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DECOMPOSITION THEOREM ON *G*-SPACES

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Abstract

In this paper, we introduce the weak G-expansivity which is a generalization of both expansivity and G-expansivity. Also, we define G-stable and G-unstable sets of a homeomorphism on a metric G-space X and investigate properties of them. Finally, we consider the decomposition theorem on G-spaces.

1. Introduction

Let X be a topological space, G be a topological group, and $\theta\colon G \times X \to X$ be a map. The triple (X, G, θ) is called a *topological G-space* if the following three conditions are satisfied:
(1) $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G;
(2) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and for all $g, h \in G$;
(3) θ is continuous.
Here, gh is the group operation on G. Simply, we denote $\theta(g, x)$ by gx and X is usually said to be a *topological G-space*.

For any subset A of X, $G(A)$ is denoted by the set $\{ga\colon g \in G,\ a \in A\}$. $G(x)$ is called a *G-orbit* of x. A subset A of X is called *G-invariant* if $G(A) = A$. A map $f\colon X \to X$ on a G-space X is said to be *pseudoequivariant* provided that $f(G(x)) = G(f(x))$ for all $x \in X$, and f is said to be *equivariant* provided that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

N. Aoki has proved the following topological decomposition theorem in 1983 ([1]), which is an extension of Smale's spectral decomposition theorem and Bowen's decomposition theorem in dynamical systems. All undefined notions can be found in [2].

Theorem 1.1 ([1]). *Let $f\colon X \to X$ be a homeomorphism on a compact metric space X and let $CR(f)$ be the chain recurrent set. If $f|_{CR(f)}\colon CR(f) \to CR(f)$ is an expansive homeomorphism with the shadowing property, then*
(1) *$CR(f)$ contains a finite sequence B_i $(1 \le i \le k)$ of f-invariant closed subsets such that*

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(a) $CR(f) = \bigcup_{i=1}^{k} B_i$ (*disjoint union*);
(b) $f|_{B_i} \colon B_i \to B_i$ *is topologically transitive*,

(2) *for each* B_i, *there exist a subset* X_p *of* B_i *and* $a > 0$ *such that*
(a) $f^a(X_p) = X_p$;
(b) $X_p \cap f^j(X_p) = \emptyset$ $(0 < j < a)$;
(c) $f^a|_{X_p} \colon X_p \to X_p$ *is topologically mixing*;
(d) $B_i = \bigcup_{j=0}^{a-1} f^j(X_p)$.

A point $x \in X$ is called a *G-periodic point* of f if there exist an integer $n > 0$ and $g \in G$ such that $f^n(x) = gx$. A point $x \in X$ is called a *G-nonwandering point* of f if for every open neighborhood U of x, there exist $n > 0$ and $g \in G$ such that $gf^n(U) \cap U \neq \emptyset$. $Per_G(f)$ (resp. $\Omega_G(f)$) is denoted by the set of all G-periodic (resp. G-nonwandering) points of f.

For a homeomorphism f on a metric G-space X, a sequence $\{x_i \in X : i \in \mathbb{Z}\}$ is called a (δ, G)-*pseudo orbit* for f provided that for each i, there exists $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$. A (δ, G)-pseudo orbit $\{x_i\}$ for f is said to be ϵ-*traced by a point* $x \in X$ provided that for each i, there exists $g_i \in G$ such that $d(f^i(x), g_i x_i) < \epsilon$.

DEFINITION 1.2 ([5]). A homeomorphism $f \colon X \to X$ has the *G-shadowing property* (GSP) provided that for any $\epsilon > 0$, there exists $\delta > 0$ such that every (δ, G)-pseudo orbit $\{x_i\}$ in X for f is ϵ-traced by a point $x \in X$.

REMARK 1.3. It was proved by E. Shah that, when X is a compact metric G-space and the orbit map $\pi \colon X \to X/G$ is a covering map, a pseudoequivariant homeomorphism f on X has the GSP if and only if the induced map $\hat{f} \colon X/G \to X/G$ has the shadowing property ([5]).

If a pseudoequivariant continuous onto map $f \colon X \to X$ has the GSP where X is a compact metric G-space with G compact, then $f|_{\Omega_G(f)}$ has the GSP ([5]).

The main purpose of this paper is to prove the following theorems on compact metric G-spaces.

Theorem A. *Let X be a compact metric G-space with G compact. If $f \colon X \to X$ is a pseudoequivariant G-expansive homeomorphism with the GSP, then $\Omega_G(f)$ contains a finite sequence B_i $(1 \le i \le n)$ of f-invariant, G-invariant, and closed subsets such that*

(1) $f|_{\Omega_G(f)}$ *is topologically* G-*transitive*;
(2) $\Omega_G(f) = \bigcup_{i=1}^{n} B_i$ (*disjoint union*);
(3) $f|_{B_i}$ *has the GSP.*

A homeomorphism $f \colon X \to X$ is said to be *topologically G-mixing* provided that for every nonempty open subsets U and V of X, there exists an integer N such that for each $n \geq N$, there is $g_n \in G$ satisfying $g_n f^n(U) \cap V \neq \emptyset$.

Theorem B. *Let $f|_{\Omega_G(f)}\colon \Omega_G(f) \to \Omega_G(f)$ be a G-expansive homeomorphism with the GSP. Then, for any f-invariant, G-invariant, open and closed subset $B \subset \Omega_G(f)$ such that $f|_B\colon B \to B$ is topologically G-transitive, there are $X_p \subset B$ and $a > 0$ such that*
(1) $f^a(X_p) = X_p$;
(2) $X_p \cap f^j(X_p) = \emptyset$ $(0 < j < a)$;
(3) $f^a|_{X_p}\colon X_p \to X_p$ *is topologically G-mixing*;
(4) $B = \bigcup_{j=0}^{a-1} f^j(X_p)$.

DEFINITION 1.4. A homeomorphism $f\colon X \to X$ on a metric G-space X is said to be *weak G-expansive* provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$ if $u \in G(x)$ and $v \in G(y)$, there exists $n = n(u, v) \in \mathbb{Z}$ such that

$$d(f^n(u),\ f^n(v)) > \delta.$$

The constant δ is called a *weak G-expansive constant* for f.

