<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Decomposition theorem on G-spaces</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Choi, Taeyoung; Kim, Junhui</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 46(1) P.87-P.104</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2009-03</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/11094/6149">http://hdl.handle.net/11094/6149</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td></td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>

*Osaka University Knowledge Archive : OUKA*

https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
DECOMPOSITION THEOREM ON G-SPACES

TAEYOUNG CHOI\textsuperscript{1} and JUNHUI KIM\textsuperscript{2}

(Received March 19, 2007, revised November 19, 2007)

Abstract

In this paper, we introduce the weak $G$-expansivity which is a generalization of both expansivity and $G$-expansivity. Also, we define $G$-stable and $G$-unstable sets of a homeomorphism on a metric $G$-space $X$ and investigate properties of them. Finally, we consider the decomposition theorem on $G$-spaces.

1. Introduction

Let $X$ be a topological space, $G$ be a topological group, and $\theta : G \times X \to X$ be a map. The triple $(X, G, \theta)$ is called a topological $G$-space if the following three conditions are satisfied:

1. $\theta(e, x) = x$ for all $x \in X$, where $e$ is the identity of $G$;
2. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and for all $g, h \in G$;
3. $\theta$ is continuous.

Here, $gh$ is the group operation on $G$. Simply, we denote $\theta(g, x)$ by $gx$ and $X$ is usually said to be a topological $G$-space.

For any subset $A$ of $X$, $G(A)$ is denoted by the set \{ga : g \in G, a \in A\}. $G(x)$ is called a $G$-orbit of $x$. A subset $A$ of $X$ is called $G$-invariant if $G(A) = A$. A map $f : X \to X$ on a $G$-space $X$ is said to be pseudoequivariant provided that $f(G(x)) = G(f(x))$ for all $x \in X$, and $f$ is said to be equivariant provided that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

N. Aoki has proved the following topological decomposition theorem in 1983 ([1]), which is an extension of Smale’s spectral decomposition theorem and Bowen’s decomposition theorem in dynamical systems. All undefined notions can be found in [2].

Theorem 1.1 ([1]). Let $f : X \to X$ be a homeomorphism on a compact metric space $X$ and let $\text{CR}(f)$ be the chain recurrent set. If $f|_{\text{CR}(f)} : \text{CR}(f) \to \text{CR}(f)$ is an expansive homeomorphism with the shadowing property, then

1. $\text{CR}(f)$ contains a finite sequence $B_i$ ($1 \leq i \leq k$) of $f$-invariant closed subsets such that

2000 Mathematics Subject Classification. Primary 54H20; Secondary 37B05.

This work was supported by the National Institute for Mathematical Sciences\textsuperscript{1} and the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-037-C00005)\textsuperscript{2}.
(a) $CR(f) = \bigcup_{i=1}^{j} B_i$ (disjoint union);
(b) $f|_{B_i}: B_i \to B_i$ is topologically transitive,
(2) for each $B_i$, there exist a subset $X_p$ of $B_i$ and $a > 0$ such that
(a) $f^a(X_p) = X_p$;
(b) $X_p \cap f^j(X_p) = \emptyset$ ($0 < j < a$);
(c) $f^a|_{X_p}: X_p \to X_p$ is topologically mixing;
(d) $B_i = \bigcup_{j=0}^{a-1} f^j(X_p).

A point $x \in X$ is called a $G$-periodic point of $f$ if there exist an integer $n > 0$ and $g \in G$ such that $f^n(x) = gx$. A point $x \in X$ is called a $G$-nonwandering point of $f$ if for every open neighborhood $U$ of $x$, there exist $n > 0$ and $g \in G$ such that $gf^n(U) \cap U \neq \emptyset$. $Per_G(f)$ (resp. $\Omega_G(f)$) is denoted by the set of all $G$-periodic (resp. $G$-nonwandering) points of $f$.

For a homeomorphism $f$ on a metric $G$-space $X$, a sequence $\{x_i \in X: i \in \mathbb{Z}\}$ is called a $(\delta, G)$-pseudo orbit for $f$ provided that for each $i$, there exists $g_i \in G$ such that $d(g_if(x_i), x_{i+1}) < \delta$. A $(\delta, G)$-pseudo orbit $\{x_i\}$ for $f$ is said to be $\epsilon$-traced by a point $x \in X$ provided that for each $i$, there exists $g_i \in G$ such that $d(f^i(x), g_i x_i) < \epsilon$.

**Definition 1.2** ([5]). A homeomorphism $f: X \to X$ has the $G$-shadowing property (GSP) provided that for any $\epsilon > 0$, there exists $\delta > 0$ such that every $(\delta, G)$-pseudo orbit $\{x_i\}$ in $X$ for $f$ is $\epsilon$-traced by a point $x \in X$.

**Remark 1.3.** It was proved by E. Shah that, when $X$ is a compact metric $G$-space and the orbit map $\pi: X \to X/G$ is a covering map, a pseudoequivariant homeomorphism $f$ on $X$ has the GSP if and only if the induced map $\tilde{f}: X/G \to X/G$ has the shadowing property ([5]).

If a pseudoequivariant continuous onto map $f: X \to X$ has the GSP where $X$ is a compact metric $G$-space with $G$ compact, then $f|_{\Omega_G(f)}$ has the GSP ([5]).

The main purpose of this paper is to prove the following theorems on compact metric $G$-spaces.

**Theorem A.** Let $X$ be a compact metric $G$-space with $G$ compact. If $f: X \to X$ is a pseudoequivariant $G$-expansive homeomorphism with the GSP, then $\Omega_G(f)$ contains a finite sequence $B_i$ ($1 \leq i \leq n$) of $f$-invariant, $G$-invariant, and closed subsets such that
(1) $f|_{\Omega_G(f)}$ is topologically $G$-transitive;
(2) $\Omega_G(f) = \bigcup_{i=1}^{n} B_i$ (disjoint union);
(3) $f|_{B_i}$ has the GSP.

A homeomorphism $f: X \to X$ is said to be topologically $G$-mixing provided that for every nonempty open subsets $U$ and $V$ of $X$, there exists an integer $N$ such that
for each $n \geq N$, there is $g_n \in G$ satisfying $g_n f^n(U) \cap V \neq \emptyset$.

**Theorem B.** Let $f|_{\Omega_G(f)} : \Omega_G(f) \to \Omega_G(f)$ be a $G$-expansive homeomorphism with the GSP. Then, for any $f$-invariant, $G$-invariant, open and closed subset $B \subset \Omega_G(f)$ such that $f|_B : B \to B$ is topologically $G$-transitive, there are $X_p \subset B$ and $a > 0$ such that

1. $f^a(X_p) = X_p$;
2. $X_p \cap f^j(X_p) = \emptyset$ (0 < $j$ < $a$);
3. $f^a|_{X_p} : X_p \to X_p$ is topologically $G$-mixing;
4. $B = \bigcup_{j=0}^{a-1} f^j(X_p)$.

