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Universal covering Calabi-Yau manifolds of the Hilbert schemes of $n$ points of Enriques surfaces

TARO HAYASHI

INTRODUCTION

Throughout this paper, we work over $\mathbb{C}$, and $n$ is an integer such that $n \geq 2$. A $K3$ surface $K$ is a compact complex surface with $\omega_K \simeq \mathcal{O}_K$ and $H^1(K, \mathcal{O}_K) = 0$. An Enriques surface $E$ is a compact complex surface with $H^1(E, \mathcal{O}_E) = 0$, $H^2(E, \mathcal{O}_E) = 0$, and $\omega_E^\otimes 2 \simeq \mathcal{O}_E$. A Calabi-Yau manifold $X$ is an $n$-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic $k$-form on $X$ for $0 < k < n$, and there is a nowhere vanishing holomorphic $n$-form on $X$. By Oguiso and Schröer [10, Theorem 3.1], the Hilbert scheme of $n$ points of an Enriques surface $E^{[n]}$ has a Calabi-Yau manifold $X$ as the universal covering space of degree 2.

In this paper, we study the Hilbert scheme of $n$ points of an Enriques surface $E^{[n]}$ and its universal covering space $X$.

Definition 0.1. For $n \geq 1$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E^{[n]}$. A variety $Y$ is called an Enriques quotient of $X$ if there is an Enriques surface $E'$ and a free involution $\tau$ of $X$ such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle \tau \rangle$. Here we call two Enriques quotients of $X$ distinct if they are not isomorphic to each other.

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Recall that when \( n = 1 \), \( E^{[1]} \) is an Enriques surface \( E \) and \( X \) is a K3 surface.

In [11, Theorem 0.1], Ohashi showed the following theorem:

**Theorem 0.2.** For any nonnegative integer \( l \), there exists a K3 surface with exactly \( 2^{l+10} \) distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a K3 surface.

Our main theorem (Theorem 0.3) is the following which is totally different from Theorem 0.2:

**Theorem 0.3.** For \( n \geq 3 \), let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), and \( X \) the universal covering space of \( E^{[n]} \). Then the number of distinct Enriques quotients of \( X \) is one.

**Remark 0.4.** When \( n = 2 \), we do not count the number of distinct Enriques quotients of \( X \). We compute the Hodge numbers of the universal covering space \( X \) of \( E^{[2]} \) (Appendix A).

In addition, we investigate the relationship between the small deformation of \( E^{[n]} \) and that of \( X \) (Theorem 0.5) and study the natural automorphisms of \( E^{[n]} \) (Theorem 0.8).

Section 2 is a preliminary section. We prepare and recall some basic facts on the Hilbert scheme of \( n \) points of a surface.

In Section 3, we show the following theorem (Theorem 0.5).

**Theorem 0.5.** For \( n \geq 2 \), let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), and \( X \) the universal covering space of \( E^{[n]} \). Then every small deformation of \( X \) is induced by that of \( E^{[n]} \).
Remark 0.6. By Fantechi [4, Theorems 0.1 and 0.3], every small deformation of $E^{[n]}$ is induced by that of $E$. Thus for $n \geq 2$, every small deformation of $X$ is induced by that of $E$.

When $n = 1$, $E^{[1]}$ is an Enriques surface $E$, and $X$ is a $K3$ surface. An Enriques surface has a 10-dimensional deformation space and a $K3$ surface has a 20-dimensional deformation space. Thus the small deformation of $X$ is much bigger than that of $E$. Our Theorem 0.5 is different from the case of $n = 1$.

In Section 4, we show the following theorem (Theorem 0.8).

Definition 0.7. For $n \geq 2$ and $S$ a smooth compact surface, any automorphism $f \in \text{Aut}(S)$ induces an automorphism $f^{[n]} \in \text{Aut}(S^{[n]})$. An automorphism $g \in \text{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \text{Aut}(S)$ such that $g = f^{[n]}$.

When $S$ is a $K3$ surface, the natural automorphisms of $S^{[n]}$ were studied by Boissière and Sarti [3]. They showed that an automorphism of $S^{[n]}$ is natural if and only if it preserves the exceptional divisor of the Hilbert-Chow morphism $[3, \text{Theorem 1}]$. We obtain Theorem 0.8 which is similar to $[3, \text{Theorem 1}]$:

Theorem 0.8. For $n \geq 2$, let $E$ be an Enriques surface, $D$ the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[2]}$. Then

i) An automorphism $f$ of $E^{[n]}$ is natural if and only if $f(D) = D$.

ii) An automorphism $g$ of $X$ is a lift of a natural automorphism of $E^{[n]}$ if and only if $g(\pi^{-1}(D)) = \pi^{-1}(D)$.

In Section 5, we show main theorem (Theorem 0.3).
In addition, let $Y$ be a smooth compact Kähler surface. For a line bundle $L$ on $Y$, by using the natural map $\Pic(Y) \to \Pic(Y[n])$, $L \mapsto L_n$, we put

$$h^{p,q}(Y[n], L_n) := \dim \mathbb{C} \mathcal{H}^q(Y[n], \Omega^p_Y \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim \mathbb{C} \mathcal{H}^q(Y, \Omega^p_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y[n], L_n)x^py^qt^n, \text{ and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q+p+k-1}y^{q+k-1}t^k} \right)^{(-1)^{p+q}h^{p,q}(Y, L)}. $$

In [2, Conjecture 1], S. Boissière conjectured that

$$A = B.$$ 

In the proof of Theorem 0.5, we obtain the counterexample to this conjecture for $Y$ an Enriques surface and $L = \Omega^2_Y$. See Appendix B for details.

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1. Preliminaries

Let $S$ be a nonsingular projective surface, $S^{[n]}$ the Hilbert scheme of $n$ points of $S$, $q : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism, and $p : S^n \to S^{(n)}$ the natural projection. We denote the exceptional divisor of $q$ by $D$. By Fogarty [5, Theorem 2.4], $S^{[n]}$ is a smooth projective variety of $\dim \mathbb{C} S^{[n]} = 2n$. We put

$$\Delta^n := \{(x_i)_{i=1}^n \in S^n : |\{x_i\}_{i=1}^n| \leq n - 1 \},$$
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\[ S^n_* := \{(x_i)_{i=1}^n \in S^n : \left| \{x_i\}_{i=1}^n \right| \geq n-1 \}, \]

\[ \Delta^n_* := \Delta^n \cap S^n_*, \text{ and} \]

\[ S_i^{[n]} := q^{-1}(p(S^n_*)). \]

When $n = 2$, $\text{Blow}_{\Delta^2} S^2/\Sigma_2 \simeq S^2[2]$, for $n \geq 3$, we have $\text{Blow}_{\Delta^n} S^n_* / \Sigma_n \simeq S^n_*^{[n]}$, and $S^n_*^{[n]} \setminus S^n_*$ is an analytic closed subset and its codimension is 2 in $S^n_*^{[n]} ([1, \text{page 767-768}])$. Here $\Sigma_n$ is the symmetric group of degree $n$ which acts naturally on $S^n$ by permuting the factors.

Let $\mu : K \to E$ be the universal covering space of $E$ where $K$ is a $K3$ surface, and $\iota$ the covering involution of $\mu$. They induces the universal covering space $\mu^n : K^n \to E^n$. For $1 \leq k \leq n$, $1 \leq i_1 < \cdots < i_k \leq n$, we define automorphisms $\iota_{i_1 \cdots i_k}$ of $K^n$ in the following way: for $x = (x_i)_{i=1}^n \in K^n$,

\[ \text{the } j\text{-th component of } \iota_{i_1 \cdots i_k}(x) = \begin{cases} \iota(x_j) & j \in \{i_1, \ldots, i_k\} \\ x_j & j \notin \{i_1, \ldots, i_k\}. \end{cases} \]

Let $G$ be the subgroup of $\text{Aut}(K^n)$ generated by $\Sigma_n$ and $\{\iota_i\}_{1 \leq i \leq n}$ and $H$ the subgroup of $\text{Aut}(K^n)$ generated by $\Sigma_n$ and $\{\iota_{ij}\}_{1 \leq i < j \leq n}$. Since $K^n/G = E^{(n)}$, $H \triangleleft G$, $|G/H| = 2$, and the codimension of $\mu^{-1}(\Delta^n)$ is two, we get the universal covering spaces

\[ p_1 : K^n \setminus \mu^{-1}(\Delta^n) \to K^n \setminus \mu^{-1}(\Delta^n)/G, \text{ and} \]

\[ p_2 : K^n \setminus \mu^{-1}(\Delta^n) \to K^n \setminus \mu^{-1}(\Delta^n)/H, \]

where $p_1$ and $p_2$ are the natural projections. For $n \geq 3$, we put

\[ K^n_\circ := (\mu^n)^{-1}(E^n_*), \]

\[ \Gamma^n_{ij} := \{(x_1)_{i=1}^n \in K^n_\circ : \iota(x_i) = x_j\}, \]

\[ \Delta^n_{ij} := \{(x_1)_{i=1}^n \in K^n_\circ : x_i = x_j\}. \]
Then we get $\mu^{-1}(\Delta^n) = \Gamma_0 \cup \Delta_0$. By the definition of $K^n_0$, $H$ acts on $K^n_0$. For an element $\tilde{x} := (\tilde{x}_i)_{i=1}^n \in \Gamma_0 \cap \Delta_0$, some $i, j, k, l$ with $k \neq l$ such that $\sigma(\tilde{x}_i) = \tilde{x}_j$ and $\tilde{x}_k = \tilde{x}_l$. Since $\sigma$ does not have fixed points. Thus $\tilde{x}_i \neq \tilde{x}_l$. Therefore $\mu^n(\tilde{x}) \notin E^n_\ast$. This is a contradiction. We obtain $\Gamma_0 \cap \Delta_0 = \emptyset$.