The weak G-expansivity is a generalization of both expansivity and G-expansivity. Here, G-expansivity has been defined by R. Das ([4]). A homeomorphism $f\colon X \to X$ is said to be *G-expansive* provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$, there exists $n \in \mathbb{Z}$ such that

$$d(f^n(u),\ f^n(v)) > \delta \quad \text{for all} \quad u \in G(x),\ v \in G(y).$$

The constant δ is called a *G-expansive constant* for f.

REMARK 1.5. R. Das proved that there is no implication between G-expansivity and expansivity by giving counterexamples ([4]).

EXAMPLE 1.6 ([4]). Consider the compact space $X = \{1/n, 1 - 1/n\colon n \in \mathbb{N}\}$ with the usual metric and let the topological group $G = \{-1, 1\}$ act on X with the action θ defined by $\theta(1, x) = x$ and $\theta(-1, x) = 1 - x$. Define a homeomorphism $f\colon X \to X$ by

$$f(x) = \begin{cases} x & \text{if} \quad x = 0,\ 1; \\ \text{next to the right of } x & \text{if} \quad x \in X \setminus \{0, 1\}. \end{cases}$$

Then f is an expansive map with expansive constant δ $(0 < \delta < 1/6)$. But, it is easy to see that for $x, y \in X \setminus \{1/2\}$ with $G(x) \neq G(y)$, there is no $n \in \mathbb{Z}$ such that

$$|f^n(u) - f^n(v)| > \delta \quad \text{for all} \quad u \in G(x),\ v \in G(y),$$

whatever $\delta > 0$ may be. This means that f is not G-expansive.

EXAMPLE 1.7 ([4]). Consider the compact space $X = \bigcup_{i=1}^{n} C_i$ with the usual metric, where each C_i is the circle in $\mathbb{R}^2$ with center the origin and radius i. Denote $G = SO(2)$ by the set of all 2×2 matrices whose determinants are ± 1 and define an action $\theta \colon G \times X \to X$ by the usual rotations on X. Then the identity map on X is G-expansive with G-expansive constant δ $(0 < \delta < 1)$.

Therefore, all properties of the following diagram are distinguished as we see in Examples 1.6 and 1.7:

$$
\begin{tikzcd}
& \text{weak } G\text{-expansive} \arrow[dl, shift right, "/" marking] \arrow[dr, shift left, "/" marking] & \\
G\text{-expansive} \arrow[ur, shift right] \arrow[rr, shift left, "/" marking] & & \text{expansive} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

DEFINITION 1.8. Let $f \colon X \to X$ be a homeomorphism of a metric G-space X. We define a *local G-stable set* $W^s_\epsilon(x)$ and a *local G-unstable set* $W^u_\epsilon(x)$ by

$$
\begin{aligned}
W^s_\epsilon(x) = \{y \in X \colon & \text{ for each } n \geq 0, \\
& \text{there is } g_n \in G \text{ such that } d(f^n(g_n x), f^n(y)) \leq \epsilon\}, \\
W^u_\epsilon(x) = \{y \in X \colon & \text{ for each } n \geq 0 \\
& \text{there is } g_n \in G \text{ such that } d(f^{-n}(g_n x), f^{-n}(y)) \leq \epsilon\}.
\end{aligned}
$$

We modify results of [3] into the following results by weakening the condition "equivariant" into "pseudoequivariant" and deleting the condition "invariant metric". A metric d on a G-space X is called an *invariant metric* provided that $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

REMARK 1.9. Let X be a compact metric G-space with G compact. If $f \colon X \to X$ is a weak G-expansive pseudoequivariant homeomorphism with weak G-expansive constant $\delta > 0$, then for every $\gamma > 0$, there is $N > 0$ such that for each $x \in X$ and for each $n \geq N$,

$$f^n(W^s_\delta(x)) \subset W^s_\gamma(f^n(x))$$

and

$$f^{-n}(W^u_\delta(x)) \subset W^u_\gamma(f^{-n}(x)).$$

Proof. We shall prove only the case of a local G-stable set because the other case can be proved similarly. To do it, suppose that there exists $\gamma > 0$ such that for all $N > 0$, there are $x \in X$ and $n \geq N$ satisfying

$$f^n(W^s_\delta(x)) \not\subset W^s_\gamma(f^n(x)).$$

Let $N > 0$. Then there are $x_1 \in X$ and $n \geq N$ satisfying

$$f^n(W^s_\delta(x_1)) \not\subset W^s_\gamma(f^n(x_1)),$$

that is, there exists $y_1 \in W^s_\delta(x_1)$ such that $f^n(y_1) \notin W^s_\gamma(f^n(x_1))$. So there exists $i \geq 0$ such that for every $h \in G$,

$$d(f^i(hf^n(x_1)),\ f^i(f^n(y_1))) > \gamma.$$

Because f is pseudoequivariant, there exists $i \geq 0$ such that for every $g \in G$,

$$d(gf^{i+n}(x_1),\ f^{i+n}(y_1)) > \gamma.$$

Take $m_1 = i + n$ and choose $N = m_1 + 1$.

Continuing the process, we can find sequences $m_n > 0$, x_n, and $y_n \in X$ such that
(1) $y_n \in W^s_\delta(x_n)$;
(2) $d(hf^{m_n}(x_n),\ f^{m_n}(y_n)) > \gamma$ for all $h \in G$;
(3) $\lim_{n\to\infty} m_n = \infty$.
It follows from $y_n \in W^s_\delta(x_n)$ that for each $i \geq -m_n$, there exists $g_{i+m_n} \in G$ such that

$$d(f^{i+m_n}(g_{i+m_n}x_n),\ f^{i+m_n}(y_n)) \leq \delta.$$

Since f is pseudoequivariant, for each g_{i+m_n}, there exists $h_{i+m_n} \in G$ such that

$$d(f^i(h_{i+m_n} f^{m_n}(x_n)),\ f^i(f^{m_n}(y_n))) = d(f^{i+m_n}(g_{i+m_n}x_n),\ f^{i+m_n}(y_n)).$$