**Definition 1.4.** A homeomorphism $f : X \to X$ on a metric $G$-space $X$ is said to be weak $G$-expansive provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$ if $u \in G(x)$ and $v \in G(y)$, there exists $n = n(u, v) \in \mathbb{Z}$ such that

$$d(f^n(u), f^n(v)) > \delta.$$  

The constant $\delta$ is called a weak $G$-expansive constant for $f$.

The weak $G$-expansivity is a generalization of both expansivity and $G$-expansivity. Here, $G$-expansivity has been defined by R. Das ([4]). A homeomorphism $f : X \to X$ is said to be $G$-expansive provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$, there exists $n \in \mathbb{Z}$ such that

$$d(f^n(u), f^n(v)) > \delta \quad \text{for all} \quad u \in G(x), \ v \in G(y).$$

The constant $\delta$ is called a $G$-expansive constant for $f$.

**Remark 1.5.** R. Das proved that there is no implication between $G$-expansivity and expansivity by giving counterexamples ([4]).

**Example 1.6 ([4]).** Consider the compact space $X = [1/n, 1-1/n : n \in \mathbb{N}]$ with the usual metric and let the topological group $G = [-1, 1]$ act on $X$ with the action $\theta$ defined by $\theta(1, x) = x$ and $\theta(-1, x) = 1 - x$. Define a homeomorphism $f : X \to X$ by

$$f(x) = \begin{cases} x & \text{if } x = 0, 1; \\ \text{next to the right of } x & \text{if } x \in X \setminus [0, 1]. \end{cases}$$

Then $f$ is an expansive map with expansive constant $\delta$ ($0 < \delta < 1/6$). But, it is easy to see that for $x, y \in X \setminus [1/2]$ with $G(x) \neq G(y)$, there is no $n \in \mathbb{Z}$ such that

$$|f^n(u) - f^n(v)| > \delta \quad \text{for all} \quad u \in G(x), \ v \in G(y),$$

whatever $\delta > 0$ may be. This means that $f$ is not $G$-expansive.
EXAMPLE 1.7 ([4]). Consider the compact space \( X = \bigcup_{i=1}^{n} C_i \) with the usual metric, where each \( C_i \) is the circle in \( \mathbb{R}^2 \) with center the origin and radius \( i \). Denote \( G = SO(2) \) by the set of all \( 2 \times 2 \) matrices whose determinants are \( \pm 1 \) and define an action \( \theta : G \times X \to X \) by the usual rotations on \( X \). Then the identity map on \( X \) is \( G \)-expansive with \( G \)-expansive constant \( \delta (0 < \delta < 1) \).

Therefore, all properties of the following diagram are distinguished as we see in Examples 1.6 and 1.7:

\[
\begin{array}{ccc}
\text{weak } G\text{-expansive} & \xrightarrow{\text{expansive}} & \text{expansive} \\
\downarrow & & \downarrow \\
G\text{-expansive} & \xleftarrow{\text{expansive}} & \text{expansive}
\end{array}
\]

DEFINITION 1.8. Let \( f : X \to X \) be a homeomorphism of a metric \( G \)-space \( X \). We define a local \( G \)-stable set \( W^{s}_{\epsilon}(x) \) and a local \( G \)-unstable set \( W^{u}_{\epsilon}(x) \) by

\[
W^{s}_{\epsilon}(x) = \{ y \in X : \text{for each } n \geq 0, \text{there is } g_n \in G \text{ such that } d(f^n(g_n x), f^n(y)) \leq \epsilon \},
\]

\[
W^{u}_{\epsilon}(x) = \{ y \in X : \text{for each } n \geq 0, \text{there is } g_n \in G \text{ such that } d(f^{-n}(g_n x), f^{-n}(y)) \leq \epsilon \}.
\]

We modify results of [3] into the following results by weakening the condition “equivariant” into “pseudoequivariant” and deleting the condition “invariant metric”. A metric \( d \) on a \( G \)-space \( X \) is called an invariant metric provided that \( d(x, y) = d(g.x, g.y) \) for all \( x, y \in X \) and \( g \in G \).

REMARK 1.9. Let \( X \) be a compact metric \( G \)-space with \( G \) compact. If \( f : X \to X \) is a weak \( G \)-expansive pseudoequivariant homeomorphism with weak \( G \)-expansive constant \( \delta > 0 \), then for every \( \gamma > 0 \), there is \( N > 0 \) such that for each \( x \in X \) and for each \( n \geq N \),

\[
f^n(W^{s}_{\delta}(x)) \subset W^{s}_{\gamma}(f^n(x))
\]

and

\[
f^{-n}(W^{u}_{\delta}(x)) \subset W^{u}_{\gamma}(f^{-n}(x)).
\]

Proof. We shall prove only the case of a local \( G \)-stable set because the other case can be proved similarly. To do it, suppose that there exists \( \gamma > 0 \) such that for all \( N > 0 \), there are \( x \in X \) and \( n \geq N \) satisfying

\[
f^n(W^{s}_{\delta}(x)) \not\subset W^{s}_{\gamma}(f^n(x)).
\]
Let $N > 0$. Then there are $x_1 \in X$ and $n \geq N$ satisfying
\[ f^n(W^s_\delta(x_1)) \not\subset W^s_\gamma(f^n(x_1)), \]
that is, there exists $y_1 \in W^s_\delta(x_1)$ such that $f^n(y_1) \notin W^s_\gamma(f^n(x_1))$. So there exists $i \geq 0$ such that for every $h \in G$,
\[ d(f^i(hf^n(x_1)), f^i(f^n(y_1))) > \gamma. \]
Because $f$ is pseudoequivariant, there exists $i \geq 0$ such that for every $g \in G$,
\[ d(gf^{i+n}(x_1), f^{i+n}(y_1)) > \gamma. \]
Take $m_1 = i + n$ and choose $N = m_1 + 1$.

Continuing the process, we can find sequences $m_n > 0$, $x_n$, and $y_n \in X$ such that
1. $y_n \in W^s_\delta(x_n)$;
2. $d(hf^{m_n}(x_n), f^{m_n}(y_n)) > \gamma$ for all $h \in G$;
3. $\lim_{n \to \infty} m_n = \infty$.

