Universal covering Calabi-Yau manifolds of the Hilbert schemes of n points of Enriques surfaces

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Introduction

Throughout this paper, we work over $\mathbb{C}$, and n is an integer such that $n \geq 2$. A $K3$ surface K is a compact complex surface with $\omega_K \simeq \mathcal{O}_K$ and $\mathrm{H}^1(K, \mathcal{O}_K) = 0$. An Enriques surface E is a compact complex surface with $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, $\mathrm{H}^2(E, \mathcal{O}_E) = 0$, and $\omega_E^{\otimes 2} \simeq \mathcal{O}_E$. A Calabi-Yau manifold X is an n-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic k-form on X for $0 < k < n$, and there is a nowhere vanishing holomorphic n-form on X. By Oguiso and Schröer [10, Theorem 3.1], the Hilbert scheme of n points of an Enriques surface $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space of degree 2.

In this paper, we study the Hilbert scheme of n points of an Enriques surface $E^{[n]}$ and its universal covering space X.

Definition 0.1. *For $n \geq 1$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. A variety Y is called an Enriques quotient of X if there is an Enriques surface E' and a free involution τ of X such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle\tau\rangle$. Here we call two Enriques quotients of X distinct if they are not isomorphic to each other.*

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Recall that when $n = 1$, $E^{[1]}$ is an Enriques surface E and X is a $K3$ surface. In [11, Theorem 0.1], Ohashi showed the following theorem:

Theorem 0.2. *For any nonnegative integer l, there exists a $K3$ surface with exactly 2^{l+10} distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a $K3$ surface.*

Our main theorem (Theorem 0.3) is the following which is totally different from Theorem 0.2:

Theorem 0.3. *For $n \geq 3$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then the number of distinct Enriques quotients of X is one.*

Remark 0.4. When $n = 2$, we do not count the number of distinct Enriques quotients of X. We compute the Hodge numbers of the universal covering space X of $E^{[2]}$ (Appendix A).

In addition, we investigate the relationship between the small deformation of $E^{[n]}$ and that of X (Theorem 0.5) and study the natural automorphisms of $E^{[n]}$ (Theorem 0.8).

Section 2 is a preliminary section. We prepare and recall some basic facts on the Hilbert scheme of n points of a surface.

In Section 3, we show the following theorem (Theorem 0.5).

Theorem 0.5. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then every small deformation of X is induced by that of $E^{[n]}$.*

Remark 0.6. By Fantechi [4, Theorems 0.1 and 0.3], every small deformation of $E^{[n]}$ is induced by that of E. Thus for $n \geq 2$, every small deformation of X is induced by that of E.

When $n = 1$, $E^{[1]}$ is an Enriques surface E, and X is a $K3$ surface. An Enriques surface has a 10-dimensional deformation space and a $K3$ surface has a 20-dimensional deformation space. Thus the small deformation of X is much bigger than that of E. Our Theorem 0.5 is different from the case of $n = 1$.

In Section 4, we show the following theorem (Theorem 0.8).

Definition 0.7. For $n \geq 2$ and S a smooth compact surface, any automorphism $f \in \mathrm{Aut}(S)$ induces an automorphism $f^{[n]} \in \mathrm{Aut}(S^{[n]})$. An automorphism $g \in \mathrm{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \mathrm{Aut}(S)$ such that $g = f^{[n]}$.

When S is a $K3$ surface, the natural automorphisms of $S^{[n]}$ were studied by Boissière and Sarti [3]. They showed that an automorphism of $S^{[n]}$ is natural if and only if it preserves the exceptional divisor of the Hilbert-Chow morphism [3, Theorem 1]. We obtain Theorem 0.8 which is similar to [3, Theorem 1]:

Theorem 0.8. *For $n \geq 2$, let E be an Enriques surface, D the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[2]}$. Then*

i) An automorphism f of $E^{[n]}$ is natural if and only if $f(D) = D$.

ii) An automorphism g of X is a lift of a natural automorphism of $E^{[n]}$ if and only if $g(\pi^{-1}(D)) = \pi^{-1}(D)$.

In Section 5, we show main theorem (Theorem 0.3).

In addition, let Y be a smooth compact Kähler surface. For a line bundle L on Y, by using the natural map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{[n]}))$, $L \mapsto L_n$, we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} \mathrm{H}^q(Y^{[n]}, \Omega^p_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega^p_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y^{[n]}, L_n) x^p y^q t^n, \text{ and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \Big(\frac{1}{1-(-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k)} \Big)^{(-1)^{p+q} h^{p,q}(Y,L)}.$$

In [2, Conjecture 1], S. Boissière conjectured that

$$A = B.$$

In the proof of Theorem 0.5, we obtain the counterexample to this conjecture for Y an Enriques surface and $L = \Omega^2_Y$. See Appendix B for details.

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1. Preliminaries

Let S be a nonsingular projective surface, $S^{[n]}$ the Hilbert scheme of n points of S, $q : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism, and $p : S^n \to S^{(n)}$ the natural projection. We denote the exceptional divisor of q by D. By Fogarty [5, Theorem 2.4], $S^{[n]}$ is a smooth projective variety of $\dim_{\mathbb{C}} S^{[n]} = 2n$. We put

$$\Delta^n := \{ (x_i)_{i=1}^n \in S^n : \ |\{x_i\}_{i=1}^n| \leq n-1 \ \},$$

$$S^n_* := \{(x_i)^n_{i=1} \in S^n : \ |\{x_i\}^n_{i=1}| \geq n-1 \ \},$$

$$\Delta^n_* := \Delta^n \cap S^n_*, \text{ and}$$

$$S^{[n]}_* := q^{-1}(p(S^n_*)),$$

When $n=2$, $\mathrm{Blow}_{\Delta^2} S^2/\Sigma_2 \simeq S^{[2]}$, for $n \geq 3$, we have $\mathrm{Blow}_{\Delta^n_*} S^n_*/\Sigma_n \simeq S^{[n]}_*$, and $S^{[n]} \backslash S^{[n]}_*$ is an analytic closed subset and its codimension is 2 in $S^{[n]}$ ([1, page 767-768]). Here Σ_n is the symmetric group of degree n which acts naturally on S^n by permuting of the factors.

Let $\mu : K \to E$ be the universal covering space of E where K is a $K3$ surface, and ι the covering involution of μ. They induces the universal covering space $\mu^n : K^n \to E^n$. For $1 \leq k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n$, we define automorphisms $\iota_{i_1 \ldots i_k}$ of K^n in the following way: for $x = (x_i)^n_{i=1} \in K^n$,

$$\text{the j-th component of } \iota_{i_1 \ldots i_k}(x) = \begin{cases} \iota(x_j) & j \in \{i_1, \ldots, i_k\} \\ x_j & j \notin \{i_1, \ldots, i_k\}. \end{cases}$$

Let G be the subgroup of $\mathrm{Aut}(K^n)$ generated by Σ_n and $\{\iota_i\}_{1 \leq i \leq n}$ and H the subgroup of $\mathrm{Aut}(K^n)$ generated by Σ_n and $\{\iota_{ij}\}_{1 \leq i < j \leq n}$. Since $K^n/G = E^{(n)}$, $H \lhd G$, $|G/H| = 2$, and the codimension of $\mu^{-1}(\Delta^n)$ is two, we get the universal covering spaces

$$p_1 : K^n \backslash \mu^{-1}(\Delta^n) \to K^n \backslash \mu^{-1}(\Delta^n)/G, \text{ and}$$

$$p_2 : K^n \backslash \mu^{-1}(\Delta^n) \to K^n \backslash \mu^{-1}(\Delta^n)/H,$$

where p_1 and p_2 are the natural projections. For $n \geq 3$, we put

$$K^n_\circ := (\mu^n)^{-1}(E^n_*),$$

$$\Gamma^{ij}_\circ := \{(x_l)^n_{l=1} \in K^n_\circ : \iota(x_i) = x_j\},$$

$$\Delta^{ij}_\circ := \{(x_l)^n_{l=1} \in K^n_\circ : x_i = x_j\},$$

$$\Gamma_\circ := \bigcup_{1\le i<j\le n} T_\circ^{i,j}, \text{ and}$$

$$\Delta_\circ := \bigcup_{1\le i<j\le n} U_\circ^{ij}.$$

Then we get $\mu^{n-1}(\Delta_*^n) = \Gamma_\circ \cup \Delta_\circ$. By the definition of $K_\circ^n$, H acts on $K_\circ^n$. For an element $\tilde{x} := (\tilde{x}_i)_{i=1}^n \in \Gamma_\circ \cap \Delta_\circ$, some i, j, k, l with $k \neq l$ such that $\sigma(\tilde{x}_i) = \tilde{x}_j$ and $\tilde{x}_k = \tilde{x}_l$. Since σ does not have fixed points. Thus $\tilde{x}_i \neq \tilde{x}_l$. Therefore $\mu^n(\tilde{x}) \notin E_*^n$. This is a contradiction. We obtain $\Gamma_\circ \cap \Delta_\circ = \emptyset$.

