Slopes and local invariants of fibered surfaces

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Introduction

Throughout this thesis, we work over the complex number field $\mathbb{C}$. Let $f\colon S \to B$ be a fibered surface of genus g, that is, a surjective morphism from a non-singular projective surface S to a non-singular projective curve B whose general fiber is a non-singular curve of genus g. We assume that the genus g is greater than 1 in the sequel. Let

$K_f = K_S - f^*K_B$ denote the relative canonical bundle of f and put $\chi_f := \deg f_*\mathcal{O}(K_f)$. We always assume that a fibered surface $f\colon S \to B$ is relatively minimal and not locally trivial, i.e., K_f is nef and χ_f is positive. The ratio $\lambda_f := K_f^2/\chi_f$ of the self-intersection number K_f^2 and χ_f is called the *slope of* f. It is well known that the slope λ_f satisfies the inequality

$$\frac{4(g-1)}{g} \leq \lambda_f \leq 12,$$

which is nowadays called the *slope inequality* for fibered surfaces. The slope λ_f attains the lower bound only if the fibration f is hyperelliptic, i.e., a general fiber F is a hyperelliptic curve ([27] and [36]). As to the upper bound, Kodaira [25] constructed examples of fibrations with slope 12, which are nowadays called *Kodaira fibrations*. Thus the inequality $\lambda_f \leq 12$ is sharp among all fibered surfaces. On the other hand, Matsusaka [32] obtained an upper bound smaller than 12 for hyperelliptic case and Xiao [38] improved this bound. In [12], upper bounds for genus 3 fibrations are studied from another point of view.

We studied in [18] primitive cyclic covering fibrations of type (g, h, n). Roughly speaking, it is a fibered surface of genus g obtained as the relatively minimal model of an n sheeted cyclic branched covering of another fibered surface of genus h. Note that hyperelliptic fibrations are nothing more than such fibrations of type $(g, 0, 2)$ and that bielliptic fibrations of genus $g \geq 6$ are those of type $(g, 1, 2)$ (cf. [8] and [15]). Here, a fibration is called *bielliptic* if a general fiber is a bielliptic curve, i.e., a non-singular projective curve obtained as a double covering of an elliptic curve. In [18], we established the lower bound of the slope for such fibrations of type (g, h, n) extending former results for $n = 2$ in [7] and [15]. Furthermore, when $h = 0$, we obtained even the upper bound (expressed as a function in g and n) which is strictly smaller than 12. Recall that known examples of Kodaira fibrations, including Kodaira's original ones [25], are presented as primitive cyclic covering fibrations with $h \geq 2$ and, in fact, there exist such fibrations for any $h \geq 2$ (see, [11] and [23]). Hence, as far as the upper bound of the slope strictly smaller than 12 concerns, the remaining case to be examined is $h = 1$.

The first purpose of the present thesis is to give an affirmative answer to the above mentioned upper bound problem by introducing numerical invariants attached to the singularities of the branch locus of the cyclic covering, and improving coarser estimates in [18]. When $h = 0$, a vertical component of the branch locus on a relatively minimal model is always a non-singular rational curve and this fact makes it much easier to handle singularities on the branch locus. On the other hand, when $h > 0$, we must pay attention to all subcurves of fibers and their singularities in a fibration of genus h, which seems quite terrible. Fortunately enough, when $h = 1$, we have Kodaira's classification of singular fibers [26] from which we know that major components are rational curves and singularities are mild. This gives us a hope to extend results for $h = 0$ to fibrations of type $(g, 1, n)$. In fact, we can show the following:

Theorem 0.1. *Assume that* $g \geq (2n-1)(3n-1)/(n+1)$. *Then, there exists a function* $\mathrm{Ind}\colon \mathcal{A}_{g,1,n} \to \mathbb{Q}_{\geq 0}$ *from the set* $\mathcal{A}_{g,1,n}$ *of all fiber germs of primitive cyclic covering fibrations of type* $(g, 1, n)$ *such that* $\mathrm{Ind}(F_p) = 0$ *for a general* $p \in B$ *and*

$$K_f^2 = \frac{12(n-1)}{2n-1}\chi_f + \sum_{p\in B} \mathrm{Ind}(F_p)$$

for any primitive cyclic covering fibration $f\colon S \to B$ *of type* $(g, 1, n)$.

Theorem 0.2. *Let* $f\colon S \to B$ *be a primitive cyclic covering fibration of type* $(g, 1, n)$. *Then*

$$\lambda_f \leq 12 - \begin{cases} \dfrac{6n^2}{(n+1)(g-1)}, & \text{if } n \geq 4, \text{ or } n = 3 \text{ and } g = 4 \\ \dfrac{24}{4g-17}, & \text{if } n = 3 \text{ and } g > 4, \\ \dfrac{2}{g-2}, & \text{if } n = 2 \text{ and } g \geq 3. \end{cases}$$

In particular, we have the slope equality and the upper bound of the slope for bielliptic fibrations. We remark that the upper bounds in Theorem 0.2 are "fiberwise" sharp as we shall see in Example 3.14. We do not know, however, whether there exist primitive cyclic covering fibrations of type $(g, 1, n)$ whose slopes attain the bounds.

The organization of the first half in the paper is as follows. In §1, we recall basic results from [18] on primitive cyclic covering fibrations and introduce some notation for the later use. In §2, we observe the local concentration of relative invariants of primitive cyclic covering fibrations of type $(g, 1, n)$ on a finite number of fiber germs and show Theorem 0.1. §3 will be devoted to the proof of Theorem 0.2. In the course of the study, we freely use Kodaira's table of singular fibers of elliptic surfaces.

Secondly, we consider fibered surfaces whose general fiber is a plane curve of degree d which are called *plane curve fibrations of degree* d. A plane curve fibration of degree 1 or 2 is a ruled surface and that of degree 3 is nothing but an elliptic surface. In the sequel, we always assume that d is greater that 3. Note that a plane curve fibration of degree 4 is nothing but a non-hyperelliptic fibration of genus 3. Let $\mathcal{A}_d$ be the set of holomorphically equivalence classes of fiber germs whose general fiber is a smooth plane curve of degree d. Then our main theorem for plane curve fibrations is as follows.

Theorem 0.3. *There exists a non-negative function* $\mathrm{Ind}_d\colon \mathcal{A}_d \to \frac{1}{d-2}\mathbb{Z}_{\geq 0}$ *such that for any relatively minimal plane curve fibration* $f\colon S \to B$ *of degree* d, *the value* $\mathrm{Ind}_d(F)$

equals to 0 *for any general fiber* F *of* f *and*

$$K_f^2 = \frac{6(d-3)}{d-2}\chi_f + \sum_{p\in B} \mathrm{Ind}_d(F_p) \tag{0.1}$$

holds, where $F_p := f^{-1}(p)$ *denotes the fiber germ over* $p \in B$.

The value $\mathrm{Ind}_d(F_p)$ is nowadays called a *Horikawa index of* F_p and the equality (0.1) a *slope equality* for plane curve fibrations of degree d (cf. [5]). In the case of $d = 4$, that is, non-hyperelliptic fibrations of genus 3, Theorem 0.3 was first obtained by Reid [33] which was generalized for fibered surfaces of odd genus g whose general fiber has maximal Clifford index by Konno [28]. The lower bound of the slope of plane curve fibrations of degree 5 was obtained by Barja-Stoppino [9].

The strategy of the proof of Theorem 0.3 is as follows. Let $\lambda_d := 6(d-3)/(d-2)$. Given a plane curve fibration $f\colon S \to B$ of degree d, we will show that there is a line bundle $\mathcal{L}$ on S such that the restriction $\mathcal{L}|_F$ to the general fiber F defines the embedding $F \subset \mathbb{P}^2$ in §4. Using the line bundle $\mathcal{L}$, we will show in §5 that the difference $K_f^2 - \lambda_d\chi_f$ can be localized on a finite number of fiber germs, that is, we can define $\mathrm{Ind}_d(F_p)$ for any fiber germ F_p of f. But it seems hard to show the non-negativity of $\mathrm{Ind}_d(F_p)$ directly from the definition. Thus we will show firstly a slope inequality $K_f^2 - \lambda_d\chi_f \geq 0$ in §6. The essential idea of the proof is to apply the Hilbert stability of the Veronese embedding (cf. [24]) to the result of Barja-Stoppino [10]. In order to deduce the non-negativity of the Horikawa index from the slope inequality, we will use an algebraization of any fiber germ in $\mathcal{A}_d$ in §7, the idea of which is due to Terasoma [35]. Roughly speaking, for an arbitrary fiber germ F in $\mathcal{A}_d$, we construct a global plane curve fibration $\overline{f}\colon \overline{S} \to \mathbb{P}^1$ of degree d whose central fiber $\overline{F} = \overline{f}^{-1}(0)$ is an "approximation" of F and any other singular fiber is an irreducible Lefschetz plane curve with one node. Since we can show that $\mathrm{Ind}_d(F_0) = 0$ for an irreducible Lefschetz fiber germ F_0 with one node, we in particular have $\mathrm{Ind}_d(F) = \mathrm{Ind}_d(\overline{F}) = K_{\overline{f}}^2 - \lambda_d\chi_{\overline{f}}$. Thus the slope inequality $K_f^2 - \lambda_d\chi_f \geq 0$ implies the non-negativity of $\mathrm{Ind}_d(F)$ for any fiber germ F in $\mathcal{A}_d$.

