(2,\mathbb{R})$ admits a Killing form $\widetilde{\mathbf{B}}$ as follows:

$$\widetilde{\mathbf{B}}(\mathbf{x},\mathbf{y}) := \frac{1}{2}\mathrm{trace}(\mathrm{ad}(\mathbf{x})\mathrm{ad}(\mathbf{y})), \tag{7}$$

where $\mathbf{x},\mathbf{y} \in \mathfrak{psl}(2,\mathbb{R})$ and the adjoint map ad maps to $\mathrm{SL}(2,\mathbb{R})$. There is a natural linear isomorphism $\psi : \mathfrak{psl}(2,\mathbb{R}) \to \mathbb{R}^3_1$ such that the Killing form $\widetilde{\mathbf{B}}$ is compatible with $\mathbf{B}$ on $\mathbb{R}^3_1$.

Hyperbolic surfaces Let $G \subset \mathbf{G}$ be a finitely generated Fuchsian group, where a *Fuchsian group*, by definition, is a discrete subgroup of $\mathbf{G}$. Then the action of G as the Möbius transformation on $\mathbb{H}$ is properly discontinuous. Throughout this thesis, we always suppose that G is isomorphic to the free group of rank b. In fact, the quotient surface $\mathbb{H}/G$ is homeomorphic to the interior of a compact orientable surface with boundary. Any component of the boundary of the surface are funnels or cusps. A funnel corresponds to a hyperbolic element of G and a cusp corresponds to a parabolic elements of G. Note that G consists of hyperbolic elements if the associated surface S has no cusp.

Finally we define the translation length $\mathbf{L}$ in $\mathbb{H}$ with respect to the hyperbolic distance d of $\mathbb{H}$; for a hyperbolic element $g \in \mathbf{G}$,

$$\mathbf{L}(g) := \inf_{\mathbf{x} \in \mathbb{H}} \{\, \mathrm{d}(g(\mathbf{x}), \mathbf{x}) \,\}, \tag{8}$$

where the action $g(\mathbf{x})$ is as the Möbius transformation.

3 Affine deformation

In the former part of this section, we define affine deformations. The study for affine deformations is mainly developed by Drumm and Goldman(for example, [15, 20, 21]). The definitions and the notations in this section are based on the papers ([8, 20]). In the latter part of this section, we refer to the result of the author about properness of affine deformations.

3.1 Affine deformation

Affine deformations Let $G \subset \mathbf{G}$ be a Fuchsian group. An *affine deformation* of G is a homomorphism $\rho : G \to \mathrm{AfIsom}(\mathbb{R}^3_1)$ such that the restriction on the linear part is the identity map. Since the concept of affine deformation is in the affine geometry, it is natural to treat affine deformations up to translational conjugacy. Let $\mathrm{H}^1(G, \mathbb{R}^3_1)$ denote the set of all affine deformations of G up to translational conjugacy.

Cocycles A *cocycle* $\mathbf{u}$ on G is a map $:G \to \mathbb{R}^3_1$, which satisfies the following cocycle condition: For any $h_1, h_2 \in \mathbf{G} = \mathrm{SO}(\mathbf{2}, \mathbf{1})^{\mathbf{0}}$,

$$\mathbf{u}(h_1 h_2) = h_1 \mathbf{u}(h_2) + \mathbf{u}(h_1). \tag{9}$$

The restriction of an affine deformation ρ on the translation part is a cocycle. Conversely, a cocycle $\mathbf{u}$ defines an affine deformation by $h \to (h, \mathbf{u}(h))$ for $h \in G$. So we may denote an affine deformation by $\rho_{\mathbf{u}}$ if the cocycle $\mathbf{u}$ is specified. Thus, the space $\mathrm{H}^1(G, \mathbb{R}^3_1)$ of affine deformations of G can be regarded as a space of all cocycles up to translational conjugacy.

