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Abstract

In this paper, we introduce the weak $G$-expansivity which is a generalization of both expansivity and $G$-expansivity. Also, we define $G$-stable and $G$-unstable sets of a homeomorphism on a metric $G$-space $X$ and investigate properties of them. Finally, we consider the decomposition theorem on $G$-spaces.

1. Introduction

Let $X$ be a topological space, $G$ be a topological group, and $\theta : G \times X \to X$ be a map. The triple $(X, G, \theta)$ is called a topological $G$-space if the following three conditions are satisfied:

1. $\theta(e, x) = x$ for all $x \in X$, where $e$ is the identity of $G$;
2. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and for all $g, h \in G$;
3. $\theta$ is continuous.

Here, $gh$ is the group operation on $G$. Simply, we denote $\theta(g, x)$ by $gx$ and $X$ is usually said to be a topological $G$-space.

For any subset $A$ of $X$, $G(A)$ is denoted by the set $\{ga : g \in G, a \in A\}$. $G(x)$ is called a $G$-orbit of $x$. A subset $A$ of $X$ is called $G$-invariant if $G(A) = A$. A map $f : X \to X$ on a $G$-space $X$ is said to be pseudoequivariant provided that $f(G(x)) = G(f(x))$ for all $x \in X$, and $f$ is said to be equivariant provided that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

N. Aoki has proved the following topological decomposition theorem in 1983 ([1]), which is an extension of Smale’s spectral decomposition theorem and Bowen’s decomposition theorem in dynamical systems. All undefined notions can be found in [2].

Theorem 1.1 ([1]). Let $f : X \to X$ be a homeomorphism on a compact metric space $X$ and let $\text{CR}(f)$ be the chain recurrent set. If $f|_{\text{CR}(f)} : \text{CR}(f) \to \text{CR}(f)$ is an expansive homeomorphism with the shadowing property, then

1. $\text{CR}(f)$ contains a finite sequence $B_i$ ($1 \leq i \leq k$) of $f$-invariant closed subsets such that

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(a) $CR(f) = \bigcup_{i=1}^{n} B_i$ (disjoint union);
(b) $f|_{B_i}: B_i \to B_i$ is topologically transitive.

(2) for each $B_i$, there exist a subset $X_p$ of $B_i$ and $a > 0$ such that

(a) $f^a(X_p) = X_p$;
(b) $X_p \cap f^j(X_p) = \emptyset$ (0 < $j < a$);
(c) $f^a|_{X_p}: X_p \to X_p$ is topologically mixing;
(d) $B_i = \bigcup_{j=0}^{a-1} f^j(X_p)$.

A point $x \in X$ is called a $G$-periodic point of $f$ if there exist an integer $n > 0$ and $g \in G$ such that $f^n(x) = gx$. A point $x \in X$ is called a $G$-nonwandering point of $f$ if for every open neighborhood $U$ of $x$, there exist $n > 0$ and $g \in G$ such that $gf^n(U) \cap U \neq \emptyset$. $Per_G(f)$ (resp. $\Omega_G(f)$) is denoted by the set of all $G$-periodic (resp. $G$-nonwandering) points of $f$.

For a homeomorphism $f$ on a metric $G$-space $X$, a sequence $\{x_i \in X : i \in \mathbb{Z}\}$ is called a $(\delta, G)$-pseudo orbit for $f$ provided that for each $i$, there exists $g_i \in G$ such that $d(g_if(x_i), x_{i+1}) < \delta$. A $(\delta, G)$-pseudo orbit $\{x_i\}$ for $f$ is said to be $\epsilon$-traced by a point $x \in X$ provided that for each $i$, there exists $g_i \in G$ such that $d(f^i(x), g_ix_i) < \epsilon$.

**Definition 1.2 ([5]).** A homeomorphism $f: X \to X$ has the $G$-shadowing property (GSP) provided that for any $\epsilon > 0$, there exists $\delta > 0$ such that every $(\delta, G)$-pseudo orbit $\{x_i\}$ in $X$ for $f$ is $\epsilon$-traced by a point $x \in X$.

**Remark 1.3.** It was proved by E. Shah that, when $X$ is a compact metric $G$-space and the orbit map $\pi: X \to X/G$ is a covering map, a pseudoequivariant homeomorphism $f$ on $X$ has the GSP if and only if the induced map $\tilde{f}: X/G \to X/G$ has the shadowing property ([5]).

If a pseudoequivariant continuous onto map $f: X \to X$ has the GSP where $X$ is a compact metric $G$-space with $G$ compact, then $f|_{\Omega_G(f)}$ has the GSP ([5]).

The main purpose of this paper is to prove the following theorems on compact metric $G$-spaces.

**Theorem A.** Let $X$ be a compact metric $G$-space with $G$ compact. If $f: X \to X$ is a pseudoequivariant $G$-expansive homeomorphism with the GSP, then $\Omega_G(f)$ contains a finite sequence $B_i$ (1 $\leq$ $i$ $\leq$ $n$) of $f$-invariant, $G$-invariant, and closed subsets such that

(1) $f|_{\Omega_G(f)}$ is topologically $G$-transitive;
(2) $\Omega_G(f) = \bigcup_{i=1}^{n} B_i$ (disjoint union);
(3) $f|_{B_i}$ has the GSP.

A homeomorphism $f: X \to X$ is said to be topologically $G$-mixing provided that for every nonempty open subsets $U$ and $V$ of $X$, there exists an integer $N$ such that
for each \( n \geq N \), there is \( g_n \in G \) satisfying \( g_n f^n(U) \cap V \neq \emptyset \).

**Theorem B.** Let \( f|_{\Omega_G(f)} : \Omega_G(f) \to \Omega_G(f) \) be a \( G \)-expansive homeomorphism with the GSP. Then, for any \( f \)-invariant, \( G \)-invariant, open and closed subset \( B \subset \Omega_G(f) \) such that \( f|_B : B \to B \) is topologically \( G \)-transitive, there are \( X_p \subset B \) and \( a > 0 \) such that

1. \( f^a(X_p) = X_p \);
2. \( X_p \cap f^j(X_p) = \emptyset \) (\( 0 < j < a \));
3. \( f^a|_{X_p} : X_p \to X_p \) is topologically \( G \)-mixing;
4. \( B = \bigcup_{j=0}^{a-1} f^j(X_p) \).