It follows from $y_n \in W^s_\delta(x_n)$ that for each $i \geq -m_n$, there exists $g_{i+m_n} \in G$ such that
\[ d(f^{i+m_n}(g_{i+m_n}, x_n), f^{i+m_n}(y_n)) \leq \delta. \]
Since $f$ is pseudoequivariant, for each $g_{i+m_n}$, there exists $h_{i+m_n} \in G$ such that
\[ d(f^i(h_{i+m_n}, f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) = d(f^{i+m_n}(g_{i+m_n}, x_n), f^{i+m_n}(y_n)). \]
Hence, for each $i \geq -m_n$,
\[ d(f^i(h_{i+m_n}, f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) \leq \delta. \]
If $f^{m_n}(x_n) \to x$, $f^{m_n}(y_n) \to y$, and $h_{i+m_n} \to h$ as $n \to \infty$, then
\[ d(f^i(hx), f^i(y)) \leq \delta \quad \text{for all} \quad i \in \mathbb{Z}. \]
Since $\delta$ is a weak $G$-expansive constant for $f$, $G(x) = G(y)$. But $d(hx, y) = \lim_{n \to \infty} d(hf^{m_n}(x_n), f^{m_n}(y_n)) \geq \gamma > 0$ for all $h \in G$ by (2). Thus $hx \neq y$ for all $h \in G$, and hence $G(x) \neq G(y)$. This is a contradiction. \hfill \Box

For a homeomorphism $f$ on a compact metric $G$-space, we define the following:
\[ W^s(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right\}, \]
\[ \lim_{n \to \infty} d(f^n(g_n x), f^n(y)) = 0 \];
\[ W^u(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right\}, \]
\[ \lim_{n \to \infty} d(f^{-n}(g_n x), f^{-n}(y)) = 0 \].
$W^s(x)$ (resp. $W^u(x)$) is called a $G$-stable set (resp. $G$-unstable set).

REM. 1.10. Let $X$ be a compact metric $G$-space with $G$ compact. If $f: X \to X$ is a weak $G$-expansive pseudoequivariant homeomorphism with weak $G$-expansive constant $\delta > 0$, then for each $\epsilon$ with $0 < \epsilon < \delta$,

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x)))$$

$$W^u(x) = \bigcup_{n \geq 0} f^n(W^u_\epsilon(f^{-n}(x))).$$

Proof. ($\subseteq$): Let $y \in W^s(x)$ and $0 < \epsilon < \delta$. Then there exists $N > 0$ such that for each $n \geq N$, we can choose $g_n \in G$ satisfying

$$d(f^n(g_n x), f^n(y)) \leq \epsilon.$$ 

Thus,

$$d(f^i(f^N(g_{i+N} x)), f^i(f^N(y))) \leq \epsilon \quad \text{for all } i \geq 0.$$ 

Since $f$ is pseudoequivariant, $f^N(y) \in W^s_\epsilon(f^N(x))$. Therefore,

$$y \in f^{-N}(W^s_\epsilon(f^N(x))) \subset \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x))).$$

($\supseteq$): Let $y \in f^{-n}(W^s_\epsilon(f^n(x)))$ for some $n \geq 0$. Then $f^n(y) \in W^s_\epsilon(f^n(x))$. It follows from Remark 1.9 that for every $\gamma > 0$ there exists $N > 0$ such that for each $x \in X$ and $m \geq N$,

$$f^{m+n}(y) \in f^m(W^s_\epsilon(f^n(x))) \subset W^s_\gamma(f^{m+n}(x)).$$

So for each $n \geq N$, we can find $g_n \in G$ such that

$$d(f^{m+n}(g_n x), f^{m+n}(y)) \leq \gamma.$$ 

Since $f$ is pseudoequivariant, $y \in W^s(x)$. The proof is completed. The case of a $G$-unstable set can be proved similarly.

2. Decomposition theorems

First we prepare the following four lemmas to show Theorem A.

Lemma 2.1 ([3]). Let $(X, G, \theta)$ be a compact metric $G$-space with $G$ compact. Then for any $\epsilon > 0$, there is a finite open cover $U = \{U_1, \ldots, U_s\}$ of $X$ such that $\text{diam}(gU_i) \leq \epsilon$ for all $g \in G$ and $i$ with $1 \leq i \leq s$. 
In Lemma 2.1, notice that, for each \( g \in G \), the open cover \( \{gU: U \in \mathcal{U}\} \) of \( X \) satisfies \( \text{diam}(hgU) \leq \epsilon \) for all \( h \in G \) and \( i \) with \( 1 \leq i \leq s \).

**Lemma 2.2.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact. If \( \mathcal{U} \) is a finite open cover of \( X \), then there exists \( \delta > 0 \) such that for each subset \( A \) of \( X \) with \( \text{diam}(A) \leq \delta \), \( A \subset gU \) for some \( U \in \mathcal{U} \) and \( g \in G \).

Proof. Suppose not. Then for every \( n > 0 \) there exists a subset \( A_n \) of \( X \) such that \( \text{diam}(A_n) \leq 1/n \) and \( A_n \not\subset gU \) for all \( U \in \mathcal{U} \) and \( g \in G \). Choose \( x_n \in A_n \) for each \( n \in \mathbb{N} \). Since \( X \) is compact, there exist a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} \to x \). We fix \( g \in G \). Then there is \( U \in \mathcal{U} \) with \( x \in gU \). Since \( X \setminus gU \) is compact, \( d(x, X \setminus gU) > 0 \). Put \( \epsilon = d(x, X \setminus gU) \) and take \( n_i > 0 \) such that \( 1/n_i < \epsilon/2 \) and \( d(x_{n_i}, x) < \epsilon/2 \). Then for any \( y \in A_{n_i} \),

\[
d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\epsilon}{2} < \epsilon.
\]

So \( y \in gU \). Therefore, \( A_{n_i} \subset gU \). This is a contradiction. \( \square \)

**Lemma 2.3.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact. Then for any \( \epsilon > 0 \), there exists \( \delta > 0 \) (\( \delta < \epsilon \)) such that

\[
d(x, y) < \delta \implies d(gs, gy) < \epsilon \quad \text{for all} \quad g \in G.
\]

Proof. Let \( \epsilon > 0 \). Then it follows from Lemma 2.1 that, for any positive \( \epsilon_1 < \epsilon \), there is a finite open cover \( \mathcal{U} \) such that \( \text{diam}(gU) \leq \epsilon_1 \) for all \( g \in G \) and \( U \in \mathcal{U} \). Also, by Lemma 2.2, there is a constant \( \delta = \delta(\mathcal{U}) > 0 \) such that for any subset \( A \) with \( \text{diam}(A) \leq \delta \), \( A \subset gU \) for some \( g \in G \) and \( U \in \mathcal{U} \). Let \( x \) and \( y \) in \( X \) with \( d(x, y) < \delta \). Then \( x, y \in g_0 U_0 \) for some \( g_0 \in G \) and \( U_0 \in \mathcal{U} \). Note that \( \{g_0 U: U \in \mathcal{U}\} \) is an open cover of \( X \). For any \( g \in G \), take \( g_1 \in G \) such that \( g_1 = g g_0 \). Then, by Lemma 2.1, \( \text{diam}(gg_0 U) \leq \epsilon_1 \), that is, \( \text{diam}(g_1 U) \leq \epsilon_1 < \epsilon \) for all \( U \in \mathcal{U} \). Since \( gs, gy \in gg_0 U_0 = g_1 U_0 \), \( d(gs, gy) < \epsilon \). \( \square \)