Hence, for each $i \geq -m_n$,

$$d(f^i(h_{i+m_n} f^{m_n}(x_n)),\ f^i(f^{m_n}(y_n))) \leq \delta.$$

If $f^{m_n}(x_n) \to x$, $f^{m_n}(y_n) \to y$, and $h_{i+m_n} \to h$ as $n \to \infty$, then

$$d(f^i(hx),\ f^i(y)) \leq \delta \quad \text{for all} \quad i \in \mathbb{Z}.$$

Since δ is a weak G-expansive constant for f, $G(x) = G(y)$. But $d(hx, y) = \lim_{n\to\infty} d(hf^{m_n}(x_n),\ f^{m_n}(y_n)) \geq \gamma > 0$ for all $h \in G$ by (2). Thus $hx \neq y$ for all $h \in G$, and hence $G(x) \neq G(y)$. This is a contradiction. □

For a homeomorphism f on a compact metric G-space, we define the following:

$$W^s(x) = \Big\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that } \lim_{n\to\infty} d(f^n(g_n x),\ f^n(y) = 0 \Big\};$$

$$W^u(x) = \Big\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that } \lim_{n\to\infty} d(f^{-n}(g_n x),\ f^{-n}(y) = 0 \Big\}.$$

$W^s(x)$ (resp. $W^u(x)$) is called a *G-stable set* (resp. *G-unstable set*).

REMARK 1.10. Let X be a compact metric G-space with G compact. If $f: X \to X$ is a weak G-expansive pseudoequivariant homeomorphism with weak G-expansive constant $\delta > 0$, then for each ϵ with $0 < \epsilon < \delta$,

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_\epsilon(f^n(x)));$$
$$W^u(x) = \bigcup_{n\geq 0} f^{n}(W^s_\epsilon(f^{-n}(x))).$$

Proof. ($\subset$): Let $y \in W^s(x)$ and $0 < \epsilon < \delta$. Then there exists $N > 0$ such that for each $n \geq N$, we can choose $g_n \in G$ satisfying

$$d(f^n(g_n x),\ f^n(y)) \leq \epsilon.$$

Thus,

$$d(f^i(f^N(g_{i+N}x)),\ f^i(f^N(y))) \leq \epsilon \quad \text{for all} \quad i \geq 0.$$

Since f is pseudoequivariant, $f^N(y) \in W^s_\epsilon(f^N(x))$. Therefore,

$$y \in f^{-N}(W^s_\epsilon(f^N(x))) \subset \bigcup_{n\geq 0} f^{-n}(W^s_\epsilon(f^n(x))).$$

($\supset$): Let $y \in f^{-n}(W^s_\epsilon(f^n(x)))$ for some $n \geq 0$. Then $f^n(y) \in W^s_\epsilon(f^n(x))$. It follows from Remark 1.9 that for every $\gamma > 0$ there exists $N > 0$ such that for each $x \in X$ and $m \geq N$,

$$f^{m+n}(y) \in f^m(W^s_\epsilon(f^n(x))) \subset W^s_\gamma(f^{m+n}(x)).$$

So for each $n \geq N$, we can find $g_n \in G$ such that

$$d(f^{m+n}(g_n x),\ f^{m+n}(y)) \leq \gamma.$$

Since f is pseudoequivariant, $y \in W^s(x)$. The proof is completed. The case of a G-unstable set can be proved similarly. $\square$

2. Decomposition theorems

First we prepare the following four lemmas to show Theorem A.

Lemma 2.1 ([3]). *Let (X, G, θ) be a compact metric G-space with G compact. Then for any $\epsilon > 0$, there is a finite open cover $\mathcal{U} = \{U_1, \ldots, U_s\}$ of X such that* $\operatorname{diam}(g\overline{U_i}) \leq \epsilon$ *for all $g \in G$ and i with $1 \leq i \leq s$.*

In Lemma 2.1, notice that, for each $g \in G$, the open cover $\{gU : U \in \mathcal{U}\}$ of X satisfies $\operatorname{diam}(hg\overline{U_i}) \leq \epsilon$ for all $h \in G$ and i with $1 \leq i \leq s$.

Lemma 2.2. *Let X be a compact metric G-space with G compact. If $\mathcal{U}$ is a finite open cover of X, then there exists $\delta > 0$ such that for each subset A of X with* $\operatorname{diam}(A) \leq \delta$, *$A \subset gU$ for some $U \in \mathcal{U}$ and $g \in G$.*

Proof. Suppose not. Then for every $n > 0$ there exists a subset A_n of X such that $\operatorname{diam}(A_n) \leq 1/n$ and $A_n \not\subset gU$ for all $U \in \mathcal{U}$ and $g \in G$. Choose $x_n \in A_n$ for each $n \in \mathbb{N}$. Since X is compact, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x$. We fix $g \in G$. Then there is $U \in \mathcal{U}$ with $x \in gU$. Since $X \setminus gU$ is compact, $d(x, X \setminus gU) > 0$. Put $\epsilon = d(x, X \setminus gU)$ and take $n_i > 0$ such that $1/n_i < \epsilon/2$ and $d(x_{n_i}, x) < \epsilon/2$. Then for any $y \in A_{n_i}$,

$$d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\epsilon}{2} < \epsilon.$$

So $y \in gU$. Therefore, $A_{n_i} \subset gU$. This is a contradiction. □

Lemma 2.3. *Let X be a compact metric G-space with G compact. Then for any $\epsilon > 0$, there exists $\delta > 0$ $(\delta < \epsilon)$ such that*

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \textit{for all} \quad g \in G.$$