In the last 3 sections, we shall focus our attention on local signatures for fibered surfaces. Here, for a closed oriented real 4-manifold X, the *signature of* X is defined to be the signature of the intersection form $H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}$, which is a symmetric bilinear form. We consider the situation that X admits a fibration $f\colon X \to B$ over a closed oriented surface B. Under some conditions, the signature of X happens to localize around a finite number of fiber germs $F_1, F_2, \ldots, F_m$:

$$\mathrm{Sign}(X) = \sum_{i=1}^{m} \sigma(F_i).$$

We call this phenomenon a *localization of the signature* and the value $\sigma(F_i)$ a *local signature of* F_i. A first example of local signatures is the one for genus 1 fibrations due to Matsumoto [30]. He also gave a local signature for Lefschetz fibrations of genus 2 in [31], which was generalized by Endo [17] for hyperelliptic fibrations. Later, Kuno [29] defined a local signature for plane curve fibrations, which includes non-hyperelliptic fibrations of genus 3. On the other hand, Horikawa [22] defined a function $\mathrm{Ind}(F)$ on the set of holomorphic fiber germs F of genus 2, which is nowadays called a Horikawa index, in order to study algebraic surfaces of general type near the Noether line. Once a Horikawa index is defined (for a certain type of holomorphic fibrations), we can define a local signature by using it, as shown in [5]. After Horikawa's work, Xiao [37] and Arakawa-Ashikaga [1] defined a Horikawa index and a local signature for hyperelliptic fibrations. Terasoma [35] showed the coincidence of Endo's local signature and Arakawa-Ashikaga's one. For non-hyperelliptic genus 3 fibrations, Reid [33] defined a Horikawa index. Similarly to Terasoma's proof, Kuno's local signature and Reid's one for non-hyperelliptic fibrations of genus 3 also coincide (cf. [4]).

In §8, we will discuss the signature of surfaces with plane curve fibrations. We can define a local signature for plane curve fibrations by using the Horikawa index in Theorem 0.3 (cf. [5]). We will show the coincidence of this local signature and Kuno's one similarly as in [35].

In §9 and §10, we construct a local signature associated with an effective divisor D on the moduli space $\mathcal{M}_g$ of smooth curves of genus g and compute some examples of local signatures for general fibrations of genus 2 or 3, which are different from Endo-Arakawa-Asikaga's one and Kuno-Reid's one. The idea of constructions is essentially due to Ashikaga-Yoshikawa [6], who called the divisor $4\lambda - \delta$ on the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g the *signature divisor* and gave a local signature by pulling back the signature divisor using a geometric meaningful effective divisor D, e.g., the Brill-Noether locus, via the moduli map of a fiber germ. Replacing D by another effective divisor, the associated local signature varies. We compute local signatures in the case that $g = 2$ and D is the bielliptic locus and that $g = 3$ and D is the locus of curves having a hyperflex.

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1 Preliminaries

In this section, we recall and state basic results for primitive cyclic covering fibrations in [18].

Definition 1.1. A relatively minimal fibration $f\colon S\to B$ of genus $g\geq 2$ is called a primitive cyclic covering fibration of type (g,h,n), if there exist a (not necessarily relatively minimal) fibration $\widetilde{\varphi}\colon \widetilde{W}\to B$ of genus $h\geq 0$, and a classical n-cyclic covering

$$\widetilde{\theta}\colon \widetilde{S}=\mathrm{Spec}_{\widetilde{W}}\left(\bigoplus_{j=0}^{n-1}\mathcal{O}_{\widetilde{W}}(-j\widetilde{\mathfrak{d}})\right)\to \widetilde{W}$$

branched over a smooth curve $\widetilde{R}\in|n\widetilde{\mathfrak{d}}|$ for some $n\geq 2$ and $\widetilde{\mathfrak{d}}\in\mathrm{Pic}(\widetilde{W})$ such that f is the relatively minimal model of $\widetilde{f}:=\widetilde{\varphi}\circ\widetilde{\theta}$.

Let $f\colon S\to B$ be a primitive cyclic covering fibration of type (g,h,n). We freely use the notation in Definition 1.1. Let $\widetilde{F}$ and $\widetilde{\Gamma}$ be general fibers of $\widetilde{f}$ and $\widetilde{\varphi}$, respectively. Then the restriction map $\widetilde{\theta}|_{\widetilde{F}}\colon \widetilde{F}\to\widetilde{\Gamma}$ is a classical n-cyclic covering branched over $\widetilde{R}\cap\widetilde{\Gamma}$. Since the genera of $\widetilde{F}$ and $\widetilde{\Gamma}$ are g and h, respectively, the Hurwitz formula gives us

$$r:=\widetilde{R}\widetilde{\Gamma}=\frac{2(g-1-n(h-1))}{n-1}. \tag{1.1}$$

Note that r is a multiple of n. Let $\widetilde{\sigma}$ be a generator of $\mathrm{Aut}(\widetilde{S}/\widetilde{W})\simeq\mathbb{Z}/n\mathbb{Z}$ and $\rho\colon\widetilde{S}\to S$ the natural birational morphism. By assumption, $\mathrm{Fix}(\widetilde{\sigma})$ is a disjoint union of smooth curves and $\widetilde{\theta}(\mathrm{Fix}(\widetilde{\sigma}))=\widetilde{R}$. Let $\varphi\colon W\to B$ be a relatively minimal model of $\widetilde{\varphi}$ and $\widetilde{\psi}\colon\widetilde{W}\to W$ the natural birational morphism. Since $\widetilde{\psi}$ is a succession of blow-ups, we can write $\widetilde{\psi}=\psi_1\circ\cdots\circ\psi_N$, where $\psi_i\colon W_i\to W_{i-1}$ denotes the blow-up at $x_i\in W_{i-1}$ $(i=1,\dots,N)$ with $W_0=W$ and $W_N=\widetilde{W}$. We define reduced curves R_i on W_i inductively as $R_{i-1}=(\psi_i)_*R_i$ starting from $R_N=\widetilde{R}$ down to $R_0=:R$. We also put $E_i=\psi_i^{-1}(x_i)$ and $m_i=\mathrm{mult}_{x_i}(R_{i-1})$ for $i=1,2,\dots,N$.

Lemma 1.2. *With the above notation, the following hold for any $i=1,\dots,N$.*

(1) *Either $m_i\in n\mathbb{Z}$ or $m_i\in n\mathbb{Z}+1$. Moreover, $m_i\in n\mathbb{Z}$ holds if and only if E_i is not contained in R_i.*

(2) $R_i=\psi_i^*R_{i-1}-n\left[\dfrac{m_i}{n}\right]E_i$, *where $[t]$ is the greatest integer not exceeding t.*

(3) *There exists $\mathfrak{d}_i\in\mathrm{Pic}(W_i)$ such that $\mathfrak{d}_i=\psi_i^*\mathfrak{d}_{i-1}$ and $R_i\sim n\mathfrak{d}_i$, $\mathfrak{d}_N=\widetilde{\mathfrak{d}}$.*

Let E be a (-1)-curve on a fiber of $\widetilde{f}$. If E is not contained in $\mathrm{Fix}(\widetilde{\sigma})$, then $L:=\widetilde{\theta}(E)$ is a (-1)-curve and $\widetilde{\theta}^*L$ is the sum of n disjoint (-1)-curves containing E. Contracting them and L, we may assume that any (-1)-curve on a fiber of $\widetilde{f}$ is contained in $\mathrm{Fix}(\widetilde{\sigma})$. Then $\widetilde{\sigma}$ induces an automorphism σ of S over B and ρ is the blow-up of all isolated fixed points of σ (cf. [18]). One sees easily that there is a one-to-one correspondence between $(-k)$-curves contained in $\mathrm{Fix}(\widetilde{\sigma})$ and $(-kn)$-curves contained in $\widetilde{R}$ via $\widetilde{\theta}$. Hence, the number of blow-ups in ρ is that of vertical $(-n)$-curves contained in $\widetilde{R}$.