**Lemma 2.4.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact and let \( f \) be a pseudoequivariant homeomorphism on \( X \). Then \( f \) has the GSP if and only if for any \( \epsilon > 0 \), we can find \( \delta > 0 \) such that for every \( (\delta, G) \)-pseudo orbit \( \{x_i\} \) of \( X \) for \( f \), there exist \( x \in X \) and \( h_i \in G \) satisfying

\[
d(f^i(h_i x), x_i) < \epsilon \quad \text{for all} \quad i \in \mathbb{Z}.
\]

Proof. Suppose that \( f \) has the GSP and let \( \epsilon > 0 \). Then, by Lemma 2.3, there exists \( \epsilon_0 > 0 \) (\( \epsilon_0 < \epsilon \)) such that for each \( x, y \in X \),

\[
d(x, y) < \epsilon_0 \implies d(gs, gy) < \epsilon \quad \text{for all} \quad g \in G.
\]
Let $\delta$ be the constant corresponding to $\epsilon_0$ in the definition of the GSP. Then every $(\delta, G)$-pseudo orbit $\{x_i\}$ of $X$ for $f$ is $\epsilon_0$-traced by a point $x \in X$, that is, for each $i$, there exists $g_i \in G$ such that

$$d(f^i(x), g_ix_i) < \epsilon_0 \quad \text{for all} \quad i \in \mathbb{Z}.\]$$

Since $f$ is pseudoequivariant, for each $g_i \in G$, there exists $h_i \in G$ such that

$$g_i^{-1}f^i(x) = f^i(h_ix).\]$$

Moreover, $d(g_i^{-1}f^i(x), x_i) < \epsilon$ and hence $d(f^i(h_ix), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

The converse can be proved similarly. \(\square\)

We have ([5]) that $f(\Omega_G(f)) = \Omega_G(f)$ and $CR_G(f) = \Omega_G(f)$ for a pseudoequivariant homeomorphism $f$ with GSP on a compact metric $G$-space $X$ where $G$ is compact.  

For $x, y \in X$ and $\delta > 0$, $x$ is said to be $(\delta, G)$-related to $y$ (denoted by $x \sim_{G} y$) if there exist finite $(\delta, G)$-pseudo orbits $\{x = x_0, x_1, \ldots, x_k = y\}$ and $\{y = y_0, y_1, \ldots, y_n = x\}$ for $f$. If for every $\delta > 0$, $x$ is $(\delta, G)$-related to $y$, then $x$ is said to be $G$-related to $y$ (denoted by $x \sim_{G} y$). A point $x$ is said to be a $G$-chain recurrent point of $f$ if $x \sim_{G} x$. $CR_G(f)$ is denoted by the set of all $G$-chain recurrent points of $f$. A homeomorphism $f: X \to X$ is called topologically $G$-transitive provided that for every nonempty open subsets $U$ and $V$ of $X$, there exist an integer $n > 0$ and $g \in G$ such that $gf^n(U) \cap V \neq \emptyset$.

**Proof of Theorem A.** Since the pseudoequivariant homeomorphism $f$ satisfies the GSP, $CR_G(f) = \Omega_G(f)$. Thus $\Omega_G(f) = \bigcup \beta_i B_\delta$ where each $B_\delta$ is an equivalence class under the relation $\sim_G$ which is defined in $CR_G(f)$.

**Claim 1.** Each $B_\delta$ is closed in $\Omega_G(f)$.

Proof. Let $x \in \overline{B_\delta}$. Then we can find a sequence $\{x_i\}$ in $B_\delta$ which converges to $x$. Let $\alpha > 0$ be given. Then there exists a finite open cover $\{U_1, \ldots, U_s\}$ of $X$ such that

$$\text{diam}(gU_i) \leq \frac{\alpha}{2} \quad \text{for all} \quad g \in G \quad \text{and} \quad i \quad \text{with} \quad 1 \leq i \leq s$$

by Lemma 2.1. So $f(x) \in U_i$ for some $i$. Choose an $\epsilon_0$-neighborhood $N_{\epsilon_0}(f(x))$ of $f(x)$ such that $N_{\epsilon_0}(f(x)) \subset U_i$. Then since $f$ is uniformly continuous, there exists $\delta_0 > 0$ such that

$$d(x, y) < \delta_0 \implies d(f(x), f(y)) < \epsilon_0.$$
Because \([x_i]\) converges to \(x\), there is \(J > 0\) such that \(d(x_J, x) < \min[\alpha/2, \delta_0]\). From the fact that \(x_J \in CR_G(f)\), we can find a \((\alpha/2, G)\)-pseudo orbit

\[x_J = y_0, y_1, \ldots, y_{k-1}, y_k = x_J.\]

So \(d(gf(y_0), y_1) < \alpha/2\) for some \(g \in G\). Also \(d(f(y_0), f(x)) < \epsilon_0\) and hence \(d(gf(y_0), gf(x)) < \alpha/2\). Thus,

\[d(gf(x), y_1) \leq d(gf(x), gf(y_0)) + d(gf(y_0), y_1) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.\]

Therefore, \([x, y_1, \ldots, y_k = x_J]\) is an \((\alpha, G)\)-pseudo orbit. It is clear that there is an \((\alpha, G)\)-pseudo orbit from \(x_J\) to \(x\) by the uniform continuity of \(f\). It follows from \(x \sim_G x_J\) that \(x \sim_G x_i\) for all \(i\) because each \(x_i \in B_{\lambda}\). Since \(\alpha\) is arbitrary, \(x \in B_{\lambda}\). Therefore, \(B_{\lambda}\) is closed.

**Claim 2.** Each \(B_{\lambda}\) is \(f\)-invariant.

**Proof.** To prove this, we firstly show that \(x \sim_G f(x)\) for all \(x \in \Omega_G(f)\). Let \(\alpha > 0\). Then there is \(\delta > 0\) (\(\delta < \alpha\)) such that

\[d(a, b) < \delta \implies d(f^2(a), f^2(b)) < \alpha.\]

Since \(x \in \Omega_G(f)\), there are \(n > 0\) and \(g \in G\) such that

\[gf^n(N_\delta(x)) \cap N_\delta(x) \neq \emptyset\]

where \(N_\delta(x)\) is a \(\delta\)-neighborhood of \(x\). Then there exists \(z \in N_\delta(x)\) such that \(gf^n(z) \in N_\delta(x)\). Hence

\[[f(x), f^2(z), \ldots, f^{n-1}(z), x]\]

is an \((\alpha, G)\)-pseudo orbit and thus, \(x \sim_G f(x)\). Since \(f\) is a homeomorphism, we can show that \(x \sim_G f^{-1}(x)\) for all \(x \in \Omega_G(f)\) similarly. Therefore, \(f(B_{\lambda}) = B_{\lambda}\) for each \(\lambda\).