Proof. Let $\epsilon > 0$. Then it follows from Lemma 2.1 that, for any positive $\epsilon_1 < \epsilon$, there is a finite open cover $\mathcal{U}$ such that $\operatorname{diam}(g\overline{U}) \leq \epsilon_1$ for all $g \in G$ and $U \in \mathcal{U}$. Also, by Lemma 2.2, there is a constant $\delta = \delta(\mathcal{U}) > 0$ such that for any subset A with $\operatorname{diam}(A) \leq \delta$, $A \subset gU$ for some $g \in G$ and $U \in U$. Let x and y in X with $d(x, y) < \delta$. Then $x, y \in g_0 U_0$ for some $g_0 \in G$ and $U_0 \in \mathcal{U}$. Note that $\{g_0 U : U \in \mathcal{U}\}$ is an open cover of X. For any $g \in G$, take $g_1 \in G$ such that $g_1 = gg_0$. Then, by Lemma 2.1, $\operatorname{diam}(gg_0\overline{U}) \leq \epsilon_1$, that is, $\operatorname{diam}(g_1\overline{U}) \leq \epsilon_1 < \epsilon$ for all $U \in \mathcal{U}$. Since $gx, gy \in gg_0 U_0 = g_1 U_0$, $d(gx, gy) < \epsilon$. □

Lemma 2.4. *Let X be a compact metric G-space with G compact and let f be a pseudoequivariant homeomorphism on X. Then f has the GSP if and only if for any $\epsilon > 0$, we can find $\delta > 0$ such that for every (δ, G)-pseudo orbit $\{x_i\}$ of X for f, there exist $x \in X$ and $h_i \in G$ satisfying*

$$d(f^i(h_i x), x_i) < \epsilon \quad \textit{for all} \quad i \in \mathbb{Z}.$$

Proof. Suppose that f has the GSP and let $\epsilon > 0$. Then, by Lemma 2.3, there exists $\epsilon_0 > 0$ $(\epsilon_0 < \epsilon)$ such that for each $x, y \in X$,

$$d(x, y) < \epsilon_0 \implies d(gx, gy) < \epsilon \quad \text{for all} \quad g \in G.$$

Let δ be the constant corresponding to ϵ_0 in the definition of the GSP. Then every (δ, G)-pseudo orbit $\{x_i\}$ of X for f is ϵ_0-traced by a point $x \in X$, that is, for each i, there exists $g_i \in G$ such that

$$d(f^i(x), g_i x_i) < \epsilon_0 \quad \text{for all} \quad i \in \mathbb{Z}.$$

Since f is pseudoequivariant, for each $g_i \in G$, there exists $h_i \in G$ such that

$$g_i^{-1} f^i(x) = f^i(h_i x).$$

Moreover, $d(g_i^{-1} f^i(x), x_i) < \epsilon$ and hence $d(f^i(h_i x), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

The converse can be proved similarly. □

We have ([5]) that $f(\Omega_G(f)) = \Omega_G(f)$ and $CR_G(f) = \Omega_G(f)$ for a pseudoequivariant homeomorphism f with GSP on a compact metric G-space X where G is compact.

For $x, y \in X$ and $\delta > 0$, x is said to be (δ, G)-*related to* y (denoted by $x \overset{\delta}{\sim}_G y$) if there exist finite (δ, G)-pseudo orbits $\{x = x_0, x_1, \dots, x_k = y\}$ and $\{y = y_0, y_1, \dots, y_n = x\}$ for f. If for every $\delta > 0$, x is (δ, G)-related to y, then x is said to be G-*related to* y (denoted by $x \sim_G y$). A point x is said to be a G-*chain recurrent point* of f if $x \sim_G x$. $CR_G(f)$ is denoted by the set of all G-chain recurrent points of f. A homeomorphism $f \colon X \to X$ is called *topologically* G-*transitive* provided that for every nonempty open subsets U and V of X, there exist an integer $n > 0$ and $g \in G$ such that $g f^n(U) \cap V \neq \emptyset$.

Proof of Theorem A. Since the pseudoequivariant homeomorphism f satisfies the GSP, $CR_G(f) = \Omega_G(f)$. Thus $\Omega_G(f) = \bigcup_\lambda B_\lambda$ where each B_λ is an equivalence class under the relation $\sim_G$ which is defined in $CR_G(f)$.

Claim 1. *Each B_λ is closed in $\Omega_G(f)$.*

Proof. Let $x \in \overline{B_\lambda}$. Then we can find a sequence $\{x_i\}$ in B_λ which converges to x. Let $\alpha > 0$ be given. Then there exists a finite open cover $\{U_1, \dots, U_s\}$ of X such that

$$\operatorname{diam}(g\overline{U_i}) \leq \frac{\alpha}{2} \quad \text{for all} \quad g \in G \quad \text{and} \quad i \quad \text{with} \quad 1 \leq i \leq s$$

by Lemma 2.1. So $f(x) \in U_i$ for some i. Choose an ϵ_0-neighborhood $N_{\epsilon_0}(f(x))$ of $f(x)$ such that $N_{\epsilon_0}(f(x)) \subset U_i$. Then since f is uniformly continuous, there exists $\delta_0 > 0$ such that

$$d(x, y) < \delta_0 \implies d(f(x), f(y)) < \epsilon_0.$$

Because $\{x_i\}$ converges to x, there is $J > 0$ such that $d(x_J, x) < \min\{\alpha/2, \delta_0\}$. From the fact that $x_J \in CR_G(f)$, we can find a $(\alpha/2, G)$-pseudo orbit

$$\{x_J = y_0, y_1, \ldots, y_{k-1}, y_k = x_J\}.$$

So $d(gf(y_0), y_1) < \alpha/2$ for some $g \in G$. Also $d(f(y_0), f(x)) < \epsilon_0$ and hence $d(gf(y_0), gf(x)) < \alpha/2$. Thus,

$$d(gf(x), y_1) \leq d(gf(x), gf(y_0)) + d(gf(y_0), y_1) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Therefore, $\{x, y_1, \ldots, y_k = x_J\}$ is an (α, G)-pseudo orbit. It is clear that there is an (α, G)-pseudo orbit from x_J to x by the uniform continuity of f. It follows from $x \overset{\alpha}{\sim}_G x_J$ that $x \overset{\alpha}{\sim}_G x_i$ for all i because each $x_i \in B_\lambda$. Since α is arbitrary, $x \in B_\lambda$. Therefore, B_λ is closed. $\square$