Lemma 1.1. *For $t \in H$ and $1 \le i < j \le n$, if $t \in H$ has a fixed point on $\Delta_\circ^{ij}$, then $t = (i,j)$ or $t = \mathrm{id}_{K^n}$.*

Proof. Let $t \in H$ be an element of H where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in \Delta_\circ^{ij}$ such that $t(\tilde{x}) = \tilde{x}$. For $t \in H$, there are ι_{ab} where $1 \le a < b \le n$ and $(j_1, \ldots, j_l) \in \Sigma_n$ such that

$$t = (j_1, \ldots, j_l) \circ \iota_{ab}.$$

From the definition of $\Delta_\circ^{ij}$, for $(x_l)_{l=1}^n \in \Delta_\circ^{ij}$,

$$\{x_1, \ldots, x_n\} \cap \{\iota(x_1), \ldots, \iota(x_n)\} = \emptyset.$$

Suppose $\iota_{ab} \neq \mathrm{id}_{K^n}$. Since $t(\tilde{x}) = \tilde{x}$, we have

$$\{\tilde{x}_1, \ldots, \tilde{x}_n\} \cap \{\iota(\tilde{x}_1), \ldots, \iota(\tilde{x}_n)\} \neq \emptyset.$$

This is a contradiction. Thus we have $t = (j_1, \ldots, j_l)$. Similarly from the definition of $\Delta_\circ^{ij}$, for $(x_l)_{l=1}^n \in \Delta_\circ^{ij}$, if $x_s = x_t$ $(1 \le s < t \le n)$, then $s = i$ and $t = j$. Thus we have $t = (i,j)$ or $t = \mathrm{id}_{K^n}$. □

Lemma 1.2. *For $t \in H$ and $1 \le i < j \le n$, if $t \in H$ has a fixed point on $\Gamma_\circ^{ij}$, then $t = \iota_{i,j} \circ (i,j)$ or $t = \mathrm{id}_{K^n}$.*

Proof. Let $t \in H$ be an element of H where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in \Gamma_\circ^{ij}$ such that $t(\tilde{x}) = \tilde{x}$. For $t \in H$, there are ι_a where $1 \le a \le n$ and $(j_1, \ldots, j_l) \in \mathcal{S}_n$ such that

$$t = (j_1 \ldots j_l) \circ \iota_a.$$

Since $(j, j+1) \circ \iota_{i,j} \circ (j, j+1) : U_{ij} \to T_{ij}$ is an isomorphism, and by Lemma 1.1, we have

$$(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = (i, j) \text{ or } \mathrm{id}_{K^n}.$$

If $(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = \mathrm{id}_{K^n}$, then $t = \mathrm{id}_{K^n}$. If $(j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ t \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) = (i, j)$, then

$$\begin{aligned} t &= (j, j+1) \circ \iota_{i,j} \circ (j, j+1) \circ (i, j) \circ (j, j+1) \circ \iota_{i,j} \circ (j, j+1) \\ &= (j, j+1) \circ \iota_{i,j} \circ (i, j+1) \circ \iota_{i,j} \circ (j, j+1) \\ &= (j, j+1) \circ \iota_{i,j+1} \circ (i, j+1) \circ (j, j+1) \\ &= \iota_{i,j} \circ (i, j). \end{aligned}$$

Thus we have $t = \iota_{i,j} \circ (i, j)$. $\square$

For the natural projection we get a unramified covering space: $K^n/H \to K^n/G = E^{(n)} = E^n/\Sigma_n$. From Lemma 1.1 and Lemma 1.2, we get a local isomorphism:

$$\theta : \mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n / H \to E_*^{[n]}.$$

Lemma 1.3. *For every* $x \in E_*^{[n]}$, $|\theta^{-1}(x)| = 2$.

Proof. For $(x_i)_{i=1}^n \in \Delta_*^n \subset E^n$ with $x_1 = x_2$, there are n elements $y_1, \ldots, y_n$ of K such that $y_1 = y_2$ and $\mu(y_i) = x_i$ for $1 \le i \le n$. Then

$$(\mu^n)^{-1}((x_i)_{i=1}^n) = \{y_1, \iota(y_1)\} \times \cdots \times \{y_n, \iota(y_n)\}.$$

Since H is generated by Σ_n and $\{\iota_{ij}\}_{1\le i<j\le n}$, for $(z_i)_{i=1}^n \in (\mu^n)^{-1}((x_i)_{i=1}^n)$ if the number of i with $z_i = y_i$ is even, then

$$(z_i)_{i=1}^n = \{\iota(y_1), \iota(y_2), y_3, \ldots, y_n\} \text{ on } K_\circ^n/H, \text{ and}$$

if the number of i with $z_i = y_i$ is odd, then

$$(z_i)_{i=1}^n = \{\iota(y_1), y_2, y_3, \ldots, y_n\} \text{ on } K_\circ^n/H.$$

Furthermore since $\iota_i \notin H$ for $1 \le i \le n$,

$$\{\iota(y_1), \iota(y_2), y_3, \ldots, y_n\} \ne \{\iota(y_1), y_2, y_3, \ldots, y_n\}, \text{ on } K_\circ^n/H.$$

Thus for every $x \in E_*^{[n]}$, we get $|\theta^{-1}(x)| = 2$. □

Proposition 1.4. *θ : $\mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n/H \to \mathrm{Blow}_{\Delta_*^n} E_*^n/\Sigma_n$ is the universal covering space, i.e. $\pi^{-1}(E_*^{[n]}) \simeq \mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n/H$. When $n = 2$, we have $X \simeq \mathrm{Blow}_{\mu^{2-1}(\Delta^2)} K^2/H$.*

Proof. Since θ is a local isomorphism, from Lemma 1.3 we get that θ is a covering map. Furthermore $\pi : \pi^{-1}(E_*^{[n]}) \to E_*^{[n]}$ is the universal covering space of degree 2, θ : $\mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n/H \to \mathrm{Blow}_{\Delta_*^n} E_*^n/\Sigma_n$ is the universal covering space. By the uniqueness of the universal covering space, we have $\pi^{-1}(E_*^{[n]}) \simeq \mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n/H$. When $n = 2$, since $E_*^2 = E^2$, $K_\circ^2 = K^2$ and $\mathrm{Blow}_{\Delta^2} E^2/\Sigma_2 \simeq E^{[2]}$, we have $X \simeq \mathrm{Blow}_{\mu^{2-1}(\Delta^2)} K^2/H$. □

Theorem 1.5. *For $n \ge 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. Then there is a birational morphism $\varphi : X \to K^n/H$ such that $\varphi^{-1}(\mu^{n-1}(\Delta^n)/H) = \pi^{-1}(D)$.*

Proof. When $n = 2$, this is proved by Proposition 1.4. From here we assume that $n \geq 3$. From Proposition 1.4, we have $\pi^{-1}(E_*^{[n]}) \simeq \mathrm{Blow}_{\mu^{n-1}(\Delta_*^n)} K_\circ^n/H$. Since the codimension of $X \backslash \pi^{-1}(E_*^{[n]})$ is 2, there is a meromorphim f of X to K^n/H which satisfies the following commutative diagram:

$$\begin{tikzcd} E_*^{[n]} \arrow[r, "q"] & E^{(n)} \\ \pi^{-1}(E_*^{[n]}) \arrow[u, "\pi"] \arrow[r, "f"] & K^n/H \arrow[u, "p"'] \end{tikzcd}$$

where $q : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $p : K^n/H \to E^{(n)}$ is the natural projection. For an ample line bundle $\mathcal{L}$ on $E^{(n)}$, since the natural projection $p : K^n/H \to E^{(n)}$ is finite, $p^*\mathcal{L}$ is ample. From the above diagram, we have $\pi^*(q^*\mathcal{L}) \mid_{\pi^{-1}(E_*^{[n]})} = f^*(p^*\mathcal{L})$. Since $X \backslash \pi^{-1}(E_*^{[n]})$ is an analytic closed subset of codimension 2 in X and $p_H^*\mathcal{L}$ is ample, there is a holomorphism φ from X to K^n/H such that $\varphi \mid_{X \backslash \pi^{-1}(F)} = f \mid_{X \backslash \pi^{-1}(F)}$. Since $f : X \setminus \pi^{-1}(D) \cong (K^n \setminus \mu^{n-1}(\Delta^n))/H$, this is a birational morphism. □

2. Proof of Theorem 0.5

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. In this section, we show Theorem 0.5 (Theorem 2.2).

