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## DECOMPOSITION THEOREM ON $G$ -SPACES

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### Abstract

In this paper, we introduce the weak  $G$ -expansivity which is a generalization of both expansivity and  $G$ -expansivity. Also, we define  $G$ -stable and  $G$ -unstable sets of a homeomorphism on a metric  $G$ -space  $X$  and investigate properties of them. Finally, we consider the decomposition theorem on  $G$ -spaces.

### 1. Introduction

Let  $X$  be a topological space,  $G$  be a topological group, and  $\theta: G \times X \rightarrow X$  be a map. The triple  $(X, G, \theta)$  is called a *topological  $G$ -space* if the following three conditions are satisfied:

- (1)  $\theta(e, x) = x$  for all  $x \in X$ , where  $e$  is the identity of  $G$ ;
- (2)  $\theta(g, \theta(h, x)) = \theta(gh, x)$  for all  $x \in X$  and for all  $g, h \in G$ ;
- (3)  $\theta$  is continuous.

Here,  $gh$  is the group operation on  $G$ . Simply, we denote  $\theta(g, x)$  by  $gx$  and  $X$  is usually said to be a *topological  $G$ -space*.

For any subset  $A$  of  $X$ ,  $G(A)$  is denoted by the set  $\{ga : g \in G, a \in A\}$ .  $G(x)$  is called a  *$G$ -orbit* of  $x$ . A subset  $A$  of  $X$  is called  *$G$ -invariant* if  $G(A) = A$ . A map  $f: X \rightarrow X$  on a  $G$ -space  $X$  is said to be *pseudoequivariant* provided that  $f(G(x)) = G(f(x))$  for all  $x \in X$ , and  $f$  is said to be *equivariant* provided that  $f(gx) = gf(x)$  for all  $x \in X$  and  $g \in G$ .

N. Aoki has proved the following topological decomposition theorem in 1983 ([1]), which is an extension of Smale's spectral decomposition theorem and Bowen's decomposition theorem in dynamical systems. All undefined notions can be found in [2].

**Theorem 1.1** ([1]). *Let  $f: X \rightarrow X$  be a homeomorphism on a compact metric space  $X$  and let  $CR(f)$  be the chain recurrent set. If  $f|_{CR(f)}: CR(f) \rightarrow CR(f)$  is an expansive homeomorphism with the shadowing property, then*

- (1)  *$CR(f)$  contains a finite sequence  $B_i$  ( $1 \leq i \leq k$ ) of  $f$ -invariant closed subsets such that*

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- (a)  $CR(f) = \bigcup_{i=1}^k B_i$  (disjoint union);
- (b)  $f|_{B_i} : B_i \rightarrow B_i$  is topologically transitive,
- (2) for each  $B_i$ , there exist a subset  $X_p$  of  $B_i$  and  $a > 0$  such that
  - (a)  $f^a(X_p) = X_p$ ;
  - (b)  $X_p \cap f^j(X_p) = \emptyset$  ( $0 < j < a$ );
  - (c)  $f^a|_{X_p} : X_p \rightarrow X_p$  is topologically mixing;
  - (d)  $B_i = \bigcup_{j=0}^{a-1} f^j(X_p)$ .

A point  $x \in X$  is called a  $G$ -periodic point of  $f$  if there exist an integer  $n > 0$  and  $g \in G$  such that  $f^n(x) = gx$ . A point  $x \in X$  is called a  $G$ -nonwandering point of  $f$  if for every open neighborhood  $U$  of  $x$ , there exist  $n > 0$  and  $g \in G$  such that  $gf^n(U) \cap U \neq \emptyset$ .  $Per_G(f)$  (resp.  $\Omega_G(f)$ ) is denoted by the set of all  $G$ -periodic (resp.  $G$ -nonwandering) points of  $f$ .

For a homeomorphism  $f$  on a metric  $G$ -space  $X$ , a sequence  $\{x_i \in X : i \in \mathbb{Z}\}$  is called a  $(\delta, G)$ -pseudo orbit for  $f$  provided that for each  $i$ , there exists  $g_i \in G$  such that  $d(g_i f(x_i), x_{i+1}) < \delta$ . A  $(\delta, G)$ -pseudo orbit  $\{x_i\}$  for  $f$  is said to be  $\epsilon$ -traced by a point  $x \in X$  provided that for each  $i$ , there exists  $g_i \in G$  such that  $d(f^i(x), g_i x_i) < \epsilon$ .

**DEFINITION 1.2** ([5]). A homeomorphism  $f: X \rightarrow X$  has the  $G$ -shadowing property (GSP) provided that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that every  $(\delta, G)$ -pseudo orbit  $\{x_i\}$  in  $X$  for  $f$  is  $\epsilon$ -traced by a point  $x \in X$ .

**REMARK 1.3.** It was proved by E. Shah that, when  $X$  is a compact metric  $G$ -space and the orbit map  $\pi: X \rightarrow X/G$  is a covering map, a pseudoequivariant homeomorphism  $f$  on  $X$  has the GSP if and only if the induced map  $\hat{f}: X/G \rightarrow X/G$  has the shadowing property ([5]).

If a pseudoequivariant continuous onto map  $f: X \rightarrow X$  has the GSP where  $X$  is a compact metric  $G$ -space with  $G$  compact, then  $f|_{\Omega_G(f)}$  has the GSP ([5]).

The main purpose of this paper is to prove the following theorems on compact metric  $G$ -spaces.

**Theorem A.** Let  $X$  be a compact metric  $G$ -space with  $G$  compact. If  $f: X \rightarrow X$  is a pseudoequivariant  $G$ -expansive homeomorphism with the GSP, then  $\Omega_G(f)$  contains a finite sequence  $B_i$  ( $1 \leq i \leq n$ ) of  $f$ -invariant,  $G$ -invariant, and closed subsets such that

- (1)  $f|_{\Omega_G(f)}$  is topologically  $G$ -transitive;
- (2)  $\Omega_G(f) = \bigcup_{i=1}^n B_i$  (disjoint union);
- (3)  $f|_{B_i}$  has the GSP.

A homeomorphism  $f: X \rightarrow X$  is said to be topologically  $G$ -mixing provided that for every nonempty open subsets  $U$  and  $V$  of  $X$ , there exists an integer  $N$  such that

for each  $n \geq N$ , there is  $g_n \in G$  satisfying  $g_n f^n(U) \cap V \neq \emptyset$ .

**Theorem B.** *Let  $f|_{\Omega_G(f)}: \Omega_G(f) \rightarrow \Omega_G(f)$  be a  $G$ -expansive homeomorphism with the GSP. Then, for any  $f$ -invariant,  $G$ -invariant, open and closed subset  $B \subset \Omega_G(f)$  such that  $f|_B: B \rightarrow B$  is topologically  $G$ -transitive, there are  $X_p \subset B$  and  $a > 0$  such that*

- (1)  $f^a(X_p) = X_p$ ;
- (2)  $X_p \cap f^j(X_p) = \emptyset$  ( $0 < j < a$ );
- (3)  $f^a|_{X_p}: X_p \rightarrow X_p$  is topologically  $G$ -mixing;
- (4)  $B = \bigcup_{j=0}^{a-1} f^j(X_p)$ .