**Claim 3.** \(\text{Per}_G(f)\) is dense in \(\Omega_G(f)\).

**Proof.** Let \(\alpha > 0\) be a \(G\)-expansive constant for \(f\) and take \(\epsilon < \alpha/2\). Since \(f\) has the GSP, there exists \(\delta > 0\) (\(\delta < \epsilon\)) such that every \((\delta, G)\)-pseudo orbit is \(\epsilon\)-traced by a point in \(X\). Since \(f\) is uniformly continuous, there exists a positive constant \(\gamma < \delta\) such that if \(d(a, b) < \gamma\), then \(d(f(a), f(b)) < \delta\). Let \(p \in \Omega_G(f)\). Then for every \(\gamma\)-neighborhood \(N_\gamma(p)\) of \(p\), there exist an integer \(n > 0\) and \(g \in G\) such that

\[gf^n(N_\gamma(p)) \cap N_\gamma(p) \neq \emptyset.\]
Choose a point $y \in g f^n(N_y(p)) \cap N_y(p)$. Since $f^{-n}(g^{-1}y) \in N_y(p)$,
\[
d(f(p), f(f^{-n}(g^{-1}y))) < \delta.
\]
Hence
\[
\{ \ldots, x_0 = p, x_1 = f^{-n+1}(g^{-1}y), x_2 = f^{-n+2}(g^{-1}y), \ldots, x_{n-1} = f^{-1}(g^{-1}y), x_n = p, \ldots \}
\]
is a $(\delta, G)$-pseudo orbit for $f$. Since $f$ has the GSP, it follows from Lemma 2.4 that, for each $i \in \mathbb{Z}$, there exist $x \in X$ and $g_i \in G$ such that
\[
d(f^i(g_i x), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.
\]
Thus,
\[
d(f^k(f^n(g_{k+n}x)), f^k(g_k x)) \leq d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_{k+n}, f^k(g_k x))
= d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_k, f^k(g_k x))
< 2\epsilon < \alpha
\]
for all $k$. Since $\alpha$ is a $G$-expansive constant for $f$,
\[
G(f^n(x)) = G(x),
\]
and hence
\[
g_0 x \in \text{Per}_G(f) \cap N_{\epsilon}(p)
\]
where $N_{\epsilon}(p)$ is an $\epsilon$-neighborhood of $p$. Therefore, $\text{Per}_G(f)$ is dense in $\Omega_G(f)$. \hfill \Box

**Claim 4.** Each $B_\lambda$ is open in $\Omega_G(f)$.

**Proof.** Let $\alpha > 0$ be a $G$-expansive constant for $f$ and let $\epsilon < \alpha$. Denote
\[
N_\delta(B_\lambda) = \{ y \in \Omega_G(f) : d(y, B_\lambda) < \delta \}
\]
where $\delta$ is the constant corresponding to $\epsilon$ in the definition of the GSP for $f|_{\Omega_G(f)}$. Then for a point $p \in N_\delta(B_\lambda) \cap \text{Per}_G(f)$, there exists $y \in B_\lambda$ such that
\[
d(y, p) < \delta.
\]
Since $f|_{\Omega_G(f)}$ has the GSP, it follows from Remark 1.10 that
\[
W^u(p) \cap W^s(y) \neq \emptyset
\]
and
\[
W^s(p) \cap W^u(y) \neq \emptyset.
\]
Here, $W^s(p)$ and $W^u(p)$ are defined on $\Omega_G(f)$. So, there exists $y_0 \in B_\lambda$ (in particular,
\( \gamma_0 \) belongs to the \( \alpha \)-limit set \( \alpha(y) \) such that \( \gamma_0 \sim p \), that is, \( p \in B_\alpha \). Therefore,

\[
B_\alpha \supset N_\delta(B_\alpha) \cap \text{Per}_G(f) \supset N_\delta(B_\alpha) \cap \text{Per}_G(f) = N_\delta(B_\alpha),
\]

that is, \( B_\alpha \) is open in \( \Omega_G(f) \).

Since \( X \) is compact and \( \Omega_G(f) \) is a closed subset of \( X \), \( \Omega_G(f) \) can be covered by finitely many \( B_\alpha \)'s, that is, \( \Omega_G(f) = \bigcup_{i=1}^n B_i \).

**Claim 5.** Each \( B_i \) is \( G \)-invariant.

**Proof.** Let \( x \in B_i \), \( g \in G \), and \( \delta > 0 \). We shall show that \( gx \in B_i \). Since \( x \in B_i \), there exists a \((\delta, G)\)-pseudo orbit \( \{x_0 = x, x_1, \ldots, x_{n-1}, x_n = x\} \). Then \( d(g_0 x_0, x_1) < \delta \) for some \( g_0 \in G \). Since \( f \) is pseudoequivariant, we can take \( h \in G \) such that \( g_0 f(x) = hf(gx) \). Thus \( \{gx, x_1, \ldots, x_{n-1}, x_n = x\} \) is a \((\delta, G)\)-pseudo orbit. By Lemma 2.3, there exists \( \gamma > 0 \) \((\gamma < \delta)\) such that

\[
d(x, y) < \gamma \implies d(gx, gy) < \delta \quad \text{for all } g \in G.
\]

Let \( \{x_0 = x, x_1, \ldots, x_{n-1}, x_n = x\} \) be a \((\gamma, G)\)-pseudo orbit. Then

\[
d(g_{n-1} f(x_{n-1}), x) < \gamma \quad \text{for some } g_{n-1} \in G
\]

and hence \( d(g g_{n-1} f(x_{n-1}), gx) < \delta \). Thus \( \{x_0 = x, x_1, \ldots, x_{n-1}, gx\} \) is a \((\delta, G)\)-pseudo orbit. Since \( \delta \) is arbitrary, \( gx \sim_G x \). Therefore, \( gx \in B_i \).