Proposition 2.1. *For $n \geq 2$, we have* $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) = 0$.

Proof. For a smooth projective manifold S, we put

$$h^{p,q}(S) := \dim_{\mathbb{C}} \mathrm{H}^q(S, \Omega_S^p) \text{ and}$$

$$h(S,x,y) := \sum_{p,q} h^{p,q}(S)x^p y^q.$$

By [7, Theorem 2] and [6, page 204], we have the equation (1):

$$\sum_{n=0}^{\infty}\sum_{p,q} h^{p,q}(E^{[n]})x^p y^q t^n = \prod_{k=1}^{\infty}\prod_{p,q=0}^{2}\Big(\frac{1}{1-(-1)^{p+q}x^{p+k-1}y^{q+k-1}t^k)}\Big)^{(-1)^{p+q}h^{p,q}(E)}.$$

Since an Enriques surface E has Hodge numbers $h^{0,0}(E) = h^{2,2}(E) = 1$, $h^{1,0}(E) = h^{0,1}(E) = 0$, $h^{2,0}(E) = h^{0,2}(E) = 0$, and $h^{1,1}(E) = 10$, the equation (1) is

$$\sum_{n=0}^{\infty}\sum_{p,q} h^{p,q}(E^{[n]})x^p y^q t^n = \prod_{k=1}^{\infty}\Big(\frac{1}{1-x^{k-1}y^{k-1}t^k}\Big)\Big(\frac{1}{1-x^k y^k t^k}\Big)^{10}\Big(\frac{1}{1-x^{k+1}y^{k+1}t^k}\Big).$$

It follows that

$$h^{p,q}(E^{[n]}) = 0 \text{ for all } p,q \text{ with } p \neq q.$$

Thus we have $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) = 0$ for $n \geq 2$. □

Theorem 2.2. *For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then every small deformation of X is induced by that of $E^{[n]}$.*

Proof. In [4, Proposition 4.2 and Theorems 0.3], Fantechi showed that for a smooth projective surface with $\mathrm{H}^0(S, T_S) = 0$ or $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, and $\mathrm{H}^1(S, \mathcal{O}_S(-K_S)) = 0$ where K_S is the canonical divisor of S,

$$\dim_{\mathbb{C}} \mathrm{H}^1(S, T_S) = \dim_{\mathbb{C}} \mathrm{H}^1(S^{[n]}, T_{S^{[n]}}).$$

Since an Enriques surface E satisfies $\mathrm{H}^0(E, T_E) = 0$ or $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, and $\mathrm{H}^1(E, \mathcal{O}_E(-K_E)) = 0$, we have $\dim_{\mathbb{C}} \mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) = 10$. Since $K_{E^{[n]}}$ is not trivial and $2K_{E^{[n]}}$ is trivial, we have

$$T_{E^{[n]}} \simeq \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}.$$

Therefore we have $\dim_{\mathbb{C}}\mathrm{H}^1(E^n, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}) = 10$. Since K_X is trivial, then we have $T_X \simeq \Omega^{2n-1}_X$. Since $\pi : X \to E^{[n]}$ is the covering map, we have

$$\mathrm{H}^k(X, \Omega^{2n-1}_X) \simeq \mathrm{H}^k(E^{[n]}, \pi_*\Omega^{2n-1}_X).$$

Since $X \simeq \mathcal{S}pec\,\mathcal{O}_{E^{[n]}} \oplus \mathcal{O}_{E^{[n]}}(K_{E^{[n]}})$ ([10, Theorem 3.1]), we have

$$\mathrm{H}^k(E^{[n]}, \pi_*\Omega^{2n-1}_X) \simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \oplus (\Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}})).$$

Thus

$$\begin{aligned}\mathrm{H}^k(X, \Omega^{2n-1}_X) &\simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \oplus (\Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}})) \\ &\simeq \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) \oplus \mathrm{H}^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}).\end{aligned}$$

Combining this with Proposition 2.1, we obtain

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{H}^1(X, \Omega^{2n-1}_X) &= \dim_{\mathbb{C}}\mathrm{H}^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}) \\ &= 10.\end{aligned}$$

Let $p : \mathcal{Y} \to U$ be the Kuranishi family of $E^{[n]}$. Since each canonical bundle of $E^{[n]}$ and E is torsion, they have unobstructed deformations ([12]). Thus U is smooth. Let $f : \mathcal{X} \to \mathcal{Y}$ be the universal covering space. Then $q : \mathcal{X} \to U$ is a flat family of X where $q := p \circ f$. By [4, Theorems 0.1 and 0.3], all small deformation of $E^{[n]}$ is induced by that of E. Thus for $u \in U$, $q^{-1}(u)$ is the universal covering space of the Hilbert scheme of n points of an Enriques surface. Then we have a commutative diagram:

$$\begin{tikzcd} T_{U,0} \arrow[r, "\rho_p"] \arrow[dr, "\rho_q"'] & \mathrm{H}^1(\mathcal{Y}_0, T_{\mathcal{Y}_0}) \arrow[r, equal] \arrow[d, "\tau"] & \mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) \arrow[d, "\pi^*"] \\ & \mathrm{H}^1(\mathcal{X}_0, T_{\mathcal{X}_0}) \arrow[r, equal] & \mathrm{H}^1(X, T_X). \end{tikzcd}$$

Since $\mathrm{H}^1(E^{[n]}, T_{E^{[n]}}) \simeq \mathrm{H}^1(X, T_X)$ by π^*, the vertical arrow τ is an isomorphism and

$$\dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{X}_u, T_{\mathcal{X}_u}) = \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{X}_u, \Omega^{2n-1}_{\mathcal{X}_u})$$

is a constant for some neighborhood of $0 \in U$, it follows that $q : \mathcal{X} \to U$ is the complete family of $\mathcal{X}_0 = X$, therefore $q : \mathcal{X} \to U$ is the versal family of $\mathcal{X}_0 = X$. Thus every small deformation of X is induced by that of $E^{[n]}$. $\square$

3. Proof of Theorem 0.8

For $n \geq 2$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and D the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$. Recall that ι is the covering involution of $\mu : K \to E$, $p_1 : K^n \setminus \mu^{n-1}(\Delta^n) \to E^{[n]} \setminus D = E^n \setminus \Delta^n / \Sigma_n = K^n \setminus \mu^{n-1}(\Delta^n)/G$ and $p_2 : K^n \setminus \mu^{n-1}(\Delta^n) \to X \setminus \pi^{-1}(D) = K^n \setminus \mu^{n-1}(\Delta^n)/H$ are the universal covering spaces where p_1 and p_2 are the natural projections. In this section, we show Theorem 0.8 (Theorem 3.2).

Lemma 3.1. *i) Let f be an automorphism of $E^{[n]} \setminus D$, and $g_1, \ldots, g_n$ automorphisms of K such that $p_1 \circ (g_1 \times \cdots \times g_n) = f \circ p_1$, where $(g_1 \times \cdots \times g_n)$ is the automorphism of K^n. Then we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for each $1 \leq i \leq n$. Moreover $g_1 \circ \iota = \iota \circ g_1$.*

ii) Let f be an automorphism of $X \setminus \pi^{-1}(D)$, and $g_1, \ldots, g_n$ automorphisms of K such that $p_2 \circ (g_1 \times \cdots \times g_n) = f \circ p_2$, where $(g_1 \times \cdots \times g_n)$ is the automorphism of K^n. Then we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for each $1 \leq i \leq n$. Moreover $g_1 \circ \iota = \iota \circ g_1$.