**DEFINITION 1.4.** A homeomorphism  $f: X \rightarrow X$  on a metric  $G$ -space  $X$  is said to be *weak  $G$ -expansive* provided that there exists  $\delta > 0$  such that for every  $x, y \in X$  with  $G(x) \neq G(y)$  if  $u \in G(x)$  and  $v \in G(y)$ , there exists  $n = n(u, v) \in \mathbb{Z}$  such that

$$d(f^n(u), f^n(v)) > \delta.$$

The constant  $\delta$  is called a *weak  $G$ -expansive constant* for  $f$ .

The weak  $G$ -expansivity is a generalization of both expansivity and  $G$ -expansivity. Here,  $G$ -expansivity has been defined by R. Das ([4]). A homeomorphism  $f: X \rightarrow X$  is said to be  *$G$ -expansive* provided that there exists  $\delta > 0$  such that for every  $x, y \in X$  with  $G(x) \neq G(y)$ , there exists  $n \in \mathbb{Z}$  such that

$$d(f^n(u), f^n(v)) > \delta \quad \text{for all } u \in G(x), v \in G(y).$$

The constant  $\delta$  is called a  *$G$ -expansive constant* for  $f$ .

**REMARK 1.5.** R. Das proved that there is no implication between  $G$ -expansivity and expansivity by giving counterexamples ([4]).

**EXAMPLE 1.6** ([4]). Consider the compact space  $X = \{1/n, 1 - 1/n: n \in \mathbb{N}\}$  with the usual metric and let the topological group  $G = \{-1, 1\}$  act on  $X$  with the action  $\theta$  defined by  $\theta(1, x) = x$  and  $\theta(-1, x) = 1 - x$ . Define a homeomorphism  $f: X \rightarrow X$  by

$$f(x) = \begin{cases} x & \text{if } x = 0, 1; \\ \text{next to the right of } x & \text{if } x \in X \setminus \{0, 1\}. \end{cases}$$

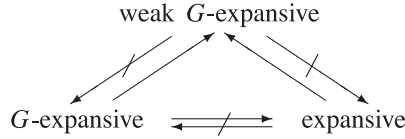
Then  $f$  is an expansive map with expansive constant  $\delta$  ( $0 < \delta < 1/6$ ). But, it is easy to see that for  $x, y \in X \setminus \{1/2\}$  with  $G(x) \neq G(y)$ , there is no  $n \in \mathbb{Z}$  such that

$$|f^n(u) - f^n(v)| > \delta \quad \text{for all } u \in G(x), v \in G(y),$$

whatever  $\delta > 0$  may be. This means that  $f$  is not  $G$ -expansive.

EXAMPLE 1.7 ([4]). Consider the compact space  $X = \bigcup_{i=1}^n C_i$  with the usual metric, where each  $C_i$  is the circle in  $\mathbb{R}^2$  with center the origin and radius  $i$ . Denote  $G = SO(2)$  by the set of all  $2 \times 2$  matrices whose determinants are  $\pm 1$  and define an action  $\theta: G \times X \rightarrow X$  by the usual rotations on  $X$ . Then the identity map on  $X$  is  $G$ -expansive with  $G$ -expansive constant  $\delta$  ( $0 < \delta < 1$ ).

Therefore, all properties of the following diagram are distinguished as we see in Examples 1.6 and 1.7:



DEFINITION 1.8. Let  $f: X \rightarrow X$  be a homeomorphism of a metric  $G$ -space  $X$ . We define a *local  $G$ -stable set*  $W_\epsilon^s(x)$  and a *local  $G$ -unstable set*  $W_\epsilon^u(x)$  by

$$\begin{aligned}
 W_\epsilon^s(x) &= \{y \in X : \text{for each } n \geq 0, \\
 &\quad \text{there is } g_n \in G \text{ such that } d(f^n(g_n x), f^n(y)) \leq \epsilon\}, \\
 W_\epsilon^u(x) &= \{y \in X : \text{for each } n \geq 0 \\
 &\quad \text{there is } g_n \in G \text{ such that } d(f^{-n}(g_n x), f^{-n}(y)) \leq \epsilon\}.
 \end{aligned}$$

We modify results of [3] into the following results by weakening the condition “equivariant” into “pseudoequivariant” and deleting the condition “invariant metric”. A metric  $d$  on a  $G$ -space  $X$  is called an *invariant metric* provided that  $d(x, y) = d(gx, gy)$  for all  $x, y \in X$  and  $g \in G$ .

REMARK 1.9. Let  $X$  be a compact metric  $G$ -space with  $G$  compact. If  $f: X \rightarrow X$  is a weak  $G$ -expansive pseudoequivariant homeomorphism with weak  $G$ -expansive constant  $\delta > 0$ , then for every  $\gamma > 0$ , there is  $N > 0$  such that for each  $x \in X$  and for each  $n \geq N$ ,

$$f^n(W_\delta^s(x)) \subset W_\gamma^s(f^n(x))$$

and

$$f^{-n}(W_\delta^u(x)) \subset W_\gamma^u(f^{-n}(x)).$$

Proof. We shall prove only the case of a local  $G$ -stable set because the other case can be proved similarly. To do it, suppose that there exists  $\gamma > 0$  such that for all  $N > 0$ , there are  $x \in X$  and  $n \geq N$  satisfying

$$f^n(W_\delta^s(x)) \not\subset W_\gamma^s(f^n(x)).$$

Let  $N > 0$ . Then there are  $x_1 \in X$  and  $n \geq N$  satisfying

$$f^n(W_\delta^s(x_1)) \not\subset W_\gamma^s(f^n(x_1)),$$

that is, there exists  $y_1 \in W_\delta^s(x_1)$  such that  $f^n(y_1) \notin W_\gamma^s(f^n(x_1))$ . So there exists  $i \geq 0$  such that for every  $h \in G$ ,

$$d(f^i(hf^n(x_1)), f^i(f^n(y_1))) > \gamma.$$

Because  $f$  is pseudoequivariant, there exists  $i \geq 0$  such that for every  $g \in G$ ,

$$d(gf^{i+n}(x_1), f^{i+n}(y_1)) > \gamma.$$

Take  $m_1 = i + n$  and choose  $N = m_1 + 1$ .

Continuing the process, we can find sequences  $m_n > 0$ ,  $x_n$ , and  $y_n \in X$  such that

- (1)  $y_n \in W_\delta^s(x_n)$ ;
- (2)  $d(hf^{m_n}(x_n), f^{m_n}(y_n)) > \gamma$  for all  $h \in G$ ;
- (3)  $\lim_{n \rightarrow \infty} m_n = \infty$ .