**Lemma 1.1.** For $t \in H$ and $1 \leq i < j \leq n$, if $t \in H$ has a fixed point on $\Delta^n_{ij}$, then $t = (i, j)$ or $t = \text{id}_{K^n}$.

**Proof.** Let $t \in H$ be an element of $H$ where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in \Delta^n_{ij}$ such that $t(\tilde{x}) = \tilde{x}$. For $t \in H$, there are $\iota_{ab}$ where $1 \leq a < b \leq n$ and $(j_1, \ldots, j_l) \in \Sigma_n$ such that

$$t = (j_1, \ldots, j_l) \circ \iota_{ab}.$$  

From the definition of $\Delta^n_{ij}$, for $(x_i)_{i=1}^n \in \Delta^n_{ij}$,

$$\{x_1, \ldots, x_n\} \cap \{t(x_1), \ldots, t(x_n)\} = \emptyset.$$

Suppose $\iota_{ab} \neq \text{id}_{K^n}$. Since $t(\tilde{x}) = \tilde{x}$, we have

$$\{\tilde{x}_1, \ldots, \tilde{x}_n\} \cap \{t(\tilde{x}_1), \ldots, t(\tilde{x}_n)\} \neq \emptyset.$$

This is a contradiction. Thus we have $t = (j_1, \ldots, j_l)$. Similarly from the definition of $\Delta^n_{ij}$, for $(x_i)_{i=1}^n \in \Delta^n_{ij}$, if $x_s = x_t$ ($1 \leq s < t \leq n$), then $s = i$ and $t = j$. Thus we have $t = (i, j)$ or $t = \text{id}_{K^n}$. 

**Lemma 1.2.** For $t \in H$ and $1 \leq i < j \leq n$, if $t \in H$ has a fixed point on $\Gamma^n_{ij}$, then $t = \iota_{i,j} \circ (i, j)$ or $t = \text{id}_{K^n}$. 

## \(\Gamma_0 := \bigcup_{1 \leq i < j \leq n} T^n_{ij}\), and \(\Delta_0 := \bigcup_{1 \leq i < j \leq n} U^n_{ij}\).
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Proof. Let $t \in H$ be an element of $H$ where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in \Gamma_0^{ij}$ such that $t(\tilde{x}) = \tilde{x}$. For $t \in H$, there are $\iota_a$ where $1 \leq a \leq n$ and $(j_1, \ldots, j_l) \in S_n$ such that

$$t = (j_1 \ldots j_l) \circ \iota_a.$$ 

Since $(j, j+1) \circ \iota_{i,j} \circ (j, j+1) : U_{ij} \to T_{ij}$ is an isomorphism, and by Lemma 1.1, we have

$$(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = (i, j) \text{ or } \text{id}_{K^n}.$$ 

If $(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = \text{id}_{K^n}$, then $t = \text{id}_{K^n}$. If

$$(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ (i, j) \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = (i, j),$$

then

$$t = (j, j+1) \circ \iota_{i,j} \circ (i, j+1) \circ \iota_{i,j} \circ (j, j+1) = (j, j+1) \circ \iota_{i,j+1} \circ (i, j+1) \circ (j, j+1) = \iota_{i,j} \circ (i, j).$$

Thus we have $t = \iota_{i,j} \circ (i, j)$. □

For the natural projection we get a unramified covering space: $K^n/H \to K^n/G = E^{[n]} = E^n/\Sigma_n$. From Lemma 1.1 and Lemma 1.2, we get a local isomorphism:

$$\theta : \text{Blow}_{\mu^n \setminus \Delta_0^n} K^n/H \to E^{[n]}_n.$$ 

Lemma 1.3. For every $x \in E^{[n]}_n$, $|\theta^{-1}(x)| = 2$.

Proof. For $(x_i)_{i=1}^n \in \Delta_0^n \subset E^n$ with $x_1 = x_2$, there are $n$ elements $y_1, \ldots, y_n$ of $K$ such that $y_1 = y_2$ and $\mu(y_i) = x_i$ for $1 \leq i \leq n$. Then

$$(\mu^n)^{-1}((x_i)_{i=1}^n) = \{y_1, \iota(y_1)\} \times \cdots \times \{y_n, \iota(y_n)\}. $$
Since \( H \) is generated by \( \Sigma_n \) and \( \{ \iota_{ij} \}_{1 \leq i < j \leq n} \), for \( (z_i)_{i=1}^n \in (\mu^n)^{-1}((x_i)_{i=1}^n) \) if the number of \( i \) with \( z_i = y_i \) is even, then

\[
(z_i)_{i=1}^n = \{ \iota(y_1), \iota(y_2), y_3, \ldots, y_n \} \text{ on } K_0^n/H, 
\]

and if the number of \( i \) with \( z_i = y_i \) is odd, then

\[
(z_i)_{i=1}^n = \{ \iota(y_1), y_2, y_3, \ldots, y_n \} \text{ on } K_0^n/H. 
\]

Furthermore since \( \iota_i \not\in H \) for \( 1 \leq i \leq n \),

\[
\{ \iota(y_1), \iota(y_2), y_3, \ldots, y_n \} \neq \{ \iota(y_1), y_2, y_3, \ldots, y_n \}, \text{ on } K_0^n/H. 
\]

Thus for every \( x \in E_*^{[n]} \), we get \( |\theta^{-1}(x)| = 2 \). \hfill \Box

**Proposition 1.4.** \( \theta : \text{Blow}_{\mu^n-1}(\Delta^n)K_0^n/H \to \text{Blow}_{\Delta^n}E_*^{[n]}/\Sigma_n \) is the universal covering space, i.e. \( \pi^{-1}(E_*^{[n]}) \cong \text{Blow}_{\mu^n-1}(\Delta^n)K_0^n/H \). When \( n = 2 \), we have \( X \cong \text{Blow}_{\mu^2-1}(\Delta^2)K^2/H \).

**Proof.** Since \( \theta \) is a local isomorphism, from Lemma 1.3 we get that \( \theta \) is a covering map. Furthermore \( \pi : \pi^{-1}(E_*^{[n]}) \to E_*^{[n]} \) is the universal covering space of degree 2, \( \theta : \text{Blow}_{\mu^n-1}(\Delta^n)K_0^n/H \to \text{Blow}_{\Delta^n}E_*^{[n]}/\Sigma_n \) is the universal covering space. By the uniqueness of the universal covering space, we have \( \pi^{-1}(E_*^{[n]}) \cong \text{Blow}_{\mu^n-1}(\Delta^n)K_0^n/H \). When \( n = 2 \), since \( E_*^2 = E^2 \), \( K_0^2 = K^2 \) and \( \text{Blow}_{\Delta^2}E^2/\Sigma_2 \cong E^{[2]} \), we have \( X \cong \text{Blow}_{\mu^2-1}(\Delta^2)K^2/H \). \hfill \Box