Proof. We show i) by contradiction. Without loss of generality, we may assume that $g_2 \neq g_1$ and $g_2 \neq g_1 \circ \iota$. Let h_1 and h_2 be two morphisms of K where $g_i \circ h_i = \mathrm{id}_K$

and $h_i \circ g_i = \mathrm{id}_K$ for $i = 1,\, 2$. We define two morphisms $A_{1,2}$ and $A_{1,2,\iota}$ from K to K^2 by

$$A_{1,2} : K \ni x \mapsto (h_1(x), h_2(x)) \in K^2$$

$$A_{1,2,\iota} : K \ni x \mapsto (h_1(x), \iota \circ h_2(x)) \in K^2.$$

Let $\Gamma_\iota := \{(x, y) :\ y = \iota(x)\}$ be the subset of K^2. Since $h_1 \neq h_2$ and $h_1 \neq \iota \circ h_2$, $A_{1,2}^{-1}(\Delta^2) \cup A_{1,2,\iota}^{-1}(\Gamma_\iota)$ do not coincide with K. Thus there is $x' \in K$ such that $A_{1,2}(x') \notin \Delta^2$ and $A_{1,2,\iota}(x') \notin \Gamma_\iota$. For $x' \in K$, we put $x_i := h_i(x') \in K$ for $i = 1,\, 2$. Then there are some elements $x_3, \ldots, x_n \in K$ such that $(x_1, \ldots, x_n) \in K^n \setminus \mu^{n-1}(\Delta^n)$. We have $g((x_1, \ldots, x_n)) \notin K^n \backslash \mu^{n-1}(\Delta^n)$ by the assumption of x_1 and x_2. It is contradiction, because g is an automorphism of $K^n \backslash \mu^{n-1}(\Delta^n)$. Thus we have $g_i = g_1$ or $g_i = g_1 \circ \iota$ for $1 \leq i \leq n$.

Let $g := g_1 \times \cdots \times g_n$. Since the covering transformation group of p is G, the liftings of f are given by $\{g \circ u : u \in G\} = \{u \circ g : u \in G\}$. Thus for $\iota_1 \circ g$, there is an element $\iota_a \circ s$ of G where $s \in \Gamma_n$ and $1 \leq a \leq n$ such that $\iota_1 \circ g = g \circ \iota_a \circ s$. If we think about the first component of $\iota_1 \circ g$, we have $s = \mathrm{id}$ and $a = 1$. Therefore $g \circ \iota \circ g^{-1} = \iota$, we have $\iota \circ g_1 = g_1 \circ \iota$. In the same way, we have ii). $\square$

Theorem 3.2. *For $n \geq 2$, let E be an Enriques surface, D the exceptional divisor of the Hilbert-Chow morphism $q : E^{[n]} \to E^{(n)}$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[2]}$. Then*

i) An automorphism f of $E^{[n]}$ is natural if and only if $f(D) = D$.

ii) An automorphism g of X is a lift of a natural automorphism of $E^{[n]}$ if and only if $g(\pi^{-1}(D)) = \pi^{-1}(D)$.

Proof. We show (1). Let $\mu : K \to E$ be the universal covering space of E. By Theorem 1.5, there is a commutative diagram

$$\begin{array}{ccc} E^{[n]} & \xrightarrow{q} & E^{(n)} \\ \pi \uparrow & & \uparrow p \\ X & \xrightarrow{\varphi} & K^n/H, \end{array}$$

where p is the natural projection and φ is a birational morphism. Since $E^{[n]}\backslash D \overset{\sim}{\to} E^n\backslash\Delta^n/\Sigma_n$, we have the universal covering spaces

$$p_1 : K^n\backslash\mu^{n-1}(\Delta^n) \to E^n\backslash\Delta^n/\Sigma_n,$$

$$p_2 : K^n\backslash\mu^{n-1}(\Delta^n) \to K^n\backslash\mu^{n-1}(\Delta^n)/H, \text{ and}$$

and the following commutative diagram:

$$\begin{array}{ccc} K^n\backslash\mu^{n-1}(\Delta^n)/H & \xrightarrow{p_3} & E^n\backslash\Delta^n/\Sigma_n \\ p_2 \uparrow & \nearrow_{p_1} & \\ K^n\backslash\mu^{n-1}(\Delta^n), & & \end{array}$$

where p_1, p_2, and p_3 are the natural projections. For $f \in$Aut$(E^{[n]})$ with $f(D) = D$, from the uniqueness of the universal covering space, f induces an automorphisms $\bar{f}$ of $K^n\backslash\mu^{n-1}(\Delta^n)$. Since K is projective and codim $\mu^{-1}(\Delta^n)$ is over 2, $\bar{f}$ is a biratioal map of K^n. By [9], $\bar{f}$ is au automorphism of K^n and there are $g_1, \ldots, g_n$ automorphisms of K such that $\bar{f} = (g_1 \times \cdots \times g_n) \circ s$ where $s \in \Sigma_n$. Since $\Sigma \subset G$, we get $f \circ p_1 = p_1 \circ (g_1 \times \cdots \times g_n)$. From Lemma 3.1, we get i). By Theorem 1.5 and the above diagram, in the same way, we get ii). □

4. Proof of Theorem 0.3

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$.

In Proposition 4.2, we shall show that for $n \geq 3$, the covering involution of $\pi : X \to E^{[n]}$ acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. In Proposition 4.5, by using Theorem 3.2 and checking the action to $\mathrm{H}^1(X, \Omega_X^{2n-1}) \cong \mathrm{H}^{2n-1,1}(X)$, we classify involutions of X which act on $\mathrm{H}^2(X, \mathbb{C})$ as the identity. We prove Theorem 0.3 (Theorem 4.7) using those results.

Lemma 4.1. *Let X be a smooth complex manifold, $Z \subset X$ a closed submanifold whose codimension is 2, $\tau : X_Z \to X$ the blow up of X along Z, $E = \tau^{-1}(Z)$ the exceptional divisor, and h the first Chern class of the line bundle $\mathcal{O}_{X_Z}(E)$.*
Then $\tau^* : \mathrm{H}^2(X, \mathbb{C}) \to \mathrm{H}^2(X_Z, \mathbb{C})$ is injective, and

$$\mathrm{H}^2(X_Z, \mathbb{C}) \simeq \mathrm{H}^2(X, \mathbb{C}) \oplus \mathbb{C}h.$$

Proof. Let $U := X \setminus Z$ be an open set of X. Then U is isomorphic to an open set $U' = X_Z \setminus E$ of X_Z. As τ gives a morphism between the pair (X_Z, U') and the pair (X, U), we have a morphism τ^* between the long exact sequence of cohomology relative to these pairs:

$$\begin{tikzcd}
\mathrm{H}^k(X,U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_{X,U}"] & \mathrm{H}^k(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & \mathrm{H}^k(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & \mathrm{H}^{k+1}(X,U,\mathbb{C}) \arrow[d, "\tau^*_{X,U}"] \\
\mathrm{H}^k(X_Z,U',\mathbb{C}) \arrow[r] & \mathrm{H}^k(X_Z,\mathbb{C}) \arrow[r] & \mathrm{H}^k(U',\mathbb{C}) \arrow[r] & \mathrm{H}^{k+1}(X_Z,U',\mathbb{C}).
\end{tikzcd}$$