Claim 2. *Each B_λ is f-invariant.*

Proof. To prove this, we firstly show that $x \sim_G f(x)$ for all $x \in \Omega_G(f)$. Let $\alpha > 0$. Then there is $\delta > 0$ $(\delta < \alpha)$ such that

$$d(a, b) < \delta \implies d(f^2(a), f^2(b)) < \alpha.$$

Since $x \in \Omega_G(f)$, there are $n > 0$ and $g \in G$ such that

$$gf^n(N_\delta(x)) \cap N_\delta(x) \neq \emptyset$$

where $N_\delta(x)$ is a δ-neighborhood of x. Then there exists $z \in N_\delta(x)$ such that $gf^n(z) \in N_\delta(x)$. Hence

$$\{f(x), f^2(z), \ldots, f^{n-1}(z), x\}$$

is an (α, G)-pseudo orbit and thus, $x \sim_G f(x)$. Since f is a homeomorphism, we can show that $x \sim_G f^{-1}(x)$ for all $x \in \Omega_G(f)$ similarly. Therefore, $f(B_\lambda) = B_\lambda$ for each λ. $\square$

Claim 3. *$Per_G(f)$ is dense in $\Omega_G(f)$.*

Proof. Let $\alpha > 0$ be a G-expansive constant for f and take $\epsilon < \alpha/2$. Since f has the GSP, there exists $\delta > 0$ $(\delta < \epsilon)$ such that every (δ, G)-pseudo orbit is ϵ-traced by a point in X. Since f is uniformly continuous, there exists a positive constant $\gamma < \delta$ such that if $d(a, b) < \gamma$, then $d(f(a), f(b)) < \delta$. Let $p \in \Omega_G(f)$. Then for every γ-neighborhood $N_\gamma(p)$ of p, there exist an integer $n > 0$ and $g \in G$ such that

$$gf^n(N_\gamma(p)) \cap N_\gamma(p) \neq \emptyset.$$

Choose a point $y \in gf^n(N_\gamma(p)) \cap N_\gamma(p)$. Since $f^{-n}(g^{-1}y) \in N_\gamma(p)$,

$$d(f(p),\ f(f^{-n}(g^{-1}y))) < \delta.$$

Hence

$$\{\dots, x_0 = p, x_1 = f^{-n+1}(g^{-1}y), x_2 = f^{-n+2}(g^{-1}y), \dots, x_{n-1} = f^{-1}(g^{-1}y), x_n = p, \dots\}$$

is a (δ, G)-pseudo orbit for f. Since f has the GSP, it follows from Lemma 2.4 that, for each $i \in \mathbb{Z}$, there exist $x \in X$ and $g_i \in G$ such that

$$d(f^i(g_i x), x_i) < \epsilon \quad \text{for all} \quad i \in \mathbb{Z}.$$

Thus,

$$\begin{aligned} d(f^k(f^n(g_{k+n}x)),\ f^k(g_k x)) &\le d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_{k+n},\ f^k(g_k x)) \\ &= d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_k,\ f^k(g_k x)) \\ &< 2\epsilon < \alpha \end{aligned}$$

for all k. Since α is a G-expansive constant for f,

$$G(f^n(x)) = G(x),$$

and hence

$$g_0 x \in Per_G(f) \cap N_\epsilon(p)$$

where $N_\epsilon(p)$ is an ϵ-neighborhood of p. Therefore, $Per_G(f)$ is dense in $\Omega_G(f)$. □

Claim 4. *Each B_λ is open in $\Omega_G(f)$.*

Proof. Let $\alpha > 0$ be a G-expansive constant for f and let $\epsilon < \alpha$. Denote

$$N_\delta(B_\lambda) = \{y \in \Omega_G(f) : d(y, B_\lambda) < \delta\}$$

where δ is the constant corresponding to ϵ in the definition of the GSP for $f|_{\Omega_G(f)}$. Then for a point $p \in N_\delta(B_\lambda) \cap Per_G(f)$, there exists $y \in B_\lambda$ such that

$$d(y, p) < \delta.$$

Since $f|_{\Omega_G(f)}$ has the GSP, it follows from Remark 1.10 that

$$W^u(p) \cap W^s(y) \neq \emptyset$$

and

$$W^s(p) \cap W^u(y) \neq \emptyset.$$

Here, $W^s(p)$ and $W^u(p)$ are defined on $\Omega_G(f)$. So, there exists $y_0 \in B_\lambda$ (in particular,

y_0 belongs to the α-limit set $\alpha(y)$) such that $y_0 \sim p$, that is, $p \in B_\lambda$. Therefore,

$$B_\lambda \supset \overline{N_\delta(B_\lambda) \cap Per_G(f)} \supset N_\delta(B_\lambda) \cap \overline{Per_G(f)} = N_\delta(B_\lambda),$$

that is, B_λ is open in $\Omega_G(f)$. $\square$

Since X is compact and $\Omega_G(f)$ is a closed subset of X, $\Omega_G(f)$ can be covered by finitely many B_λ's, that is, $\Omega_G(f) = \bigcup_{i=1}^n B_i$.

Claim 5. *Each B_i is G-invariant.*

Proof. Let $x \in B_i$, $g \in G$, and $\delta > 0$. We shall show that $gx \in B_i$. Since $x \in B_i$, there exists a (δ, G)-pseudo orbit $\{x_0 = x, x_1, \ldots, x_{n-1}, x_n = x\}$. Then $d(g_0 f(x), x_1) < \delta$ for some $g_0 \in G$. Since f is pseudoequivariant, we can take $h \in G$ such that $g_0 f(x) = hf(gx)$. Thus $\{gx, x_1, \ldots, x_{n-1}, x_n = x\}$ is a (δ, G)-pseudo orbit. By Lemma 2.3, there exists $\gamma > 0$ ($\gamma < \delta$) such that

$$d(x, y) < \gamma \implies d(gx, gy) < \delta \quad \text{for all} \quad g \in G.$$

Let $\{x_0 = x, x_1, \ldots, x_{n-1}, x_n = x\}$ be a (γ, G)-pseudo orbit. Then

$$d(g_{n-1} f(x_{n-1}), x) < \gamma \quad \text{for some} \quad g_{n-1} \in G$$

and hence $d(gg_{n-1} f(x_{n-1}), gx) < \delta$. Thus $\{x_0 = x, x_1, \ldots, x_{n-1}, gx\}$ is a (δ, G)-pseudo orbit. Since δ is arbitrary, $gx \sim_G x$. Therefore, $gx \in B_i$. $\square$