**Theorem 1.5.** For \( n \geq 2 \), let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), and \( \pi : X \to E^{[n]} \) the universal covering space of \( E^{[n]} \). Then there is a birational morphism \( \varphi : X \to K^n/H \) such that \( \varphi^{-1}(\mu^{n-1}(\Delta^n)/H) = \pi^{-1}(D) \).
**Proof.** When \( n = 2 \), this is proved by Proposition 1.4. From here we assume that \( n \geq 3 \). From Proposition 1.4, we have \( \pi^{-1}(E^*[n]) \simeq \text{Blow}_{\mu^{-1}(\Delta^n)}K^n/H \). Since the codimension of \( X \backslash \pi^{-1}(E^*[n]) \) is 2, there is a meromorphic \( f \) of \( X \) to \( K^n/H \) which satisfies the following commutative diagram:

\[
\begin{array}{ccc}
E^*[n] & \overset{q}{\longrightarrow} & E^{(n)} \\
\downarrow{\pi} & & \downarrow{p} \\
\pi^{-1}(E^*[n]) & \overset{f}{\longrightarrow} & K^n/H
\end{array}
\]

where \( q : E^*[n] \rightarrow E^{(n)} \) is the Hilbert-Chow morphism, and \( p : K^n/H \rightarrow E^{(n)} \) is the natural projection. For an ample line bundle \( \mathcal{L} \) on \( E^{(n)} \), since the natural projection \( p : K^n/H \rightarrow E^{(n)} \) is finite, \( p^*\mathcal{L} \) is ample. From the above diagram, we have \( \pi^*(q^*\mathcal{L}) \mid_{\pi^{-1}(E^*[n])} = f^*(p^*\mathcal{L}) \). Since \( X \backslash \pi^{-1}(E^*[n]) \) is an analytic closed subset of codimension 2 in \( X \) and \( p^*H^1(E^{[n]},\mathcal{L}) \) is ample, there is a holomorphism \( \varphi \) from \( X \) to \( K^n/H \) such that \( \varphi \mid_{X \backslash \pi^{-1}(D)} = f \mid_{X \backslash \pi^{-1}(D)} \). Since \( f : X \backslash \pi^{-1}(D) \cong (K^n \backslash \mu^{-1}(\Delta^n))/H, \) this is a birational morphism. \( \square \)

2. **Proof of Theorem 0.5**

Let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), and \( \pi : X \rightarrow E^{[n]} \) the universal covering space of \( E^{[n]} \). In this section, we show Theorem 0.5 (Theorem 2.2).

**Proposition 2.1.** For \( n \geq 2 \), we have \( \dim \mathcal{C}^1(E^{[n]},\Omega^{2n-1}_{E^{[n]}}) = 0 \).

**Proof.** For a smooth projective manifold \( S \), we put

\[
h^{p,q}(S) := \dim \mathcal{C}^q(S,\Omega^p_S)
\]
\[ h(S, x, y) := \sum_{p, q} h^{p,q}(S)x^p y^q. \]

By [7, Theorem 2] and [6, page 204], we have the equation (1):

\[
\sum_{n=0}^{\infty} \sum_{p, q} h^{p,q}(E[n])x^p y^q t^n = \prod_{k=1}^{2} \prod_{p, q=0}^{2} \left( \frac{1}{1 - \left( -1 \right)^{p+q} x^{p+k-1} y^{q+k-1} t^k} \right)^{(-1)^{p+q} h^{p,q}(E)}. 
\]

Since an Enriques surface \( E \) has Hodge numbers \( h^{0,0}(E) = h^{2,2}(E) = 1, \) \( h^{1,0}(E) = h^{0,1}(E) = 0, \) \( h^{2,0}(E) = h^{0,2}(E) = 0, \) and \( h^{1,1}(E) = 10, \) the equation (1) is

\[
\sum_{n=0}^{\infty} \sum_{p, q} h^{p,q}(E[n])x^p y^q t^n = \prod_{k=1}^{\infty} \left( \frac{1}{1 - x^{k-1} y^{k-1} t^k} \right) \left( \frac{1}{1 - x^{k} y^{k} t^k} \right)^{10} \left( \frac{1}{1 - x^{k+1} y^{k+1} t^k} \right).
\]

It follows that

\[ h^{p,q}(E[n]) = 0 \text{ for all } p, q \text{ with } p \neq q. \]

Thus we have \( \dim_{\mathbb{C}} H^1(E[n], \Omega_{E[n]}^{2n-1}) = 0 \) for \( n \geq 2. \)

**Theorem 2.2.** For \( n \geq 2, \) let \( E \) be an Enriques surface, \( E[n] \) the Hilbert scheme of \( n \) points of \( E, \) and \( X \) the universal covering space of \( E[n]. \) Then every small deformation of \( X \) is induced by that of \( E[n]. \)

**Proof.** In [4, Proposition 4.2 and Theorems 0.3], Fantechi showed that for a smooth projective surface with \( H^0(S, T_S) = 0 \) or \( H^1(S, \mathcal{O}_S) = 0, \) and \( H^1(S, \mathcal{O}_S(-K_S)) = 0 \) where \( K_S \) is the canonical divisor of \( S, \)

\[ \dim_{\mathbb{C}} H^1(S, T_S) = \dim_{\mathbb{C}} H^1(S[n], T_{S[n]}) \]

Since an Enriques surface \( E \) satisfies \( H^0(E, T_E) = 0 \) or \( H^1(E, \mathcal{O}_E) = 0, \) and \( H^1(E, \mathcal{O}_E(-K_E)) = 0, \) we have \( \dim_{\mathbb{C}} H^1(E[n], T_{E[n]}) = 10. \) Since \( K_{E[n]} \) is not trivial and \( 2K_{E[n]} \) is trivial, we have

\[ T_{E[n]} \cong \Omega_{E[n]}^{2n-1} \otimes K_{E[n]}. \]
Therefore we have $\dim \mathbb{C} H^1(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]}) = 10$. Since $K_X$ is trivial, then we have $T_X \simeq \Omega_X^{2n-1}$. Since $\pi : X \to E[n]$ is the covering map, we have

$$H^k(X, \Omega_X^{2n-1}) \simeq H^k(E[n], \pi_* \Omega_X^{2n-1}).$$

Since $X \simeq Spec \mathcal{O}_{E[n]} \oplus \mathcal{O}_{E[n]}(K_{E[n]})$ ([10, Theorem 3.1]), we have

$$H^k(E[n], \pi_* \Omega_X^{2n-1}) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus (\Omega^{2n-1}_{E[n]} \otimes K_{E[n]})).$$

Thus

$$H^k(X, \Omega_X^{2n-1}) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus (\Omega^{2n-1}_{E[n]} \otimes K_{E[n]})) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus H^k(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]})).$$

Combining this with Proposition 2.1, we obtain

$$\dim \mathbb{C} H^1(X, \Omega_X^{2n-1}) = \dim \mathbb{C} H^1(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]})$$

$$= 10.$$

Let $p : \mathcal{Y} \to U$ be the Kuranishi family of $E[n]$. Since each canonical bundle of $E[n]$ and $E$ is torsion, they have unobstructed deformations ([12]). Thus $U$ is smooth.

Let $f : \mathcal{X} \to \mathcal{Y}$ be the universal covering space. Then $q : \mathcal{X} \to U$ is a flat family of $X$ where $q := p \circ f$. By [4, Theorems 0.1 and 0.3], all small deformation of $E[n]$ is induced by that of $E$. Thus for $u \in U$, $q^{-1}(u)$ is the universal covering space of the Hilbert scheme of $n$ points of an Enriques surface. Then we have a commutative diagram:

$$
\begin{array}{ccc}
T_{U,0} & \overset{\rho_q}{\longrightarrow} & H^1(\mathcal{Y}_0, T_{\mathcal{Y}_0}) \\
\Bigg\downarrow & \downarrow \tau & \downarrow \pi^* \\
H^1(\mathcal{X}_0, T_{\mathcal{X}_0}) & \overset{\rho_p}{\longrightarrow} & H^1(X, T_X).
\end{array}
$$
Since $H^1(E^{[n]}, T_{E^{[n]}}) \simeq H^1(X, T_X)$ by $\pi^*$, the vertical arrow $\tau$ is an isomorphism and
\[
\dim \mathbb{C}H^1(X_u, T_{X_u}) = \dim \mathbb{C}H^1(X_u, \Omega^{2n-1}_{X_u})
\]
is a constant for some neighborhood of $0 \in U$, it follows that $q : X \to U$ is the complete family of $X_0 = X$, therefore $q : X \to U$ is the versal family of $X_0 = X$.

Thus every small deformation of $X$ is induced by that of $E^{[n]}$.

\[\square\]

3. Proof of Theorem 0.8

For $n \geq 2$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and $D$ the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$. Recall that $\iota$ is the covering involution of $\mu : K \to E$, $p_1 : K^n \setminus \mu^{n-1}(\Delta^n) \to E^{[n]} \setminus D = E^n \setminus \Delta^n / \Sigma_n = K^n \setminus \mu^{n-1}(\Delta^n) / G$ and $p_2 : K^n \setminus \mu^{n-1}(\Delta^n) \to X \setminus \pi^{-1}(D) = K^n \setminus \mu^{n-1}(\Delta^n) / H$ are the universal covering spaces where $p_1$ and $p_2$ are the natural projections. In this section, we show Theorem 0.8 (Theorem 3.2).