It follows from  $y_n \in W_\delta^s(x_n)$  that for each  $i \geq -m_n$ , there exists  $g_{i+m_n} \in G$  such that

$$d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)) \leq \delta.$$

Since  $f$  is pseudoequivariant, for each  $g_{i+m_n}$ , there exists  $h_{i+m_n} \in G$  such that

$$d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) = d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)).$$

Hence, for each  $i \geq -m_n$ ,

$$d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) \leq \delta.$$

If  $f^{m_n}(x_n) \rightarrow x$ ,  $f^{m_n}(y_n) \rightarrow y$ , and  $h_{i+m_n} \rightarrow h$  as  $n \rightarrow \infty$ , then

$$d(f^i(hx), f^i(y)) \leq \delta \quad \text{for all } i \in \mathbb{Z}.$$

Since  $\delta$  is a weak  $G$ -expansive constant for  $f$ ,  $G(x) = G(y)$ . But  $d(hx, y) = \lim_{n \rightarrow \infty} d(hf^{m_n}(x_n), f^{m_n}(y_n)) \geq \gamma > 0$  for all  $h \in G$  by (2). Thus  $hx \neq y$  for all  $h \in G$ , and hence  $G(x) \neq G(y)$ . This is a contradiction.  $\square$

For a homeomorphism  $f$  on a compact metric  $G$ -space, we define the following:

$$W^s(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right. \\ \left. \lim_{n \rightarrow \infty} d(f^n(g_n x), f^n(y)) = 0 \right\};$$

$$W^u(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right. \\ \left. \lim_{n \rightarrow \infty} d(f^{-n}(g_n x), f^{-n}(y)) = 0 \right\}.$$

$W^s(x)$  (resp.  $W^u(x)$ ) is called a  $G$ -stable set (resp.  $G$ -unstable set).

REMARK 1.10. Let  $X$  be a compact metric  $G$ -space with  $G$  compact. If  $f: X \rightarrow X$  is a weak  $G$ -expansive pseudoequivariant homeomorphism with weak  $G$ -expansive constant  $\delta > 0$ , then for each  $\epsilon$  with  $0 < \epsilon < \delta$ ,

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x)));$$

$$W^u(x) = \bigcup_{n \geq 0} f^n(W_\epsilon^s(f^{-n}(x))).$$

Proof. (C): Let  $y \in W^s(x)$  and  $0 < \epsilon < \delta$ . Then there exists  $N > 0$  such that for each  $n \geq N$ , we can choose  $g_n \in G$  satisfying

$$d(f^n(g_n x), f^n(y)) \leq \epsilon.$$

Thus,

$$d(f^i(f^N(g_{i+N}x)), f^i(f^N(y))) \leq \epsilon \quad \text{for all } i \geq 0.$$

Since  $f$  is pseudoequivariant,  $f^N(y) \in W_\epsilon^s(f^N(x))$ . Therefore,

$$y \in f^{-N}(W_\epsilon^s(f^N(x))) \subset \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x))).$$

(D): Let  $y \in f^{-n}(W_\epsilon^s(f^n(x)))$  for some  $n \geq 0$ . Then  $f^n(y) \in W_\epsilon^s(f^n(x))$ . It follows from Remark 1.9 that for every  $\gamma > 0$  there exists  $N > 0$  such that for each  $x \in X$  and  $m \geq N$ ,

$$f^{m+n}(y) \in f^m(W_\epsilon^s(f^n(x))) \subset W_\gamma^s(f^{m+n}(x)).$$

So for each  $n \geq N$ , we can find  $g_n \in G$  such that

$$d(f^{m+n}(g_n x), f^{m+n}(y)) \leq \gamma.$$

Since  $f$  is pseudoequivariant,  $y \in W^s(x)$ . The proof is completed. The case of a  $G$ -unstable set can be proved similarly.  $\square$

## 2. Decomposition theorems

First we prepare the following four lemmas to show Theorem A.

**Lemma 2.1** ([3]). *Let  $(X, G, \theta)$  be a compact metric  $G$ -space with  $G$  compact. Then for any  $\epsilon > 0$ , there is a finite open cover  $\mathcal{U} = \{U_1, \dots, U_s\}$  of  $X$  such that  $\text{diam}(g\overline{U}_i) \leq \epsilon$  for all  $g \in G$  and  $i$  with  $1 \leq i \leq s$ .*

In Lemma 2.1, notice that, for each  $g \in G$ , the open cover  $\{gU : U \in \mathcal{U}\}$  of  $X$  satisfies  $\text{diam}(hg\overline{U}_i) \leq \epsilon$  for all  $h \in G$  and  $i$  with  $1 \leq i \leq s$ .

**Lemma 2.2.** *Let  $X$  be a compact metric  $G$ -space with  $G$  compact. If  $\mathcal{U}$  is a finite open cover of  $X$ , then there exists  $\delta > 0$  such that for each subset  $A$  of  $X$  with  $\text{diam}(A) \leq \delta$ ,  $A \subset gU$  for some  $U \in \mathcal{U}$  and  $g \in G$ .*

*Proof.* Suppose not. Then for every  $n > 0$  there exists a subset  $A_n$  of  $X$  such that  $\text{diam}(A_n) \leq 1/n$  and  $A_n \not\subset gU$  for all  $U \in \mathcal{U}$  and  $g \in G$ . Choose  $x_n \in A_n$  for each  $n \in \mathbb{N}$ . Since  $X$  is compact, there exist a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that  $x_{n_i} \rightarrow x$ . We fix  $g \in G$ . Then there is  $U \in \mathcal{U}$  with  $x \in gU$ . Since  $X \setminus gU$  is compact,  $d(x, X \setminus gU) > 0$ . Put  $\epsilon = d(x, X \setminus gU)$  and take  $n_i > 0$  such that  $1/n_i < \epsilon/2$  and  $d(x_{n_i}, x) < \epsilon/2$ . Then for any  $y \in A_{n_i}$ ,

$$d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\epsilon}{2} < \epsilon.$$

So  $y \in gU$ . Therefore,  $A_{n_i} \subset gU$ . This is a contradiction.  $\square$

**Lemma 2.3.** *Let  $X$  be a compact metric  $G$ -space with  $G$  compact. Then for any  $\epsilon > 0$ , there exists  $\delta > 0$  ( $\delta < \epsilon$ ) such that*