By Thom isomorphism, the tubular neighborhood Theorem, and Excision theorem, we have

$$\mathrm{H}^q(Z, \mathbb{C}) \simeq \mathrm{H}^{q+4}(X, U, \mathbb{C}), \text{ and}$$

$$\mathrm{H}^q(E, \mathbb{C}) \simeq \mathrm{H}^{q+2}(X_Z, U', \mathbb{C}).$$

In particular, we have

$$\mathrm{H}^l(X, U, \mathbb{C}) = 0 \text{ for } l = 0, 1, 2, 3, \text{ and}$$

$$\mathrm{H}^j(X_Z, U', \mathbb{C}) = 0 \text{ for } l = 0, 1.$$

Thus we have

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\tau^*_{X,U}"] & \mathrm{H}^1(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & \mathrm{H}^1(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & 0 \arrow[d, "\tau^*_{X,U}"] \\
0 \arrow[r] & \mathrm{H}^1(X_Z,\mathbb{C}) \arrow[r] & \mathrm{H}^1(U',\mathbb{C}) \arrow[r] & \mathrm{H}^0(E,\mathbb{C}),
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\tau^*_{X,U}"] & \mathrm{H}^2(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & \mathrm{H}^2(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & 0 \arrow[d, "\tau^*_{X,U}"] \\
\mathrm{H}^0(E,\mathbb{C}) \arrow[r] & \mathrm{H}^2(X_Z,\mathbb{C}) \arrow[r] & \mathrm{H}^2(U',\mathbb{C}) \arrow[r] & \mathrm{H}^3(X_Z,U',\mathbb{C}).
\end{tikzcd}$$

Since $\tau\ |_{U'}\colon U' \xrightarrow{\sim} U$, we have isomorphisms $\tau_U^* : \mathrm{H}^k(U,\mathbb{C}) \simeq \mathrm{H}^k(U',\mathbb{C})$. Thus we have

$$\dim_{\mathbb{C}}\mathrm{H}^2(X_Z,\mathbb{C}) = \dim_{\mathbb{C}}\mathrm{H}^2(X,\mathbb{C}) + 1, \text{ and}$$

$$\tau^* : \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(X_Z,\mathbb{C}) \text{ is injective,}$$

and therefore we obtain

$$\mathrm{H}^2(X_Z,\mathbb{C}) \simeq \mathrm{H}^2(X,\mathbb{C}) \oplus \mathbb{C}h.$$

□

Proposition 4.2. *Suppose $n \geq 3$. For the covering involution ρ of the universal covering space $\pi : X \to E^{[n]}$, the induced map $\rho^* : \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(X,\mathbb{C})$ is the identity.*

Proof. Since the codimension of $X \backslash \pi^{-1}(E_*^{[n]})$ is 2, we get

$$\mathrm{H}^2(X,\mathbb{C}) \cong \mathrm{H}^2(X \setminus \pi^{-1}(F),\mathbb{C}).$$

By Proposition 2.6, $X \backslash \pi^{-1}(E_*^{[n]}) \simeq \mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n / H$.

Let $\tau : \mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n \to K_\circ^n$ be the blow up of $K_\circ^n$ along $\mu^{n-1}(\Delta^n)$,

$$h_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n}(\tau^{-1}(\Delta_\circ^{ij})),$$

and

$$k_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n}(\tau^{-1}(\Gamma_\circ^{ij})).$$

By Lemma 4.1, we have

$$\mathrm{H}^2(\mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n, \mathbb{C}) \cong \mathrm{H}^2(K^n, \mathbb{C}) \oplus \left(\bigoplus_{1 \le i < j \le n} \mathbb{C} h_{ij} \right) \oplus \left(\bigoplus_{1 \le i < j \le n} \mathbb{C} k_{ij} \right).$$

Since $n \geq 3$, there is an isomorphism

$$(j, j+1) \circ \sigma_{ij} \circ (j, j+1) : \Delta_\circ^{ij} \xrightarrow{\sim} \Gamma_\circ^{ij}.$$

Thus we have $\dim_{\mathbb{C}} \mathrm{H}^2(\mathrm{Blow}_{\mu^{n-1}(\Delta^n)} K_\circ^n / H, \mathbb{C}) = 11$, i.e. $\dim_{\mathbb{C}} \mathrm{H}^2(X, \mathbb{C}) = 11$. Since $\mathrm{H}^2(E^{[n]}, \mathbb{C}) = \mathrm{H}^2(X, \mathbb{C})^{\rho^*}$, ρ^* is the identity. $\square$

Since K^n/H is normal, $\pi^{-1}(E)$ is the exceptional divisor (Theorem 2.5) and X is a Calabi-Yau, we have that for an automorphism f of X, $f(\pi^{-1}(D)) = \pi^{-1}(D)$ if and only if $f^* \mathcal{O}_X(\pi^{-1}(D)) = \mathcal{O}_X(\pi^{-1}(D))$ in $\mathrm{Pic}(X)$.

Definition 4.3. Let S be a smooth surface. An automorphism φ of S is numerically trivial if the induced automorphism φ^* of the cohomology ring over $\mathbb{Q}$, $\mathrm{H}^*(S, \mathbb{Q})$ is the identity.

We suppose that an Enriques surface E has numerically trivial involutions. By [8, Proposition 1.1], there is just one numerically trivial involution of E, denoted υ. For υ, there are just two involutions of K which are liftings of υ, one acts on

$\mathrm{H}^0(K,\Omega_K^2)$ as the identity, and another acts on $\mathrm{H}^0(K,\Omega_K^2)$ as $-\mathrm{id}_{\mathrm{H}^0(K,\Omega_K^2)}$, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$.

Let $\upsilon^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς', respectively:

$$\begin{array}{ccc} E^{[n]} & \xrightarrow{\upsilon^{[n]}} & E^{[n]} \\ {\scriptstyle\pi}\uparrow & & {\scriptstyle\pi}\uparrow \\ X & \xrightarrow{\varsigma\,(\varsigma')} & X. \end{array}$$

Then they satisfies $\varsigma = \varsigma' \circ \rho$ where ρ is the covering involution of $\pi : X \longrightarrow E^{[n]}$ and the each order of ς and ς' is 2. From here, we classify involutions acting on $\mathrm{H}^2(X,\mathbb{C})$ as the identity by checking the action to $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$.

Lemma 4.4. $\dim_{\mathbb{C}}\mathrm{H}^{2n-1,1}(K^n/H,\mathbb{C}) = 10$.

Proof. Let ι be the covering involution of $\mu : K \to E$. Put

$$\mathrm{H}^{p,q}_{\pm}(K,\mathbb{C}) := \{\alpha \in \mathrm{H}^{p,q}(K,\mathbb{C}) : \iota^*(\alpha) = \pm\alpha\} \text{ and}$$

$$h^{p,q}_{\pm}(K) := \dim_{\mathbb{C}}\mathrm{H}^{p,q}_{\pm}(K,\mathbb{C}).$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1,\ h^{1,1}(K) = 20,$$

$$h^{0,0}_+(K) = 1,\ h^{1,0}_+(K) = 0,\ h^{2,0}_+(K) = 0\ h^{1,1}_+(K) = 10,$$

$$h^{0,0}_-(K) = 0,\ h^{1,0}_-(K) = 0,\ h^{2,0}_-(K) = 1, \text{ and } h^{2,0}_-(K) = 10.$$

Let

$$\Lambda := \{(s_1,\cdots,s_n,t_1,\cdots,t_n) \in \mathbb{Z}^{2n}_{\geq 0} : \Sigma^n_{i=1}s_i = 2n-1,\ \Sigma^n_{j=1}t_j = 1\}.$$

From the Künneth Theorem, we have

$$\mathrm{H}^{2n-1,1}(K^n,\mathbb{C}) \simeq \bigoplus_{(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\Lambda} \left(\bigotimes_{i=1}^{n} \mathrm{H}^{s_i,t_i}(K,\mathbb{C})\right).$$

We take a base α of $\mathrm{H}^{2,0}(K,\mathbb{C})$ and a base $\{\beta_i\}_{i=1}^{20}$ of $\mathrm{H}^{1,1}(K,\mathbb{C})$ such that $\{\beta_i\}_{i=1}^{10}$ is a base of $\mathrm{H}^{1,1}_{-}(K,\mathbb{C})$ and $\{\beta_i\}_{i=11}^{20}$ is a base of $\mathrm{H}^{1,1}_{+}(K,\mathbb{C})$. Let

$$\tilde{\beta}_i := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \beta_i$ for $j = i$, and

$$\gamma_i := \bigoplus_{j=1}^{n} \tilde{\beta}_j.$$

Then $\{\gamma_i\}_{i=1}^{20}$ is a base of $\mathrm{H}^{2n-1,1}(K^n,\mathbb{C})^{\mathcal{S}_n}$. Since $\iota^*\alpha = -\alpha$, $\iota^*\beta_i = -\beta_i$ for $1 \leq i \leq 10$, and $\iota^*\beta_i = \beta_i$ for $11 \leq i \leq 20$, we obtain

$$\iota_{ij}^*\gamma_i = \gamma_i \text{ for } 1 \leq i \leq 10, \text{ and}$$

$$\iota_{ij}^*\gamma_i = -\gamma_i \text{ for } 11 \leq i \leq 20.$$

Since $\mathrm{H}^{2n-1,1}(K^n/H,\mathbb{C}) \simeq \mathrm{H}^{2n-1,1}(K^n,\mathbb{C})^H$ and $H = \langle \mathcal{S}_n, \{\sigma_{ij}\}_{1\leq i<j\leq n}\rangle$, we obtain

$$\mathrm{H}^{2n-1,1}(K^n/H,\mathbb{C}) = \bigoplus_{i=1}^{10} \mathbb{C}\gamma_i.$$