Lemma 3.1. i) Let $f$ be an automorphism of $E^{[n]} \setminus D$, and $g_1, \ldots, g_n$ automorphisms of $K$ such that $p_1 \circ (g_1 \times \cdots \times g_n) = f \circ p_1$, where $(g_1 \times \cdots \times g_n)$ is the automorphism of $K^n$. Then we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for each $1 \leq i \leq n$. Moreover $g_1 \circ \iota = \iota \circ g_1$.

ii) Let $f$ be an automorphism of $X \setminus \pi^{-1}(D)$, and $g_1, \ldots, g_n$ automorphisms of $K$ such that $p_2 \circ (g_1 \times \cdots \times g_n) = f \circ p_2$, where $(g_1 \times \cdots \times g_n)$ is the automorphism of $K^n$. Then we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for each $1 \leq i \leq n$. Moreover $g_1 \circ \iota = \iota \circ g_1$.

Proof. We show i) by contradiction. Without loss of generality, we may assume that $g_2 \neq g_1$ and $g_2 \neq g_1 \circ \iota$. Let $h_1$ and $h_2$ be two morphisms of $K$ where $g_i \circ h_i = \id_K$
and $h_i \circ g_i = \text{id}_K$ for $i = 1, 2$. We define two morphisms $A_{1,2}$ and $A_{1,2,\iota}$ from $K$ to $K^2$ by

\[
A_{1,2} : K \ni x \mapsto (h_1(x), h_2(x)) \in K^2
\]

\[
A_{1,2,\iota} : K \ni x \mapsto (h_1(x), \iota \circ h_2(x)) \in K^2.
\]

Let $\Gamma_\iota := \{(x, y) : y = \iota(x)\}$ be the subset of $K^2$. Since $h_1 \neq h_2$ and $h_1 \neq \iota \circ h_2$, $A_{1,2,\iota}^{-1}(\Delta^2) \cup A_{1,2}^{-1}(\Gamma_\iota)$ do not coincide with $K$. Thus there is $x' \in K$ such that $A_{1,2}(x') \notin \Delta^2$ and $A_{1,2,\iota}(x') \notin \Gamma_\iota$. For $x' \in K$, we put $x_i := h_i(x') \in K$ for $i = 1, 2$. Then there are some elements $x_3, \ldots, x_n \in K$ such that $(x_1, \ldots, x_n) \in K^n \setminus \mu^{n-1}(\Delta^n)$. We have $g((x_1, \ldots, x_n)) \notin K^n \setminus \mu^{n-1}(\Delta^n)$ by the assumption of $x_1$ and $x_2$. It is contradiction, because $g$ is an automorphism of $K^n \setminus \mu^{n-1}(\Delta^n)$. Thus we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for $1 \leq i \leq n$.

Let $g := g_1 \times \cdots \times g_n$. Since the covering transformation group of $p$ is $G$, the liftings of $f$ are given by $\{g \circ u : u \in G\} = \{u \circ g : u \in G\}$. Thus for $\iota_1 \circ g$, there is an element $\iota_a \circ s$ of $G$ where $s \in \Gamma_n$ and $1 \leq a \leq n$ such that $\iota_1 \circ g = g \circ \iota_a \circ s$. If we think about the first component of $\iota_1 \circ g$, we have $s = \text{id}$ and $a = 1$. Therefore $g \circ \iota \circ g^{-1} = \iota$, we have $\iota \circ g_1 = g_1 \circ \iota$. In the same way, we have ii).

\[\square\]

**Theorem 3.2.** For $n \geq 2$, let $E$ be an Enriques surface, $D$ the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. Then

i) An automorphism $f$ of $E^{[n]}$ is natural if and only if $f(D) = D$.

ii) An automorphism $g$ of $X$ is a lift of a natural automorphism of $E^{[n]}$ if and only if $g(\pi^{-1}(D)) = \pi^{-1}(D)$.
Proof. We show (1). Let $\mu : K \to E$ be the universal covering space of $E$. By Theorem 1.5, there is a commutative diagram

$$
\begin{array}{ccc}
E^{[n]} & \xrightarrow{q} & E^{(n)} \\
\pi \downarrow & & \downarrow p \\
X & \xrightarrow{\varphi} & K^n/H,
\end{array}
$$

where $p$ is the natural projection and $\varphi$ is a birational morphism. Since $E^{[n]} \setminus D \cong E^n \setminus \Delta^n / \Sigma_n$, we have the universal covering spaces

$$p_1 : K^n \setminus \mu^{n-1}(\Delta^n) \to E^n \setminus \Delta^n / \Sigma_n,$$

$$p_2 : K^n \setminus \mu^{n-1}(\Delta^n) \to K^n \setminus \mu^{n-1}(\Delta^n) / H,$$

and the following commutative diagram:

$$
\begin{array}{ccc}
K^n \setminus \mu^{n-1}(\Delta^n) / H & \xrightarrow{p_3} & E^n \setminus \Delta^n / \Sigma_n \\
\downarrow p_2 & & \downarrow p_1 \\
K^n \setminus \mu^{n-1}(\Delta^n), & & 
\end{array}
$$

where $p_1$, $p_2$, and $p_3$ are the natural projections. For $f \in \text{Aut}(E^{[n]})$ with $f(D) = D$, from the uniqueness of the universal covering space, $f$ induces an automorphisms $\tilde{f}$ of $K^n \setminus \mu^{n-1}(\Delta^n)$. Since $K$ is projective and codim $\mu^{-1}(\Delta^n)$ is over 2, $\tilde{f}$ is a birational map of $K^n$. By [9], $\tilde{f}$ is an automorphism of $K^n$ and there are $g_1, \ldots, g_n$ automorphisms of $K$ such that $\tilde{f} = (g_1 \times \cdots \times g_n) \circ s$ where $s \in \Sigma_n$. Since $\Sigma \subset G$, we get $f \circ p_1 = p_1 \circ (g_1 \times \cdots \times g_n)$. From Lemma 3.1, we get i). By Theorem 1.5 and the above diagram, in the same way, we get ii). \qed

4. Proof of Theorem 0.3

Let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. 

In Proposition 4.2, we shall show that for \( n \geq 3 \), the covering involution of \( \pi : X \to E^{[n]} \) acts on \( H^2(X, \mathbb{C}) \) as the identity. In Proposition 4.5, by using Theorem 3.2 and checking the action to \( H^1(X, \Omega^n_X) \cong H^{2n-1}(X) \), we classify involutions of \( X \) which act on \( H^2(X, \mathbb{C}) \) as the identity. We prove Theorem 0.3 (Theorem 4.7) using those results.

**Lemma 4.1.** Let \( X \) be a smooth complex manifold, \( Z \subset X \) a closed submanifold whose codimension is 2, \( \tau : X_Z \to X \) the blow up of \( X \) along \( Z \), \( E = \tau^{-1}(Z) \) the exceptional divisor, and \( h \) the first Chern class of the line bundle \( \mathcal{O}_{X_Z}(E) \).

Then \( \tau^*: H^2(X, \mathbb{C}) \to H^2(X_Z, \mathbb{C}) \) is injective, and

\[
H^2(X_Z, \mathbb{C}) \cong H^2(X, \mathbb{C}) \oplus \mathbb{C}h.
\]

**Proof.** Let \( U := X \setminus Z \) be an open set of \( X \). Then \( U \) is isomorphic to an open set \( U' = X_Z \setminus E \) of \( X_Z \). As \( \tau \) gives a morphism between the pair \((X_Z, U')\) and the pair \((X, U)\), we have a morphism \( \tau^* \) between the long exact sequence of cohomology relative to these pairs:

\[
\begin{array}{cccc}
H^k(X, \mathbb{C}) & \to & H^k(U, \mathbb{C}) & \to & H^{k+1}(X, \mathbb{C}) \\
\tau^* & \downarrow & \tau^* & \downarrow & \tau^* \\
H^k(X_Z, \mathbb{C}) & \to & H^k(X_Z, \mathbb{C}) & \to & H^{k+1}(X_Z, \mathbb{C}).
\end{array}
\]

By Thom isomorphism, the tubular neighborhood Theorem, and Excision theorem, we have

\[
H^q(Z, \mathbb{C}) \cong H^{q+4}(X, U, \mathbb{C}), \quad \text{and}
\]

\[
H^q(E, \mathbb{C}) \cong H^{q+2}(X_Z, U', \mathbb{C}).
\]