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$

*Proof.* Let  $\epsilon > 0$ . Then it follows from Lemma 2.1 that, for any positive  $\epsilon_1 < \epsilon$ , there is a finite open cover  $\mathcal{U}$  such that  $\text{diam}(g\overline{U}) \leq \epsilon_1$  for all  $g \in G$  and  $U \in \mathcal{U}$ . Also, by Lemma 2.2, there is a constant  $\delta = \delta(\mathcal{U}) > 0$  such that for any subset  $A$  with  $\text{diam}(A) \leq \delta$ ,  $A \subset gU$  for some  $g \in G$  and  $U \in \mathcal{U}$ . Let  $x$  and  $y$  in  $X$  with  $d(x, y) < \delta$ . Then  $x, y \in g_0U_0$  for some  $g_0 \in G$  and  $U_0 \in \mathcal{U}$ . Note that  $\{g_0U : U \in \mathcal{U}\}$  is an open cover of  $X$ . For any  $g \in G$ , take  $g_1 \in G$  such that  $g_1 = gg_0$ . Then, by Lemma 2.1,  $\text{diam}(gg_0\overline{U}) \leq \epsilon_1$ , that is,  $\text{diam}(g_1\overline{U}) \leq \epsilon_1 < \epsilon$  for all  $U \in \mathcal{U}$ . Since  $gx, gy \in gg_0U_0 = g_1U_0$ ,  $d(gx, gy) < \epsilon$ .  $\square$

**Lemma 2.4.** *Let  $X$  be a compact metric  $G$ -space with  $G$  compact and let  $f$  be a pseudoequivariant homeomorphism on  $X$ . Then  $f$  has the GSP if and only if for any  $\epsilon > 0$ , we can find  $\delta > 0$  such that for every  $(\delta, G)$ -pseudo orbit  $\{x_i\}$  of  $X$  for  $f$ , there exist  $x \in X$  and  $h_i \in G$  satisfying*

$$d(f^i(h_i x), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.$$

*Proof.* Suppose that  $f$  has the GSP and let  $\epsilon > 0$ . Then, by Lemma 2.3, there exists  $\epsilon_0 > 0$  ( $\epsilon_0 < \epsilon$ ) such that for each  $x, y \in X$ ,

$$d(x, y) < \epsilon_0 \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$



Let  $\delta$  be the constant corresponding to  $\epsilon_0$  in the definition of the GSP. Then every  $(\delta, G)$ -pseudo orbit  $\{x_i\}$  of  $X$  for  $f$  is  $\epsilon_0$ -traced by a point  $x \in X$ , that is, for each  $i$ , there exists  $g_i \in G$  such that

$$d(f^i(x), g_i x_i) < \epsilon_0 \quad \text{for all } i \in \mathbb{Z}.$$

Since  $f$  is pseudoequivariant, for each  $g_i \in G$ , there exists  $h_i \in G$  such that

$$g_i^{-1} f^i(x) = f^i(h_i x).$$

Moreover,  $d(g_i^{-1} f^i(x), x_i) < \epsilon$  and hence  $d(f^i(h_i x), x_i) < \epsilon$  for all  $i \in \mathbb{Z}$ .

The converse can be proved similarly.  $\square$

We have ([5]) that  $f(\Omega_G(f)) = \Omega_G(f)$  and  $CR_G(f) = \Omega_G(f)$  for a pseudoequivariant homeomorphism  $f$  with GSP on a compact metric  $G$ -space  $X$  where  $G$  is compact.

For  $x, y \in X$  and  $\delta > 0$ ,  $x$  is said to be  $(\delta, G)$ -related to  $y$  (denoted by  $x \overset{\delta}{\sim}_G y$ ) if there exist finite  $(\delta, G)$ -pseudo orbits  $\{x = x_0, x_1, \dots, x_k = y\}$  and  $\{y = y_0, y_1, \dots, y_n = x\}$  for  $f$ . If for every  $\delta > 0$ ,  $x$  is  $(\delta, G)$ -related to  $y$ , then  $x$  is said to be  $G$ -related to  $y$  (denoted by  $x \sim_G y$ ). A point  $x$  is said to be a  $G$ -chain recurrent point of  $f$  if  $x \sim_G x$ .  $CR_G(f)$  is denoted by the set of all  $G$ -chain recurrent points of  $f$ . A homeomorphism  $f: X \rightarrow X$  is called *topologically  $G$ -transitive* provided that for every nonempty open subsets  $U$  and  $V$  of  $X$ , there exist an integer  $n > 0$  and  $g \in G$  such that  $gf^n(U) \cap V \neq \emptyset$ .

**Proof of Theorem A.** Since the pseudoequivariant homeomorphism  $f$  satisfies the GSP,  $CR_G(f) = \Omega_G(f)$ . Thus  $\Omega_G(f) = \bigcup_{\lambda} B_{\lambda}$  where each  $B_{\lambda}$  is an equivalence class under the relation  $\sim_G$  which is defined in  $CR_G(f)$ .

**Claim 1.** *Each  $B_{\lambda}$  is closed in  $\Omega_G(f)$ .*

*Proof.* Let  $x \in \overline{B_{\lambda}}$ . Then we can find a sequence  $\{x_i\}$  in  $B_{\lambda}$  which converges to  $x$ . Let  $\alpha > 0$  be given. Then there exists a finite open cover  $\{U_1, \dots, U_s\}$  of  $X$  such that

$$\text{diam}(g\overline{U}_i) \leq \frac{\alpha}{2} \quad \text{for all } g \in G \quad \text{and } i \quad \text{with } 1 \leq i \leq s$$

by Lemma 2.1. So  $f(x) \in U_i$  for some  $i$ . Choose an  $\epsilon_0$ -neighborhood  $N_{\epsilon_0}(f(x))$  of  $f(x)$  such that  $N_{\epsilon_0}(f(x)) \subset U_i$ . Then since  $f$  is uniformly continuous, there exists  $\delta_0 > 0$  such that

$$d(x, y) < \delta_0 \implies d(f(x), f(y)) < \epsilon_0.$$

Because  $\{x_i\}$  converges to  $x$ , there is  $J > 0$  such that  $d(x_J, x) < \min\{\alpha/2, \delta_0\}$ . From the fact that  $x_J \in CR_G(f)$ , we can find a  $(\alpha/2, G)$ -pseudo orbit

$$\{x_J = y_0, y_1, \dots, y_{k-1}, y_k = x_J\}.$$

So  $d(gf(y_0), y_1) < \alpha/2$  for some  $g \in G$ . Also  $d(f(y_0), f(x)) < \epsilon_0$  and hence  $d(gf(y_0), gf(x)) < \alpha/2$ . Thus,

$$d(gf(x), y_1) \leq d(gf(x), gf(y_0)) + d(gf(y_0), y_1) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Therefore,  $\{x, y_1, \dots, y_k = x_J\}$  is an  $(\alpha, G)$ -pseudo orbit. It is clear that there is an  $(\alpha, G)$ -pseudo orbit from  $x_J$  to  $x$  by the uniform continuity of  $f$ . It follows from  $x \overset{\alpha}{\sim}_G x_J$  that  $x \overset{\alpha}{\sim}_G x_i$  for all  $i$  because each  $x_i \in B_\lambda$ . Since  $\alpha$  is arbitrary,  $x \in B_\lambda$ . Therefore,  $B_\lambda$  is closed.  $\square$