Claim 6. *$f|_{B_i}$ has the GSP.*

Proof. Let $0 < \epsilon < \min\{d(B_i, B_j) \colon i \neq j,\ 1 \leq i, j \leq n\}$ be given. Since $f|_{\Omega_G(f)}$ has the GSP, there exists $\delta < \epsilon$ such that every (δ, G)-pseudo orbit $\{x_k\} \subset B_i$ is ϵ-traced by a point $x \in \Omega_G(f)$. This means that, for each k, there exists $g_k \in G$ such that

$$d(f^k(x), g_k x_k) < \epsilon.$$

Since B_i is G-invariant and $x_0 \in B_i$, $g_0 x_0 \in B_i$. Therefore $x \in B_i$. $\square$

Claim 7. *$f|_{B_i}$ is topologically G-transitive.*

Proof. Let U and V be nonempty open subsets of B_i. Take $x \in U$ and $y \in V$. Then $x \sim_G y$. Let $N_\epsilon(x)$ and $N_\epsilon(y)$ be ϵ-neighborhoods of x and y respectively such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. Choose a positive $\epsilon_1 < \epsilon$ such that

$$d(a, b) < \epsilon_1 \implies d(ga, gb) < \epsilon \quad \text{for all} \quad g \in G.$$

Since $f|_{B_i}$ has the GSP, there exists $\delta_1 > 0$ such that every (δ_1, G)-pseudo orbit in B_i is ϵ_1-traced by a point in B_i. Thus, a (δ_1, G)-pseudo orbit $\{x_0 = x, \ldots, x_n = y\} \subset B_i$ from x to y is ϵ_1-traced by a point $z \in B_i$. In particular,

$$d(z, g_0 x) < \epsilon_1 \quad \text{and} \quad d(f^n(z), g_n y) < \epsilon_1 \quad \text{for some} \quad g_0, g_n \in G.$$

Since $d(g_0^{-1}z, x) < \epsilon$ and $d(g_n^{-1}f^n(z), y) < \epsilon$,

$$g_0^{-1}z \in N_\epsilon(x) \subset U$$

and

$$g_n^{-1}f^n(z) \subset N_\epsilon(y) \subset V.$$

Since $f^n(g_0^{-1}z) \in f^n(U)$ and f is pseudoequivariant,

$$g_1 f^n(z) \in f^n(U) \quad \text{for some} \quad g_1 \in G.$$

Choose $g \in G$ such that $gg_1 = g_n^{-1}$. Then $g_n^{-1}f^n(z) \in gf^n(U)$. Therefore, $gf^n(U) \cap V \neq \emptyset$. $\square$

We next prepare the following three lemmas to complete Theorem B.

Lemma 2.5. *Let $f: X \to X$ be a pseudoequivariant homeomorphism on a compact metric G-space X with G compact. Then*

$$W^i(x) = W^i(p) \quad \textit{for any} \quad x \in W^i(p) \quad (i = s, u).$$

Proof. We shall prove only the case $i = s$. Let $y \in W^s(x)$ and let $\epsilon > 0$. Since $y \in W^s(x)$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies that

$$d(f^n(h_n x),\ f^n(y)) < \frac{\epsilon}{2} \quad \text{for some} \quad h_n \in G.$$

Let $\delta > 0$ be the constant satisfying the following:

$$d(x, y) < \delta \implies d(gx, gy) < \frac{\epsilon}{2} \quad \text{for all} \quad g \in G.$$

Since $x \in W^s(p)$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies that

$$d(f^n(g'_n p),\ f^n(x)) < \delta \quad \text{for some} \quad g'_n \in G.$$

Hence for some $h'_n \in G$ with $h'_n f^n(x) = f^n(h_n x)$,

$$d(h'_n f^n(g'_n p), h'_n f^n(x)) < \frac{\epsilon}{2}.$$

Since $h_n' f^n(g_n' p) = f^n(g_n p)$ for some $g_n \in G$,

$$d(f^n(g_n p),\ f^n(h_n x)) < \frac{\epsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies that

$$d(f^n(g_n p),\ f^n(y)) \leq d(f^n(g_n p),\ f^n(h_n x)) + d(f^n(h_n x),\ f^n(y)) < \epsilon.$$

Therefore, $W^s(x) \subset W^s(p)$. Similarly, one can prove $W^s(p) \subset W^s(x)$. $\square$

Lemma 2.6. *Let $f: X \to X$ be a pseudoequivariant homeomorphism on a compact metric G-space X with G compact and let $x \in W^i(p)$. Then*

$$gx \in W^i(p) \quad \text{for every} \quad g \in G,$$

and hence

$$G(W^i(p)) = W^i(p) \quad (i = s, u).$$

Proof. Let $x \in W^s(p)$, $g \in G$ and let $\epsilon > 0$. Then there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(gx, gy) < \epsilon$ for all $g \in G$. Since for each $n \in \mathbb{Z}$, we have $g_n \in G$ such that

$$\lim_{n \to \infty} d(f^n(g_n p),\ f^n(x))) = 0,$$

that is, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(f^n(g_n p),\ f^n(x)) < \delta.$$

Hence, for $h_n' \in G$ with $h_n' f^n(x) = f^n(gx)$,

$$d(h_n' f^n(g_n p), h_n' f^n(x)) < \epsilon.$$

Let $h_n' f^n(g_n p) = f^n(h_n p)$. Then

$$d(f^n(h_n p),\ f^n(gx)) < \epsilon.$$

Therefore, $gx \in W^s(p)$. Similarly, one can prove the statement for the case $i = u$. $\square$

Lemma 2.7. *Let $f: X \to X$ be a pseudoequivariant homeomorphism on a compact metric G-space X with G compact. Then for any $\epsilon > 0$, there exists a positive* *constant* $\delta < \epsilon$ *satisfying the following*: *if* $x \in W^u_\delta(y)$, *then for all* $g \in G$,

(1) $$gx \in W^u_\epsilon(y)$$

and

(2) $$gy \in W^u_\epsilon(x).$$

Proof. Let $\epsilon > 0$. Then, by Lemma 2.3, there exists a positive constant $\delta < \epsilon$ such that

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all} \quad g \in G.$$

Let $x \in W^u_\delta(y)$ and let $g \in G$. Then for each $n \geq 0$, there exists $g_n \in G$ such that

$$d(f^{-n}(x),\ f^{-n}(g_n y)) < \delta.$$

(1) Take $g'_n \in G$ such that $g'_n f^{-n}(x) = f^{-n}(gx)$. Then

$$d(f^{-n}(gx),\ g'_n f^{-n}(g_n y)) < \epsilon.$$

Since f is pseudoequivariant, $gx \in W^u_\epsilon(y)$.