**Claim 6.** \( f|_{B_i} \) has the GSP.

**Proof.** Let \( 0 < \epsilon < \min\{d(B_i, B_j) : i \neq j, 1 \leq i, j \leq n\} \) be given. Since \( f|_{\Omega_G(f)} \) has the GSP, there exists \( \delta < \epsilon \) such that every \((\delta, G)\)-pseudo orbit \( \{x_i\} \subset B_i \) is \( \epsilon \)-traced by a point \( x \in \Omega_G(f) \). This means that, for each \( k \), there exists \( g_k \in G \) such that

\[
d(f^k(x), g_k x_k) < \epsilon.
\]

Since \( B_i \) is \( G \)-invariant and \( x_0 \in B_i \), \( g_0 x_0 \in B_i \). Therefore \( x \in B_i \).

**Claim 7.** \( f|_{B_i} \) is topologically \( G \)-transitive.

**Proof.** Let \( U \) and \( V \) be nonempty open subsets of \( B_i \). Take \( x \in U \) and \( y \in V \). Then \( x \sim_G y \). Let \( N_\epsilon(x) \) and \( N_\epsilon(y) \) be \( \epsilon \)-neighborhoods of \( x \) and \( y \) respectively such that \( N_\epsilon(x) \subset U \) and \( N_\epsilon(y) \subset V \). Choose a positive \( \epsilon_1 < \epsilon \) such that

\[
d(a, b) < \epsilon_1 \implies d(ga, gb) < \epsilon \quad \text{for all } g \in G.
\]
Since $f|_{B_i}$ has the GSP, there exists $\delta_1 > 0$ such that every $(\delta_1, G)$-pseudo orbit in $B_i$ is $\epsilon_1$-traced by a point in $B_i$. Thus, a $(\delta_1, G)$-pseudo orbit $\{x_0 = x, \ldots, x_n = y\} \subset B_i$ from $x$ to $y$ is $\epsilon_1$-traced by a point $z \in B_i$. In particular,

$$d(z, g_0x) < \epsilon_1 \quad \text{and} \quad d(f^n(z), g_ny) < \epsilon_1 \quad \text{for some} \quad g_0, g_n \in G.$$ 

Since $d(g_0^{-1}z, x) < \epsilon$ and $d(g_n^{-1}f^n(z), y) < \epsilon$,

$$g_0^{-1}z \in N_\epsilon(x) \subset U$$

and

$$g_n^{-1}f^n(z) \in N_\epsilon(y) \subset V.$$ 

Since $f^n(g_0^{-1}z) \in f^n(U)$ and $f$ is pseudoequivariant,

$$g_1f^n(z) \in f^n(U) \quad \text{for some} \quad g_1 \in G.$$ 

Choose $g \in G$ such that $gg_1 = g_n^{-1}$. Then $g_n^{-1}f^n(z) \in gf^n(U)$. Therefore, $gf^n(U) \cap V \neq \emptyset$. \hfill $\Box$

We next prepare the following three lemmas to complete Theorem B.

**Lemma 2.5.** Let $f : X \to X$ be a pseudoequivariant homeomorphism on a compact metric $G$-space $X$ with $G$ compact. Then

$$W^i(x) = W^i(p) \quad \text{for any} \quad x \in W^i(p) \quad (i = s, u).$$

Proof. We shall prove only the case $i = s$. Let $y \in W^s(x)$ and let $\epsilon > 0$. Since $y \in W^s(x)$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies that

$$d(f^n(h_nx), f^n(y)) < \epsilon 2 \quad \text{for some} \quad h_n \in G.$$ 

Let $\delta > 0$ be the constant satisfying the following:

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon 2 \quad \text{for all} \quad g \in G.$$ 

Since $x \in W^s(p)$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies that

$$d(f^n(g_n'p), f^n(x)) < \delta \quad \text{for some} \quad g_n' \in G.$$ 

Hence for some $h_n' \in G$ with $h_n'f^n(x) = f^n(h_nx)$,

$$d(h_n'f^n(g_n'p), h_n'f^n(x)) < \epsilon 2.$$
Since \( h_n^i f^n(g_n^i p) = f^n(g_n p) \) for some \( g_n \in G \),
\[
d(f^n(g_n p), f^n(h_n x)) < \frac{\epsilon}{2}.
\]

Take \( N = \max \{ N_1, N_2 \} \). Then \( n \geq N \) implies that
\[
d(f^n(g_n p), f^n(y)) \leq d(f^n(g_n p), f^n(h_n x)) + d(f^n(h_n x), f^n(y)) < \epsilon.
\]

Therefore, \( W^s(x) \subset W^s(p) \). Similarly, one can prove \( W^u(p) \subset W^u(x) \).

\[\tag*{\Box}\]

**Lemma 2.6.** Let \( f : X \to X \) be a pseudoequivariant homeomorphism on a compact metric \( G \)-space \( X \) with \( G \) compact and let \( x \in W^i(p) \). Then
\[
gx \in W^i(p) \quad \text{for every} \quad g \in G,
\]
and hence
\[
G(W^i(p)) = W^i(p) \quad (i = s, u).
\]

Proof. Let \( x \in W^i(p), \; g \in G \) and let \( \epsilon > 0 \). Then there is \( \delta > 0 \) such that if \( d(x, y) < \delta \), then \( d(gx, gy) < \epsilon \) for all \( g \in G \). Since for each \( n \in \mathbb{Z} \), we have \( g_n \in G \) such that
\[
\lim_{n \to \infty} d(f^n(g_n p), f^n(x)) = 0,
\]
that is, there exists \( N \in \mathbb{N} \) such that
\[
n \geq N \implies d(f^n(g_n p), f^n(x)) < \delta.
\]

Hence, for \( h'_n \in G \) with \( h'_n f^n(x) = f^n(gx) \),
\[
d(h'_n f^n(g_n p), h'_n f^n(x)) < \epsilon.
\]

Let \( h'_n f^n(g_n p) = f^n(h_n p) \). Then
\[
d(f^n(h_n p), f^n(gx)) < \epsilon.
\]

Therefore, \( gx \in W^i(p) \). Similarly, one can prove the statement for the case \( i = u \). \[\tag*{\Box}\]

**Lemma 2.7.** Let \( f : X \to X \) be a pseudoequivariant homeomorphism on a compact metric \( G \)-space \( X \) with \( G \) compact. Then for any \( \epsilon > 0 \), there exists a positive
constant \( \delta < \epsilon \) satisfying the following: if \( x \in W^u_\delta(y) \), then for all \( g \in G \),

(1) \( gx \in W^u_\epsilon(y) \)

and

(2) \( gy \in W^u_\epsilon(x) \).

Proof. Let \( \epsilon > 0 \). Then, by Lemma 2.3, there exists a positive constant \( \delta < \epsilon \) such that

\[
d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.
\]

Let \( x \in W^u_\delta(y) \) and let \( g \in G \). Then for each \( n \geq 0 \), there exists \( g_n \in G \) such that

\[
d(f^{-n}(x), f^{-n}(g_n y)) < \delta.
\]

(1) Take \( g'_n \in G \) such that \( g'_n f^{-n}(x) = f^{-n}(gx) \). Then

\[
d(f^{-n}(gx), g'_n f^{-n}(g_n y)) < \epsilon.
\]

Since \( f \) is pseudoequivariant, \( gx \in W^u_\epsilon(y) \).