**Claim 2.** *Each  $B_\lambda$  is  $f$ -invariant.*

*Proof.* To prove this, we firstly show that  $x \sim_G f(x)$  for all  $x \in \Omega_G(f)$ . Let  $\alpha > 0$ . Then there is  $\delta > 0$  ( $\delta < \alpha$ ) such that

$$d(a, b) < \delta \implies d(f^2(a), f^2(b)) < \alpha.$$

Since  $x \in \Omega_G(f)$ , there are  $n > 0$  and  $g \in G$  such that

$$gf^n(N_\delta(x)) \cap N_\delta(x) \neq \emptyset$$

where  $N_\delta(x)$  is a  $\delta$ -neighborhood of  $x$ . Then there exists  $z \in N_\delta(x)$  such that  $gf^n(z) \in N_\delta(x)$ . Hence

$$\{f(x), f^2(z), \dots, f^{n-1}(z), x\}$$

is an  $(\alpha, G)$ -pseudo orbit and thus,  $x \sim_G f(x)$ . Since  $f$  is a homeomorphism, we can show that  $x \sim_G f^{-1}(x)$  for all  $x \in \Omega_G(f)$  similarly. Therefore,  $f(B_\lambda) = B_\lambda$  for each  $\lambda$ .  $\square$

**Claim 3.**  *$Per_G(f)$  is dense in  $\Omega_G(f)$ .*

*Proof.* Let  $\alpha > 0$  be a  $G$ -expansive constant for  $f$  and take  $\epsilon < \alpha/2$ . Since  $f$  has the GSP, there exists  $\delta > 0$  ( $\delta < \epsilon$ ) such that every  $(\delta, G)$ -pseudo orbit is  $\epsilon$ -traced by a point in  $X$ . Since  $f$  is uniformly continuous, there exists a positive constant  $\gamma < \delta$  such that if  $d(a, b) < \gamma$ , then  $d(f(a), f(b)) < \delta$ . Let  $p \in \Omega_G(f)$ . Then for every  $\gamma$ -neighborhood  $N_\gamma(p)$  of  $p$ , there exist an integer  $n > 0$  and  $g \in G$  such that

$$gf^n(N_\gamma(p)) \cap N_\gamma(p) \neq \emptyset.$$

Choose a point  $y \in gf^n(N_\gamma(p)) \cap N_\gamma(p)$ . Since  $f^{-n}(g^{-1}y) \in N_\gamma(p)$ ,

$$d(f(p), f(f^{-n}(g^{-1}y))) < \delta.$$

Hence

$$\{\dots, x_0 = p, x_1 = f^{-n+1}(g^{-1}y), x_2 = f^{-n+2}(g^{-1}y), \dots, x_{n-1} = f^{-1}(g^{-1}y), x_n = p, \dots\}$$

is a  $(\delta, G)$ -pseudo orbit for  $f$ . Since  $f$  has the GSP, it follows from Lemma 2.4 that, for each  $i \in \mathbb{Z}$ , there exist  $x \in X$  and  $g_i \in G$  such that

$$d(f^i(g_i x), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.$$

Thus,

$$\begin{aligned} d(f^k(f^n(g_{k+n}x)), f^k(g_k x)) &\leq d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_{k+n}, f^k(g_k x)) \\ &= d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_k, f^k(g_k x)) \\ &< 2\epsilon < \alpha \end{aligned}$$

for all  $k$ . Since  $\alpha$  is a  $G$ -expansive constant for  $f$ ,

$$G(f^n(x)) = G(x),$$

and hence

$$g_0 x \in \text{Per}_G(f) \cap N_\epsilon(p)$$

where  $N_\epsilon(p)$  is an  $\epsilon$ -neighborhood of  $p$ . Therefore,  $\text{Per}_G(f)$  is dense in  $\Omega_G(f)$ .  $\square$

**Claim 4.** *Each  $B_\lambda$  is open in  $\Omega_G(f)$ .*

*Proof.* Let  $\alpha > 0$  be a  $G$ -expansive constant for  $f$  and let  $\epsilon < \alpha$ . Denote

$$N_\delta(B_\lambda) = \{y \in \Omega_G(f) : d(y, B_\lambda) < \delta\}$$

where  $\delta$  is the constant corresponding to  $\epsilon$  in the definition of the GSP for  $f|_{\Omega_G(f)}$ . Then for a point  $p \in N_\delta(B_\lambda) \cap \text{Per}_G(f)$ , there exists  $y \in B_\lambda$  such that

$$d(y, p) < \delta.$$

Since  $f|_{\Omega_G(f)}$  has the GSP, it follows from Remark 1.10 that

$$W^u(p) \cap W^s(y) \neq \emptyset$$

and

$$W^s(p) \cap W^u(y) \neq \emptyset.$$

Here,  $W^s(p)$  and  $W^u(p)$  are defined on  $\Omega_G(f)$ . So, there exists  $y_0 \in B_\lambda$  (in particular,

$y_0$  belongs to the  $\alpha$ -limit set  $\alpha(y)$  such that  $y_0 \sim p$ , that is,  $p \in B_\lambda$ . Therefore,

$$B_\lambda \supset \overline{N_\delta(B_\lambda) \cap \text{Per}_G(f)} \supset N_\delta(B_\lambda) \cap \overline{\text{Per}_G(f)} = N_\delta(B_\lambda),$$

that is,  $B_\lambda$  is open in  $\Omega_G(f)$ .  $\square$

Since  $X$  is compact and  $\Omega_G(f)$  is a closed subset of  $X$ ,  $\Omega_G(f)$  can be covered by finitely many  $B_\lambda$ 's, that is,  $\Omega_G(f) = \bigcup_{i=1}^n B_i$ .

**Claim 5.** *Each  $B_i$  is  $G$ -invariant.*

*Proof.* Let  $x \in B_i$ ,  $g \in G$ , and  $\delta > 0$ . We shall show that  $gx \in B_i$ . Since  $x \in B_i$ , there exists a  $(\delta, G)$ -pseudo orbit  $\{x_0 = x, x_1, \dots, x_{n-1}, x_n = x\}$ . Then  $d(g_0 f(x), x_1) < \delta$  for some  $g_0 \in G$ . Since  $f$  is pseudoequivariant, we can take  $h \in G$  such that  $g_0 f(x) = hf(gx)$ . Thus  $\{gx, x_1, \dots, x_{n-1}, x_n = x\}$  is a  $(\delta, G)$ -pseudo orbit. By Lemma 2.3, there exists  $\gamma > 0$  ( $\gamma < \delta$ ) such that

$$d(x, y) < \gamma \implies d(gx, gy) < \delta \quad \text{for all } g \in G.$$

Let  $\{x_0 = x, x_1, \dots, x_{n-1}, x_n = x\}$  be a  $(\gamma, G)$ -pseudo orbit. Then

$$d(g_{n-1} f(x_{n-1}), x) < \gamma \quad \text{for some } g_{n-1} \in G$$

and hence  $d(gg_{n-1} f(x_{n-1}), gx) < \delta$ . Thus  $\{x_0 = x, x_1, \dots, x_{n-1}, gx\}$  is a  $(\delta, G)$ -pseudo orbit. Since  $\delta$  is arbitrary,  $gx \sim_G x$ . Therefore,  $gx \in B_i$ .  $\square$