**Definition 1.4.** A homeomorphism \( f : X \to X \) on a metric \( G \)-space \( X \) is said to be weak \( G \)-expansive provided that there exists \( \delta > 0 \) such that for every \( x, y \in X \) with \( G(x) \neq G(y) \) if \( u \in G(x) \) and \( v \in G(y) \), there exists \( n = n(u, v) \in \mathbb{Z} \) such that

\[
d(f^n(u), f^n(v)) > \delta.
\]

The constant \( \delta \) is called a weak \( G \)-expansive constant for \( f \).

The weak \( G \)-expansivity is a generalization of both expansivity and \( G \)-expansivity. Here, \( G \)-expansivity has been defined by R. Das ([4]). A homeomorphism \( f : X \to X \) is said to be \( G \)-expansive provided that there exists \( \delta > 0 \) such that for every \( x, y \in X \) with \( G(x) \neq G(y) \), there exists \( n \in \mathbb{Z} \) such that

\[
d(f^n(u), f^n(v)) > \delta \quad \text{for all} \quad u \in G(x), \ v \in G(y).
\]

The constant \( \delta \) is called a \( G \)-expansive constant for \( f \).

**Remark 1.5.** R. Das proved that there is no implication between \( G \)-expansivity and expansivity by giving counterexamples ([4]).

**Example 1.6** ([4]). Consider the compact space \( X = [1/\sqrt{n}, 1-1/\sqrt{n} : n \in \mathbb{N}] \) with the usual metric and let the topological group \( G = [-1, 1] \) act on \( X \) with the action \( \theta \) defined by \( \theta(1, x) = x \) and \( \theta(-1, x) = 1 - x \). Define a homeomorphism \( f : X \to X \) by

\[
f(x) = \begin{cases} 
    x & \text{if} \quad x = 0, 1; \\
    \text{next to the right of} \ x & \text{if} \quad x \in X \setminus [0, 1].
\end{cases}
\]

Then \( f \) is an expansive map with expansive constant \( \delta \) (\( 0 < \delta < 1/6 \)). But, it is easy to see that for \( x, y \in X \setminus [1/2] \) with \( G(x) \neq G(y) \), there is no \( n \in \mathbb{Z} \) such that

\[
|f^n(u) - f^n(v)| > \delta \quad \text{for all} \quad u \in G(x), \ v \in G(y),
\]

whatever \( \delta > 0 \) may be. This means that \( f \) is not \( G \)-expansive.
EXAMPLE 1.7 ([4]). Consider the compact space $X = \bigcup_{i=1}^{\infty} C_i$ with the usual metric, where each $C_i$ is the circle in $\mathbb{R}^2$ with center the origin and radius $i$. Denote $G = SO(2)$ by the set of all $2 \times 2$ matrices whose determinants are $\pm 1$ and define an action $\theta: G \times X \to X$ by the usual rotations on $X$. Then the identity map on $X$ is $G$-expansive with $G$-expansive constant $\delta$ ($0 < \delta < 1$).

Therefore, all properties of the following diagram are distinguished as we see in Examples 1.6 and 1.7:

\begin{equation*}
\begin{array}{c}
\text{weak } G \text{-expansive} \\
\text{expansive}
\end{array}
\end{equation*}

DEFINITION 1.8. Let $f: X \to X$ be a homeomorphism of a metric $G$-space $X$. We define a local $G$-stable set $W^s_\epsilon(x)$ and a local $G$-unstable set $W^u_\epsilon(x)$ by

$W^s_\epsilon(x) = \{ y \in X : \text{for each } n \geq 0, \text{ there is } g_n \in G \text{ such that } d(f^n(g_nx), f^n(y)) \leq \epsilon \}$,

$W^u_\epsilon(x) = \{ y \in X : \text{for each } n \geq 0 \text{ there is } g_n \in G \text{ such that } d(f^{-n}(g_nx), f^{-n}(y)) \leq \epsilon \}$.

We modify results of [3] into the following results by weakening the condition “equivariant” into “pseudoequivariant” and deleting the condition “invariant metric”. A metric $d$ on a $G$-space $X$ is called an invariant metric provided that $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

REMARK 1.9. Let $X$ be a compact metric $G$-space with $G$ compact. If $f: X \to X$ is a weak $G$-expansive pseudoequivariant homeomorphism with weak $G$-expansive constant $\delta > 0$, then for every $\gamma > 0$, there is $N > 0$ such that for each $x \in X$ and for each $n \geq N$,

$f^n(W^s_\delta(x)) \subset W^s_\gamma(f^n(x))$

and

$f^{-n}(W^u_\delta(x)) \subset W^u_\gamma(f^{-n}(x))$.

Proof. We shall prove only the case of a local $G$-stable set because the other case can be proved similarly. To do it, suppose that there exists $\gamma > 0$ such that for all $N > 0$, there are $x \in X$ and $n \geq N$ satisfying

$f^n(W^s_\delta(x)) \not\subset W^s_\gamma(f^n(x))$.
Let $N > 0$. Then there are $x_1 \in X$ and $n \geq N$ satisfying
\[ f^n(W^s_\delta(x_1)) \not\subset W^s_\gamma(f^n(x_1)), \]
that is, there exists $y_1 \in W^s_\delta(x_1)$ such that $f^n(y_1) \not\in W^s_\gamma(f^n(x_1))$. So there exists $i \geq 0$ such that for every $h \in G$,
\[ d(f^i(hf^n(x_1)), f^i(f^n(y_1))) > \gamma. \]
Because $f$ is pseudoequivariant, there exists $i \geq 0$ such that for every $g \in G$,
\[ d(gf^{i+n}(x_1), f^{i+n}(y_1)) > \gamma. \]
Take $m_1 = i + n$ and choose $N = m_1 + 1$.

Continuing the process, we can find sequences $m_n > 0$, $x_n$, and $y_n \in X$ such that
1. $y_n \in W^s_\delta(x_n)$;
2. $d(hf^{m_n}(x_n), f^{m_n}(y_n)) > \gamma$ for all $h \in G$;
3. $\lim_{n \to \infty} m_n = \infty$.