Principal vector A hyperbolic element $g \in \mathbf{G}$ has three distinct fixed lines L_{att}, L_{res} and L_{pri} passing the origin in $\mathbb{R}^3_1$; L_{att} (*resp.* L_{rep}) is tangent to $\mathbb{H}$ and has an eigenvalue larger (*resp.* smaller) than 1. The last line L_{pri} is a space-like line and its eigenvalue is just 1.

Let us take future-pointing vectors $\mathbf{X}^+_g$ and $\mathbf{X}^-_g$ in the lines L_{att} and L_{res} respectively. We can define a unique vector $\mathbf{X}^0_g$ of g with the following two conditions satisfied, regardless of the choice of $\mathbf{X}^+_g$ and $\mathbf{X}^-_g$:

$$\mathbf{Det}\{\mathbf{X}^0_g, \mathbf{X}^-_g, \mathbf{X}^+_g\} > 0 \ ; \tag{10}$$

$$\mathbf{B}(\mathbf{X}^0_g, \mathbf{X}^0_g) = 1. \tag{11}$$

We call $\mathbf{X}^0_g$ the *principal vector* of g.

Margulis invariant Margulis [25] introduced an invariant with respect to affine deformations, which is recently called the *Margulis invariant*. The Margulis invariant is a function $\alpha_{\mathbf{u}} : G \to \mathbb{R}$, which assigns $g \in G$ to the translation length of $\rho_{\mathbf{u}}(g)$ in $\mathbb{R}^3_1$; for an affine deformation $\rho_{\mathbf{u}}$ and $g \in \mathbf{G}$,

$$\alpha_{\mathbf{u}}(g) := \mathbf{B}(\rho(g)(\mathbf{x}) - \mathbf{x}, \mathbf{X}^0_g), \tag{12}$$

where any $\mathbf{x} \in \mathbb{R}^3_1$. In fact, the choice of $\mathbf{x} \in \mathbb{R}^3_1$ is independent of the value of this function(*cf.* **3** of [25]). The following lemma tells us that the Margulis invariant is one of the fundamental invariants in the theory of affine deformations.

Lemma 4 ([25, 21, 7])**.** *Two affine deformations $\rho_{\mathbf{u}}$ and $\rho_{\tilde{\mathbf{u}}}$ of G are same class in $\mathrm{H}^1(G, \mathbb{R}^3_1)$ if and only if $\alpha_{\mathbf{u}}(g) = \alpha_{\tilde{\mathbf{u}}}(g)$ holds for every $g \in G$.*

Namely the image of the Margulis invariant determines a conjugacy class of an cocycle in $\mathrm{H}^1(G, \mathbb{R}^3_1)$. With only this lemma, we need to check all values of the image in order to estimate the conjugacy classes of cocycles. A necessity of affine twist cocycles derives from this idea.

3.2 Proper affine deformation

Here the author refers to proper affine deformations. A *proper* affine deformation ρ is an affine deformation, whose image acts on the Minkowski space-time $\mathbb{R}^3_1$ properly discontinuously. Then the quotient manifold is a (geodesically) complete flat Lorentz 3-manifold.

Though the works [12, 13, 14] by J.Danciger, F.Gueritaud and F.Kassel revealed the classification of proper affine deformations, the author constructs the proper affine deformations in the case of the punctured sphere. This result is an application of the works [8, 10] by Charette, Drumm and Godlman of proper affine deformations to other surfaces. Their works are to construct proper affine deformations and its fundamental domain *concretely*. Our application to the surface gives a part of proper affine deformations with respect to the parametrization in Theorem 1. This application is mentioned as the examples in Section 6 of [26].

4 Review of infinitesimal deformation

In this section, we introduce some topics of a theory of infinitesimal deformations of hyperbolic structures, which is mainly based on two papers [24] and [28].

4.1 Relationship between $\mathbb{R}^3_1$ and $\mathfrak{psl}(2,\mathbb{R})$

The theories here are introduced in the works [24] of Goldman and Margulis. By using the identification between the Lorentzian space-time $\mathbb{R}^3_1$ and the Lie algebra $\mathfrak{psl}(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$, they showed the relation between the Margulis invariant and the first derivariation of the translation length $\mathbf{L}$ in the hyperbolic geometry.