Thus we get $\dim_{\mathbb{C}}\mathrm{H}^{2n-1,1}(K^n/H,\mathbb{C}) = 10$. □

Recall that $p : K^n \setminus \mu^{n-1}(\Delta^n) \to E^{[n]} \setminus D = E^n \setminus \Delta^n/\Sigma_n$ is the universal covering space.

Proposition 4.5. *We suppose that E has a numerically trivial involution, denoted υ. Let $\upsilon^{[n]}$ be the natural automorphism of $E^{[n]}$ which is induced by υ. Since the* *degree of* $\pi : X \to E^{[n]}$ *is* 2*, there are just two involutions* ζ *and* ζ' *of* X *which are lifts of* $\upsilon^{[n]}$*. Then* ς *and* ς' *do not act on* $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ *as* $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$.

Proof. Since $\upsilon^{[n]}(D) = D$, $\upsilon^{[n]}|_{E^{[n]}\setminus D}$ is an automorphism of $E^{[n]} \setminus D$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \mu^{n-1}(\Delta^n)$ such that $\upsilon^{[n]} \circ p = p \circ g$:

$$\begin{array}{ccc} E^{[n]} \setminus D & \xrightarrow{\upsilon^{[n]}} & E^{[n]} \setminus D \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle p} \\ K^n \setminus \mu^{n-1}(\Delta^n) & \xrightarrow{g} & K^n \setminus \mu^{n-1}(\Delta^n). \end{array}$$

By Proposition 3.1, there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$ for each $1 \le i \le n$, $g_i = g_1$ or $g_i = g_1 \circ \iota$, and $g_1 \circ \iota = \iota \circ g_1$. By Theorem 1.5, we get $K^n \setminus \mu^{n-1}(\Delta^n)/H \simeq X \setminus \pi^{-1}(D)$. Put

$$\upsilon_{+,even} := u_1 \times \cdots \times u_n$$

where

$$u_i = \upsilon_+ \text{ or } u_i = \upsilon_- \text{ and the number of } i \text{ with } u_i = \upsilon_+ \text{ is even.}$$

$\upsilon_{+,even}$ is an automorphism of K^n and induces an automorphism $\widetilde{\upsilon_{+,even}}$ of $K^n \setminus \mu^{n-1}(\Delta^n)/H$. We define automorphisms $\widetilde{\upsilon_{+,odd}}$, $\widetilde{\upsilon_{-,even}}$, and $\widetilde{\upsilon_{-,odd}}$ of $K^n \setminus \mu^{n-1}(\Delta^n)/H$ in the same way. Since $\sigma_{ij} \in H$ for $1 \le i < j \le n$, and $\upsilon_+ = \upsilon_- \circ \iota$, if n is odd,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,even}},\ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,odd}},\ \text{and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}},$$

and if n is even,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,odd}},\ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,even}},\ \text{and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}}.$$

Since $\upsilon^{(n)} \circ \pi_E = \pi_E \circ \upsilon^{[n]}$ and $K^n \setminus \mu^{n-1}(\Delta^n)/H \simeq X \setminus \pi^{-1}(D)$, we have $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,odd}}$ and $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,even}}$ where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $\upsilon^{(n)}$ is the automorphism of $E^{(n)}$ induced by υ. Since the degree of π is 2, we have $\{\varsigma, \varsigma'\} = \{\widetilde{\upsilon_{+,odd}}, \widetilde{\upsilon_{+,even}}\}$. By [8, page 386-389], there is an element $\alpha_{\pm} \in \mathrm{H}^{1,1}_{-}(K, \mathbb{C})$ such that $\upsilon_+^*(\alpha_\pm) = \pm\alpha_\pm$. We fix a basis α of $\mathrm{H}^{2,0}(K, \mathbb{C})$, and let

$$\widetilde{\alpha_{\pm}}_i := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \alpha_\pm$ for $j = i$, and

$$\widetilde{\alpha_\pm} := \bigoplus_{j=1}^{n} \widetilde{\alpha_{\pm}}_i.$$

Since there is a birational map $\varphi : K^n \to X$ by Theorem 1.5, and by the definition of $\widetilde{\upsilon_{+,odd}}$ and $\widetilde{\upsilon_{+,even}}$, we have

$$\widetilde{\upsilon_{+,odd}}^*(\varphi^*(\widetilde{\alpha_+})) = \varphi^*(\widetilde{\alpha_+}) \text{ and } \widetilde{\upsilon_{+,even}}^*(\varphi^*(\widetilde{\alpha_-})) = \varphi^*(\widetilde{\alpha_-}).$$

Thus ς and ς' do not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. $\square$

Definition 4.6. For $n \geq 1$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. A variety Y is called an Enriques quotient of X if there is an Enriques surface E' and a free involution τ of X such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle\tau\rangle$. Here we call two Enriques quotients of X distinct if they are not isomorphic to each other.

Theorem 4.7. *For $n \geq 3$, let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then the number of distinct Enriques quotients of X is one.*

Proof. Let ρ be the covering involution of $\pi : X \to E^{[n]}$ for $n \geq 3$. Since for $n \geq 3$ $\dim_{\mathbb{C}} \mathrm{H}^2(E^{[n]}, \mathbb{C})$ $=\dim_{\mathbb{C}} \mathrm{H}^2(X, \mathbb{C}) = 11$, $\dim_{\mathbb{C}} \mathrm{H}^{2n-1,1}(E'^{[n]}, \mathbb{C}) = 0$, and $\dim_{\mathbb{C}} \mathrm{H}^{2n-1,1}(X, \mathbb{C}) = 10$, we obtain that ρ^* acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, and $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$.

Let φ be an involution of X, which acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity and on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. By Theorem 3.2, for φ, there is an automorphism ϕ of E such that φ is a lift of $\phi^{[n]}$ where $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Furthermore since the order of ϕ is at most 2, the order of φ is 2. Since $\phi^{[n]} \circ \pi = \pi \circ \varphi$, $\phi^{[n]*}$ acts on $\mathrm{H}^2(E^{[n]}, \mathbb{C})$ as the identity. Thus ϕ^* acts on $\mathrm{H}^2(E, \mathbb{C})$ as the identity. If E does not have numerically trivial automorphisms, then $\phi = \mathrm{id}_E$. Thus $\varphi = \rho$.

We assume that ϕ does not the identity map. Then ϕ is numerically trivial. Then $\phi = \upsilon$ and $\varphi \in \{\zeta, \zeta'\}$. By Proposition 4.5, we obtain that φ does not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as $-\mathrm{id}_{\mathrm{H}^{2n-1,1}(X,\mathbb{C})}$. This is a contradiction. Thus $\phi = \mathrm{id}_E$, and we get $\varphi = \rho$. This proves the theorem. $\square$

Theorem 4.8. *For $n \geq 2$, let $\pi : X \to E^{[n]}$ be the universal covering space. For any automorphism φ of X, if φ^* is acts on $\mathrm{H}^*(X, \mathbb{C}) := \bigoplus_{i=0}^{2n} \mathrm{H}^i(X, \mathbb{C})$ as the identity, then $\varphi = \mathrm{id}_X$.*

Proof. By Theorem 3.2, for φ, there is an automorphism ϕ of E such that φ is a lift of $\phi^{[n]}$ where $\phi^{[n]}$ is the natural automorphism of $E^{[n]}$ induced by ϕ. Since φ^* acts on $\mathrm{H}^2(X, \mathbb{C})$ as the identity, ϕ^* acts on $\mathrm{H}^2(E, \mathbb{C})$ as the identity. From [8, page 386-389] the order of ϕ is at most 4.