It follows from $y_n \in W^s_\delta(x_n)$ that for each $i \geq -m_n$, there exists $g_{i+m_n} \in G$ such that
\[ d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)) \leq \delta. \]
Since $f$ is pseudoequivariant, for each $g_{i+m_n}$, there exists $h_{i+m_n} \in G$ such that
\[ d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) = d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)). \]
Hence, for each $i \geq -m_n$,
\[ d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) \leq \delta. \]
If $f^{m_n}(x_n) \to x$, $f^{m_n}(y_n) \to y$, and $h_{i+m_n} \to h$ as $n \to \infty$, then
\[ d(f^i(hx), f^i(y)) \leq \delta \quad \text{for all } i \in \mathbb{Z}. \]
Since $\delta$ is a weak $G$-expansive constant for $f$, $G(x) = G(y)$. But $d(hx, y) = \lim_{n \to \infty} d(hf^{m_n}(x_n), f^{m_n}(y_n)) \geq \gamma > 0$ for all $h \in G$ by (2). Thus $hx \neq y$ for all $h \in G$, and hence $G(x) \neq G(y)$. This is a contradiction. \qed

For a homeomorphism $f$ on a compact metric $G$-space, we define the following:
\[
W^s(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that } \lim_{n \to \infty} d(f^n(g_nx), f^n(y)) = 0 \right\};
\]
\[
W^u(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that } \lim_{n \to \infty} d(f^{-n}(g_nx), f^{-n}(y)) = 0 \right\}.
\]
\( W^s(x) \) (resp. \( W^u(x) \)) is called a *G-stable set* (resp. *G-unstable set*).

**Remark 1.10.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact. If \( f : X \to X \) is a weak \( G \)-expansive pseudoequivariant homeomorphism with weak \( G \)-expansive constant \( \delta > 0 \), then for each \( \epsilon \) with \( 0 < \epsilon < \delta \),

\[
W^s(x) = \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x))); \\
W^u(x) = \bigcup_{n \geq 0} f^n(W^u_\epsilon(f^{-n}(x))).
\]

**Proof.** (\( \subseteq \)) Let \( y \in W^s(x) \) and \( 0 < \epsilon < \delta \). Then there exists \( N > 0 \) such that for each \( n \geq N \), we can choose \( g_n \in G \) satisfying

\[
d(f^n(g_nx), f^n(y)) \leq \epsilon.
\]

Thus,

\[
d(f^i(f^N(g_{i+N}x)), f^i(f^N(y))) \leq \epsilon \quad \text{for all} \quad i \geq 0.
\]

Since \( f \) is pseudoequivariant, \( f^N(y) \in W^s_\epsilon(f^N(x)) \). Therefore,

\[
y \in f^{-N}(W^s_\epsilon(f^N(x))) \subseteq \bigcup_{n \geq 0} f^{-n}(W^s_\epsilon(f^n(x))).
\]

(\( \supseteq \)): Let \( y \in f^{-n}(W^s_\epsilon(f^n(x))) \) for some \( n \geq 0 \). Then \( f^n(y) \in W^s_\epsilon(f^n(x)) \). It follows from Remark 1.9 that for every \( \gamma > 0 \) there exists \( N > 0 \) such that for each \( x \in X \) and \( m \geq N \),

\[
f^{m+n}(y) \in f^m(W^s_\epsilon(f^n(x))) \subseteq W^s_\gamma(f^{m+n}(x)).
\]

So for each \( n \geq N \), we can find \( g_n \in G \) such that

\[
d(f^{m+n}(g_nx), f^{m+n}(y)) \leq \gamma.
\]

Since \( f \) is pseudoequivariant, \( y \in W^s(x) \). The proof is completed. The case of a \( G \)-unstable set can be proved similarly.

\[
2. \text{ Decomposition theorems}
\]

First we prepare the following four lemmas to show Theorem A.

**Lemma 2.1** ([3]). Let \( (X, G, \theta) \) be a compact metric \( G \)-space with \( G \) compact. Then for any \( \epsilon > 0 \), there is a finite open cover \( \mathcal{U} = \{U_1, \ldots, U_s\} \) of \( X \) such that \( \text{diam}(gU_i) \leq \epsilon \) for all \( g \in G \) and \( i \) with \( 1 \leq i \leq s \).
In Lemma 2.1, notice that, for each \( g \in G \), the open cover \( \{gU : U \in \mathcal{U}\} \) of \( X \) satisfies \( \text{diam}(hgU) \leq \varepsilon \) for all \( h \in G \) and \( i \) with \( 1 \leq i \leq s \).

**Lemma 2.2.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact. If \( \mathcal{U} \) is a finite open cover of \( X \), then there exists \( \delta > 0 \) such that for each subset \( A \) of \( X \) with \( \text{diam}(A) \leq \delta \), \( A \subseteq gU \) for some \( U \in \mathcal{U} \) and \( g \in G \).

Proof. Suppose not. Then for every \( n > 0 \) there exists a subset \( A_n \) of \( X \) such that \( \text{diam}(A_n) \leq 1/n \) and \( A_n \nsubseteq gU \) for all \( U \in \mathcal{U} \) and \( g \in G \). Choose \( x_n \in A_n \) for each \( n \in \mathbb{N} \). Since \( X \) is compact, there exist a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} \to x \). We fix \( g \in G \). Then there is \( U \in \mathcal{U} \) with \( x \in gU \). Since \( X \setminus gU \) is compact, \( d(x, X \setminus gU) > 0 \). Put \( \varepsilon = d(x, X \setminus gU) \) and take \( n_i > 0 \) such that \( 1/n_i < \varepsilon/2 \) and \( d(x_{n_i}, x) < \varepsilon/2 \). Then for any \( y \in A_{n_i} \),

\[
    d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\varepsilon}{2} < \varepsilon.
\]

So \( y \in gU \). Therefore, \( A_{n_i} \subseteq gU \). This is a contradiction.

**Lemma 2.3.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact. Then for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) \((\delta < \varepsilon)\) such that

\[
    d(x, y) < \delta \implies d(gx, gy) < \varepsilon \quad \text{for all} \quad g \in G.
\]

Proof. Let \( \varepsilon > 0 \). Then it follows from Lemma 2.1 that, for any positive \( \varepsilon_1 < \varepsilon \), there is a finite open cover \( \mathcal{U} \) such that \( \text{diam}(gU) \leq \varepsilon_1 \) for all \( g \in G \) and \( U \in \mathcal{U} \). Also, by Lemma 2.2, there is a constant \( \delta = \delta(\mathcal{U}) > 0 \) such that for any subset \( A \) with \( \text{diam}(A) \leq \delta \), \( A \subseteq gU \) for some \( g \in G \) and \( U \in \mathcal{U} \). Let \( x \) and \( y \) in \( X \) with \( d(x, y) < \delta \). Then \( x, y \in g_0U_0 \) for some \( g_0 \in G \) and \( U_0 \in \mathcal{U} \). Note that \( \{g_0U : U \in \mathcal{U}\} \) is an open cover of \( X \). For any \( g \in G \), take \( g_1 \in G \) such that \( g_1 = g g_0 \). Then, by Lemma 2.1, \( \text{diam}(gg_0U) \leq \varepsilon_1 \), that is, \( \text{diam}(g_1U) \leq \varepsilon_1 < \varepsilon \) for all \( U \in \mathcal{U} \). Since \( gx, gy \in gg_0U_0 = g_1U_0, d(gx, gy) < \varepsilon \).