Infinitesimal deformations An *infinitesimal deformation* of a hyperbolic surface S is an tangent vector of the Teichmüller space of S at S.

Let G be a finitely generated Fuchsian group. A cocycle is regarded as the infinitesimal deformation of $\sigma \in G$ thorough the following equation: by $\mathbf{u}(\sigma) \in \mathfrak{psl}(2,\mathbb{R})$,

$$\sigma^{\mathbf{u}}(s) := \sigma \exp(s\mathbf{u}(\sigma) + o(s)), \tag{13}$$

where s is in the small interval containing 0.

Margulis invariant Under the equation (13), a value of the Margulis invariant may be interpreted as half of the first derivariation of translation length $\mathbf{L}$ in the hyperbolic geometry.

Lemma 5 (Lemma 2 in Section 6 of [24]). *The following equation holds:*

$$\left.\frac{d}{ds}\right|_{s=0} \mathbf{L}(\sigma^{\mathbf{u}}(s)) = 2\,\alpha_{\mathbf{u}}(\sigma) \tag{14}$$

for any cocycle $\mathbf{u}$ *and* $\sigma \in G$.

4.2 Infinitesimal twist deformation

Here we introduce a topic associated with Fenchel-Nielsen twist deformations. The infinitesimal deformation of a twist deformation is estimated by the formula (mentioned as in Section 3 of [28], which combines the first derivation of translation length of hyperbolic metrics with the angles.

Wolpert formula Let $g \in G$ be a hyperbolic element corresponding to the closed geodesic on $S = \mathbb{H}/G$. The Fenchel-Niesen twist deformation along g can be defined, and then we consider its infinitesimal deformation.

Lemma 6 (Section 3 in [28]). *We take any* $\sigma \in G$*, which corresponds to a closed curve. The first derivariation of the length of* σ *along* g *is equal to*

$$\sum_{p \in g \cap \sigma} \cos\theta_p, \tag{15}$$

where θ_p *is the angle between the tangents at* p *of* g *and* σ*. The angle at* p *is chosen by the segment of the left side of* g *along* σ*.*

5 Study of affine twist deformations

Let S be a hyperbolic surface S with boundaries. We take a separating simple closed curve g on S.

The cocycles corresponding to the infinitesimal twist deformations along g can be represented explicitly in Section 5.1, and we call the cocycle an *affine twist cocycle*. We apply the affine twist cocycles to the affine deformation in Subsection 5.2. We will discuss in Theorem 3 whether the affine twist cocycle truly corresponds to the representation of the infinitesimal twist deformation. Indeed, we will observe an analogue between the Teichmüller theory and the theory of the affine deformations by comparing between the relations studied by Goldman and Margulis with the formula by Wolpert (*cf.* Subsection 5.3).

5.1 Definition of affine twist cocycle

Let g be a separating simple closed curve on S as above, and set $S - h = S_1 \cup S_2$. We only consider this decomposition when both S_1 and S_2 are compatible with hyperbolic metrics.

Definition 7 (Definition 4.2 in [26] and Section 4 of [27]). *A* canonical affine twist cocycle $\mathbf{AT}_g$ along g *is a cocycle on* S*, which is defined as follows:*

$$\mathbf{AT}_g(\sigma) := \begin{cases} \mathbf{0}, & \sigma \in \pi_1(S_1) \\ \mathbf{X}_g^0 - \sigma \mathbf{X}_g^0, & \sigma \in \pi_1(S_2) \\ \text{defined by the cocycle condition}, & \text{Otherwise.} \end{cases} \tag{16}$$

Definition 8. *A cocycle* $\mathbf{v}$ *is an* affine twist cocycle *along* g *on* S *if the* *following equation*

$$\mathbf{v} = \kappa \mathbf{AT}_g \tag{17}$$

holds for a certain $\kappa \in \mathbb{R}$.