If the order of ϕ is 2, by Proposition 4.5 φ does not act on $\mathrm{H}^{2n-1,1}(X, \mathbb{C})$ as the identity. This is a contradiction.

If the order of ϕ is 4, then φ^2 is a lift of ${\phi^{[n]}}^2 = {\phi^2}^{[n]}$. Thus by the above, φ^2 does not act on $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ as the identity. This is a contradiction. Thus we have $\phi = \mathrm{id}_E$ and $\varphi \in \{\mathrm{id}_X, \rho\}$. Since ρ does not act on $\mathrm{H}^{2n-1,1}(X,\mathbb{C})$ as the identity, we have $\varphi = \mathrm{id}_X$. $\square$

Corollary 4.9. *For $n \geq 2$, let $\pi : X \to E^{[n]}$ be the universal covering space. For any two automorphisms f and g of X, if $f^* = g^*$ on $\mathrm{H}^*(X,\mathbb{C})$, then $f = g$.*

Theorem 4.10. *For $n \geq 3$, let E be an Enriques surfaces, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space. Then there is an exact sequence:*

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(X) \to \mathrm{Aut}(E^{[n]}) \to 0.$$

Proof. Let f be an automorphism f of X. We put $g = f^{-1} \circ \rho \circ f$. Since for $n \geq 3$ ρ^* acts on $H^2(X,\mathbb{C})$ as the identity and on $H^{2n-1,1}(X)$ as $-\mathrm{id}_{H^{2n-1,1}(X)}$, we get that $g^* = \rho^*$ as automorphisms of $H^2(X,\mathbb{C}) \oplus H^{2n-1,1}(X)$. Like the proof of Theorem 4.8, we have $g = \rho$, i.e. $f \circ \rho = \rho \circ f$. Thus f induces a automorphism of $E^{[n]}$, and we have an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(X) \to \mathrm{Aut}(E^{[n]}) \to 0.$$

$\square$

5. Appendix A

We compute the Hodge number of the universal covering space X of $E^{[2]}$. Let ι be the covering involution of $\mu : K \to E$, and $\tau : \mathrm{Blow}_{\Delta\cup\Gamma} K^2 \to K^2$ the natural map, where $\Gamma = \{(x,y) \in K^2 : y = \iota(x)\}$ and $\Delta = \{(x,x) \in K^2\}$. By Proposition

1.4, we have

$$X \simeq \mathrm{Blow}_{\Delta\cup\Gamma} K^2/H.$$

We put

$$D_\Delta := \tau^{-1}(\Delta) \text{ and}$$

$$D_\Gamma := \tau^{-1}(\Gamma).$$

For two inclusions

$$j_{D_\Delta} : D_\Delta \hookrightarrow \mathrm{Blow}_{\Delta\cup\Gamma} K^2, \text{ and}$$

$$j_{D_\Gamma} : D_\Gamma \hookrightarrow \mathrm{Blow}_{\Delta\cup\Gamma} K^2,$$

let j_{*D_Δ} be the Gysin morphism

$$j_{*D_\Delta} : \mathrm{H}^p(D_\Delta, \mathbb{C}) \to \mathrm{H}^{p+2}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C}),$$

j_{*D_Γ} the Gysin morphism

$$j_{*D_\Gamma} : \mathrm{H}^p(D_\Gamma, \mathbb{C}) \to \mathrm{H}^{p+2}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C}), \text{and}$$

$$\psi := \tau^* + j_{*D_\Delta} \circ \tau|^*_{D_\Delta} + j_{*D_\Gamma} \circ \tau|^*_{D_\Gamma}$$

the morphism from $\mathrm{H}^p(K^2,\mathbb{C}) \oplus \mathrm{H}^{p-2}(\Delta,\mathbb{C}) \oplus \mathrm{H}^{p-2}(\Gamma,\mathbb{C})$ to $\mathrm{H}^p(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C})$. From [13, Theorem 7.31], we have isomorphisms of Hodge structures by ψ:

$$\mathrm{H}^k(K^2,\mathbb{C}) \oplus \mathrm{H}^{k-2}(\Delta,\mathbb{C}) \oplus \mathrm{H}^{k-2}(\Gamma,\mathbb{C}) \simeq \mathrm{H}^k(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C}).$$

Furthermore, for automorphism f of K, let $\bar{f}$ (resp. $\bar{f}_\iota$) be the automorphism of $\mathrm{Blow}_{\Delta\cup\Gamma} K^2$ which is induced by $f \times f$ (resp. $f \times (f \circ \iota)$, f_Δ the automorphism of Δ which is induced by $f \times f$, f_Γ the automorphism of Γ which is induced by $f \times f$, and $\tilde{f}$ the isomorphism from Γ to Δ which is induced by $f \times (f \circ \iota)$. For $\alpha \in \mathrm{H}^*(K^2,\mathbb{C})$, $\beta \in \mathrm{H}^*(\Delta,\mathbb{C})$, and $\gamma \in \mathrm{H}^*(\Gamma,\mathbb{C})$, we obtain

$$\bar{f}^*(\tau^*\alpha) = \tau^*((f \times f)^*\alpha),$$

$$\bar{f}^*(j_{*D_\Delta} \circ \tau|_{D_\Delta}^* \beta) = j_{*D_\Delta} \circ \tau|_{D_\Delta}^* (f_\Delta^* \beta),$$

$$\bar{f}^*(j_{*D_\Gamma} \circ \tau|_{D_\Gamma}^* \gamma) = j_{*D_\Gamma} \circ \tau|_{D_\Gamma}^* (f_\Gamma^* \gamma),$$

$$\bar{f}_\sigma^*(\tau^* \alpha) = \tau^*((f \times (f \circ \iota)^* \alpha),$$

$$\bar{f}_\iota^*(j_{*D_\Delta} \circ \tau|_{D_\Delta}^* \beta) = j_{*D_\Gamma} \circ \tau|_{D_\Delta}^* (\tilde{f}^* \beta),$$

in $\mathrm{H}^*(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C})$.

Theorem 5.1. *For the universal covering space* $\pi : X \to E^{[2]}$, *we have* $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, *and* $h^{2,2}(X) = 131$.

Proof. Since $X \simeq \mathrm{Blow}_{\Delta\cup\Gamma} K^2/H$, we have

$$h^{p,q}(X) = \dim_{\mathbb{C}}\{\alpha \in \mathrm{H}^{p,q}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2, \mathbb{C}) : \ h^*\alpha = \alpha \text{ for } h \in H\}.$$

Let ι be the covering involution of $\mu : K \to E$. We put

$$\mathrm{H}_{\pm}^{p,q}(K, \mathbb{C}) := \{\alpha \in \mathrm{H}^{p,q}(K, \mathbb{C}) : \iota^*(\alpha) = \pm\alpha\} \text{ and}$$

$$h_{\pm}^{p,q}(K) := \dim_{\mathbb{C}} \mathrm{H}_{\pm}^{p,q}(K, \mathbb{C}).$$

From $E = K/\langle\iota\rangle$, we have

$$\mathrm{H}^{p,q}(E, \mathbb{C}) \simeq \mathrm{H}_{+}^{p,q}(K, \mathbb{C}).$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1, \text{ and } h^{1,1}(K) = 20, \text{ and}$$

$$h_{+}^{0,0}(K) = 1,\ h_{+}^{1,0}(K) = 0,\ h_{+}^{2,0}(K) = 0, \text{ and } h_{+}^{1,1}(K) = 10, \text{ and}$$

$$h_{-}^{0,0}(K) = 0,\ h_{-}^{1,0}(K) = 0,\ h_{-}^{2,0}(K) = 1, \text{ and } h_{-}^{2,0}(K) = 10.$$