**Lemma 2.4.** Let \( X \) be a compact metric \( G \)-space with \( G \) compact and let \( f \) be a pseudoequivariant homeomorphism on \( X \). Then \( f \) has the GSP if and only if for any \( \varepsilon > 0 \), we can find \( \delta > 0 \) such that for every \((\delta, G)\)-pseudo orbit \( \{x_i\} \) of \( X \) for \( f \), there exist \( x \in X \) and \( h_i \in G \) satisfying

\[
    d(f^i(h_i x), x_i) < \varepsilon \quad \text{for all} \quad i \in \mathbb{Z}.
\]

Proof. Suppose that \( f \) has the GSP and let \( \varepsilon > 0 \). Then, by Lemma 2.3, there exists \( \varepsilon_0 > 0 \) \((\varepsilon_0 < \varepsilon)\) such that for each \( x, y \in X \),

\[
    d(x, y) < \varepsilon_0 \implies d(gx, gy) < \varepsilon \quad \text{for all} \quad g \in G.
\]
Let \( \delta \) be the constant corresponding to \( \epsilon_0 \) in the definition of the GSP. Then every \((\delta, G)\)-pseudo orbit \( \{x_i\} \) of \( X \) for \( f \) is \( \epsilon_0 \)-traced by a point \( x \in X \), that is, for each \( i \), there exists \( g_i \in G \) such that

\[
d(f^i(x), g_i x_i) < \epsilon_0 \quad \text{for all} \quad i \in \mathbb{Z}.
\]

Since \( f \) is pseudoequivariant, for each \( g_i \in G \), there exists \( h_i \in G \) such that

\[
g_i^{-1} f^i(x) = f^i(h_i x).
\]

Moreover, \( d(g_i^{-1} f^i(x), x_i) < \epsilon \) and hence \( d(f^i(h_i x), x_i) < \epsilon \) for all \( i \in \mathbb{Z} \).

The converse can be proved similarly. \( \square \)

We have ([5]) that \( f(\Omega_G(f)) = \Omega_G(f) \) and \( CR_G(f) = \Omega_G(f) \) for a pseudoequivariant homeomorphism \( f \) with GSP on a compact metric \( G \)-space \( X \) where \( G \) is compact.

For \( x, y \in X \) and \( \delta > 0 \), \( x \) is said to be \((\delta, G)\)-related to \( y \) (denoted by \( x \sim_G y \)) if there exist finite \((\delta, G)\)-pseudo orbits \( \{x = x_0, x_1, \ldots, x_k = y\} \) and \( \{y = y_0, y_1, \ldots, y_n = x\} \) for \( f \). If for every \( \delta > 0 \), \( x \) is \((\delta, G)\)-related to \( y \), then \( x \) is said to be \( G \)-related to \( y \) (denoted by \( x \sim_G y \)). A point \( x \) is said to be a \( G \)-chain recurrent point of \( f \) if \( x \sim_G x \). \( CR_G(f) \) is denoted by the set of all \( G \)-chain recurrent points of \( f \). A homeomorphism \( f: X \to X \) is called topologically \( G \)-transitive provided that for every nonempty open subsets \( U \) and \( V \) of \( X \), there exist an integer \( n > 0 \) and \( g \in G \) such that \( g f^n(U) \cap V \neq \emptyset \).

**Proof of Theorem A.** Since the pseudoequivariant homeomorphism \( f \) satisfies the GSP, \( CR_G(f) = \Omega_G(f) \). Thus \( \Omega_G(f) = \bigcup \beta B_\delta \) where each \( B_\delta \) is an equivalence class under the relation \( \sim_G \) which is defined in \( CR_G(f) \).

**Claim 1.** Each \( B_\delta \) is closed in \( \Omega_G(f) \).

Proof. Let \( x \in \overline{B_\delta} \). Then we can find a sequence \( \{x_i\} \) in \( B_\delta \) which converges to \( x \). Let \( \alpha > 0 \) be given. Then there exists a finite open cover \( \{U_1, \ldots, U_s\} \) of \( X \) such that

\[
\text{diam}(gU_i) \leq \frac{\alpha}{2} \quad \text{for all} \quad g \in G \quad \text{and} \quad i \quad \text{with} \quad 1 \leq i \leq s
\]

by Lemma 2.1. So \( f(x) \in U_i \) for some \( i \). Choose \( \epsilon_0 \)-neighborhood \( N_{\epsilon_0}(f(x)) \) of \( f(x) \) such that \( N_{\epsilon_0}(f(x)) \subset U_i \). Then since \( f \) is uniformly continuous, there exists \( \delta_0 > 0 \) such that

\[
d(x, y) < \delta_0 \implies d(f(x), f(y)) < \epsilon_0.
\]
Because \( \{x_i\} \) converges to \( x \), there is \( J > 0 \) such that \( d(x_J, x) < \min(\alpha/2, \delta_0) \). From the fact that \( x_J \in \text{CR}_G(f) \), we can find a \((\alpha/2, G)\)-pseudo orbit

\[ [x_J = y_0, y_1, \ldots, y_{k-1}, y_k = x_J]. \]

So \( d(gf(y_0), y_1) < \alpha/2 \) for some \( g \in G \). Also \( d(f(y_0), f(x)) < \epsilon_0 \) and hence \( d(gf(y_0), gf(x)) < \alpha/2 \). Thus,

\[
d(gf(x), y_1) \leq d(gf(x), gf(y_0)) + d(gf(y_0), y_1) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
\]

Therefore, \( \{x, y_1, \ldots, y_k = x_J\} \) is an \((\alpha, G)\)-pseudo orbit. It is clear that there is an \((\alpha, G)\)-pseudo orbit from \( x_J \) to \( x \) by the uniform continuity of \( f \). It follows from \( x \sim_G x_J \) that \( x \sim_G x_i \) for all \( i \) because each \( x_i \in B_\lambda \). Since \( \alpha \) is arbitrary, \( x \in B_\lambda \).

Therefore, \( B_\lambda \) is closed.