Pants-decomposition Suppose that S is homeomorphic to a punctured sphere. We fix a pants-decomposition of S:

$$S = \sqcup_k S_k, \tag{18}$$

where S_k is each pair of pants in the pants-decomposition of S. Let the associated Fuchsian groups denoted by $\{P_k\}_k$ respectively.

Lemma 9 (Section 4 of [26])**.** *Let S be a punctured sphere with $(b+1)$ boundaries($b \geq 3$). For a given pants-decomposition, there exists a sequence of standard affine twist cocycles which are defined along the dividing curves.*

Proof. We define the pants-decomposition (18) as $S = S_1 \cup \cdots \cup S_{b-1}$ such that S_k and S_{k+1} share a boundary curve ($k = 1, \ldots, b-2$). Let the boundary curve denote $g_k (k = 1, \ldots, b-2)$. Then we define a sequence of canonical affine twists $\mathbf{AT}_{g_k}$ by the following:

$$\mathbf{AT}_{g_k}(\sigma) := \begin{cases} \mathbf{0}, & \sigma \in \pi_1(S_1 \cup \cdots \cup S_k) \\ \mathbf{X}^0_{g_k} - \sigma \mathbf{X}^0_{g_k}, & \sigma \in \pi_1(S_{k+1} \cup \cdots \cup S_{b-1}) \end{cases} \tag{19}$$

and a image for a general loop σ in S is defined by the cocycle condition. $\square$

The case that S have genus is discussed In [27].

5.2 Application to Lorentzian deformations

Here we treat Theorem 1. Let P be the associated Fuchsian group, and then P can be represented as

$$P := \langle g_1, g_2, g_3 \mid g_1 g_2 g_3 = id \rangle, \tag{20}$$

where g_1, g_2 and g_3 represent three boundary curves of the pair of pants.

Lemma 10 (Theorem A in [8])**.** *There exists a canonical linear isomorphism $\Phi : \mathbb{R}^3 \to \mathrm{H}^1(P, \mathbb{R}^3_1)$, which satisfies: For $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ and $\mathbf{u} := \Phi(\alpha_1, \alpha_2, \alpha_3)$, the equations*

$$\alpha_i = \alpha_{\mathbf{u}}(g_i) \tag{21}$$

hold when $i = 1, 2, 3$.

Proof. By using vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{B}(\mathbf{v}_i, \mathbf{X}^0_{g_i}) = 0 (i = 1, 2)$, a cocycle $\mathbf{u}$ is represented by

$$\begin{aligned} \mathbf{u}(g_1) &= \alpha_1 \mathbf{X}^0_{g_1} + \mathbf{v}_1 & (22) \\ \mathbf{u}(g_2) &= \alpha_2 \mathbf{X}^0_{g_2} + \mathbf{v}_2. & (23) \end{aligned}$$

The cocycle condition indicates the relation

$$\mathbf{u}(g_3) = -g_2^{-1} g_1^{-1} (\alpha_1 \mathbf{X}^0_{g_1} + \mathbf{v}_1) - g_2^{-1} (\alpha_2 \mathbf{X}^0_{g_2} + \mathbf{v}_2). \tag{24}$$

If we represent, by using $\mathbf{v}_3$ with $\mathbf{B}(\mathbf{X}_{g_3}, \mathbf{v}_3) = 0$, as

$$\mathbf{u}(g_3) = \alpha_3 \mathbf{X}^0_{g_3} + \mathbf{v}_3, \tag{25}$$

then a direct calculation find a triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of the vectors, which satisfies the equation of (24) and (25). This correspondence defines a linear map to the set of cocycles. Furthermore, this linear map is naturally extended to a linear map $\Phi : \mathbb{R}^3 \to \mathrm{H}^1(P, \mathbb{R}^3_1)$. Lemma 4 says that Φ is injective. Since $\mathrm{H}^1(P, \mathbb{R}^3_1)$ is three dimensional, the map Φ is surjective. $\square$

Second, by extending Lemma 10 to an affine deformation on G, whose associated surface is homeomorphic to a punctured sphere, we construct the following standard cocycles; with respect to the fixed pants-decomposition, we assign each of the boundary curves to a real value respectively. Let the set of the values be $\mathbf{a} \in \mathbb{R}^{2b-1}$.