From Lemma 1.2, we have

$$K_{\widetilde{\varphi}} = \widetilde{\psi}^* K_{\varphi} + \sum_{i=1}^{N} \mathbf{E}_i, \tag{1.2}$$

$$\widetilde{\mathfrak{d}} = \widetilde{\psi}^* \mathfrak{d} - \sum_{i=1}^{N} \left[\frac{m_i}{n}\right] \mathbf{E}_i, \tag{1.3}$$

where $\mathbf{E}_i$ denotes the total transform of E_i. Since

$$K_{\widetilde{S}} = \widetilde{\theta}^* \left(K_{\widetilde{W}} + (n-1)\widetilde{\mathfrak{d}}\right)$$

and

$$\chi(\mathcal{O}_{\widetilde{S}}) = n\chi(\mathcal{O}_{\widetilde{W}}) + \frac{1}{2}\sum_{j=1}^{n-1} j\widetilde{\mathfrak{d}}(j\widetilde{\mathfrak{d}} + K_{\widetilde{W}}),$$

we get

$$K_{\widetilde{f}}^2 = n(K_{\widetilde{\varphi}}^2 + 2(n-1)K_{\widetilde{\varphi}}\widetilde{\mathfrak{d}} + (n-1)^2\widetilde{\mathfrak{d}}^2), \tag{1.4}$$

$$\chi_{\widetilde{f}} = n\chi_{\widetilde{\varphi}} + \frac{1}{2}\sum_{j=1}^{n-1} j\widetilde{\mathfrak{d}}(j\widetilde{\mathfrak{d}} + K_{\widetilde{\varphi}}). \tag{1.5}$$

Definition 1.3 (Singularity index α). Let k be a positive integer. For $p \in B$, we consider all the singular points (including infinitely near ones) of R on the fiber Γ_p of $\varphi\colon W \to B$ over p. We let $\alpha_k(F_p)$ be the number of singular points of multiplicity either kn or $kn+1$ among them, and call it the *k-th singularity index* of F_p, the fiber of $f\colon S \to B$ over p. Clearly, we have $\alpha_k(F_p) = 0$ except for a finite number of $p \in B$. We put $\alpha_k = \sum_{p\in B} \alpha(F_p)$ and call it the *k-th singularity index* of f.

Let D_1 be the sum of all $\widetilde{\varphi}$-vertical $(-n)$-curves contained in $\widetilde{R}$ and put $\widetilde{R}_0 = \widetilde{R} - D_1$. We denote by $\alpha_0(F_p)$ the ramification index of $\widetilde{\varphi}|_{\widetilde{R}_0}\colon \widetilde{R}_0 \to B$ over p, that is, the ramification index of $\widetilde{\varphi}|_{(\widetilde{R}_0)_h}\colon (\widetilde{R}_0)_h \to B$ over p minus the sum of the topological Euler number of irreducible components of $(\widetilde{R}_0)_v$ over p. Then $\alpha_0(F_p) = 0$ except for a finite number of $p \in B$, and we have

$$\sum_{p\in B} \alpha_0(F_p) = (K_{\widetilde{\varphi}} + \widetilde{R}_0)\widetilde{R}_0$$

by definition. We put $\alpha_0 = \sum_{p\in B} \alpha(F_p)$ and call it the *0-th singularity index* of f.

Let $\varepsilon(F_p)$ be the number of (-1)-curves contained in $\widetilde{F}_p$, and put $\varepsilon = \sum_{p\in B}\varepsilon(F_p)$. This is no more than the number of blowing-ups appearing in $\rho\colon \widetilde{S}\to S$.

From (1.2) and (1.3), we have

$$\begin{aligned}(K_{\widetilde{\varphi}}+\widetilde{R})\widetilde{R} &= \left(\widetilde{\psi}^*(K_\varphi+R)+\sum_{i=1}^{N}\left(1-n\left[\frac{m_i}{n}\right]\right)\mathbf{E}_i\right)\left(\widetilde{\psi}^*R-n\left[\frac{m_i}{n}\right]\mathbf{E}_i\right)\\ &=(K_\varphi+R)R-\sum_{i=1}^{N}n\left[\frac{m_i}{n}\right]\left(n\left[\frac{m_i}{n}\right]-1\right)\\ &=(K_\varphi+R)R-n\sum_{k\geq 1}k(nk-1)\alpha_k. \end{aligned} \tag{1.6}$$

On the other hand, we have

$$(K_{\widetilde{\varphi}}+\widetilde{R})\widetilde{R}=(K_{\widetilde{\varphi}}+\widetilde{R}_0)\widetilde{R}_0+D_1(K_{\widetilde{\varphi}}+D_1)=\alpha_0-2\varepsilon. \tag{1.7}$$

Hence,

$$(K_\varphi+R)R=n\sum_{k\geq 1}k(nk-1)\alpha_k+\alpha_0-2\varepsilon. \tag{1.8}$$

by (1.6) and (1.7). Since $K_f^2=K_{\widetilde{f}}^2+\varepsilon$, $\chi_{\widetilde{f}}=\chi_f$, (1.2), (1.3), (1.4) and (1.5), we get

$$K_f^2=nK_\varphi^2+2(n-1)K_\varphi R+\frac{(n-1)^2}{n}R^2-\sum_{k\geq 1}((n-1)k-1)^2\alpha_k+\varepsilon \tag{1.9}$$

and

$$\chi_f=n\chi_\varphi+\frac{(n-1)(2n-1)}{12n}R^2+\frac{n-1}{4}K_\varphi R-\frac{n(n-1)}{12}\sum_{k\geq 1}((2n-1)k^2-3k)\alpha_k. \tag{1.10}$$

From (1.8), (1.9), (1.10) and Noether's formula, we have

$$e_f=ne_\varphi+n\sum_{k\geq 1}\alpha_k+(n-1)\alpha_0-(2n-1)\varepsilon. \tag{1.11}$$

We define some notation for the later use. For a vertical divisor T and $p\in B$, we denote by $T(p)$ the greatest subdivisor of T consisting of components of the fiber over p. Then $T=\sum_{p\in B}T(p)$. We consider a family $\{L^i\}_i$ of vertical irreducible curves in $\widetilde{R}$ over p satisfying:

(i) L^1 is the proper transform of an irreducible curve Γ^1 contained in the fiber Γ_p or a (-1)-curve E^1 appearing in $\widetilde{\psi}$.

(ii) For $i \geq 2$, L^i is the proper transform of an irreducible curve Γ^i contained in the fiber Γ_p intersecting Γ^k for some $k < i$ or an exceptional (-1)-curve E^i that contracts to a point x^i on C^k (or on its proper transform) for some $k < i$, where we define C^j to be E^j or Γ^j according to whether L^j is the proper transform of which curve.

(iii) $\{L^i\}_i$ is the largest among those satisfying (i) and (ii).

The set of all vertical irreducible curves in $\widetilde{R}$ over p is decomposed into the disjoint union of such families uniquely. We denote it as

$$\widetilde{R}_v(p) = D^1(p) + \cdots + D^{\eta_p}(p), \quad D^t(p) = \sum_{k \geq 1} L^{t,k}$$

where η_p denotes the number of the decomposition and $\{L^{t,k}\}_k$ satisfies (i), (ii), (iii). Let $C^{t,k}$ be the exceptional curve or the component of the fiber Γ_p the proper transform of which is $L^{t,k}$. Let $D'^t(p)$ be the sum of all irreducible components of $D^t(p)$ which are the proper transforms of curves contained in Γ_p and $D''^t(p) = D^t(p) - D'^t(p)$. Let η'_p be the cardinality of the set $\{t = 1, \ldots, \eta_p | D'^t(p) \neq 0\}$ and $\eta''_p = \eta_p - \eta'_p$.