\textbf{Claim 2.} Each \( B_\lambda \) is \( f \)-invariant.

Proof. To prove this, we firstly show that \( x \sim_G f(x) \) for all \( x \in \Omega_G(f) \). Let \( \alpha > 0 \). Then there is \( \delta > 0 \) (\( \delta < \alpha \)) such that

\[
d(a, b) < \delta \implies d(f^2(a), f^2(b)) < \alpha.
\]

Since \( x \in \Omega_G(f) \), there are \( n > 0 \) and \( g \in G \) such that

\[ gf^n(N_\delta(x)) \cap N_\delta(x) \neq \emptyset \]

where \( N_\delta(x) \) is a \( \delta \)-neighborhood of \( x \). Then there exists \( z \in N_\delta(x) \) such that \( gf^n(z) \in N_\delta(x) \). Hence

\[ \{ f(x), f^2(z), \ldots, f^{n-1}(z), x \} \]

is an \((\alpha, G)\)-pseudo orbit and thus, \( x \sim_G f(x) \). Since \( f \) is a homeomorphism, we can show that \( x \sim_G f^{-1}(x) \) for all \( x \in \Omega_G(f) \) similarly. Therefore, \( f(B_\lambda) = B_\lambda \) for each \( \lambda \).

\textbf{Claim 3.} \( \text{Per}_G(f) \) is dense in \( \Omega_G(f) \).

Proof. Let \( \alpha > 0 \) be a \( G \)-expansive constant for \( f \) and take \( \epsilon < \alpha/2 \). Since \( f \) has the GSP, there exists \( \delta > 0 \) (\( \delta < \epsilon \)) such that every \((\delta, G)\)-pseudo orbit is \( \epsilon \)-traced by a point in \( X \). Since \( f \) is uniformly continuous, there exists a positive constant \( \gamma < \delta \) such that if \( d(a, b) < \gamma \), then \( d(f(a), f(b)) < \delta \). Let \( p \in \Omega_G(f) \). Then for every \( \gamma \)-neighborhood \( N_\gamma(p) \) of \( p \), there exist an integer \( n > 0 \) and \( g \in G \) such that

\[ gf^n(N_\gamma(p)) \cap N_\gamma(p) \neq \emptyset. \]
Choose a point \( y \in g^f(N_\rho(p)) \cap N_\rho(p) \). Since \( f^{-n}(g^{-1}y) \in N_\rho(p) \),

\[
d(f(p), f(f^{-n}(g^{-1}y))) < \delta.
\]

Hence

\[
\{ \ldots, x_0 = p, x_1 = f^{-n+1}(g^{-1}y), x_2 = f^{-n+2}(g^{-1}y), \ldots, x_{n-1} = f^{-1}(g^{-1}y), x_n = p, \ldots \}
\]
is a \((\delta, G)\)-pseudo orbit for \( f \). Since \( f \) has the GSP, it follows from Lemma 2.4 that, for each \( i \in \mathbb{Z} \), there exist \( x \in X \) and \( g_i \in G \) such that

\[
d(f^i(g_i x), x_i) < \epsilon \quad \text{for all} \quad i \in \mathbb{Z}.
\]

Thus,

\[
d(f^k(f^n(g_{k+n}x)), f^k(g_k x)) \leq d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_{k+n}, f^k(g_k x))
\]
\[
= d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_k, f^k(g_k x))
\]
\[
< 2\epsilon < \alpha
\]

for all \( k \). Since \( \alpha \) is a \( G \)-expansive constant for \( f \),

\[
G(f^n(x)) = G(x),
\]

and hence

\[
g_0 x \in \text{Per}_G(f) \cap N_{\rho}(p)
\]

where \( N_{\rho}(p) \) is an \( \epsilon \)-neighborhood of \( p \). Therefore, \( \text{Per}_G(f) \) is dense in \( \Omega_G(f) \). \( \square \)

**Claim 4.** Each \( B_\lambda \) is open in \( \Omega_G(f) \).

**Proof.** Let \( \alpha > 0 \) be a \( G \)-expansive constant for \( f \) and let \( \epsilon < \alpha \). Denote

\[
N_\delta(B_\lambda) = \{ y \in \Omega_G(f) : d(y, B_\lambda) < \delta \}
\]

where \( \delta \) is the constant corresponding to \( \epsilon \) in the definition of the GSP for \( f|_{\Omega_G(f)} \).

Then for a point \( p \in N_\delta(B_\lambda) \cap \text{Per}_G(f) \), there exists \( y \in B_\lambda \) such that

\[
d(y, p) < \delta.
\]

Since \( f|_{\Omega_G(f)} \) has the GSP, it follows from Remark 1.10 that

\[
W^u(p) \cap W^s(y) \neq \emptyset
\]

and

\[
W^s(p) \cap W^u(y) \neq \emptyset.
\]

Here, \( W^s(p) \) and \( W^u(p) \) are defined on \( \Omega_G(f) \). So, there exists \( y_0 \in B_\lambda \) (in particular,
\( y_0 \) belongs to the \( \alpha \)-limit set \( \alpha(y) \) such that \( y_0 \sim p \), that is, \( p \in B_\lambda \). Therefore,

\[
B_\lambda \ni N_\delta(B_\lambda) \cap \overline{\text{Per}_G(f)} \supset N_\delta(B_\lambda) \cap \overline{\text{Per}_G(f)} = N_\delta(B_\lambda),
\]

that is, \( B_\lambda \) is open in \( \Omega_G(f) \).

Since \( X \) is compact and \( \Omega_G(f) \) is a closed subset of \( X \), \( \Omega_G(f) \) can be covered by finitely many \( B_\lambda \)'s, that is, \( \Omega_G(f) = \bigcup_{i=1}^n B_i \).

**Claim 5.** Each \( B_i \) is \( G \)-invariant.

Proof. Let \( x \in B_i \), \( g \in G \), and \( \delta > 0 \). We shall show that \( gx \in B_i \). Since \( x \in B_i \), there exists a \((\delta, G)\)-pseudo orbit \([x_0 = x, x_1, \ldots, x_{n-1}, x_n = x]\). Then \( d(g_0 f(x), x_1) < \delta \) for some \( g_0 \in G \). Since \( f \) is pseudoequivariant, we can take \( h \in G \) such that \( g_0 f(x) = hf(gx) \). Thus \([gx, x_1, \ldots, x_{n-1}, x_n = x]\) is a \((\delta, G)\)-pseudo orbit. By Lemma 2.3, there exists \( \gamma > 0 \) (\( \gamma < \delta \)) such that

\[
d(x, y) < \gamma \implies d(gx, gy) < \delta \text{ for all } g \in G.
\]

Let \([x_0 = x, x_1, \ldots, x_{n-1}, x_n = x]\) be a \((\gamma, G)\)-pseudo orbit. Then

\[
d(g_{n-1} f(x_{n-1}), x) < \gamma \text{ for some } g_{n-1} \in G
\]

and hence \( d(gg_{n-1} f(x_{n-1}), gx) < \delta \). Thus \([x_0 = x, x_1, \ldots, x_{n-1}, gx]\) is a \((\delta, G)\)-pseudo orbit. Since \( \delta \) is arbitrary, \( gx \sim_G x \). Therefore, \( gx \in B_i \).