Proposition 11 (Lemma 3.2 in [26])**.** *There is a linear injection from* $\mathbb{R}^{2b-1}$ *to the set of cocycles on* G*, such that every value of* $\mathbf{a}$ *is equal to the value of the Margulis invariant corresponding to the corresponding closed curve. We denote this correspondence by* $\mathbf{u}^{\mathbf{a}}_0$ *for* $\mathbf{a} \in \mathbb{R}^{2b-1}$*.*

Proof. We remind the pants-decompositon of Lemma 9. We denote by P_k the Fuchsian group associated with $S_k (k = 1, \ldots, b-1)$. For the surface S_1, we define a cocycle $\mathbf{u}_1$ by Lemma 10 and the given three values on S_1. There are many choices of this cocycle, however, in fact, the ambiguity is only by translations . We fix $\mathbf{u}_1$ from translation conjugacy.

Next we define a cocycle on $S_1 \cup S_2$. We choose a cocycle $\mathbf{u}'_2$ on P_2 from same discussion about $\mathbf{u}_1$. This cocycle has same Margulis invariant on the shared boundary curve g_1 by definition of the construction of $\mathbf{u}_1$ and $\mathbf{u}_2$. By a suitable translation of $\mathbf{u}_2$, we can equate the vectors of $\mathbf{u}_1(g_1)$ with $\mathbf{u}_2'(g_1)$. We redefine $\mathbf{u}_2$ by satisfying the above condition. Then a cocycle $\mathbf{u}_2$ on $S_1 \cup S_2$ are defined as follows:

$$\mathbf{u}_2(\sigma) := \begin{cases} \mathbf{u}_1(\sigma), & \sigma \in P_1 \\ \mathbf{u}_2(\sigma), & \sigma \in P_2, \end{cases} \tag{26}$$

and a image for a general loop σ in $S_1 \cup S_2$ is defined by the cocycle condition.

We can define a cocycle $\mathbf{u}_0^{\mathbf{a}}$ on S inductively from the above discussions. Finally a linearity of the relation $(\mathbf{a} \to \mathbf{u}_0^{\mathbf{a}})$ follows the linearity of the construction of the cocycles of each pair of pants. □

Now we can prove Theorem 1.

Proof of Theorem 1. For the given the pants-decomposition of S and the values $(\mathbf{a}, \mathbf{t}) \in \mathbb{R}^{2b-1} \times \mathbb{R}^{b-2}$, we will construct a cocycle on G satisfying the conditions of Theorem 1.

From Proposition 11, we define the cocycle $\mathbf{u}_0^{\mathbf{a}}$ on G for $\mathbf{a} \in \mathbb{R}^{2b-1}$. Let the sequence of the affine twist cocycles along the dividing curves g_k be $\mathbf{AT}_{g_k}$ in Lemma 9. Then we can define the cocycle of the form (1) in Theorem 1;

$$(\mathbf{a}, \mathbf{t}) \to \mathbf{u}_0^{\mathbf{a}} + \sum_{k=1}^{b-2} t_k \mathbf{AT}_{g_k}, \tag{27}$$

where $t_k \in \mathbf{t} = (t_1, \ldots, t_{b-2})$. Indeed, this is a *cocycle* because of the linear sum of the cocycles. Furthermore, the map (27) is linear because of Proposition 11.