Definition 1.4 (Index j). Let $j_{b,a}(F_p)$ (resp. $j^t_{b,a}(F_p)$, $j'^t_{b,a}(F_p)$, $j''^t_{b,a}(F_p)$) be the number of irreducible curves of genus b with self-intersection number $-an$ contained in $\widetilde{R}_v(p)$ (resp. $D^t(p)$, $D'^t(p)$, $D''^t(p)$). Put

$$j^t_{\bullet,a}(F_p) = \sum_{b \geq 0} j^t_{b,a}(F_p), \quad j^t_{b,\bullet}(F_p) = \sum_{a \geq 0} j^t_{b,a}(F_p), \quad j_{b,a}(F_p) = \sum_{t \geq 1} j^t_{b,a}(F_p).$$

Similarly, we define $j^t(F_p) = j^t_{\bullet,\bullet}(F_p)$, $j'^t_{\bullet,a}(F_p)$, $j''^t_{\bullet,a}(F_p)$, etc. Clearly, we have $j''_{b,\bullet}(F_p) = 0$ for any $b \geq 1$ by the definition of $D''^t(p)$.

Rearranging the index if necessary, we may assume that $D'^t(p) = \sum_{k=1}^{j'^t(F_p)} L^{t,k}$, $D''^t(p) = \sum_{k=j'^t(F_p)+1}^{j^t(F_p)} L^{t,k}$. Put $L'^{t,k} = L^{t,k}$, $L''^{t,k} = L^{t,j'^t(F_p)+k}$, $C'^{t,k} = C^{t,k}$, $C''^{t,k} = C^{t,j'^t(F_p)+k}$.

Let $\alpha_0^+(F_p)$ be the ramification index of $\widetilde{\varphi} : \widetilde{R}_h \to B$ over p and put $\alpha_0^-(F_p) = \alpha_0(F_p) - \alpha_0^+(F_p)$. It is clear that $\varepsilon(F_p) = j_{0,1}(F_p)$ and $\alpha_0^-(F_p) = \sum_{b \geq 0}(2b-2)j_{b,\bullet}(F_p) + 2\varepsilon(F_p)$.

Let $\overline{\eta}_p$ be the number of $t = 1, \ldots, \eta_p$ such that $j^t(F_p) = j''^t_{0,1}(F_p)$ and $\widehat{\eta}_p = \eta''_p - \overline{\eta}_p$.

Definition 1.5 (Vertical type singularity). Let x be a singular point of R. For $t = 1, \ldots, \eta_p$ and $u \geq 1$, x is a (t,u)-*vertical type singularity* or simply a u-vertical type singularity if the number of $C^{t,k}$'s whose proper transforms pass through x is u. If x is a (t,u)-vertical type singularity and the multiplicity of it belongs to $n\mathbb{Z}$ (resp. $n\mathbb{Z}+1$), we call it a (t,u)-vertical $n\mathbb{Z}$ type singularity (resp. (t,u)-vertical $n\mathbb{Z}+1$ type singularity).

Let $\iota^{t,(u)}(F_p)$, $\kappa^{t,(u)}(F_p)$ respectively be the number of (t,u)-vertical $n\mathbb{Z}$, $n\mathbb{Z}+1$ type singularities over p and put

$$\iota^t(F_p) = \sum_{u\geq 1}(u-1)\iota^{t,(u)}(F_p), \quad \kappa^t(F_p) = \sum_{u\geq 1}(u-1)\kappa^{t,(u)}(F_p),$$

$\iota(F_p) = \sum_{t=1}^{\eta_p}\iota^t(F_p)$ and $\kappa(F_p) = \sum_{t=1}^{\eta_p}\kappa^t(F_p)$. Let $\iota_k^{t,(u)}(F_p)$, $\kappa_k^{t,(u)}(F_p)$ respectively be the number of (t,u)-vertical type singularities with multiplicity kn, $kn+1$ and we define $\iota_k^t(F_p)$, $\kappa_k^t(F_p)$, $\iota_k(F_p)$ and $\kappa_k(F_p)$ similarly.

Definition 1.6 (Indices α', α''). We say that a singular point x of R is *involved in* $D^t(p)$ if there exists $C^{t,k}$ such that it or its proper transform passes through x or it contracts to x. A singular point x of R is involved in $\widetilde{R}_v(p)$ if it is involved in $D^t(p)$ for some t. Let $\alpha'_k(F_p)$ (resp. $\alpha''_k(F_p)$) denote the number of singularities with multiplicity kn or $kn+1$ over p not involved in $\widetilde{R}_v(p)$ (resp. involved in $\widetilde{R}_v(p)$). Clearly, we have $\alpha_k(F_p) = \alpha'_k(F_p) + \alpha''_k(F_p)$. Let $\alpha''^t_k(F_p)$ denote the number of singularities with multiplicity kn or $kn+1$ over p involved in $D^t(p)$. Then, we have $\alpha''_k(F_p) = \sum_{t=1}^{\eta_p}\alpha''^t_k(F_p)$ by the definition of the decomposition $\widetilde{R}_v(p) = D^1(p) + \cdots + D^{\eta_p}(p)$. Let $\alpha_k^{n\mathbb{Z}}(F_p)$, $\alpha_k^{n\mathbb{Z}+1}(F_p)$ respectively denote the number of singularities with multiplicity kn, $kn+1$ over p. Similarly, we define $\alpha''^{n\mathbb{Z}}_k(F_p)$, $\alpha''^{n\mathbb{Z}+1}_k(F_p)$, etc.

Definition 1.7 (Singularity of type $(i \to i)$). Suppose that $n=2$. If the exceptional curve E_x of the blow-up at a singularity x of R with multiplicity $2k+1$ contains only one singularity y, then the multiplicity at y is $2k+2$ and E_x contributes to $j''_{0,1}(F_p)$. Conversely, the exceptional curve E contributing to $j''_{0,1}(F_p)$ has such a pair (x,y). Then we call the pair (x,y) a singularity of type $(2k+1 \to 2k+1)$ (cf. [37]). Let $\alpha_{(2k+1\to 2k+1)}(F_p)$ be the number of singularities of type $(2k+1 \to 2k+1)$ over p (i.e., $s_{2k+1}(F_p)$ in the notation of [37]). Then we have

$$j''_{0,1}(F_p) = \sum_{k\geq 1}\alpha_{(2k+1\to 2k+1)}(F_p). \tag{1.12}$$

We decompose

$$\alpha_{(2k+1\to 2k+1)}(F_p) = \alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p) + \alpha^{\mathrm{co}}_{(2k+1\to 2k+1)}(F_p)$$

and

$$\alpha^{\mathrm{co}}_{(2k+1\to 2k+1)}(F_p) = \alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p) + \alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)$$

as follows. Let $\alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p)$ be the number of singularities of type $(2k+1 \to 2k+1)$ over p at which any local branch of R_h intersects the fiber over p transversely. Let $\alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p)$ (resp. $\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)$) be the number of singularities (x,y) of type

$(2k+1 \to 2k+1)$ over p such that the proper transform of the vertical component passing through x also passes through y and is not contained in R (resp. is contained in R).

Notation 1.8. For a condition or a Roman numeral $\mathcal{P}$, we put $\delta_{\mathcal{P}} = 1$ if the condition $\mathcal{P}$ holds or Γ_p is a singular fiber of type $\mathcal{P}$, and $\delta_{\mathcal{P}} = 0$ otherwise.

Let $C = C^{t,k}$ and assume that it is smooth. If C is on W_i, we drop the index and set $R = R_i$ for simplicity. Let $R' = R - C$. Let $x_1, \dots, x_l$ be all the points of $C \cap R'$. We put $x_{i,1} = x_i$ and $m_{i,1} = m_i$. We define $\psi_{i,1} \colon W_{i,1} \to W$ to be the blow-up at $x_{i,1}$ and put $E_{i,1} = \psi_{i,1}^{-1}(x_{i,1})$ and $R_{i,1} = \psi_{i,1}^{*}R - n[m_{i,1}/n]E_{i,1}$. Inductively, we define $x_{i,j}$, $m_{i,j}$ to be the intersection point of the proper transform of C and $E_{i,j-1}$, the multiplicity of $R_{i,j-1}$ at $x_{i,j}$, and if $m_{i,j} > 1$, we define $\psi_{i,j} \colon W_{i,j} \to W_{i,j-1}$, $E_{i,j}$ and $R_{i,j}$ to be the blow-up at $x_{i,j}$, the exceptional curve for $\psi_{i,j}$ and $R_{i,j} = \psi_{i,j}^{*}R_{i,j-1} - n[m_{i,j}/n]E_{i,j}$, respectively. Put $i_{\mathrm{bm}} = \max\{j \mid m_{i,j} > 1\}$, that is, the number of blowing-ups occuring over x_i. We may assume that $i_{\mathrm{bm}} \geq (i+1)_{\mathrm{bm}}$ for $i = 1, \dots, l-1$ after rearranging the index if necessary. Put $t = R'C$ and $c = \sum_{i=1}^{l} i_{\mathrm{bm}}$. If C is a fiber Γ of φ, t is the number of branch points r. If C is an exceptional curve, t is the multiplicity of R at the point to which C is contracted. Clearly, c is the number of blow-ups on C. Set $d_{i,j} = [m_{i,j}/n]$. Then the following lemmas hold (cf. [18]).