**Claim 6.**  *$f|_{B_i}$  has the GSP.*

*Proof.* Let  $0 < \epsilon < \min\{d(B_i, B_j) : i \neq j, 1 \leq i, j \leq n\}$  be given. Since  $f|_{\Omega_G(f)}$  has the GSP, there exists  $\delta < \epsilon$  such that every  $(\delta, G)$ -pseudo orbit  $\{x_k\} \subset B_i$  is  $\epsilon$ -traced by a point  $x \in \Omega_G(f)$ . This means that, for each  $k$ , there exists  $g_k \in G$  such that

$$d(f^k(x), g_k x_k) < \epsilon.$$

Since  $B_i$  is  $G$ -invariant and  $x_0 \in B_i$ ,  $g_0 x_0 \in B_i$ . Therefore  $x \in B_i$ .  $\square$

**Claim 7.**  *$f|_{B_i}$  is topologically  $G$ -transitive.*

*Proof.* Let  $U$  and  $V$  be nonempty open subsets of  $B_i$ . Take  $x \in U$  and  $y \in V$ . Then  $x \sim_G y$ . Let  $N_\epsilon(x)$  and  $N_\epsilon(y)$  be  $\epsilon$ -neighborhoods of  $x$  and  $y$  respectively such that  $N_\epsilon(x) \subset U$  and  $N_\epsilon(y) \subset V$ . Choose a positive  $\epsilon_1 < \epsilon$  such that

$$d(a, b) < \epsilon_1 \implies d(ga, gb) < \epsilon \quad \text{for all } g \in G.$$

Since  $f|_{B_i}$  has the GSP, there exists  $\delta_1 > 0$  such that every  $(\delta_1, G)$ -pseudo orbit in  $B_i$  is  $\epsilon_1$ -traced by a point in  $B_i$ . Thus, a  $(\delta_1, G)$ -pseudo orbit  $\{x_0 = x, \dots, x_n = y\} \subset B_i$  from  $x$  to  $y$  is  $\epsilon_1$ -traced by a point  $z \in B_i$ . In particular,

$$d(z, g_0x) < \epsilon_1 \quad \text{and} \quad d(f^n(z), g_ny) < \epsilon_1 \quad \text{for some} \quad g_0, g_n \in G.$$

Since  $d(g_0^{-1}z, x) < \epsilon$  and  $d(g_n^{-1}f^n(z), y) < \epsilon$ ,

$$g_0^{-1}z \in N_\epsilon(x) \subset U$$

and

$$g_n^{-1}f^n(z) \in N_\epsilon(y) \subset V.$$

Since  $f^n(g_0^{-1}z) \in f^n(U)$  and  $f$  is pseudoequivariant,

$$g_1f^n(z) \in f^n(U) \quad \text{for some} \quad g_1 \in G.$$

Choose  $g \in G$  such that  $gg_1 = g_n^{-1}$ . Then  $g_n^{-1}f^n(z) \in gf^n(U)$ . Therefore,  $gf^n(U) \cap V \neq \emptyset$ .  $\square$

We next prepare the following three lemmas to complete Theorem B.

**Lemma 2.5.** *Let  $f: X \rightarrow X$  be a pseudoequivariant homeomorphism on a compact metric  $G$ -space  $X$  with  $G$  compact. Then*

$$W^i(x) = W^i(p) \quad \text{for any} \quad x \in W^i(p) \quad (i = s, u).$$

*Proof.* We shall prove only the case  $i = s$ . Let  $y \in W^s(x)$  and let  $\epsilon > 0$ . Since  $y \in W^s(x)$ , there exists  $N_1 \in \mathbb{N}$  such that  $n \geq N_1$  implies that

$$d(f^n(h_nx), f^n(y)) < \frac{\epsilon}{2} \quad \text{for some} \quad h_n \in G.$$

Let  $\delta > 0$  be the constant satisfying the following:

$$d(x, y) < \delta \implies d(gx, gy) < \frac{\epsilon}{2} \quad \text{for all} \quad g \in G.$$

Since  $x \in W^s(p)$ , there exists  $N_2 \in \mathbb{N}$  such that  $n \geq N_2$  implies that

$$d(f^n(g'_np), f^n(x)) < \delta \quad \text{for some} \quad g'_n \in G.$$

Hence for some  $h'_n \in G$  with  $h'_nf^n(x) = f^n(h_nx)$ ,

$$d(h'_nf^n(g'_np), h'_nf^n(x)) < \frac{\epsilon}{2}.$$

Since  $h'_n f^n(g'_n p) = f^n(g_n p)$  for some  $g_n \in G$ ,

$$d(f^n(g_n p), f^n(h_n x)) < \frac{\epsilon}{2}.$$

Take  $N = \max\{N_1, N_2\}$ . Then  $n \geq N$  implies that

$$d(f^n(g_n p), f^n(y)) \leq d(f^n(g_n p), f^n(h_n x)) + d(f^n(h_n x), f^n(y)) < \epsilon.$$

Therefore,  $W^s(x) \subset W^s(p)$ . Similarly, one can prove  $W^s(p) \subset W^s(x)$ .  $\square$

**Lemma 2.6.** *Let  $f: X \rightarrow X$  be a pseudoequivariant homeomorphism on a compact metric  $G$ -space  $X$  with  $G$  compact and let  $x \in W^i(p)$ . Then*

$$gx \in W^i(p) \text{ for every } g \in G,$$

and hence

$$G(W^i(p)) = W^i(p) \quad (i = s, u).$$

*Proof.* Let  $x \in W^s(p)$ ,  $g \in G$  and let  $\epsilon > 0$ . Then there is  $\delta > 0$  such that if  $d(x, y) < \delta$ , then  $d(gx, gy) < \epsilon$  for all  $g \in G$ . Since for each  $n \in \mathbb{Z}$ , we have  $g_n \in G$  such that

$$\lim_{n \rightarrow \infty} d(f^n(g_n p), f^n(x)) = 0,$$

that is, there exists  $N \in \mathbb{N}$  such that

$$n \geq N \implies d(f^n(g_n p), f^n(x)) < \delta.$$

Hence, for  $h'_n \in G$  with  $h'_n f^n(x) = f^n(gx)$ ,

$$d(h'_n f^n(g_n p), h'_n f^n(x)) < \epsilon.$$

Let  $h'_n f^n(g_n p) = f^n(h_n p)$ . Then

$$d(f^n(h_n p), f^n(gx)) < \epsilon.$$

Therefore,  $gx \in W^s(p)$ . Similarly, one can prove the statement for the case  $i = u$ .  $\square$