(2) Take $g'_n \in G$ such that $g'_n f^{-n}(g_n y) = f^{-n}(gy)$. Then

$$d(g'_n f^{-n}(x),\ f^{-n}(gy)) < \epsilon.$$

Since f is pseudoequivariant, $gy \in W^u_\epsilon(x)$ for all $g \in G$. □

Proof of Theorem B. Let $\epsilon > 0$ be a constant which is less than the G-expansive constant for $f|_B$ and let $\delta > 0$ ($\delta < \epsilon$) be the constant corresponding to ϵ in the definition of the GSP. Let $X_p = \overline{W^u(p) \cap B}$ for $p \in B \cap Per_G(f)$. We can see directly from Lemmas 2.3 and 2.6 that X_p is G-invariant, that is, if $x \in X_p$, then $gx \in X_p$ for all $g \in G$.

Claim 1. X_p *is open in* B.

Proof. Since $p \in Per_G(f)$, we have an integer $m > 0$ and $g_1 \in G$ such that $g_1 f^m(p) = p$. Denote $N_\delta(X_p) = \{y \in B : d(y, X_p) < \delta\}$. Let $q \in N_\delta(X_p) \cap Per_G(f)$. Then there is $x \in W^u(p) \cap B$ with $d(q, x) < \delta$. Note that $g_2 f^n(q) = q$ for some integer $n > 0$ and $g_2 \in G$. Since $f|_B$ has the GSP, the (δ, G)-pseudo orbit

$$\{\dots,\ f^{-2}(x),\ f^{-1}(x), q,\ f(q),\ f^2(q), \dots\}$$

is ϵ-traced by a point $x' \in B$, that is, for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that
(a) $d(x', h_0 q) < \epsilon$;
(b) $d(f^t(x'), h_t f^t(q)) < \epsilon$ $(t > 0)$;
(c) $d(f^{-t}(x'), h_{-t} f^{-t}(x)) < \epsilon$ $(t > 0)$.
Hence, it follows from Remark 1.10 that $x' \in W^s(q) \cap W^u(x) \cap B$.

Since f is pseudoequivariant and $p \in Per_G(f)$, for each $k \in \mathbb{Z}$, we have $g_{kmn} \in G$ such that $f^{kmn}(g_{kmn} p) = p$. Since $W^u(x) = W^u(p) = W^u(g_{kmn} p)$ by Lemmas 2.5 and 2.6,

$$f^{kmn}(x') \in f^{kmn}(W^u(g_{kmn} p)) = W^u(f^{kmn}(g_{kmn} p)) = W^u(p).$$

Since $q \in W^s(x')$, for each $k \in \mathbb{Z}$, one can find $h_{kmn} \in G$ such that

$$\lim_{k\to\infty} d(h_{kmn} f^{kmn}(x'), f^{kmn}(q)) = 0.$$

Take $i_{kmn} \in G$ such that $i_{kmn}(h_{kmn})^{-1} f^{kmn}(q) = q$. Then

$$\lim_{k\to\infty} d(i_{kmn} f^{kmn}(x'), i_{kmn}(h_{kmn})^{-1} f^{kmn}(q)) = \lim_{k\to\infty} d(i_{kmn} f^{kmn}(x'), q) = 0.$$

Hence, $q \in \overline{W^u(p) \cap B} = X_p$ because $i_{kmn} f^{kmn}(x') \in W^u(p)$ for each $k \in \mathbb{Z}$ by Lemma 2.6. Therefore,

$$X_p \supset \overline{N_\delta(X_p) \cap Per_G(f)} \supset N_\delta(X_p) \cap \overline{Per_G(f)} = N_\delta(X_p),$$

that is, X_p is open in B. □

Note that $f(X_p) = f(\overline{W^u(p) \cap B}) = \overline{f(W^u(p)) \cap f(B)} = \overline{W^u(f(p)) \cap B} = X_{f(p)}$. Since $X_p = X_{gp}$ for any $g \in G$ and $g_1 f^m(p) = p$,

$$f^m(X_p) = X_{f^m(p)} = X_{g_1 f^m(p)} = X_p.$$

Take the smallest integer $a > 0$ such that $a \leq m$ and $f^a(X_p) = X_p$.

Claim 2. $B = \bigcup_{j=0}^{a-1} f^j(X_p)$.

Proof. Let $y \in B$. Since $f|_B$ is topologically G-transitive, for each $1/n$-neighborhood $N_{1/n}(y)$ of y, there are $k > 0$ and $h_n \in G$ such that $h_n N_{1/n}(y) \cap f^k(X_p) \neq \emptyset$. So $h_n N_{1/n}(y) \cap \left(\bigcup_{j=0}^{a-1} f^j(X_p)\right) \neq \emptyset$ for each $n \in \mathbb{N}$. We may assume that $h_n \to h \in G$ because G is compact. Since $\bigcup_{j=0}^{a-1} f^j(X_p)$ is closed in B, $hy \in \bigcup_{j=0}^{a-1} f^j(X_p)$. Since $G(f^j(X_p)) = G(X_{f^j(p)}) = X_{f^j(p)} = f^j(X_p)$, we have $y \in \bigcup_{j=0}^{a-1} f^j(X_p)$. □

Claim 3. $X_p = X_q$ *for* $q \in X_p \cap Per_G(f)$.