Lemma 1.9 ([18]). *We have*

$$\frac{t+c}{n} = \sum_{i=1}^{l} \sum_{j=1}^{i_{\mathrm{bm}}} d_{i,j}. \tag{1.13}$$

This is a special case of the following lemma:

Lemma 1.10. *Let $f \colon S \to B$ be a primitive cyclic covering fibration of type (g, h, n). Let C be a curve contained in R, L the proper transform of C on $\widetilde{W}$ and x_1, x_2,..., x_c all the singularities of R on C (including infinitely near ones). Put $m_i = \mathrm{mult}_{x_i}(R)$, $k_i = \mathrm{mult}_{x_i}(C)$ and $d_i = [m_i/n]$. Then, we have*

$$\frac{RC - L^2}{n} = \sum_{i=1}^{c} k_i d_i.$$

Proof. We may assume that ψ_i is the blow-up at x_i for $i = 1, \dots, c$. Then, we can write $L = \widetilde{\psi}^{*}C - \sum_{i=1}^{c} k_i \mathbf{E}_i$ and $\widetilde{R} = \widetilde{\psi}^{*}R - \sum_{i=1}^{N} nd_i \mathbf{E}_i$. Thus, we have $\widetilde{R}L = RC - \sum_{i=1}^{c} nk_i d_i$. On the other hand, since $\widetilde{R} - L$ and L are disjoint, we have $L^2 = \widetilde{R}L$. From these equalities, the assertion follows. □

Lemma 1.11 ([18]). *The following hold.*

(1) *When* $n \geq 3$*, then* $m_{i,j} \geq m_{i,j+1}$*. When* $n = 2$*, then* $m_{i,j} + 1 \geq m_{i,j+1}$ *with equality holds only if* $m_{i,j-1} \in 2\mathbb{Z}$ *(if* $j > 1$*) and* $m_{i,j} \in 2\mathbb{Z} + 1$*.*

(2) *If* $m_{i,j-1} \in n\mathbb{Z} + 1$ *and* $m_{i,j} \in n\mathbb{Z}$*, then* $m_{i,j} > m_{i,j+1}$*.*

(3) $m_{i,i_{\mathrm{bm}}} \in n\mathbb{Z}$*.*

Definition 1.12. By using the datum $\{m_{i,j}\}$, one can construct a diagram as in Table 1. We call it the *singularity diagram* of C.

Table1 singularity diagram

#		
$(x_{1,1_{\mathrm{bm}}}, m_{1,1_{\mathrm{bm}}})$		
$\cdots$		#
		$(x_{l,l_{\mathrm{bm}}}, m_{l,l_{\mathrm{bm}}})$
	$\cdots$	$\cdots$
$(x_{1,1}, m_{1,1})$	$\cdots$	$(x_{l,1}, m_{l,1})$

On the top of the i-th column (indicated by # in Table 1), we write $\# = (i_{\max} - i_{\mathrm{bm}})$ if $i_{\mathrm{bm}} < i_{\max}$ and leave it blank when $i_{\mathrm{bm}} = i_{\max}$. We say that the singularity diagram of C is *of type* 0 (resp. *of type* 1) if $C \not\subset R$ (resp. $C \subset R$).

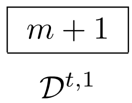
Definition 1.13. Let $\mathcal{D}^{t,k}$ be the singularity diagram of $C^{t,k}$. We call $\mathcal{D}^{t,1}, \mathcal{D}^{t,2}, \ldots, \mathcal{D}^{t,j^t(F_p)}$ a *sequence of singularity diagrams associated with* $D^t(p)$.

Then the following lemma is clear.

Lemma 1.14. *Let* $\mathcal{D}^{t,1}, \mathcal{D}^{t,2}, \ldots, \mathcal{D}^{t,j^t(F_p)}$ *be a sequence of singularity diagrams associated with* $D^t(p)$*. Let* $l^{t,k} := \#(R' \cap C^{t,k})$ *and* $(x^{t,k}_{i,j}, m^{t,k}_{i,j})$*,* $i = 1, \ldots, l^{t,k}$*,* $j = 1, \ldots, i_{\mathrm{bm}}$ *denote entries of* $\mathcal{D}^{t,k}$*. Let* $(x^{t,p}_{i,j}, m^{t,p}_{i,j})$ *be a singularity on* $C^{t,p}$ *such that* $m^{t,p}_{i,j} \in n\mathbb{Z} + 1$*, and* $m^{t,p}_{i,j-1} \in n\mathbb{Z}$ *when* $j > 1$*. Let* $q > p$ *be the integer such that* $C^{t,q}$ *is the exceptional curve for the blow-up at* $x^p_{i,j}$*. Then, for every* $1 \leq p' \leq p$*,* i'*,* j' *satisfying* $(x^{t,p'}_{i',j'}, m^{t,p'}_{i',j'}) = (x^{t,p}_{i,j}, m^{t,p}_{i,j})$*, the diagram* $\mathcal{D}^{t,q}$ *has* $(x^{t,p'}_{i',j'+1}, m^{t,p'}_{i',j'+1})$ *as an entry in the bottom row.*

Example 1.15. Suppose that t contributes to $\overline{\eta}_p$. Then $C^{t,k} = C''^{t,k}$ is a (-1)-curve and blown up $n-1$ times for any k.

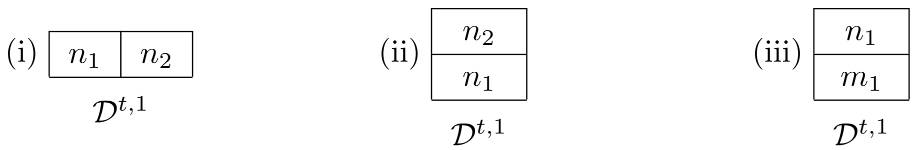
(1) If $n = 2$, then the point to which $C^{t,1}$ is contracted is a singularity of type $(m \to m)$ for some odd integer m. Indeed, $R'C^{t,1} = m$ and from Lemma 1.9, the singularity diagram of $C^{t,1}$ is the following:



where we drop the symbol indicating the singular point on $C^{t,1}$ for simplicity. Since $m+1$ is even, we have $j^t(F_p)=1$. This observation gives us

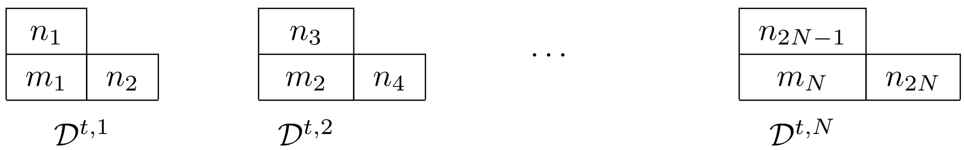
$$\overline{\eta}_p = \sum_{k\geq 1}\left(\alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p) + \alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p)\right). \tag{1.14}$$

(2) Suppose that $n=3$. Let m be the multiplicity of the singular point to which $C^{t,1}$ is contracted. Then $R'C^{t,1}=m$ and from Lemma 1.9, all possible singularity diagram of $C^{t,1}$ are the following:



where the integers $n_i \in 3\mathbb{Z}$ and $m_i \in 3\mathbb{Z}+1$ satisfy that $m+2=n_1+n_2$ in the case (i), $m+2=n_1+n_2$ and $n_2\leq n_1$ in the case (ii), $m+3=m_1+n_1$ and $n_1<m_1$ in the case (iii). If the diagram $\mathcal{D}^{t,1}$ is (i) or (ii), then $j^t(F_p)=1$ since there are no $3\mathbb{Z}+1$ type singularities on $C^{t,1}$. If the diagram $\mathcal{D}^{t,1}$ is (iii), then $j^t(F_p)>1$ and the singularity diagram $\mathcal{D}^{t,2}$ of $C^{t,2}$ which is obtained by the blow-up at the singularity with multiplicity m_1 is (i) or (ii) from Lemma 1.14. Thus we have $j^t(F_p)=2$.