(2) Take \( g'_n \in G \) such that \( g'_n f^{-n}(g_n y) = f^{-n}(gy) \). Then

\[
d(g'_n f^{-n}(x), f^{-n}(gy)) < \epsilon.
\]

Since \( f \) is pseudoequivariant, \( gy \in W^u_\epsilon(x) \) for all \( g \in G \). \( \Box \)

Proof of Theorem B. Let \( \epsilon > 0 \) be a constant which is less than the \( G \)-expansive constant for \( f|_B \) and let \( \delta > 0 \) (\( \delta < \epsilon \)) be the constant corresponding to \( \epsilon \) in the definition of the GSP. Let \( X_p = \overline{W^u(p)} \cap B \) for \( p \in B \cap Per_G(f) \). We can see directly from Lemmas 2.3 and 2.6 that \( X_p \) is \( G \)-invariant, that is, if \( x \in X_p \), then \( gx \in X_p \) for all \( g \in G \).

Claim 1. \( X_p \) is open in \( B \).

Proof. Since \( p \in Per_G(f) \), we have an integer \( m > 0 \) and \( g_1 \in G \) such that \( g_1 f^m(p) = p \). Denote \( N_\delta(X_p) = \{ y \in B : d(y, X_p) < \delta \} \). Let \( q \in N_\delta(X_p) \cap Per_G(f) \). Then there is \( x \in W^u(p) \cap B \) with \( d(q, x) < \delta \). Note that \( g_2 f^n(q) = q \) for some integer \( n > 0 \) and \( g_2 \in G \). Since \( f|_B \) has the GSP, the \( (\delta, G) \)-pseudo orbit

\[
[\ldots, f^{-2}(x), f^{-1}(x), q, f(q), f^2(q), \ldots]
\]
is \(\epsilon\)-traced by a point \(x' \in B\), that is, for each \(t \in \mathbb{Z}\), there exists \(h_t \in G\) such that
(a) \(d(x', h_0 q) < \epsilon\);
(b) \(d(f^t(x'), h_t f^t(q)) < \epsilon\) \((t > 0)\);
(c) \(d(f^{-t}(x'), h_{-t} f^{-t}(x)) < \epsilon\) \((t > 0)\).

Hence, it follows from Remark 1.10 that \(x' \in W^s(q) \cap W^u(x) \cap B\).

Since \(f\) is pseudoequivariant and \(p \in \text{Per}_G(f)\), for each \(k \in \mathbb{Z}\), we have \(g_{k,m,n} \in G\) such that \(f^{kmn}(g_{k,m,n} p) = p\). Since \(W^u(x) = W^u(p) = W^u(g_{k,m,n} p)\) by Lemmas 2.5 and 2.6,

\[
f^{kmn}(x') \in f^{kmn}(W^u(g_{k,m,n} p)) = W^u(f^{kmn}(g_{k,m,n} p)) = W^u(p).
\]

Since \(q \in W^s(x')\), for each \(k \in \mathbb{Z}\), one can find \(h_{k,m,n} \in G\) such that

\[
\lim_{k \to \infty} d(h_{k,m,n} f^{kmn}(x'), f^{kmn}(q)) = 0.
\]

Take \(i_{k,m,n} \in G\) such that \(i_{k,m,n}(h_{k,m,n})^{-1} f^{kmn}(q) = q\). Then

\[
\lim_{k \to \infty} d(i_{k,m,n} f^{kmn}(x'), i_{k,m,n}(h_{k,m,n})^{-1} f^{kmn}(q)) = \lim_{k \to \infty} d(i_{k,m,n} f^{kmn}(x'), q) = 0.
\]

Hence, \(q \in \overline{W^u(p)} \cap B = X_p\) because \(i_{k,m,n} f^{kmn}(x') \in W^u(p)\) for each \(k \in \mathbb{Z}\) by Lemma 2.6. Therefore,

\[
X_p \supset N_\delta(X_p) \cap \text{Per}_G(f) \supset N_\delta(X_p) \cap \text{Per}_G(f) = N_\delta(X_p),
\]

that is, \(X_p\) is open in \(B\).

Note that \(f(X_p) = f(W^u(p) \cap B) = f(W^u(p)) \cap f(B) = W^u(f(p)) \cap B = X_{f(p)}\).

Since \(X_p = X_{g_{k,m,n}}\) for any \(g \in G\) and \(g_{k,m,n} f^{m}(p) = p\),

\[
f^{m}(X_p) = X_{f^{m}(p)} = X_{g_{k,m,n} f^{m}(p)} = X_p.
\]

Take the smallest integer \(a > 0\) such that \(a \leq m\) and \(f^{a}(X_p) = X_p\).

**Claim 2.** \(B = \bigcup_{j=0}^{a-1} f^{j}(X_p)\).

Proof. Let \(y \in B\). Since \(f|_B\) is topologically \(G\)-transitive, for each \(1/n\)-neighborhood \(N_{1/n}(y)\) of \(y\), there are \(k > 0\) and \(h_n \in G\) such that \(h_n N_{1/n}(y) \cap f^k(X_p) \neq \emptyset\). So \(h_n N_{1/n}(y) \cap \bigcup_{j=0}^{a-1} f^j(X_p) \neq \emptyset\) for each \(n \in \mathbb{N}\). We may assume that \(h_n \to h \in G\) because \(G\) is compact. Since \(\bigcup_{j=0}^{a-1} f^j(X_p)\) is closed in \(B\), \(hy \in \bigcup_{j=0}^{a-1} f^j(X_p)\). Since \(G(f^j(X_p)) = G(X_{f^j(p)}) = X_{f^j(p)} = f^j(X_p)\), we have \(y \in \bigcup_{j=0}^{a-1} f^j(X_p)\). \(\square\)
Claim 3. \( X_p = X_q \) for \( q \in X_p \cap \text{Per}_G(f) \).