Proof. Let $q \in X_p \cap Per_G(f)$ and suppose m and n are G-periodic numbers of p and q respectively. Since $N_\delta(X_p) = X_p$ for the constant $\delta > 0$ in the above of Claim 1, $W^u_\delta(q) \subset X_p$. We firstly show that $p \in X_q$. Suppose that $p \notin X_q$. Then $d(K, X_q) > 0$ where $K = X_p \setminus X_q$. Since $q \in X_p = \overline{W^u(p) \cap B}$, there exists $z \in W^u(p) \cap B$ such that $d(z, q) < d(K, X_q)$. Since $z \in X_p$ and $z \notin K$, $z \in X_q$. Furthermore, for each $j \in \mathbb{Z}$, there exists $g'_{mnj} \in G$ such that

$$\lim_{j\to\infty} d(f^{-mnj}(z), f^{-mnj}(g'_{mnj}p)) = 0.$$

For each $j \in \mathbb{Z}$, choose $g_{mnj} \in G$ with $g_{mnj} f^{-mnj}(g'_{mnj}p) = p$. Then we have

$$\lim_{j\to\infty} d(g_{mnj} f^{-mnj}(z), p) = 0.$$

So $g_{mnj} f^{-mnj}(z) \notin X_q$ for sufficiently large j. Hence,

$$h_{mnj} z \notin f^{mnj}(X_q) = X_q$$

for $h_{mnj} \in G$ with $g_{mnj} f^{-mnj}(z) = f^{-mnj}(h_{mnj} z)$. Thus, $z \notin X_q$. This is a contradiction. Therefore, $p \in X_q$.

Let $y \in W^u(q)$ and let $0 < \delta_1 < \delta_2 < \delta_3 = \delta$ such that

$$d(x, y) < \delta_i \implies d(gx, gy) < \delta_{i+1} \quad \text{for all} \quad g \in G \ (i = 1, 2).$$

Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $d(f^{-k}(y), f^{-k}(h_k q) < \delta_1$ for some $h_k \in G$. Choose $j \in \mathbb{N}$ with $mnj \geq N$. Then

$$d((f^{-i} \circ f^{-mnj})(y), (f^{-i} \circ f^{-mnj})(h_{mnj+i}q)) < \delta_1 \quad \text{for all} \quad i \geq 0,$$

that is,

$$f^{-mnj}(y) \in W^u_{\delta_1}(f^{-mnj}(q)).$$

By Lemma 2.7 (2), $gf^{-mnj}(q) \in W^u_{\delta_2}(f^{-mnj}(y))$ for all $g \in G$. Since $q \in Per_G(f)$, we have $q \in W^u_{\delta_2}(f^{-mnj}(y))$. Again, by Lemma 2.7 (2), $gf^{-mnj}(y) \in W^u_\delta(q)$ for all $g \in G$. In particular, $f^{-mnj}(y) \in W^u_\delta(q)$. This means that $y \in f^{mnj}(W^u_\delta(q))$ for some $j \geq 0$. So $W^u(q) \subset \bigcup_{j\geq 0} f^{mnj}(W^u_\delta(q))$. Therefore,

$$X_q = \overline{W^u(q) \cap B} \subset \overline{\bigcup_{j\geq 0} f^{mnj}(W^u_\delta(q)) \cap B} \subset \overline{X_p \cap B} = X_p \cap B = X_p.$$

Similarly, we have $X_p \subset X_q$. $\square$

Claim 4. $X_p \cap f^j(X_p) = \emptyset$ *for* $0 < j < a$.

Proof. Suppose $X_p \cap f^j(X_p) \neq \emptyset$ for some j. Since $X_p \cap f^j(X_p)$ is open in B, we can find $q \in X_p \cap f^j(X_p) \cap Per_G(f)$. Then $X_q = X_p = f^j(X_p)$, which is a contradiction to the choice of the integer a. □

Claim 5. $f^a|_{X_p}$ *is topologically* G*-mixing.*

Proof. Let U and V be non-empty open subsets of X_p and let $q \in V \cap Per_G(f)$. Then $f^{aj}(q) \in X_p \cap Per_G(f)$ for all $j \in \mathbb{Z}$. Since $X_p = X_{f^{aj}(q)}$ for all $j \in \mathbb{Z}$,

$$U \cap W^u(f^{aj}(q)) = U \cap (W^u(f^{aj}(q)) \cap B) \neq \emptyset \quad \text{for all} \quad j \in \mathbb{Z}.$$

Let $n > 0$ be a G-periodic number of q. Then for each j such that $0 \leq j \leq n-1$, there exists $z_j \in U \cap W^u(f^{aj}(q))$. Since f is pseudoequivariant, we may take this statement: for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that

$$\lim_{t\to\infty} d(f^{-ant}(z_j),\ f^{aj}(h_t f^{-ant}(q))) = 0.$$

For each $t \in \mathbb{Z}$, choose $g_t \in G$ such that $g_t f^{aj}(h_t f^{-ant}(q)) = f^{aj}(q)$. Then we have

$$\lim_{t\to\infty} d(g_t f^{-ant}(z_j),\ f^{aj}(q)) = 0,$$

and thus

$$\lim_{t\to\infty} g_t f^{-ant}(z_j) = f^{aj}(q).$$

Since $f^{aj}(q) \in f^{aj}(V)$, for each j with $0 \leq j \leq n-1$, we may choose $N_j > 0$ such that for all $t \geq N_j$,

$$g_t f^{-ant}(z_j) \in f^{aj}(V).$$

Let $M = \max\{N_j : 0 \leq j \leq n-1\}$. For each $t \geq M$, we get $t = ns + j$. If $s \geq M$, then

$$f^{-at}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j)) \in V$$

for each $i_s \in G$ such that $f^{-ans-aj}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j))$. Hence,

$$i_s z_j \in f^{at}(V) \quad \text{if} \quad s \geq M \quad (\text{that is, } t \geq nM).$$

Thus, it follows from $z_j \in U$ that there exists $k_t \in G$ such that

$$k_t f^{at}(V) \cap U \neq \emptyset \quad \text{for each} \quad t \geq nM.$$

Therefore, $f^a|_{X_p}$ is topologically G-mixing. □

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