(3) When $n\geq 4$, then the number $j^t(F_p)$ is not bounded. For example, we can consider the following sequence of singularity diagrams associated with $D^t(p)$ when $n=4$:



where $n_k\in 4\mathbb{Z}$, $m_k\in 4\mathbb{Z}+1$ and $C^{t,k}$, $k\geq 2$ is the exceptional curve obtained by the blow-up at the multiplicity on $C^{t,k-1}$ with multiplicity m_{k-1}. From Lemma 1.9, we have $m_k+4=m_{k+1}+n_{2k+1}+n_{2k+2}$ for any $k\geq 1$.

Recall that the gonality $\mathrm{gon}(C)$ of a non-singular projective curve C is the minimum of the degree of morphisms onto $\mathbb{P}^1$. The gonality $\mathrm{gon}(f)$ of a fibered surface $f\colon S \to B$ is defined to be that of a general fiber (cf. [28]).

Proposition 1.16. *Let $\theta\colon F \to \Gamma$ be a totally ramified covering of degree n between smooth projective curves branched over r points. If $r \geq 2n\,\mathrm{gon}(\Gamma)$, then $\mathrm{gon}(F) = n\,\mathrm{gon}(\Gamma)$. In particular, the gonality of a primitive cyclic covering fibration of type (g, h, n) is $n\,\mathrm{gon}(\varphi)$, when $r \geq 2n\,\mathrm{gon}(\varphi)$.*

Proof. Assume contrary that F has a morphism onto $\mathbb{P}^1$ of degree $k < n\,\mathrm{gon}(\Gamma)$. This together with the covering $\theta\colon F \to \Gamma$ defines a morphism $\Phi : F \to \mathbb{P}^1 \times \Gamma$. If Φ is of degree m onto the image $\Phi(F)$, then m is a common divisor of n, k and the arithmetic genus of $\Phi(F)$ is $(n/m-1)k/m + (h-1)n/m + 1$ by the genus formula. Now, let F' be the normalization of $\Phi(F)$. Since the covering $F \to \Gamma$ factors through F', we see that the induced covering $F' \to \Gamma$ of degree n/m is a totally ramified covering branched over r points. Then, by the Hurwitz formula, we have $2g(F')-2 = (2h-2)n/m+(n/m-1)r$. Since the genus $g(F')$ of F' is not bigger than the arithmetic genus of $\Phi(F)$, we get $r \leq 2(k/m)$ when $n > m$, which is impossible, since $r \geq 2n\,\mathrm{gon}(\Gamma)$ and $k < n\,\mathrm{gon}(\Gamma)$. Thus, we get $n = m$. Then F' is isomorphic to Γ and therefore the morphism $F \to \mathbb{P}^1$ factors through Γ. Hence we have $k \geq n\,\mathrm{gon}(\Gamma)$ by the difinition of the gonality of Γ, which contradicts $k < n\,\mathrm{gon}(\Gamma)$. A more careful study shows that any gonality pencil of F is the pull-back of a gonality pencil of Γ when $r > 2n\,\mathrm{gon}(\Gamma)$. □

2 Primitive cyclic covering fibrations of an elliptic surface

Let $f\colon S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. Since $\varphi\colon W \to B$ is a relatively minimal elliptic surface, K_φ is numerically equivalent to $\left(\chi_\varphi + \sum_{p\in B}\left(1 - \frac{1}{m_p}\right)\right)\Gamma$ by the canonical bundle formula, where m_p denotes the multiplicity of the fiber Γ_p of φ over p. In particular, we have $K_\varphi^2 = 0$. For $p \in B$, we put $\nu(F_p) = 1 - 1/m_p$ and $\nu = \sum_{p\in B}\nu(F_p)$. Then, we have $K_\varphi R = (\chi_\varphi + \nu)r$. Combining these equalities with (1.8), (1.9), (1.10) and (1.11), we get the following lemma:

Lemma 2.1. *The following equalities hold.*

$$K_f^2 = \sum_{k\geq 1}\left((n+1)(n-1)k - n\right)\alpha_k + \frac{(n-1)^2}{n}(\alpha_0 - 2\varepsilon)$$

$$
\begin{aligned}
&+\frac{(n+1)(n-1)r}{n}(\chi_\varphi+\nu)+\varepsilon.\\
\chi_f &= \frac{1}{12}(n-1)(n+1)\sum_{k\geq 1}k\alpha_k+\frac{(n-1)(2n-1)}{12n}(\alpha_0-2\varepsilon)\\
&+\frac{(n+1)(n-1)r}{12n}(\chi_\varphi+\nu)+n\chi_\varphi.\\
e_f &= (n-1)\alpha_0+n\sum_{k\geq 1}\alpha_k-(2n-1)\varepsilon+12n\chi_\varphi.
\end{aligned}
$$

For $p\in B$, we put $\chi_\varphi(F_p)=e_\varphi(\Gamma_p)/12$ and

$$
\begin{aligned}
K_f^2(F_p) &= \sum_{k\geq 1}\left((n+1)(n-1)k-n\right)\alpha_k(F_p)+\frac{(n-1)^2}{n}(\alpha_0(F_p)-2\varepsilon(F_p))\\
&+\frac{(n+1)(n-1)r}{n}(\chi_\varphi(F_p)+\nu(F_p))+\varepsilon(F_p),\\
\chi_f(F_p) &= \frac{1}{12}(n-1)(n+1)\sum_{k\geq 1}k\alpha_k(F_p)+\frac{(n-1)(2n-1)}{12n}(\alpha_0(F_p)-2\varepsilon(F_p))\\
&+\frac{(n+1)(n-1)r}{12n}(\chi_\varphi(F_p)+\nu(F_p))+n\chi_\varphi(F_p),\\
e_f(F_p) &= (n-1)\alpha_0(F_p)+n\sum_{k\geq 1}\alpha_k(F_p)-(2n-1)\varepsilon(F_p)+12n\chi_\varphi(F_p).
\end{aligned}
$$

Then, the following slope equality holds:

Theorem 2.2. *Let $f\colon S\to B$ be a primitive cyclic covering fibration of type $(g,1,n)$. Then*

$$
K_f^2=\lambda_{g,1,n}\chi_f+\sum_{p\in B}\mathrm{Ind}(F_p),
$$

where $\lambda_{g,1,n}:=12(n-1)/(2n-1)$ and $\mathrm{Ind}(F_p)$ is defined by

$$
\begin{aligned}
\mathrm{Ind}(F_p) =& n\sum_{k\geq 1}\left(\frac{(n+1)(n-1)}{2n-1}k-1\right)\alpha_k(F_p)+\frac{n-1}{2n-1}\left((n+1)r-12n\right)\chi_\varphi(F_p)\\
&+\frac{(n+1)(n-1)r}{2n-1}\nu(F_p)+\varepsilon(F_p).
\end{aligned}
$$

Moreover, if $r\geq \dfrac{12n}{n+1}$, then $\mathrm{Ind}(F_p)$ is non-negative for any $p\in B$.

Proof. Since $K_f^2 = \sum_{p\in B} K_f^2(F_p)$, $\chi_f = \sum_{p\in B} \chi_f(F_p)$ and $K_f^2(F_p) - \lambda_{g,1,n}\chi_f(F_p) = \mathrm{Ind}(F_p)$, the claim follows. □

For an oriented compact real 4-dimensional manifold X, the signature $\mathrm{Sign}(X)$ of X is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form on $H^2(X)$. From Lemma 2.1, we observe the local concentration of $\mathrm{Sign}(S)$ to a finite number of fiber germs.