In particular, we have

\[
H^l(X, U, \mathbb{C}) = 0 \quad \text{for} \quad l = 0, 1, 2, 3, \quad \text{and}
\]
Thus we have

\[
\begin{array}{c}
0 \to H^1(X, \mathbb{C}) \to H^1(U, \mathbb{C}) \to 0 \\
\tau_{X,U} \downarrow \quad \tau_{X} \downarrow \quad \tau_{U} \downarrow \quad \tau_{X,U} \\
0 \to H^1(X_Z, \mathbb{C}) \to H^1(U', \mathbb{C}) \to H^0(E, \mathbb{C}),
\end{array}
\]

and

\[
\begin{array}{c}
0 \to H^2(X, \mathbb{C}) \to H^2(U, \mathbb{C}) \to 0 \\
\tau_{X,U} \downarrow \quad \tau_{X} \downarrow \quad \tau_{U} \downarrow \quad \tau_{X,U} \\
H^0(E, \mathbb{C}) \to H^2(X_Z, \mathbb{C}) \to H^2(U', \mathbb{C}) \to H^3(X_Z, U', \mathbb{C}).
\end{array}
\]

Since \( \tau_{U'} : U' \to U \), we have isomorphisms \( \tau_{U}^* : H^k(U, \mathbb{C}) \simeq H^k(U', \mathbb{C}) \). Thus we have

\[\dim_{\mathbb{C}} H^2(X_Z, \mathbb{C}) = \dim_{\mathbb{C}} H^2(X, \mathbb{C}) + 1, \text{ and} \]

\[\tau^* : H^2(X, \mathbb{C}) \to H^2(X_Z, \mathbb{C}) \text{ is injective,} \]

and therefore we obtain

\[H^2(X_Z, \mathbb{C}) \simeq H^2(X, \mathbb{C}) \oplus \mathbb{C}h.\]

\[\square\]

**Proposition 4.2.** Suppose \( n \geq 3 \). For the covering involution \( \rho \) of the universal covering space \( \pi : X \to E^{[n]} \), the induced map \( \rho^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C}) \) is the identity.

**Proof.** Since the codimension of \( X \setminus \pi^{-1}(E^{[n]}) \) is 2, we get

\[H^2(X, \mathbb{C}) \cong H^2(X \setminus \pi^{-1}(F), \mathbb{C}).\]
By Proposition 2.6, $X \setminus \pi^{-1}(E_n) \simeq \text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma / H$.

Let $\tau : \text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma \to K^n_\sigma$ be the blow up of $K^n_\sigma$ along $\mu^{-1}(\Delta^n)$,

$$h_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma}(\tau^{-1}(\Delta^n_{ij})).$$

and

$$k_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma}(\tau^{-1}(\Gamma^n_{ij})).$$

By Lemma 4.1, we have

$$H^2(\text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma, \mathbb{C}) \cong H^2(K^n, \mathbb{C}) \oplus \left( \bigoplus_{1 \leq i < j \leq n} \mathbb{C} h_{ij} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq n} \mathbb{C} k_{ij} \right).$$

Since $n \geq 3$, there is an isomorphism

$$(j, j + 1) \circ \sigma_{ij} \circ (j, j + 1) : \Delta^n_{ij} \xrightarrow{\sim} \Gamma^n_{ij}.$$

Thus we have $\dim_{\mathbb{C}} H^2(\text{Blow}_{\mu^{-1}(\Delta^n)} K^n_\sigma / H, \mathbb{C}) = 11$, i.e. $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 11$. Since $H^2(E[n], \mathbb{C}) = H^2(X, \mathbb{C})^{\rho^*}$, $\rho^*$ is the identity. \qed

Since $K^n / H$ is normal, $\pi^{-1}(E)$ is the exceptional divisor (Theorem 2.5) and $X$ is a Calabi-Yau, we have that for an automorphism $f$ of $X$, $f(\pi^{-1}(D)) = \pi^{-1}(D)$ if and only if $f^*\mathcal{O}_X(\pi^{-1}(D)) = \mathcal{O}_X(\pi^{-1}(D))$ in $\text{Pic}(X)$.

**Definition 4.3.** Let $S$ be a smooth surface. An automorphism $\varphi$ of $S$ is numerically trivial if the induced automorphism $\varphi^*$ of the cohomology ring over $\mathbb{Q}$, $H^*(S, \mathbb{Q})$ is the identity.

We suppose that an Enriques surface $E$ has numerically trivial involutions. By [8, Proposition 1.1], there is just one numerically trivial involution of $E$, denoted $\nu$. For $\nu$, there are just two involutions of $K$ which are liftings of $\nu$, one acts on
$H^0(K, \Omega^2_K)$ as the identity, and another acts on $H^0(K, \Omega^2_K)$ as $-\text{id}_{H^0(K, \Omega^2_K)}$, we denote by $\nu_+$ and $\nu_-$, respectively. Then they satisfies $\nu_+ = \nu_- \circ \sigma$.

Let $\nu^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by $\nu$. For $\nu^{[n]}$, there are just two automorphisms of $X$ which are liftings of $\nu^{[n]}$, denoted $\varsigma$ and $\varsigma'$, respectively:

\[
\begin{array}{ccc}
E^{[n]} & \xrightarrow{\nu^{[n]}} & E^{[n]} \\
\pi & & \pi \\
X & \xrightarrow{\varsigma (\varsigma')} & X.
\end{array}
\]

Then they satisfies $\varsigma = \varsigma' \circ \rho$ where $\rho$ is the covering involution of $\pi : X \longrightarrow E^{[n]}$ and the each order of $\varsigma$ and $\varsigma'$ is 2. From here, we classify involutions acting on $H^2(X, \mathbb{C})$ as the identity by checking the action to $H^{2n-1,1}(X, \mathbb{C})$.

**Lemma 4.4.** $\dim_{\mathbb{C}} H^{2n-1,1}(K^n/H, \mathbb{C}) = 10$.

**Proof.** Let $\iota$ be the covering involution of $\mu : K \rightarrow E$. Put

\[
H^p,q_{\pm}(K, \mathbb{C}) := \{ \alpha \in H^p,q(K, \mathbb{C}) : \iota^*(\alpha) = \pm \alpha \} \text{ and }
\]

\[
h^p,q_{\pm}(K) := \dim_{\mathbb{C}} H^p,q_{\pm}(K, \mathbb{C}).
\]

Since $K$ is a $K3$ surface, we have

\[
h^{0,0}(K) = 1, \quad h^{1,0}(K) = 0, \quad h^{2,0}(K) = 1, \quad h^{1,1}(K) = 20,
\]

\[
h^0_+(K) = 1, \quad h^1_+(K) = 0, \quad h^2_+(K) = 0, \quad h^{1,1}_+(K) = 10,
\]

\[
h^0_-(K) = 0, \quad h^1_-(K) = 0, \quad h^2_-(K) = 1, \quad h^{2,0}_-(K) = 10.
\]

Let

\[
\Lambda := \{ (s_1, \ldots, s_n, t_1, \ldots, t_n) \in \mathbb{Z}_{\geq 0}^n : \Sigma_{i=1}^n s_i = 2n - 1, \Sigma_{j=1}^n t_j = 1 \}.
\]
From the Künneth Theorem, we have

\[ H^{2n-1,1}(K^n, \mathbb{C}) \simeq \bigoplus_{(s_1, \ldots, s_n, t_1, \ldots, t_n) \in A} \bigotimes_{i=1}^n H^{s_i, t_i}(K, \mathbb{C}). \]

We take a base \( \alpha \) of \( H^{2,0}(K, \mathbb{C}) \) and a base \( \{ \beta_i \}_{i=1}^{20} \) of \( H^{1,1}(K, \mathbb{C}) \) such that \( \{ \beta_i \}_{i=1}^{10} \) is a base of \( H^{1,1}_-(K, \mathbb{C}) \) and \( \{ \beta_i \}_{i=11}^{20} \) is a base of \( H^{1,1}_+(K, \mathbb{C}) \). Let

\[ \tilde{\beta}_i := \bigotimes_{j=1}^n \epsilon_j \]

where \( \epsilon_j = \alpha \) for \( j \neq i \) and \( \epsilon_j = \beta_i \) for \( j = i \), and

\[ \gamma_i := \bigoplus_{j=1}^n \tilde{\beta}_j. \]

Then \( \{ \gamma_i \}_{i=1}^{20} \) is a base of \( H^{2n-1,1}(K^n, \mathbb{C})^{S_n} \). Since \( \iota^* \alpha = -\alpha \), \( \iota^* \beta_i = -\beta_i \) for \( 1 \leq i \leq 10 \), and \( \iota^* \beta_i = \beta_i \) for \( 11 \leq i \leq 20 \), we obtain

\[ \iota_{ij}^* \gamma_i = \gamma_i \text{ for } 1 \leq i \leq 10, \text{ and} \]

\[ \iota_{ij}^* \gamma_i = -\gamma_i \text{ for } 11 \leq i \leq 20. \]

Since \( H^{2n-1,1}(K^n/H, \mathbb{C}) \simeq H^{2n-1,1}(K^n, \mathbb{C})^H \) and \( H = \langle S_n, \{ \sigma_{ij} \}_{1 \leq i < j \leq n} \rangle \), we obtain

\[ H^{2n-1,1}(K^n/H, \mathbb{C}) = \bigoplus_{i=1}^{10} \mathbb{C} \gamma_i. \]

Thus we get \( \dim_{\mathbb{C}} H^{2n-1,1}(K^n/H, \mathbb{C}) = 10. \) \( \square \)

Recall that \( p : K^n \setminus \mu^{n-1}(\Delta^n) \to E^{[n]} \setminus D = E^n \setminus \Delta^n / \Sigma_n \) is the universal covering space.