**Claim 6.** \( f \mid_{B_i} \) has the GSP.

Proof. Let \( 0 < \epsilon < \min\{d(B_i, B_j) : i \neq j, 1 \leq i, j \leq n\} \) be given. Since \( f \mid_{\Omega_G(f)} \) has the GSP, there exists \( \delta < \epsilon \) such that every \((\delta, G)\)-pseudo orbit \([x_k] \subset B_i \) is \( \epsilon \)-traced by a point \( x \in \Omega_G(f) \). This means that, for each \( k \), there exists \( g_k \in G \) such that

\[
d(f^k(x), g_k x_k) < \epsilon.
\]

Since \( B_i \) is \( G \)-invariant and \( x_0 \in B_i \), \( g_0 x_0 \in B_i \). Therefore \( x \in B_i \).

**Claim 7.** \( f \mid_{B_i} \) is topologically \( G \)-transitive.

Proof. Let \( U \) and \( V \) be nonempty open subsets of \( B_i \). Take \( x \in U \) and \( y \in V \). Then \( x \sim_G y \). Let \( N_\epsilon(x) \) and \( N_\epsilon(y) \) be \( \epsilon \)-neighborhoods of \( x \) and \( y \) respectively such that \( N_\epsilon(x) \subset U \) and \( N_\epsilon(y) \subset V \). Choose a positive \( \epsilon_1 < \epsilon \) such that

\[
d(a, b) < \epsilon_1 \implies d(ga, gb) < \epsilon \text{ for all } g \in G.
\]
Since \( f|_{B_i} \) has the GSP, there exists \( \delta_1 > 0 \) such that every \((\delta_1, G)\)-pseudo orbit in \( B_i \) is \( \epsilon_1 \)-traced by a point in \( B_i \). Thus, a \((\delta_1, G)\)-pseudo orbit \( \{x_0 = x, \ldots, x_n = y\} \subset B_i \) from \( x \) to \( y \) is \( \epsilon_1 \)-traced by a point \( z \in B_i \). In particular,

\[
d(z, g_0x) < \epsilon_1 \quad \text{and} \quad d(f^n(z), g_ny) < \epsilon_1 \quad \text{for some} \quad g_0, g_n \in G.
\]

Since \( d(g_0^{-1}z, x) < \epsilon \) and \( d(g_n^{-1}f^n(z), y) < \epsilon \),

\[
g_0^{-1}z \in N_\epsilon(x) \subset U
\]

and

\[
g_n^{-1}f^n(z) \in N_\epsilon(y) \subset V.
\]

Since \( f^n(g_0^{-1}z) \in f^n(U) \) and \( f \) is pseudoequivariant,

\[
g_1f^n(z) \in f^n(U) \quad \text{for some} \quad g_1 \in G.
\]

Choose \( g \in G \) such that \( gg_1 = g_n^{-1} \). Then \( g_n^{-1}f^n(z) \in gf^n(U) \). Therefore, \( gf^n(U) \cap V \neq \emptyset \).

We next prepare the following three lemmas to complete Theorem B.

**Lemma 2.5.** Let \( f: X \to X \) be a pseudoequivariant homeomorphism on a compact metric \( G \)-space \( X \) with \( G \) compact. Then

\[
W^i(x) = W^i(p) \quad \text{for any} \quad x \in W^i(p) \quad (i = s, u).
\]

Proof. We shall prove only the case \( i = s \). Let \( y \in W^s(x) \) and let \( \epsilon > 0 \). Since \( y \in W^s(x) \), there exists \( N_1 \in \mathbb{N} \) such that \( n \geq N_1 \) implies that

\[
d(f^n(h_nx), f^n(y)) < \frac{\epsilon}{2} \quad \text{for some} \quad h_n \in G.
\]

Let \( \delta > 0 \) be the constant satisfying the following:

\[
d(x, y) < \delta \implies d(gx, gy) < \frac{\epsilon}{2} \quad \text{for all} \quad g \in G.
\]

Since \( x \in W^s(p) \), there exists \( N_2 \in \mathbb{N} \) such that \( n \geq N_2 \) implies that

\[
d(f^n(g_n'p), f^n(x)) < \delta \quad \text{for some} \quad g_n' \in G.
\]

Hence for some \( h_n' \in G \) with \( h_n'f^n(x) = f^n(h_nx) \),

\[
d(h_n f^n(g_n'p), h_n' f^n(x)) < \frac{\epsilon}{2}.
\]
Since \( h_n f^n(g_n p) = f^n(g_n p) \) for some \( g_n \in G \),

\[
d(f^n(g_n p), f^n(h_n x)) < \frac{\epsilon}{2}.
\]

Take \( N = \max\{N_1, N_2\} \). Then \( n \geq N \) implies that

\[
d(f^n(g_n p), f^n(y)) \leq d(f^n(g_n p), f^n(h_n x)) + d(f^n(h_n x), f^n(y)) < \epsilon.
\]

Therefore, \( W^s(x) \subset W^s(p) \). Similarly, one can prove \( W^u(p) \subset W^u(x) \).