Corollary 2.3 (cf. [5])**.** *Let $f\colon S \to B$ be a primitive cyclic covering fibration of type $(g,1,n)$. Then*

$$\mathrm{Sign}(S) = \sum_{p\in B} \sigma(F_p),$$

where $\sigma(F_p)$ is defined by

$$\begin{aligned}\sigma(F_p) =& n\sum_{k\geq 1}\left(\frac{(n+1)(n-1)}{3}k-1\right)\alpha_k(F_p) + \left(\frac{(n-1)(n+1)r}{3n} - 8n\right)\chi_\varphi(F_p) \\ &+ \frac{(n+1)(2n-1)}{3n}\varepsilon(F_p) + \frac{(n+1)(n-1)r}{3n}\nu(F_p) - \frac{(n+1)(n-1)}{3n}\alpha_0(F_p).\end{aligned}$$

Proof. By the index theorem (cf. [20, p. 126]), we have

$$\mathrm{Sign}(S) = \sum_{p+q\equiv 0(\mathrm{mod}2)} h^{p,q}(S) = K_f^2 - 8\chi_f.$$

On the other hand, we can see that

$$\sigma(F_p) = K_f^2(F_p) - 8\chi_f(F_p)$$

by a computation. □

3 Upper bound of the slope

In this section, we prove the following theorem:

Theorem 3.1. *Let $f\colon S \to B$ be a primitive cyclic covering fibration of type $(g,1,n)$.*
(1) *If $n \geq 4$ or $n = 3$ and $g = 4$, then we have*

$$K_f^2 \leq \left(12 - \frac{12n^2}{r(n-1)(n+1)}\right)\chi_f.$$

(2) *If $n = 3$ and $g \geq 7$, then we have*

$$K_f^2 \leq \left(12 - \frac{24}{4g-17}\right)\chi_f.$$

(3) *If $n = 2$ and $g \geq 3$, then we have*

$$K_f^2 \leq \left(12 - \frac{2}{g-2}\right)\chi_f.$$

Corollary 3.2. *Let $f\colon S \to B$ be a relatively minimal bielliptic fibered surface of genus $g \geq 3$. Then, we have*

$$K_f^2 \leq \left(12 - \frac{2}{g-2}\right)\chi_f.$$

Proof. Let $f\colon S \to B$ be a relatively minimal fibered surface of genus g whose general fiber F is a double cover of a smooth curve Γ of genus h. If $g > 4h+1$, an involution of the general fiber F of f over Γ is unique. Then, the fibration f has a global involution since it is relatively minimal (cf. [15]). Hence f is a primitive cyclic covering fibration of type $(g, h, 2)$. In particular, a relatively minimal bielliptic fibered surface of genus $g \geq 6$ is a primitive cyclic covering fibration of type $(g, 1, 2)$. In the case of $g \leq 5$, we use the semi-stable reduction. We may assume that the slope λ_f is greater than 8. Taking a suitable base change $B' \to B$, we get the base change fibration $f'\colon S' \to B'$ which is semi-stable and the bielliptic involution on F extends to a global involution, that is, primitive cyclic covering fibration of type $(g, 1, 2)$. From Theorem 3.1 (3), we have $\lambda_{f'} \leq 12 - 2/(g-2)$. On the other hand, we have $\lambda_f \leq \lambda_{f'}$ from [34]. Thus the claim holds. □

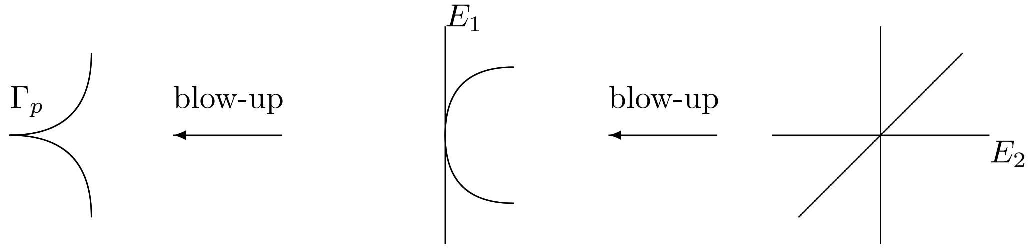
In particular, any bielliptic fibration is not a Kodaira fibration. Namely, the following holds.

Corollary 3.3. *Let $B_g \subset M_g$ be the bielliptic locus on the moduli space M_g of smooth curves of genus g. Then B_g contains no complete subvarieties of positive dimension.*

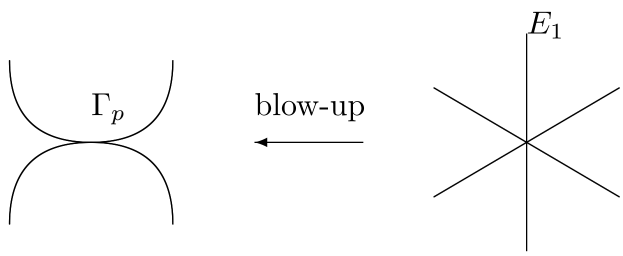
Let $f\colon S \to B$ be a primitive cyclic covering fibration of type $(g, 1, n)$. We fix $p \in B$. Let $m = m_p$ be the multiplicity of the fiber Γ_p of φ over p. Since $h = 1$, we have $j_{b,\bullet}(F_p) = 0$ for any $b \geq 2$. From the classification of singular fibers of relatively minimal elliptic surfaces ([26]), we have the following lemma for u-vertical type singularities:

Lemma 3.4. *There exist no u-vertical type singularities of R for $u \geq 4$. All possible 3-vertical type singularities are as follows.*

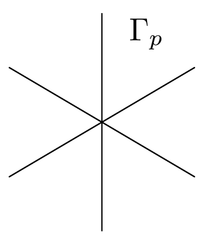
Type (II) : *Γ_p is a singular fiber of type* (II) *in the Kodaira's table* ([26])(*i.e., it is a singular rational curve with one cusp*) *and it is contained in R. The cusp on Γ_p is a* *singularity of type* $n\mathbb{Z}+1$ *and the singularity at which the proper transform of* Γ_p *and the exceptional curve* E_1 *for the blow-up at the cusp intersect is also of type* $n\mathbb{Z}+1$*. Then, the proper transforms of* Γ_p *and* E_1 *and the exceptional curve* E_2 *for the blow-up at this singularity form a* 3*-vertical type singularity.*



Type (III) : Γ_p *is a singular fiber of type* (III) *in the Kodaira's table* (*i.e., it consists of two nonsingular rational curves intersecting each other at one point of order two*) *and it is contained in* R*. The singularity on* Γ_p *is a singularity of type* $n\mathbb{Z}+1$*. Then, the proper transforms of* Γ_p *and the exceptional curve* E_1 *for the blow-up at this singularity form a* 3*-vertical type singularity.*



Type (IV) : Γ_p *is a singular fiber of type* (IV) *in the Kodaira's table* (*i.e. it consists of three nonsingular rational curves intersecting one another at one point transversely*) *and it is contained in* R*. The singularity on* Γ_p *is a* 3*-vertical type singularity.*



In particular, we have $\iota^{(u)}(F_p)=\kappa^{(u)}(F_p)=0$ *for* $u\geq 4$ *and* $0\leq \iota^{(3)}(F_p)+\kappa^{(3)}(F_p)\leq 1$*.*

Next, we give a lower bound of $\alpha_0^+(F_p)$ by using $\iota(F_p)$ and $\kappa(F_p)$.

Lemma 3.5. *We have*

$$\alpha_0^+(F_p) \geq \left(1 - \frac{1}{m}\right) r + (n-2)(\iota(F_p) + 2\kappa(F_p)) + \beta_p.$$

where $\beta_p := \delta_{n\neq 2}\left((n-7)\delta_{\mathrm{II}} - (n+1)\delta_{\mathrm{III}} - 2\delta_{\mathrm{IV}}\right)\iota^{(3)}(F_p) + \delta_{n=2}\sum_{k\geq 1} 2k\alpha^{\mathrm{co}}_{(2k+1\to 2k+1)}(F_p)$.

Proof. Let $\widetilde{\Gamma}_p = m\widetilde{D}_p$ and $\widetilde{D}_p = \sum_i m_i G_i$ the irreducible decomposition. Then we have

$$\begin{aligned}\alpha_0^+(F_p) &= r - \#(\mathrm{Supp}(\widetilde{R}_h) \cap \mathrm{Supp}(\widetilde{\Gamma}_p)) \\ &\geq r - \sum_i \widetilde{R}_h G_i \\ &= \left(1 - \frac{1}{m}\right) r + \sum_i (m_i - 1)\widetilde{R}_h G_i.\end{aligned}$$

For a $(t,2)$ or $(t,3)$-vertical $n\mathbb{Z}$ type singularity x, we denote by E_x^t the exceptional curve for the blow-up at x. Let m_x^t be the multiplicity of $\widetilde{D}_p$ along $\widehat{E}_x^t$, the proper transform of E_x^t on $\widetilde{W}$. Then, we have

$$\sum_i (m_i - 1)\widetilde{R}_h G_i \geq \sum_{t=1}^{\eta_p} \sum_{x:(t,\,u)\ n\mathbb{Z},\ u\,\geq\,2} (m_x^t - 1)\widetilde{R}_h \widehat{E}_x^t.$$