**Proposition 4.5.** We suppose that \( E \) has a numerically trivial involution, denoted \( \nu \). Let \( \nu^{[n]} \) be the natural automorphism of \( E^{[n]} \) which is induced by \( \nu \). Since the
degree of $\pi : X \to E^{[n]}$ is 2, there are just two involutions $\zeta$ and $\zeta'$ of $X$ which are lifts of $\nu^{[n]}$. Then $\zeta$ and $\zeta'$ do not act on $H^{2n-1,1}(X, \mathbb{C})$ as $-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}$.

Proof. Since $\nu^{[n]}(D) = D$, $\nu^{[n]}|_{E^{[n]} \setminus D}$ is an automorphism of $E^{[n]} \setminus D$. By the uniqueness of the universal covering space, there is an automorphism $g$ of $K^n \setminus \mu^{n-1}(\Delta^n)$ such that $\nu^{[n]} \circ p = p \circ g$:

$$
\begin{array}{ccc}
E^{[n]} \setminus D & \xrightarrow{\nu^{[n]}} & E^{[n]} \setminus D \\
p & & p \\
K^n \setminus \mu^{n-1}(\Delta^n) & \xrightarrow{g} & K^n \setminus \mu^{n-1}(\Delta^n).
\end{array}
$$

By Proposition 3.1, there are some automorphisms $g_i$ of $K$ such that $g = g_1 \times \cdots \times g_n$ for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \iota$, and $g_1 \circ \iota = \iota \circ g_1$. By Theorem 1.5, we get $K^n \setminus \mu^{n-1}(\Delta^n)/H \simeq X \setminus \pi^{-1}(D)$. Put

$$
\nu_{+,\text{even}} := u_1 \times \cdots \times u_n
$$

where

$$
u_i = \nu_+ \text{ or } \nu_i = \nu_- \text{ and the number of } i \text{ with } \nu_i = \nu_+ \text{ is even.}
$$

$\nu_{+,\text{even}}$ is an automorphism of $K^n$ and induces an automorphism $\nu_{+,\text{even}}$ of $K^n \setminus \mu^{n-1}(\Delta^n)/H$. We define automorphisms $\nu_{+,\text{odd}}, \nu_{-,\text{even}},$ and $\nu_{-,\text{odd}}$ of $K^n \setminus \mu^{n-1}(\Delta^n)/H$ in the same way. Since $\sigma_{ij} \in H$ for $1 \leq i < j \leq n$, and $\nu_+ = \nu_- \circ \iota$, if $n$ is odd,

$$
\nu_{+,\text{odd}} = \nu_{-,\text{even}}, \quad \nu_{+,\text{even}} = \nu_{-,\text{odd}}, \quad \text{and} \quad \nu_{+,\text{odd}} \neq \nu_{+,\text{even}},
$$

and if $n$ is even,

$$
\nu_{+,\text{odd}} = \nu_{-,\text{odd}}, \quad \nu_{+,\text{even}} = \nu_{-,\text{even}}, \quad \text{and} \quad \nu_{+,\text{odd}} \neq \nu_{+,\text{even}}.$$


Universal covering Calabi-Yau manifolds of $E^{[n]}$

Since $v^{(n)} \circ \pi_E = \pi_E \circ v^{[n]}$ and $K^n \setminus \mu^{n-1}(\Delta^n)/H \simeq X \setminus \pi^{-1}(D)$, we have $v^{[n]} \circ \pi = \pi \circ v^{+, odd}$ and $v^{[n]} \circ \pi = \pi \circ v^{+, even}$ where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $v^{(n)}$ is the automorphism of $E^{(n)}$ induced by $v$. Since the degree of $\pi$ is 2, we have \{ς, ς’\} = \{v^{+, odd}, v^{+, even}\}. By [8, page 386-389], there is an element $\alpha_{\pm} \in H^{1,1}(K, \mathbb{C})$ such that $v^{+}_*(\alpha_{\pm}) = \pm \alpha_{\pm}$. We fix a basis $\alpha$ of $H^{2,0}(K, \mathbb{C})$, and let

$$\widetilde{\alpha}_{\pm, i} := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \alpha_{\pm}$ for $j = i$, and

$$\widetilde{\alpha}_{\pm} := \bigoplus_{j=1}^{n} \widetilde{\alpha}_{\pm, j}.$$  

Since there is a birational map $\varphi : K^n \to X$ by Theorem 1.5, and by the definition of $v^{+, odd}$ and $v^{+, even}$, we have

$$v^{+, odd}^*(\varphi^*(\widetilde{\alpha}_+)) = \varphi^*(\widetilde{\alpha}_+) \quad \text{and} \quad v^{+, even}^*(\varphi^*(\widetilde{\alpha}_-)) = \varphi^*(\widetilde{\alpha}_-).$$

Thus $\varsigma$ and $\varsigma'$ do not act on $H^{2n-1,1}(X, \mathbb{C})$ as $-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}$. \hfill \Box

**Definition 4.6.** For $n \geq 1$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E^{[n]}$. A variety $Y$ is called an Enriques quotient of $X$ if there is an Enriques surface $E'$ and a free involution $\tau$ of $X$ such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle \tau \rangle$. Here we call two Enriques quotients of $X$ distinct if they are not isomorphic to each other.

**Theorem 4.7.** For $n \geq 3$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E^{[n]}$. Then the number of distinct Enriques quotients of $X$ is one.
Proof. Let \( \rho \) be the covering involution of \( \pi : X \to E^{[n]} \) for \( n \geq 3 \). Since for \( n \geq 3 \) \( \dim_{\mathbb{C}} H^2(E^{[n]}, \mathbb{C}) = \dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 11 \), \( \dim_{\mathbb{C}} H^{2n-1,1}(E^{[n]}, \mathbb{C}) = 0 \), and \( \dim_{\mathbb{C}} H^{2n-1,1}(X, \mathbb{C}) = 10 \), we obtain that \( \rho^* \) acts on \( H^2(X, \mathbb{C}) \) as the identity, and \( H^{2n-1,1}(X, \mathbb{C}) \) as \( -\text{id}_{H^{2n-1,1}(X, \mathbb{C})} \).

Let \( \varphi \) be an involution of \( X \), which acts on \( H^2(X, \mathbb{C}) \) as the identity and on \( H^{2n-1,1}(X, \mathbb{C}) \) as \( -\text{id}_{H^{2n-1,1}(X, \mathbb{C})} \). By Theorem 3.2, for \( \varphi \), there is an automorphism \( \phi \) of \( E \) such that \( \varphi \) is a lift of \( \phi^{[n]} \) where \( \phi^{[n]} \) is the natural automorphism of \( E^{[n]} \) induced by \( \phi \). Furthermore since the order of \( \phi \) is at most 2, the order of \( \varphi \) is 2.

Since \( \phi^{[n]} \circ \pi = \pi \circ \varphi \), \( \phi^{[n]} \) acts on \( H^2(E^{[n]}, \mathbb{C}) \) as the identity. Thus \( \phi^* \) acts on \( H^2(E, \mathbb{C}) \) as the identity. If \( E \) does not have numerically trivial automorphisms, then \( \phi = \text{id}_E \). Thus \( \varphi = \rho \).

We assume that \( \phi \) does not the identity map. Then \( \phi \) is numerically trivial. Then \( \phi = \nu \) and \( \varphi \in \{ \zeta, \zeta' \} \). By Proposition 4.5, we obtain that \( \varphi \) does not act on \( H^{2n-1,1}(X, \mathbb{C}) \) as \( -\text{id}_{H^{2n-1,1}(X, \mathbb{C})} \). This is a contradiction. Thus \( \phi = \text{id}_E \), and we get \( \varphi = \rho \). This proves the theorem.

Theorem 4.8. For \( n \geq 2 \), let \( \pi : X \to E^{[n]} \) be the universal covering space. For any automorphism \( \varphi \) of \( X \), if \( \varphi^* \) is acts on \( H^*(X, \mathbb{C}) := \bigoplus_{i=0}^{2n} H^i(X, \mathbb{C}) \) as the identity, then \( \varphi = \text{id}_X \).