The quotient by the translations naturally extends the relation (27) to the map from $\mathbb{R}^{3b-3}$ to $\mathrm{H}^1(G, \mathbb{R}^3_1)$. Let us prove that this map is injective; assume that

$$\mathbf{u}_0^{\mathbf{a}} + \sum_{k=1}^{b-2} t_k \mathbf{AT}_{g_k} = \mathbf{u}_0^{\mathbf{a}'} + \sum_{k=1}^{b-2} t'_k \mathbf{AT}_{g_k} \tag{28}$$

for $(\mathbf{a}, \mathbf{t})$ and $(\mathbf{a}', \mathbf{t}') \in \mathbb{R}^{3b-3}$. Since $\mathbf{a}$ and $\mathbf{a}'$ indicate the values of the Margulis invariant of the cocycle, we have $\mathbf{a} = \mathbf{a}'$ and then $\mathbf{u}_0^{\mathbf{a}} = \mathbf{u}_0^{\mathbf{a}'}$ by Lemma 4. Thus we obtain

$$\sum_{k=1}^{b-2} t_k \mathbf{AT}_{g_k} = \sum_{k=1}^{b-2} t'_k \mathbf{AT}_{g_k}. \tag{29}$$

By the definition of the sequence of the affine twists, we find $t_k = t'_k$, and then $\mathbf{t} = \mathbf{t}'$. Then the map is an injection as desired. The direct calculation indicates that the extended map of (27) is maximal rank. $\square$

5.3 Application to infinitesimal deformations

After we introduce a corollary of Theorem 2, we prove Theorem 2.

By the linearity of the Margulis invariant, Theorem 2 indicates the following result.

Corollary 12 ([26, 27])**.** *For any cocycle* $\mathbf{u}$ *on* G*, we can represent* $\mathbf{u}$ *as* $\mathbf{u}_0^{\mathbf{a}} + \sum_{k=1}^{b-2} t_k \mathrm{AT}_k$ *up to translation conjugacy. Let* g_k *be the dividing curve in the pants-decomposition of Lemma 9, and* σ *be a simple closed curve. Then*

$$\frac{1}{2} \frac{d}{ds}\bigg|_{s=0} \mathbf{L}(\sigma^{\mathbf{u}}(s)) = \alpha_{\mathbf{u}}(\sigma) + \sum_{k=1}^{b-2} t_k \sum_{p_k \in g_k \cap \sigma} \cos\theta_{p_k} \tag{30}$$

holds.

The following lemma ties the Margulis invariant and the cosines, which is a core of the proof of Theorem 2. The relation (31) is a basic relation in the Lorentzian geometry.

Lemma 13. *Let h_1, h_2 be hyperbolic elements in $\mathbf{G}$. Suppose that the unique invariant lines which h_1, h_2 have on $\mathbb{H}$ are crossing. Then the equation*

$$\mathbf{B}(\mathbf{X}^0_{h_1}, \mathbf{X}^0_{h_2}) = \cos\theta_p \tag{31}$$

holds, where θ_p is an angle between the tangent vectors of h_1 and h_2 at the intersection p.

Proof. Since the equation (31) is invariant under conjugation, we may set the intersection p as ${}^{\mathrm{t}}(0,0,1)$. Then the vectors $\mathbf{X}^0_{h_1}$ and $\mathbf{X}^0_{h_2}$ can be represented as ${}^{\mathrm{t}}(1,0,0)$ and ${}^{\mathrm{t}}(\cos\theta_p, \sin\theta_p, 0)$ respectively. A direct calculation indicates the equation (31) as desired. □

Finally we give the proof of Theorem 2 here.

Proof of Theorem 2. Take a loop σ in S. The decomposition of S along the dividing curve g defines two surfaces S_1 and S_2. Note that the affine twist cocycle $\mathbf{AT}_g$ follows the representation (16). The equation (14) in Lemma 5 gives the following:

$$\frac{1}{2}\left.\frac{d}{ds}\right|_{s=0} \mathbf{L}(\sigma^{\mathbf{AT}_g}(s)) = \alpha_{\mathbf{AT}_g}(\sigma). \tag{32}$$