Proof. Let \( q \in X_p \cap \text{Per}_G(f) \) and suppose \( m \) and \( n \) are \( G \)-periodic numbers of \( p \) and \( q \) respectively. Since \( N_q(X_p) = X_p \), the constant \( \delta > 0 \) in the above of Claim 1, \( W^u_q(q) \subset X_p \). We firstly show that \( p \in X_q \). Suppose that \( p \notin X_q \). Then \( d(K, X_q) > 0 \) where \( K = X_p \setminus X_q \). Since \( q \in X_p = W^u(p) \cap B \), there exists \( z \in W^u(p) \cap B \) such that \( d(z, q) < d(K, X_q) \). Since \( z \in X_p \) and \( z \notin K, z \in X_q \). Furthermore, for each \( j \in \mathbb{Z} \), there exists \( g^j_{mnj} \in G \) such that

\[
\lim_{j \to \infty} d(f^{-mnj}(z), f^{-mnj}(g^j_{mnj} p)) = 0.
\]

For each \( j \in \mathbb{Z} \), choose \( g_{mnj} \in G \) with \( g_{mnj} f^{-mnj}(g^j_{mnj} p) = p \). Then we have

\[
\lim_{j \to \infty} d(g_{mnj} f^{-mnj}(z), p) = 0.
\]

So \( g_{mnj} f^{-mnj}(z) \notin X_q \) for sufficiently large \( j \). Hence,

\[
h_{mnj} z \notin f^{-mnj}(X_q) = X_q
\]

for \( h_{mnj} \in G \) with \( g_{mnj} f^{-mnj}(z) = f^{-mnj}(h_{mnj} z) \). Thus, \( z \notin X_q \). This is a contradiction. Therefore, \( p \in X_q \).

Let \( y \in W^u(q) \) and let \( 0 < \delta_1 < \delta_2 < \delta_3 = \delta \) such that

\[
d(x, y) < \delta_1 \implies d(gx, gy) < \delta_{i+1} \quad \text{for all} \quad g \in G \ (i = 1, 2).
\]

Then there exists \( N \in \mathbb{N} \) such that if \( k \geq N \), then \( d(f^{-k}(y), f^{-k}(h_{mnj} z)) < \delta_1 \) for some \( h_{mnj} \in G \). Choose \( j \in \mathbb{N} \) with \( mnj \geq N \). Then

\[
d((f^{-i} \circ f^{-mnj})(y), (f^{-i} \circ f^{-mnj})(h_{mnj+i} q)) < \delta_1 \quad \text{for all} \quad i \geq 0,
\]

that is,

\[
f^{-mnj}(y) \in W^u_{\delta_1}(f^{-mnj}(q)).
\]

By Lemma 2.7 (2), \( g f^{-mnj}(q) \in W^u_{\delta_2}(f^{-mnj}(y)) \) for all \( g \in G \). Since \( q \in \text{Per}_G(f) \), we have \( q \in W^u_{\delta_2}(f^{-mnj}(y)) \). Again, by Lemma 2.7 (2), \( g f^{-mnj}(y) \in W^u_{\delta_2}(q) \) for all \( g \in G \). In particular, \( f^{-mnj}(y) \in W^u_{\delta_2}(q) \). This means that \( y \in f^{-mnj}(W^u_{\delta_2}(q)) \) for some \( j \geq 0 \). So \( W^u(q) \subset \bigcup_{j \geq 0} f^{-mnj}(W^u_{\delta_2}(q)) \). Therefore,

\[
X_q = W^u(q) \cap B \subset \bigcup_{j \geq 0} f^{-mnj}(W^u_{\delta_2}(q)) \cap B \subset X_p \cap B = X_p \cap B = X_p.
\]

Similarly, we have \( X_p \subset X_q \). \qed
Claim 4. $X_p \cap f^j(X_p) = \emptyset$ for $0 < j < a$.

Proof. Suppose $X_p \cap f^j(X_p) \neq \emptyset$ for some $j$. Since $X_p \cap f^j(X_p)$ is open in $B$, we can find $q \in X_p \cap f^j(X_p) \cap \text{Per}_G(f)$. Then $X_q = X_p = f^j(X_p)$, which is a contradiction to the choice of the integer $a$. \qed

Claim 5. $f^a|_{X_p}$ is topologically $G$-mixing.

Proof. Let $U$ and $V$ be non-empty open subsets of $X_p$ and let $q \in V \cap \text{Per}_G(f)$. Then $f^{aj}(q) \in X_p \cap \text{Per}_G(f)$ for all $j \in \mathbb{Z}$. Since $X_p = X_{f^{\circ t}(q)}$ for all $j \in \mathbb{Z}$,

$$U \cap W^u(f^{aj}(q)) = U \cap (W^u(f^{aj}(q)) \cap B) \neq \emptyset \quad \text{for all} \quad j \in \mathbb{Z}.$$ 

Let $n > 0$ be a $G$-periodic number of $q$. Then for each $j$ such that $0 \leq j \leq n - 1$, there exists $z_j \in U \cap W^u(f^{aj}(q))$. Since $f$ is pseudoequivariant, we may take this statement: for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that

$$\lim_{t \to \infty} d(f^{-ant}(z_j), f^{aj}(h_t, f^{-ant}(q))) = 0.$$ 

For each $t \in \mathbb{Z}$, choose $g_t \in G$ such that $g_t f^{aj}(h_t, f^{-ant}(q)) = f^{aj}(q)$. Then we have

$$\lim_{t \to \infty} d(g_t f^{-ant}(z_j), f^{aj}(q)) = 0,$$

and thus

$$\lim_{t \to \infty} g_t f^{-ant}(z_j) = f^{aj}(q).$$

Since $f^{aj}(q) \in f^{aj}(V)$, for each $j$ with $0 \leq j \leq n - 1$, we may choose $N_j > 0$ such that for all $t \geq N_j$,

$$g_t f^{-ant}(z_j) \in f^{aj}(V).$$

Let $M = \max\{N_j : 0 \leq j \leq n - 1\}$. For each $t \geq M$, we get $t = ns + j$. If $s \geq M$, then

$$f^{-al}(i_s z_j) = f^{-aj}(g_s f^{-ant}(z_j)) \in V$$

for each $i_s \in G$ such that $f^{-ans-aj}(i_s z_j) = f^{-aj}(g_s f^{-ant}(z_j))$. Hence,

$$i_s z_j \in f^{al}(V) \quad \text{if} \quad s \geq M \quad \text{(that is,} \quad t \geq nM).$$

Thus, it follows from $z_j \in U$ that there exists $k_t \in G$ such that

$$k_t f^{al}(V) \cap U \neq \emptyset \quad \text{for each} \quad t \geq nM.$$

Therefore, $f^a|_{X_p}$ is topologically $G$-mixing. \qed
ACKNOWLEDGEMENT. The authors would like to thank the referee for very detailed comments in improving the exposition of the paper.

References


Taeyoung Choi
Department of Mathematics
Chungnam National University
Daejeon 305–764
South Korea

Current address:
National Institute for Mathematical Sciences
628 Daeduk-Boulevard Yuseong-gu
Daejeon
South Korea
e-mail: shadowcty@gmail.com

Junhui Kim
Division of Mathematics & Informational Statistics
Wonkwang University
Iksan 570–749
South Korea
e-mail: junhikim@wonkwang.ac.kr