Recall that H is generated by $\mathcal{S}_2$ and $\iota_{1,2}$. Since $\iota \times \iota(\Delta) = \Delta$ and $\iota \times \iota(\Gamma) = \Gamma$, from $E = K/\langle \iota \rangle$ we have $\Delta/H \simeq E$ and $\Gamma/H \simeq E$. Thus we have

$$h^{0,0}(\Delta/H) = 1,\, h^{1,0}(\Delta/H) = 0,\, h^{2,0}(\Delta/H) = 0,\, h^{1,1}(\Delta/H) = 10,$$

$$h^{0,0}(\Gamma/H) = 1,\, h^{1,0}(\Gamma/H) = 0,\, h^{2,0}(\Gamma/H) = 0,\, \text{and}\, h^{1,1}(\Gamma/H) = 10.$$

From the Künneth Theorem, we have

$$\mathrm{H}^{p,q}(K^2, \mathbb{C}) \simeq \bigoplus_{s+u=p, t+v=q} \mathrm{H}^{s,t}(K, \mathbb{C}) \otimes \mathrm{H}^{u,v}(K, \mathbb{C}), \text{ and}$$

$$\mathrm{H}^{p,q}(K^2/H, \mathbb{C}) \simeq \{\alpha \in \mathrm{H}^{p,q}(K^2, \mathbb{C}) : s^*(\alpha) = \alpha \text{ for } s \in \Sigma_2 \text{ and } \iota_{1,2}^*(\alpha) = \alpha\}.$$

Thus we obtain

$$h^{0,0}(K^2/H) = 1,\, h^{1,0}(K^2/H) = 0,\, h^{2,0}(K^2/H) = 0,\, h^{1,1}(K^2/H) = 10,$$

$$h^{3,0}(K^2/H) = 0,\, h^{2,1}(K^2/H) = 0,\, h^{4,0}(K^2/H) = 1,$$

$$h^{3,1}(K^2/H) = 10, \text{ and}\, h^{2,2}(K^2/H) = 111.$$

We fix a basis β of $\mathrm{H}^{2,0}(K, \mathbb{C})$ and a basis $\{\gamma_i\}_{i=1}^{10}$ of $\mathrm{H}^{1,1}_{-}(K, \mathbb{C})$, then we have

$$\mathrm{H}^{3,1}(K^2/H, \mathbb{C}) \simeq \bigoplus_{i=1}^{10} \mathbb{C}(\beta \otimes \gamma_i + \gamma_i \otimes \beta).$$

By the above equation, we have

$$h^{0,0}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 1,\, h^{1,0}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 0,$$

$$h^{2,0}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 0,\, h^{1,1}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 12,$$

$$h^{3,0}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 0,\, h^{2,1}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 0,$$

$$h^{4,0}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 1,\, h^{3,1}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 10, \text{ and}$$

$$h^{2,2}(\mathrm{Blow}_{\Delta\cup\Gamma} K^2/H) = 131.$$

Thus we obtain $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$. $\square$

6. Appendix B

Now we show that the conjecture in [2, Conjecture 1] is not established for Y an Enriques surface and $L = \Omega_Y^2$.

Let Y be a smooth compact Kähler surface. Recall that $Y^{[n]}$ is the Hilbert scheme of n points of Y, $\pi_Y : Y^{[n]} \to Y^{(n)}$ the Hilbert-Chow morphism, and $p_Y : Y^n \to Y^{(n)}$ the natural projection. For a line bundle L on Y, there is a unique line bundle $\mathcal{L}$ on $Y^{(n)}$ such that $p_Y^*\mathcal{L} = \bigotimes_{i=1}^n p^{i*}L$. By using pull back we have the natural map

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(Y^{[n]}),\ L \mapsto L_n := \pi_Y^*\mathcal{L}.$$

we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} \mathrm{H}^q(Y^{[n]}, \Omega^p_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega^p_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y^{[n]}, L_n) x^p y^q t^n, \text{ and}$$

$$B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \Big(\frac{1}{1 - (-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k)} \Big)^{(-1)^{p+q} h^{p,q}(Y,L)}.$$

Then in [2, Conjecture 1] S. Boissière conjectured that

$$A = B.$$

For Y an Enriques surface and $L = \Omega_Y^2$, as in the proof on Theorem 2.2 and the Serre duality, we have

$$\begin{aligned} h^{2n-1,1}(Y^{[n]}, (\Omega_Y^2)_n) &= \dim_{\mathbb{C}} \mathrm{H}^1(Y^{[n]}, \Omega^{2n-1}_{Y^{[n]}} \otimes \Omega^{2n}_{Y^{[n]}}) \\ &= \dim_{\mathbb{C}} \mathrm{H}^1(Y^{[n]}, T_{Y^{[n]}}) \\ &= 10. \end{aligned}$$

for $n \geq 2$. It follows that the coefficient of x^3yt^2 of A is 10.

We show that the coefficient of x^3yt^2 of B is not 10.

$$h^{0,0}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \Omega_Y^2) = 0.$$

$$h^{0,1}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \Omega_Y^2) = 0.$$

$$h^{0,2}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \mathcal{O}_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \Omega_Y^2) = 1.$$

By Serre duality, we get

$$\Omega_Y \otimes \Omega_Y^2 \simeq T_Y.$$

Since Y is an Enriques surface, we have

$$h^{1,0}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \Omega_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, T_Y) = 0.$$

$$h^{1,1}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \Omega_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, T_Y) = 10.$$

$$h^{1,2}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \Omega_Y \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, T_Y) = 0.$$

Since Y is an Enriques surface, we obtain

$$\Omega_Y^2 \otimes \Omega_Y^2 \simeq \mathcal{O}_Y.$$

$$h^{2,0}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^0(Y, \mathcal{O}_Y) = 1.$$

$$h^{2,1}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^1(Y, \mathcal{O}_Y) = 0.$$

$$h^{2,2}(Y, \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim_{\mathbb{C}} \mathrm{H}^2(Y, \mathcal{O}_Y) = 0.$$

Thus we obtain

$$\begin{aligned}
B &= \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \Big(\frac{1}{1 - (-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k} \Big)^{(-1)^{p+q} h^{p,q}(E, \Omega_E^2)} \\
&= \prod_{k=1}^{\infty} \Big(\frac{1}{1 - x^{k-1} y^{k+1} t^k} \Big) \Big(\frac{1}{1 - x^k y^k t^k} \Big)^{10} \Big(\frac{1}{1 - x^{k+1} y^{k-1} t^k} \Big) \\
&= \prod_{k=1}^{\infty} \Big(\sum_{a=0}^{\infty} (x^{k-1} y^{k+1} t^k)^a \Big) \Big(\sum_{b=0}^{\infty} (x^k y^k t^k)^b \Big)^{10} \Big(\sum_{c=0}^{\infty} (x^{k+1} y^{k-1} t^k)^c \Big).
\end{aligned}$$

Thus we have

$$B \equiv \prod_{k=1}^{2}(1+x^{k-1}y^{k+1}t^k + x^{2k-2}y^{2k+2}t^{2k}) \times (1+x^k y^k t^k + x^{2k}y^{2k}t^{2k})^{10} \times$$

$$(1+x^{k+1}y^{k-1}t^k + x^{2k+2}y^{2k-2}t^{2k}) \ (mod\ t^3)$$

$$\equiv \Big((1+y^2t+y^4t^2)\times(1+xy^3t^2)\Big)\times$$

$$\Big((1+10(xyt+x^2y^2t^2)+45(xyt+x^2y^2t^2)^2)\times(1+x^2y^2t^2)\Big)\times$$

$$\Big((1+x^2t+x^4t^2)\times(1+x^3yt^2)\Big)\ (mod\ t^3)$$

$$\equiv \Big(1+y^2t+(xy^3+y^4)t^2\Big)\times\Big(1+10xyt+56x^2y^2t^2\Big)\times$$

$$\Big(1+x^2t+(x^3y+x^4)t^2\Big)\ (mod\ t^3)$$

$$\equiv \Big(1+(10xy+y^2)t+(56x^2y^2+11xy^3+y^4)t^2\Big)\times$$

$$\Big(1+x^2t+(x^3y+x^4)t^2\Big)\ (mod\ t^3)$$

$$\equiv 1+(x^2+10xy+y^2)t+(x^4+11x^3y+56x^2y^2+11xy^3+y^4)t^2\ (mod\ t^3)$$

Therefore the coefficient of x^3yt^2 of B is 11. The conjecture in [2, Conjecture 1] is not established for Y an Enriques surface and $L=\Omega_Y^2$.

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