**Lemma 2.6.** Let \( f : X \to X \) be a pseudoequivariant homeomorphism on a compact metric \( G \)-space \( X \) with \( G \) compact and let \( x \in W^i(p) \). Then

\[
gx \in W^i(p) \quad \text{for every} \quad g \in G,
\]

and hence

\[
G(W^i(p)) = W^i(p) \quad (i = s, u).
\]

**Proof.** Let \( x \in W^i(p), \ g \in G \) and let \( \epsilon > 0 \). Then there is \( \delta > 0 \) such that if \( d(x, y) < \delta \), then \( d(gx, gy) < \epsilon \) for all \( g \in G \). Since for each \( n \in \mathbb{Z} \), we have \( g_n \in G \) such that

\[
\lim_{n \to \infty} d(f^n(g_n p), f^n(x)) = 0,
\]

that is, there exists \( N \in \mathbb{N} \) such that

\[
n \geq N \quad \Rightarrow \quad d(f^n(g_n p), f^n(x)) < \delta.
\]

Hence, for \( h'_n \in G \) with \( h'_n f^n(x) = f^n(gx) \),

\[
d(h'_n f^n(g_n p), h'_n f^n(x)) < \epsilon.
\]

Let \( h'_n f^n(g_n p) = f^n(h_n p) \). Then

\[
d(f^n(h_n p), f^n(gx)) < \epsilon.
\]

Therefore, \( gx \in W^s(p) \). Similarly, one can prove the statement for the case \( i = u \). □

**Lemma 2.7.** Let \( f : X \to X \) be a pseudoequivariant homeomorphism on a compact metric \( G \)-space \( X \) with \( G \) compact. Then for any \( \epsilon > 0 \), there exists a positive
constant $\delta < \epsilon$ satisfying the following: if $x \in W^u_\delta(y)$, then for all $g \in G$,

(1) $gx \in W^u_\epsilon(y)$

and

(2) $gy \in W^u_\epsilon(x)$.

Proof. Let $\epsilon > 0$. Then, by Lemma 2.3, there exists a positive constant $\delta < \epsilon$ such that

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \text{ for all } g \in G.$$ 

Let $x \in W^u_\delta(y)$ and let $g \in G$. Then for each $n \geq 0$, there exists $g_n \in G$ such that

$$d(f^{-n}(x), f^{-n}(gx)) < \delta.$$ 

(1) Take $g'_n \in G$ such that $g'_n f^{-n}(x) = f^{-n}(gx)$. Then

$$d(f^{-n}(gx), g'_n f^{-n}(gy)) < \epsilon.$$ 

Since $f$ is pseudoequivariant, $gx \in W^u_\epsilon(y)$.

(2) Take $g'_n \in G$ such that $g'_n f^{-n}(gy) = f^{-n}(gy)$. Then

$$d(g'_n f^{-n}(x), f^{-n}(gy)) < \epsilon.$$ 

Since $f$ is pseudoequivariant, $gy \in W^u_\epsilon(x)$ for all $g \in G$.

Proof of Theorem B. Let $\epsilon > 0$ be a constant which is less than the $G$-expansive constant for $f|_B$ and let $\delta > 0$ ($\delta < \epsilon$) be the constant corresponding to $\epsilon$ in the definition of the GSP. Let $X_p = \overline{W^u(p)} \cap B$ for $p \in B \cap \text{Per}_G(f)$. We can see directly from Lemmas 2.3 and 2.6 that $X_p$ is $G$-invariant, that is, if $x \in X_p$, then $gx \in X_p$ for all $g \in G$.

Claim 1. $X_p$ is open in $B$.

Proof. Since $p \in \text{Per}_G(f)$, we have an integer $m > 0$ and $g_1 \in G$ such that $g_1 f^m(p) = p$. Denote $N_\delta(X_p) = \{ y \in B : d(y, X_p) < \delta \}$. Let $q \in N_\delta(X_p) \cap \text{Per}_G(f)$. Then there is $x \in W^u(p) \cap B$ with $d(q, x) < \delta$. Note that $g_2 f^n(q) = q$ for some integer $n > 0$ and $g_2 \in G$. Since $f|_B$ has the GSP, the $(\delta, G)$-pseudo orbit

$$\{ \ldots, f^{-2}(x), f^{-1}(x), q, f(q), f^2(q), \ldots \}$$
Since $q \in W^s(x')$, for each $k \in \mathbb{Z}$, one can find $h_{kmn} \in G$ such that
\[
\lim_{k \to \infty} d(h_{kmn} f_{kmn}(x'), f_{kmn}(q)) = 0.
\]
Take $i_{kmn} \in G$ such that $i_{kmn}(h_{kmn})^{-1} f_{kmn}(q) = q$. Then
\[
\lim_{k \to \infty} d(i_{kmn} f_{kmn}(x'), i_{kmn}(h_{kmn})^{-1} f_{kmn}(q)) = \lim_{k \to \infty} d(i_{kmn} f_{kmn}(x'), q) = 0.
\]
Hence, $q \in \overline{W^u(p) \cap B} = X_p$ because $i_{kmn} f_{kmn}(x') \in W^u(p)$ for each $k \in \mathbb{Z}$ by Lemma 2.6. Therefore,
\[
X_p \supset N_{\delta}(X_p) \cap \Per_G(f) \supset N_{\delta}(X_p) \cap \Per_G(f) = N_{\delta}(X_p),
\]
that is, $X_p$ is open in $B$. \(\square\)

Note that $f(X_p) = f(W^u(p) \cap B) = \overline{f(W^u(p)) \cap f(B)} = \overline{W^u(f(p)) \cap B} = X_{f(p)}$. Since $X_p = X_{g \cdot f^m(p)}$ for any $g \in G$ and $g \cdot f^m(p) = p$,
\[
f^m(X_p) = X_{f^m(p)} = X_{g \cdot f^m(p)} = X_p.
\]
Take the smallest integer $a > 0$ such that $a \leq m$ and $f^a(X_p) = X_p$.

**Claim 2.** $B = \bigcup_{j=0}^{n-1} f^j(X_p)$.

**Proof.** Let $y \in B$. Since $f|_B$ is topologically $G$-transitive, for each $1/n$-neighborhood $N_{1/n}(y)$ of $y$, there are $k > 0$ and $h_n \in G$ such that $h_n N_{1/n}(y) \cap f^k(X_p) \neq \emptyset$. So $h_n N_{1/n}(y) \cap \bigcup_{j=0}^{n-1} f^j(X_p) \neq \emptyset$ for each $n \in \mathbb{N}$. We may assume that $h_n \to h \in G$ because $G$ is compact. Since $\bigcup_{j=0}^{n-1} f^j(X_p)$ is closed in $B$, $hy \in \bigcup_{j=0}^{n-1} f^j(X_p)$. Since $G(f^j(X_p)) = G(X_{f^j(p)}) = X_{f^j(p)} = f^j(X_p)$, we have $y \in \bigcup_{j=0}^{n-1} f^j(X_p)$.
Claim 3. $X_p = X_q$ for $q \in X_p \cap \text{Per}_G(f)$.