**Lemma 2.7.** *Let  $f: X \rightarrow X$  be a pseudoequivariant homeomorphism on a compact metric  $G$ -space  $X$  with  $G$  compact. Then for any  $\epsilon > 0$ , there exists a positive*

constant  $\delta < \epsilon$  satisfying the following: if  $x \in W_\delta^u(y)$ , then for all  $g \in G$ ,

$$(1) \quad gx \in W_\epsilon^u(y)$$

and

$$(2) \quad gy \in W_\epsilon^u(x).$$

*Proof.* Let  $\epsilon > 0$ . Then, by Lemma 2.3, there exists a positive constant  $\delta < \epsilon$  such that

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$

Let  $x \in W_\delta^u(y)$  and let  $g \in G$ . Then for each  $n \geq 0$ , there exists  $g_n \in G$  such that

$$d(f^{-n}(x), f^{-n}(g_n y)) < \delta.$$

(1) Take  $g'_n \in G$  such that  $g'_n f^{-n}(x) = f^{-n}(gx)$ . Then

$$d(f^{-n}(gx), g'_n f^{-n}(g_n y)) < \epsilon.$$

Since  $f$  is pseudoequivariant,  $gx \in W_\epsilon^u(y)$ .

(2) Take  $g'_n \in G$  such that  $g'_n f^{-n}(g_n y) = f^{-n}(gy)$ . Then

$$d(g'_n f^{-n}(x), f^{-n}(gy)) < \epsilon.$$

Since  $f$  is pseudoequivariant,  $gy \in W_\epsilon^u(x)$  for all  $g \in G$ . □

**Proof of Theorem B.** Let  $\epsilon > 0$  be a constant which is less than the  $G$ -expansive constant for  $f|_B$  and let  $\delta > 0$  ( $\delta < \epsilon$ ) be the constant corresponding to  $\epsilon$  in the definition of the GSP. Let  $X_p = \overline{W^u(p)} \cap B$  for  $p \in B \cap \text{Per}_G(f)$ . We can see directly from Lemmas 2.3 and 2.6 that  $X_p$  is  $G$ -invariant, that is, if  $x \in X_p$ , then  $gx \in X_p$  for all  $g \in G$ .

**Claim 1.**  $X_p$  is open in  $B$ .

*Proof.* Since  $p \in \text{Per}_G(f)$ , we have an integer  $m > 0$  and  $g_1 \in G$  such that  $g_1 f^m(p) = p$ . Denote  $N_\delta(X_p) = \{y \in B : d(y, X_p) < \delta\}$ . Let  $q \in N_\delta(X_p) \cap \text{Per}_G(f)$ . Then there is  $x \in W^u(p) \cap B$  with  $d(q, x) < \delta$ . Note that  $g_2 f^n(q) = q$  for some integer  $n > 0$  and  $g_2 \in G$ . Since  $f|_B$  has the GSP, the  $(\delta, G)$ -pseudo orbit

$$\{\dots, f^{-2}(x), f^{-1}(x), q, f(q), f^2(q), \dots\}$$

is  $\epsilon$ -traced by a point  $x' \in B$ , that is, for each  $t \in \mathbb{Z}$ , there exists  $h_t \in G$  such that

- (a)  $d(x', h_0 q) < \epsilon$ ;
- (b)  $d(f^t(x'), h_t f^t(q)) < \epsilon$  ( $t > 0$ );
- (c)  $d(f^{-t}(x'), h_{-t} f^{-t}(q)) < \epsilon$  ( $t > 0$ ).

Hence, it follows from Remark 1.10 that  $x' \in W^s(q) \cap W^u(x) \cap B$ .

Since  $f$  is pseudoequivariant and  $p \in \text{Per}_G(f)$ , for each  $k \in \mathbb{Z}$ , we have  $g_{kmn} \in G$  such that  $f^{kmn}(g_{kmn}p) = p$ . Since  $W^u(x) = W^u(p) = W^u(g_{kmn}p)$  by Lemmas 2.5 and 2.6,

$$f^{kmn}(x') \in f^{kmn}(W^u(g_{kmn}p)) = W^u(f^{kmn}(g_{kmn}p)) = W^u(p).$$

Since  $q \in W^s(x')$ , for each  $k \in \mathbb{Z}$ , one can find  $h_{kmn} \in G$  such that

$$\lim_{k \rightarrow \infty} d(h_{kmn} f^{kmn}(x'), f^{kmn}(q)) = 0.$$

Take  $i_{kmn} \in G$  such that  $i_{kmn}(h_{kmn})^{-1} f^{kmn}(q) = q$ . Then

$$\lim_{k \rightarrow \infty} d(i_{kmn} f^{kmn}(x'), i_{kmn}(h_{kmn})^{-1} f^{kmn}(q)) = \lim_{k \rightarrow \infty} d(i_{kmn} f^{kmn}(x'), q) = 0.$$

Hence,  $q \in \overline{W^u(p) \cap B} = X_p$  because  $i_{kmn} f^{kmn}(x') \in W^u(p)$  for each  $k \in \mathbb{Z}$  by Lemma 2.6. Therefore,

$$X_p \supset \overline{N_\delta(X_p) \cap \text{Per}_G(f)} \supset N_\delta(X_p) \cap \overline{\text{Per}_G(f)} = N_\delta(X_p),$$

that is,  $X_p$  is open in  $B$ . □

Note that  $f(X_p) = f(\overline{W^u(p) \cap B}) = \overline{f(W^u(p)) \cap f(B)} = \overline{W^u(f(p)) \cap B} = X_{f(p)}$ . Since  $X_p = X_{g_1 p}$  for any  $g_1 \in G$  and  $g_1 f^m(p) = p$ ,

$$f^m(X_p) = X_{f^m(p)} = X_{g_1 f^m(p)} = X_p.$$

Take the smallest integer  $a > 0$  such that  $a \leq m$  and  $f^a(X_p) = X_p$ .

**Claim 2.**  $B = \bigcup_{j=0}^{a-1} f^j(X_p)$ .

*Proof.* Let  $y \in B$ . Since  $f|_B$  is topologically  $G$ -transitive, for each  $1/n$ -neighborhood  $N_{1/n}(y)$  of  $y$ , there are  $k > 0$  and  $h_n \in G$  such that  $h_n N_{1/n}(y) \cap f^k(X_p) \neq \emptyset$ . So  $h_n N_{1/n}(y) \cap (\bigcup_{j=0}^{a-1} f^j(X_p)) \neq \emptyset$  for each  $n \in \mathbb{N}$ . We may assume that  $h_n \rightarrow h \in G$  because  $G$  is compact. Since  $\bigcup_{j=0}^{a-1} f^j(X_p)$  is closed in  $B$ ,  $h y \in \bigcup_{j=0}^{a-1} f^j(X_p)$ . Since  $G(f^j(X_p)) = G(X_{f^j(p)}) = X_{f^j(p)} = f^j(X_p)$ , we have  $y \in \bigcup_{j=0}^{a-1} f^j(X_p)$ . □