By the definition of the Marglis invariant $\alpha_{\mathbf{AT}_g}(\sigma)$, we have

$$\alpha_{\mathbf{AT}_g}(\sigma) = \mathbf{B}(g\mathbf{x} + \mathbf{AT}_g(\sigma) - \mathbf{x}, \mathbf{X}^0_g) \tag{33}$$

for $\mathbf{x} \in \mathbb{R}^3_1$. In fact, the vector $g\mathbf{x} - \mathbf{x}$ is in the orthogonal plane of $\mathbf{X}^0_g$ in $\mathbb{R}^3_1$. Thus we will prove that $\mathbf{B}(\mathbf{AT}_g(\sigma), \mathbf{X}^0_g)$ is equal to the right side of the equation (3). We complete the proof in the case that the loop σ has the following form

$$\sigma = \sigma^1_1\sigma^2_1\sigma^1_2\sigma^2_2 \cdots \sigma^1_n\sigma^2_n, \tag{34}$$

where the loop σ^ξ_j is in S_ξ $(1 \leq j \leq n)$. Even if σ has the other forms, the same discussion gives the proof of the case. By the cocycle condition and $\mathbf{AT}_g(\sigma^1_j) = \mathbf{0}$, we have

$$\mathbf{AT}_g(\sigma) = \sum_{j=1}^{n} \sigma^1_1 \cdots \sigma^1_j \mathbf{AT}_g(\sigma^2_j). \tag{35}$$

Furthermore

$$\begin{aligned} \alpha_{\mathbf{AT}_g}(\sigma) &= \mathbf{B}(\mathbf{X}^0_\sigma, \sum_{j=1}^{n} \sigma^1_1 \cdots \sigma^1_j \mathbf{AT}_g(\sigma^2_j)) & (36)\\ &= \sum_{j=1}^{n} \mathbf{B}((\sigma^1_1 \cdots \sigma^1_j)^{-1}\mathbf{X}^0_\sigma, \mathbf{AT}_g(\sigma^2_j)). & (37) \end{aligned}$$

Note that, for every $\phi \in \mathbf{G}$, the following relation holds:

$$\phi\mathbf{X}^0_\sigma = \mathbf{X}^0_{\phi\sigma\phi^{-1}}. \tag{38}$$

Remind the definition of the affine twist cocycle:

$$\mathbf{AT}_g(\sigma_j^2) = \mathbf{X}_g^0 - \sigma_j^2 \mathbf{X}_g^0. \tag{39}$$

These two relations imply that the equation (37) corresponds to

$$\sum_{j=1}^{n} \mathbf{B}(\mathbf{X}^0_{\sigma_j^2\sigma_{j+1}^1\sigma_{j+1}^2\cdots\sigma_n^2\sigma_1^1\cdots\sigma_j^1}, \mathbf{X}_g^0) - \sum_{j=1}^{n} \mathbf{B}(\mathbf{X}^0_{\sigma_{j+1}^1\sigma_{j+1}^2\cdots\sigma_n^2\sigma_1^1\cdots\sigma_j^1\sigma_j^2}, \mathbf{X}_g^0). \tag{40}$$

Since these vectors in the above equation are principal vectors, Lemma 13 says that the equation is a sum of the cosines. Notice the orientation of the vectors. The former terms $\mathbf{B}(\mathbf{X}^0_{\sigma_j^2\sigma_{j+1}^1\sigma_{j+1}^2\cdots\sigma_n^2\sigma_1^1\cdots\sigma_j^1}, \mathbf{X}_g^0)$ are equal to $\cos\theta_{p_j}$ respectively, where the point $p_j \in S$ is the intersection between the loops $\sigma_j^2\sigma_{j+1}^1\sigma_{j+1}^2\cdots\sigma_j^1$ and g. The latter terms satisfy

$$\mathbf{B}(\mathbf{X}^0_{\sigma_{j+1}^1\cdots\sigma_j^2}, \mathbf{X}_g^0) = \cos(\pi - \theta_{q_j}) = -\cos\theta_{q_j}, \tag{41}$$

where the point $q_j \in S$ is the intersection between the loops $\sigma_{j+1}^1\cdots\sigma_j^2$ and g. Thus we obtain

$$\alpha_{\mathbf{AT}_g}(\sigma) = \sum_{j=1}^{n} \cos\theta_{p_j} + \sum_{j=1}^{n} \cos\theta_{q_j}, \tag{42}$$

and the points $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are all intersections of the loops σ and g. $\square$

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