Proof. By Theorem 3.2, for \( \varphi \), there is an automorphism \( \phi \) of \( E \) such that \( \varphi \) is a lift of \( \phi^{[n]} \) where \( \phi^{[n]} \) is the natural automorphism of \( E^{[n]} \) induced by \( \phi \). Since \( \varphi^* \) acts on \( H^2(X, \mathbb{C}) \) as the identity, \( \phi^* \) acts on \( H^2(E, \mathbb{C}) \) as the identity. From [8, page 386-389] the order of \( \phi \) is at most 4.

If the order of \( \phi \) is 2, by Proposition 4.5 \( \varphi \) does not act on \( H^{2n-1,1}(X, \mathbb{C}) \) as the identity. This is a contradiction.
If the order of $\phi$ is 4, then $\phi^2$ is a lift of $\phi^{[n]}_2 = \phi^2[n]$. Thus by the above, $\phi^2$ does not act on $H^{2n-1,1}(X, \mathbb{C})$ as the identity. This is a contradiction. Thus we have $\phi = \text{id}_E$ and $\varphi \in \{\text{id}_X, \rho\}$. Since $\rho$ does not act on $H^{2n-1,1}(X, \mathbb{C})$ as the identity, we have $\varphi = \text{id}_X$. $\square$

**Corollary 4.9.** For $n \geq 2$, let $\pi : X \to E[n]$ be the universal covering space. For any two automorphisms $f$ and $g$ of $X$, if $f^* = g^*$ on $H^*(X, \mathbb{C})$, then $f = g$.

**Theorem 4.10.** For $n \geq 3$, let $E$ be an Enriques surfaces, $E[n]$ the Hilbert scheme of $n$ points of $E$, $\pi : X \to E[n]$ the universal covering space. Then there is an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X) \to \text{Aut}(E[n]) \to 0.$$ 

**Proof.** Let $f$ be an automorphism $f$ of $X$. We put $g = f^{-1} \circ \rho \circ f$. Since for $n \geq 3$, $\rho^*$ acts on $H^2(X, \mathbb{C})$ as the identity and on $H^{2n-1,1}(X)$ as $-\text{id}_{H^{2n-1,1}(X)}$, we get that $g^* = \rho^*$ as automorphisms of $H^2(X, \mathbb{C}) \oplus H^{2n-1,1}(X)$. Like the proof of Theorem 4.8, we have $g = \rho$, i.e. $f \circ \rho = \rho \circ f$. Thus $f$ induces a automorphism of $E[n]$, and we have an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X) \to \text{Aut}(E[n]) \to 0.$$ 

$\square$

5. **Appendix A**

We compute the Hodge number of the universal covering space $X$ of $E[2]$. Let $\iota$ be the covering involution of $\mu : K \to E$, and $\tau : \text{Blow}_{\Delta, \Gamma} K^2 \to K^2$ the natural map, where $\Gamma = \{(x, y) \in K^2 : y = \iota(x)\}$ and $\Delta = \{(x, x) \in K^2\}$. By Proposition
1.4, we have

\[ X \simeq \text{Blow}_{\Delta \cup \Gamma} K^2 / H. \]

We put

\[ D_\Delta := \tau^{-1}(\Delta) \quad \text{and} \quad D_\Gamma := \tau^{-1}(\Gamma). \]

For two inclusions

\[ j_{D_\Delta} : D_\Delta \hookrightarrow \text{Blow}_{\Delta \cup \Gamma} K^2, \quad \text{and} \]
\[ j_{D_\Gamma} : D_\Gamma \hookrightarrow \text{Blow}_{\Delta \cup \Gamma} K^2, \]

let \( j_{*D_\Delta} \) be the Gysin morphism

\[ j_{*D_\Delta} : H^p(D_\Delta, \mathbb{C}) \to H^{p+2} \text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C}, \]

\( j_{*D_\Gamma} \) the Gysin morphism

\[ j_{*D_\Gamma} : H^p(D_\Gamma, \mathbb{C}) \to H^{p+2} \text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C}, \]

\[ \psi := \tau^* + j_{*D_\Delta} \circ \tau|_{D_\Delta}^* + j_{*D_\Gamma} \circ \tau|_{D_\Gamma}^* \]

the morphism from \( H^p(K^2, \mathbb{C}) \oplus H^{p-2}(\Delta, \mathbb{C}) \oplus H^{p-2}(\Gamma, \mathbb{C}) \) to \( H^p(\text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C}) \).

From [13, Theorem 7.31], we have isomorphisms of Hodge structures by \( \psi \):

\[ H^k(K^2, \mathbb{C}) \oplus H^{k-2}(\Delta, \mathbb{C}) \oplus H^{k-2}(\Gamma, \mathbb{C}) \simeq H^k(\text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C}). \]

Furthermore, for automorphism \( f \) of \( K \), let \( \tilde{f} \) (resp. \( \tilde{f}_i \)) be the automorphism of \( \text{Blow}_{\Delta \cup \Gamma} K^2 \) which is induced by \( f \times f \) (resp. \( f \times (f \circ i) \), \( f_\Delta \) the automorphism of \( \Delta \) which is induced by \( f \times f \), \( f_\Gamma \) the automorphism of \( \Gamma \) which is induced by \( f \times f \), and \( \tilde{f} \) the isomorphism from \( \Gamma \) to \( \Delta \) which is induced by \( f \times (f \circ i) \). For \( \alpha \in H^*(K^2, \mathbb{C}) \), \( \beta \in H^*(\Delta, \mathbb{C}) \), and \( \gamma \in H^*(\Gamma, \mathbb{C}) \), we obtain

\[ \tilde{f}^* (\tau^* \alpha) = \tau^*(f \times f)^* \alpha, \]
Universal covering Calabi-Yau manifolds of $E^{[n]}$

\[
\tilde{f}^*(j_\ast D_{\Delta} \circ \tau |_{\Delta \ast} \beta) = j_\ast D_{\Delta} \circ \tau |_{\Delta \ast} (f_\ast \beta),
\]

\[
\tilde{f}^*(j_\ast D_{\Gamma} \circ \tau |_{\Gamma \ast} \gamma) = j_\ast D_{\Gamma} \circ \tau |_{\Gamma \ast} (f_\ast \gamma),
\]

\[
\tilde{f}_\tau^*(\tau^* \alpha) = \tau^*((f \times (f \circ \iota)^\ast \alpha),
\]

\[
\tilde{f}_\tau^*(j_\ast D_{\Delta} \circ \tau |_{\Delta \ast} \beta) = j_\ast D_{\Delta} \circ \tau |_{\Delta \ast} (\tilde{f}_\ast \beta),
\]

in $H^*(\text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C})$.

**Theorem 5.1.** For the universal covering space $\pi : X \to E^{[2]}$, we have $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$.

**Proof.** Since $X \simeq \text{Blow}_{\Delta \cup \Gamma} K^2 / H$, we have

\[
h^{p,q}(X) = \dim_{\mathbb{C}} \{ \alpha \in H^{p,q}(\text{Blow}_{\Delta \cup \Gamma} K^2, \mathbb{C}) : h^\ast \alpha = \alpha \text{ for } h \in H \}.
\]

Let $\iota$ be the covering involution of $\mu : K \to E$. We put

\[
H_\pm^{p,q}(K, \mathbb{C}) := \{ \alpha \in H^{p,q}(K, \mathbb{C}) : \iota^\ast (\alpha) = \pm \alpha \}
\]

and

\[
h_\pm^{p,q}(K) := \dim_{\mathbb{C}} H_\pm^{p,q}(K, \mathbb{C}).
\]

From $E = K / \langle \iota \rangle$, we have

\[
H^{p,q}(E, \mathbb{C}) \simeq H_+^{p,q}(K, \mathbb{C}).
\]

Since $K$ is a $K3$ surface, we have

\[
h^{0,0}(K) = 1, \ h^{1,0}(K) = 0, \ h^{2,0}(K) = 1, \text{ and } h^{1,1}(K) = 20, \text{ and}
\]

\[
h_+^{0,0}(K) = 1, \ h_+^{1,0}(K) = 0, \ h_+^{2,0}(K) = 0, \text{ and } h_+^{1,1}(K) = 10, \text{ and}
\]

\[
h_-^{0,0}(K) = 0, \ h_-^{1,0}(K) = 0, \ h_-^{2,0}(K) = 1, \text{ and } h_-^{2,0}(K) = 10.
\]
Recall that \( H \) is generated by \( S_2 \) and \( \iota_{1,2} \). Since \( \iota \times \iota(\Delta) = \Delta \) and \( \iota \times \iota(\Gamma) = \Gamma \), from \( E = K/\langle \iota \rangle \) we have \( \Delta/H \simeq E \) and \( \Gamma/H \simeq E \). Thus we have

\[
h_{0,0}(\Delta/H) = 1, \ h_{1,0}(\Delta/H) = 0, \ h_{2,0}(\Delta/H) = 0, \ h_{1,1}(\Delta/H) = 10,
\]

\[
h_{0,0}(\Gamma/H) = 1, \ h_{1,0}(\Gamma/H) = 0, \ h_{2,0}(\Gamma/H) = 0, \text{ and } h_{1,1}(\Gamma/H) = 10.
\]