**Claim 3.**  $X_p = X_q$  for  $q \in X_p \cap \text{Per}_G(f)$ .

*Proof.* Let  $q \in X_p \cap \text{Per}_G(f)$  and suppose  $m$  and  $n$  are  $G$ -periodic numbers of  $p$  and  $q$  respectively. Since  $N_\delta(X_p) = X_p$  for the constant  $\delta > 0$  in the above of Claim 1,  $W_\delta^u(q) \subset X_p$ . We firstly show that  $p \in X_q$ . Suppose that  $p \notin X_q$ . Then  $d(K, X_q) > 0$  where  $K = X_p \setminus X_q$ . Since  $q \in X_p = \overline{W^u(p) \cap B}$ , there exists  $z \in W^u(p) \cap B$  such that  $d(z, q) < d(K, X_q)$ . Since  $z \in X_p$  and  $z \notin K$ ,  $z \in X_q$ . Furthermore, for each  $j \in \mathbb{Z}$ , there exists  $g'_{mnj} \in G$  such that

$$\lim_{j \rightarrow \infty} d(f^{-mnj}(z), f^{-mnj}(g'_{mnj}p)) = 0.$$

For each  $j \in \mathbb{Z}$ , choose  $g_{mnj} \in G$  with  $g_{mnj}f^{-mnj}(g'_{mnj}p) = p$ . Then we have

$$\lim_{j \rightarrow \infty} d(g_{mnj}f^{-mnj}(z), p) = 0.$$

So  $g_{mnj}f^{-mnj}(z) \notin X_q$  for sufficiently large  $j$ . Hence,

$$h_{mnj}z \notin f^{mnj}(X_q) = X_q$$

for  $h_{mnj} \in G$  with  $g_{mnj}f^{-mnj}(z) = f^{-mnj}(h_{mnj}z)$ . Thus,  $z \notin X_q$ . This is a contradiction. Therefore,  $p \in X_q$ .

Let  $y \in W^u(q)$  and let  $0 < \delta_1 < \delta_2 < \delta_3 = \delta$  such that

$$d(x, y) < \delta_i \implies d(gx, gy) < \delta_{i+1} \quad \text{for all } g \in G \quad (i = 1, 2).$$

Then there exists  $N \in \mathbb{N}$  such that if  $k \geq N$ , then  $d(f^{-k}(y), f^{-k}(h_kq)) < \delta_1$  for some  $h_k \in G$ . Choose  $j \in \mathbb{N}$  with  $mnj \geq N$ . Then

$$d((f^{-i} \circ f^{-mnj})(y), (f^{-i} \circ f^{-mnj})(h_{mnj+i}q)) < \delta_1 \quad \text{for all } i \geq 0,$$

that is,

$$f^{-mnj}(y) \in W_{\delta_1}^u(f^{-mnj}(q)).$$

By Lemma 2.7 (2),  $gf^{-mnj}(q) \in W_{\delta_2}^u(f^{-mnj}(y))$  for all  $g \in G$ . Since  $q \in \text{Per}_G(f)$ , we have  $q \in W_{\delta_2}^u(f^{-mnj}(y))$ . Again, by Lemma 2.7 (2),  $gf^{-mnj}(y) \in W_{\delta}^u(q)$  for all  $g \in G$ . In particular,  $f^{-mnj}(y) \in W_{\delta}^u(q)$ . This means that  $y \in f^{mnj}(W_{\delta}^u(q))$  for some  $j \geq 0$ . So  $W^u(q) \subset \bigcup_{j \geq 0} f^{mnj}(W_{\delta}^u(q))$ . Therefore,

$$X_q = \overline{W^u(q) \cap B} \subset \overline{\bigcup_{j \geq 0} f^{mnj}(W_{\delta}^u(q)) \cap B} \subset \overline{X_p \cap B} = X_p \cap B = X_p.$$

Similarly, we have  $X_p \subset X_q$ . □

**Claim 4.**  $X_p \cap f^j(X_p) = \emptyset$  for  $0 < j < a$ .

*Proof.* Suppose  $X_p \cap f^j(X_p) \neq \emptyset$  for some  $j$ . Since  $X_p \cap f^j(X_p)$  is open in  $B$ , we can find  $q \in X_p \cap f^j(X_p) \cap \text{Per}_G(f)$ . Then  $X_q = X_p = f^j(X_p)$ , which is a contradiction to the choice of the integer  $a$ .  $\square$

**Claim 5.**  $f^a|_{X_p}$  is topologically  $G$ -mixing.

*Proof.* Let  $U$  and  $V$  be non-empty open subsets of  $X_p$  and let  $q \in V \cap \text{Per}_G(f)$ . Then  $f^{aj}(q) \in X_p \cap \text{Per}_G(f)$  for all  $j \in \mathbb{Z}$ . Since  $X_p = X_{f^{aj}(q)}$  for all  $j \in \mathbb{Z}$ ,

$$U \cap W^u(f^{aj}(q)) = U \cap (W^u(f^{aj}(q)) \cap B) \neq \emptyset \quad \text{for all } j \in \mathbb{Z}.$$

Let  $n > 0$  be a  $G$ -periodic number of  $q$ . Then for each  $j$  such that  $0 \leq j \leq n-1$ , there exists  $z_j \in U \cap W^u(f^{aj}(q))$ . Since  $f$  is pseudoequivariant, we may take this statement: for each  $t \in \mathbb{Z}$ , there exists  $h_t \in G$  such that

$$\lim_{t \rightarrow \infty} d(f^{-ant}(z_j), f^{aj}(h_t f^{-ant}(q))) = 0.$$

For each  $t \in \mathbb{Z}$ , choose  $g_t \in G$  such that  $g_t f^{aj}(h_t f^{-ant}(q)) = f^{aj}(q)$ . Then we have

$$\lim_{t \rightarrow \infty} d(g_t f^{-ant}(z_j), f^{aj}(q)) = 0,$$

and thus

$$\lim_{t \rightarrow \infty} g_t f^{-ant}(z_j) = f^{aj}(q).$$

Since  $f^{aj}(q) \in f^{aj}(V)$ , for each  $j$  with  $0 \leq j \leq n-1$ , we may choose  $N_j > 0$  such that for all  $t \geq N_j$ ,

$$g_t f^{-ant}(z_j) \in f^{aj}(V).$$

Let  $M = \max\{N_j : 0 \leq j \leq n-1\}$ . For each  $t \geq M$ , we get  $t = ns + j$ . If  $s \geq M$ , then

$$f^{-at}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j)) \in V$$

for each  $i_s \in G$  such that  $f^{-ans-aj}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j))$ . Hence,

$$i_s z_j \in f^{at}(V) \quad \text{if } s \geq M \quad (\text{that is, } t \geq nM).$$

Thus, it follows from  $z_j \in U$  that there exists  $k_t \in G$  such that

$$k_t f^{at}(V) \cap U \neq \emptyset \quad \text{for each } t \geq nM.$$

Therefore,  $f^a|_{X_p}$  is topologically  $G$ -mixing.  $\square$

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