Proof. Let $q \in X_p \cap \text{Per}_G(f)$ and suppose $m$ and $n$ are $G$-periodic numbers of $p$ and $q$ respectively. Since $N_\delta(X_p) = X_p$ for the constant $\delta > 0$ in the above of Claim 1, $W^u_\delta(q) \subset X_p$. We firstly show that $p \in X_q$. Suppose that $p \notin X_q$. Then $d(K, X_q) > 0$ where $K = X_p \setminus X_q$. Since $q \in X_p = W^u(p) \cap B$, there exists $z \in W^u(p) \cap B$ such that $d(z, q) < d(K, X_q)$. Since $z \in X_p$ and $z \notin K$, $z \in X_q$. Furthermore, for each $j \in \mathbb{Z}$, there exists $g_{mnj} \in G$ such that

$$
\lim_{j \to \infty} d(f^{-mnj}(z), f^{-mnj}(g_{mnj} p)) = 0.
$$

For each $j \in \mathbb{Z}$, choose $g_{mnj} \in G$ with $g_{mnj} f^{-mnj}(g_{mnj} p) = p$. Then we have

$$
\lim_{j \to \infty} d(g_{mnj} f^{-mnj}(z), p) = 0.
$$

So $g_{mnj} f^{-mnj}(z) \notin X_q$ for sufficiently large $j$. Hence,

$$
h_{mnj} z \notin f^{-mnj}(X_q) = X_q
$$

for $h_{mnj} \in G$ with $g_{mnj} f^{-mnj}(z) = f^{-mnj}(h_{mnj} z)$. Thus, $z \notin X_q$. This is a contradiction. Therefore, $p \in X_q$.

Let $y \in W^u(q)$ and let $0 < \delta_1 < \delta_2 < \delta_3 = \delta$ such that

$$
d(x, y) < \delta_1 \implies d(gx, gy) < \delta_{i+1} \quad \text{for all} \quad g \in G \ (i = 1, 2).
$$

Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $d(f^{-k}(y), f^{-k}(h_k q)) < \delta_1$ for some $h_k \in G$. Choose $j \in \mathbb{N}$ with $mnj \geq N$. Then

$$
d((f^{-i} \circ f^{-mnj})(y), (f^{-i} \circ f^{-mnj})(h_{mnj+i} q)) < \delta_1 \quad \text{for all} \quad i \geq 0,
$$

that is,

$$
f^{-mnj}(y) \in W^u_\delta(f^{-mnj}(q)).
$$

By Lemma 2.7 (2), $g f^{-mnj}(q) \in W^u_\delta(f^{-mnj}(y))$ for all $g \in G$. Since $q \in \text{Per}_G(f)$, we have $q \in W^u_\delta(f^{-mnj}(y))$. Again, by Lemma 2.7 (2), $g f^{-mnj}(y) \in W^u_\delta(q)$ for all $g \in G$. In particular, $f^{-mnj}(y) \in W^u_\delta(q)$. This means that $y \in f^{mnj}(W^u_\delta(q))$ for some $j \geq 0$. So $W^u(q) \subset \bigcup_{j \geq 0} f^{mnj}(W^u_\delta(q))$. Therefore,

$$
X_q = \overline{W^u(q) \cap B} \subset \bigcup_{j \geq 0} f^{mnj}(W^u_\delta(q)) \cap B \subset \overline{X_p \cap B} = X_p \cap B = X_p.
$$

Similarly, we have $X_p \subset X_q$. \qed
Claim 4. $X_p \cap f^j(X_p) = \emptyset$ for $0 < j < a$.

Proof. Suppose $X_p \cap f^j(X_p) \neq \emptyset$ for some $j$. Since $X_p \cap f^j(X_p)$ is open in $B$, we can find $q \in X_p \cap f^j(X_p) \cap \text{Per}_G(f)$. Then $X_q = X_p = f^j(X_p)$, which is a contradiction to the choice of the integer $a$. \hfill \square

Claim 5. $f^a|_{X_p}$ is topologically $G$-mixing.

Proof. Let $U$ and $V$ be non-empty open subsets of $X_p$ and let $q \in V \cap \text{Per}_G(f)$. Then $f^{aj}(q) \in X_p \cap \text{Per}_G(f)$ for all $j \in \mathbb{Z}$. Since $X_p = X_{f^s(q)}$ for all $j \in \mathbb{Z}$,

$$U \cap W^u(f^{aj}(q)) = U \cap (W^u(f^{aj}(q)) \cap B) \neq \emptyset \quad \text{for all} \quad j \in \mathbb{Z}.$$ 

Let $n > 0$ be a $G$-periodic number of $q$. Then for each $j$ such that $0 \leq j \leq n-1$, there exists $z_j \in U \cap W^u(f^{aj}(q))$. Since $f$ is pseudoequivariant, we may take this statement: for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that

$$\lim_{t \to \infty} d(f^{-\text{ant}}(z_j), f^{aj}(h_t,f^{-\text{ant}}(q))) = 0.$$ 

For each $t \in \mathbb{Z}$, choose $g_t \in G$ such that $g_t f^{aj}(h_t,f^{-\text{ant}}(q)) = f^{aj}(q)$. Then we have

$$\lim_{t \to \infty} d(g_t f^{-\text{ant}}(z_j), f^{aj}(q)) = 0,$$

and thus

$$\lim_{t \to \infty} g_t f^{-\text{ant}}(z_j) = f^{aj}(q).$$

Since $f^{aj}(q) \in f^{aj}(V)$, for each $j$ with $0 \leq j \leq n-1$, we may choose $N_j > 0$ such that for all $t \geq N_j$,

$$g_t f^{-\text{ant}}(z_j) \in f^{aj}(V).$$

Let $M = \max\{N_j : 0 \leq j \leq n-1\}$. For each $t \geq M$, we get $t = ns + j$. If $s \geq M$, then

$$f^{-al}(i_z z_j) = f^{-aj}(g_s f^{-\text{ant}}(z_j)) \in V$$

for each $i_s \in G$ such that $f^{-al}(i_z z_j) = f^{-aj}(g_s f^{-\text{ant}}(z_j))$. Hence,

$$i_z z_j \in f^{al}(V) \quad \text{if} \quad s \geq M \quad \text{(that is, } t \geq nM).$$

Thus, it follows from $z_j \in U$ that there exists $k_t \in G$ such that

$$k_t f^{al}(V) \cap U \neq \emptyset \quad \text{for each} \quad t \geq nM.$$ 

Therefore, $f^a|_{X_p}$ is topologically $G$-mixing. \hfill \square
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