From the Künneth Theorem, we have

\[
H^{p,q}(K^2, \mathbb{C}) \simeq \bigoplus_{s+u=p,t+v=q} H^{s,t}(K, \mathbb{C}) \otimes H^{u,v}(K, \mathbb{C}),
\]

and

\[
H^{p,q}(K^2/H, \mathbb{C}) \simeq \{ \alpha \in H^{p,q}(K^2, \mathbb{C}) : s^*(\alpha) = \alpha \text{ for } s \in \Sigma_2 \text{ and } \iota_{1,2}^*(\alpha) = \alpha \}.
\]

Thus we obtain

\[
h_{0,0}(K^2/H) = 1, \ h_{1,0}(K^2/H) = 0, \ h_{2,0}(K^2/H) = 0, \ h_{1,1}(K^2/H) = 10,
\]

\[
h_{3,0}(K^2/H) = 0, \ h_{2,1}(K^2/H) = 0, \ h_{4,0}(K^2/H) = 1,
\]

\[
h_{3,1}(K^2/H) = 10, \text{ and } h_{2,2}(K^2/H) = 111.
\]

We fix a basis \( \beta \) of \( H^{2,0}(K, \mathbb{C}) \) and a basis \( \{ \gamma_i \}_{i=1}^{10} \) of \( H^{1,1}(K, \mathbb{C}) \), then we have

\[
H^{3,1}(K^2/H, \mathbb{C}) \simeq \bigoplus_{i=1}^{10} \mathbb{C}(\beta \otimes \gamma_i + \gamma_i \otimes \beta).
\]

By the above equation, we have

\[
h_{0,0}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 1, \ h_{1,0}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 0,
\]

\[
h_{2,0}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 0, \ h_{1,1}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 12,
\]

\[
h_{3,0}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 0, \ h_{2,1}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 0,
\]

\[
h_{4,0}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 1, \ h_{3,1}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 10, \text{ and } h_{2,2}(\text{Blow}_{\Delta\cup \Gamma} K^2/H) = 131.
\]

Thus we obtain

\[
h_{0,0}(X) = 1, \ h_{1,0}(X) = 0, \ h_{2,0}(X) = 0, \ h_{1,1}(X) = 12, \ h_{3,0}(X) = 0, \ h_{2,1}(X) = 0, \ h_{4,0}(X) = 1, \ h_{3,1}(X) = 10, \text{ and } h_{2,2}(X) = 131.
\]

\[\square\]
6. Appendix B

Now we show that the conjecture in [2, Conjecture 1] is not established for $Y$ an Enriques surface and $L = \Omega^2_Y$.

Let $Y$ be a smooth compact Kähler surface. Recall that $Y^{[n]}$ is the Hilbert scheme of $n$ points of $Y$, $\pi_Y : Y^{[n]} \to Y^{(n)}$ the Hilbert-Chow morphism, and $p_Y : Y^n \to Y^{(n)}$ the natural projection. For a line bundle $L$ on $Y$, there is a unique line bundle $L$ on $Y^{(n)}$ such that $p_Y^* L = \bigotimes_{i=1}^n p_i^* L$. By using pull back we have the natural map

$$\text{Pic}(Y) \to \text{Pic}(Y^{[n]}), \quad L \mapsto L_n := \pi_Y^* L.$$

we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} H^q(Y^{[n]}, \Omega^n_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim_{\mathbb{C}} H^q(Y, \Omega^n_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y^{[n]}, L_n)x^p y^q t^n, \quad \text{and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q+1} y^{q+k-1} t^{k}} \right)^{(-1)^{p+q} h^{p,q}(Y, L)}.$$
for \( n \geq 2 \). It follows that the coefficient of \( x^3yt^2 \) of \( A \) is 10.

We show that the coefficient of \( x^3yt^2 \) of \( B \) is not 10.

\[
\begin{align*}
\text{By Serre duality, we get} & \qquad \Omega_Y \otimes \Omega_Y^2 \simeq T_Y. \\
\text{Since } Y \text{ is an Enriques surface, we have} & \qquad h^{1,0}(Y, \Omega_Y^2) = \dim \mathbb{C} H^0(Y, \Omega_Y \otimes \Omega_Y^2) = \dim \mathbb{C} H^0(Y, \mathcal{O}_Y) = 0. \\
& \qquad h^{1,1}(Y, \Omega_Y^2) = \dim \mathbb{C} H^1(Y, \Omega_Y \otimes \Omega_Y^2) = \dim \mathbb{C} H^1(Y, \mathcal{O}_Y) = 10. \\
& \qquad h^{1,2}(Y, \Omega_Y^2) = \dim \mathbb{C} H^2(Y, \Omega_Y \otimes \Omega_Y^2) = \dim \mathbb{C} H^2(Y, \mathcal{O}_Y) = 0.
\end{align*}
\]

Thus we obtain

\[
\begin{align*}
\text{Since } Y \text{ is an Enriques surface, we obtain} & \qquad \Omega_Y^2 \otimes \Omega_Y^2 \simeq \mathcal{O}_Y. \\
& \qquad h^{2,0}(Y, \Omega_Y^2) = \dim \mathbb{C} H^0(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} H^0(Y, \mathcal{O}_Y) = 1. \\
& \qquad h^{2,1}(Y, \Omega_Y^2) = \dim \mathbb{C} H^1(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} H^1(Y, \mathcal{O}_Y) = 0. \\
& \qquad h^{2,2}(Y, \Omega_Y^2) = \dim \mathbb{C} H^2(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} H^2(Y, \mathcal{O}_Y) = 0.
\end{align*}
\]

Thus we obtain

\[
B = \prod_{k=1}^{\infty} \prod_{p, q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q}x^{k+1}y^{k-1}t^{k+1}y^{k-1}t^{k}} \right)^{(-1)^{p+q}h^{p, q}(E, \Omega_E^2)}
\]

\[
= \prod_{k=1}^{\infty} \left( \frac{1}{1 - x^{k-1}y^{k+1}t^{k}} \right) \left( \frac{1}{1 - x^{k}y^{k}t^{k}} \right)^{10} \left( \frac{1}{1 - x^{k+1}y^{k-1}t^{k}} \right)
\]

\[
= \prod_{k=1}^{\infty} \left( \sum_{a=0}^{\infty} (x^{k-1}y^{k+1}t^{k})^a \right) \left( \sum_{b=0}^{\infty} (x^{k}y^{k}t^{k})^b \right)^{10} \left( \sum_{c=0}^{\infty} (x^{k+1}y^{k-1}t^{k})^c \right).
\]
Thus we have

\[ B \equiv \prod_{k=1}^{2} (1 + x^{k-1}y^{k+1}t^{k} + x^{2k-2}y^{2k+2}t^{2k}) \times (1 + x^{k}y^{k}t^{k} + x^{2k}y^{2k}t^{2k})^{10} \times \\
(1 + x^{k+1}y^{k-1}t^{k} + x^{2k+2}y^{2k-2}t^{2k}) \pmod{t^{3}} \\
\equiv \left(1 + y^{2}t + y^{4}t^{2}\right) \times \\
\left(1 + 10(xyt + x^{2}y^{2}t^{2}) + 45(xyt + x^{2}y^{2}t^{2})^{2}\right) \times \\
\left(1 + x^{2}t + x^{4}t^{2}\right) \pmod{t^{3}} \\
\equiv \left(1 + y^{2}t + (xy^3 + y^4)t^{2}\right) \times \left(1 + 10xyt + 56x^{2}y^{2}t^{2}\right) \times \\
\left(1 + x^{2}t + (x^{3}y + x^{4})t^{2}\right) \pmod{t^{3}} \\
\equiv \left(1 + (10xy + y^{2})t + (56x^{2}y^{2} + 11xy^{3} + y^{4})t^{2}\right) \times \\
\left(1 + x^{2}t + (x^{3}y + x^{4})t^{2}\right) \pmod{t^{3}} \\
\equiv 1 + (x^{2} + 10xy + y^{2})t + (x^{4} + 11x^{3}y + 56x^{2}y^{2} + 11xy^{3} + y^{4})t^{2} \pmod{t^{3}} \]

Therefore the coefficient of \(x^{3}yt^{2}\) of \(B\) is 11. The conjecture in [2, Conjecture 1] is not established for \(Y\) an Enriques surface and \(L = \Omega^{2}_{Y}\).

References


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