If there exists a singular point of type $n\mathbb{Z}$ on E_x^t, we replace E_x^t with the exceptional curve E obtained by blowing up at this point. Repeating this procedure, we may assume that there exist no singular points of type $n\mathbb{Z}$ on E_x^t. If there exists a singular point of type $n\mathbb{Z}+1$ on E_x^t, the proper transform of the exceptional curve obtained by blowing up at this point belongs to other $D^u(p)$. Since the multiplicity of $\widetilde{\Gamma}_p$ along it is not less than $m_x^t > 1$, we do not have to consider this situation. Thus, we may assume that there exist no singular points on E_x^t and we have $\widetilde{R}_h\widehat{E}_x^t \geq n-2$ if x is a 2-vertical type singularity, and $\widetilde{R}_h\widehat{E}_x^t \geq n-3$ if x is a 3-vertical type singularity. We can see that $\sum_{x:(t,\,2)\ n\mathbb{Z}} m_x^t \geq 2\iota^t(F_p) + 2\kappa^t(F_p)$ for any t with $\iota^{t,(3)}(F_p) = 0$. Thus, if $\iota^{(3)}(F_p) = 0$, we have

$$\alpha_0^+(F_p) \geq \left(1 - \frac{1}{m}\right) r + (n-2)(\iota(F_p) + 2\kappa(F_p)).$$

If $\iota^{(3)}(F_p) = 1$, then $m = 1$ and Γ_p is of type (II), (III) or (IV) from Lemma 3.4. We may assume that $D'^1(p) \neq 0$. Let x_0 be the 3-vertical $n\mathbb{Z}$ type singularity over p. Suppose that Γ_p is of type (II). Then, we can see that $m_{x_0}^1 = 6$ and $\sum_{x:(1,\,2)\ n\mathbb{Z}} m_x^1 \geq$

$2\iota^{1,(2)}(F_p) + 2(\kappa^1(F_p) - 1)$. Then we have

$$\begin{aligned}\alpha_0^+(F_p) \geq & \left(1 - \frac{1}{m}\right) r + 5(n-3) + (n-2)(\iota^{1,(2)}(F_p) + 2(\kappa^1(F_p) - 1)) \\ & + (n-2)\sum_{t=2}^{\eta_p}(\iota^t(F_p) + 2\kappa^t(F_p)) \\ = & \left(1 - \frac{1}{m}\right) r + n - 7 + (n-2)(\iota(F_p) + 2\kappa(F_p)).\end{aligned}$$

Suppose that Γ_p is of type (Ⅲ). Then, we can see that $m_{x_0}^1 = 4$ and $\sum_{x:(1,\,2)\ n\mathbb{Z}} m_x^1 \geq 2\iota^{1,(2)}(F_p) + 2(\kappa^1(F_p) - 1)$. Then we have

$$\begin{aligned}\alpha_0^+(F_p) \geq & \left(1 - \frac{1}{m}\right) r + 3(n-3) + (n-2)(\iota^{1,(2)}(F_p) + 2(\kappa^1(F_p) - 1)) \\ & + (n-2)\sum_{t=2}^{\eta_p}(\iota^t(F_p) + 2\kappa^t(F_p)) \\ = & \left(1 - \frac{1}{m}\right) r - n - 1 + (n-2)(\iota(F_p) + 2\kappa(F_p)).\end{aligned}$$

Suppose that Γ_p is of type (Ⅳ). Then, we can see that $m_{x_0}^1 = 3$ and $\sum_{x:(1,\,2)\ n\mathbb{Z}} m_x^1 \geq 2\iota^{1,(2)}(F_p) + 2\kappa^1(F_p)$. Then we have

$$\begin{aligned}\alpha_0^+(F_p) \geq & \left(1 - \frac{1}{m}\right) r + 2(n-3) + (n-2)(\iota^{1,(2)}(F_p) + 2\kappa^1(F_p)) \\ & + (n-2)\sum_{t=2}^{\eta_p}(\iota^t(F_p) + 2\kappa^t(F_p)) \\ = & \left(1 - \frac{1}{m}\right) r - 2 + (n-2)(\iota(F_p) + 2\kappa(F_p)).\end{aligned}$$

Suppose that $n = 2$. For a $(2k+1 \to 2k+1)$ singularity (x, y), let E_y denote the exceptional curve for the blow-up at y and m_y the multiplicity of $\widetilde{\Gamma}_p$ along $\widehat{E}_y$, the proper transform of E_y. Then we have

$$\sum_i (m_i - 1)\widetilde{R}_h G_i \geq \sum_{k \geq 1} \sum_{(x,y):(2k\,+\,1\,\to\,2k\,+\,1)} (m_y - 1)\widetilde{R}_h \widehat{E}_y.$$

By an argument similar to the above, we may assume that there are no singular points on E_y. Then we have $\widetilde{R}_h \widehat{E}_y = 2k$ for any $(2k+1 \to 2k+1)$ singularity (x, y). On the other hand, we have $m_y \geq 2$ for any $(2k+1 \to 2k+1)$ singularity (x, y) involved in $\alpha^{\mathrm{co}}_{(2k+1 \to 2k+1)}(F_p)$. Thus, we obtain $\sum_{(x,y):(2k\,+\,1\,\to\,2k\,+\,1)} (m_y - 1)\widetilde{R}_h \widehat{E}_y \geq 2k\alpha^{\mathrm{co}}_{(2k+1 \to 2k+1)}(F_p)$.

□

We can translate the index α'' into other indices as follows.

Lemma 3.6. *The following equalities hold.*

$$\sum_{k\geq 1}\alpha''_k(F_p)=\eta''_p+\sum_{a\geq 1}anj'_{1,a}(F_p)+\sum_{a\geq 1}(an-2-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'_{0,a}(F_p) \qquad (3.1)$$
$$+\sum_{a\geq 1}(an-1)j''_{0,a}(F_p)-\iota(F_p)-\kappa(F_p).$$

$$\sum_{k\geq 1}k\alpha''_k(F_p)=\gamma_p+\sum_{a\geq 1}aj_{\bullet,a}(F_p)+\sum_{k\geq 1}k\left(\alpha_k^{n\mathbb{Z}+1}(F_p)-\iota_k(F_p)-\kappa_k(F_p)\right). \qquad (3.2)$$

where $\gamma_p=\sum_{t=1}^{\eta_p}\gamma_p^t$ and γ_p^t is defined to be the following (1), (2), (3):
(1) $\gamma_p^t=d^{t,1}$ if $D'^t(p)=0$, where $m^{t,1}$ is the multiplicity of the singularity to which $C^{t,1}$ contracts and $d^{t,1}=[m^{t,1}/n]$.
(2) $\gamma_p^t=\sum_{k=1}^{j'^t(F_p)}RC'^{t,k}/n$ if $D'^t(p)\neq 0$ and any $C'^{t,k}$ is smooth.
(3) $\gamma_p^t=r/mn-d'^{t,1}$ if $C'^{t,1}=(\Gamma_p)_{\mathrm{red}}$ is singular, where $m'^{t,1}$ is the multiplicity of the singular point of R which is singular for $C'^{t,1}$ and $d'^{t,1}=[m'^{t,1}/n]$.

Proof. Let $a'^{t,k}$, $a''^{t,k}$ be the integers such that $(L'^{t,k})^2=-a'^{t,k}n$, $(L''^{t,k})^2=-a''^{t,k}n$. If $C'^{t,k}$ is smooth, then $C'^{t,k}$ is blown up $a'^{t,k}n+(C'^{t,k})^2$ times. If $C'^{t,k}$ is a singular rational curve, then $C'^{t,k}$ is blown up $a'^{t,k}n-3$ times. Since $C''^{t,k}$ is a (-1)-curve, $C''^{t,k}$ is blown up $a''^{t,k}n-1$ times. Hence, if $D'^t(p)\neq 0$ and every $C'^{t,k}$ is smooth (resp. $C'^{t,1}$ is singular rational), the number of singular points associated with $D^t(p)$ is $\sum_k\left(a'^{t,k}n+(C'^{t,k})^2\right)+$ (resp. $\sum_k\left(a'^{t,k}n-3\right)+$) $\sum_k\left(a''^{t,k}n-1\right)-\iota^t(F_p)-\kappa^t(F_p)$. Namely, we have

$$\sum_{k\geq 1}\alpha''^t_k(F_p)=\sum_{a\geq 1}anj'^t_{1,a}(F_p)+\sum_{a\geq 1}(an-2-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'^t_{0,a}(F_p)+\sum_{a\geq 1}(an-1)j''^t_{0,a}(F_p)-\iota^t(F_p)-\kappa^t(F_p).$$

If $D'^t(p)=0$, we have

$$\sum_{k\geq 1}\alpha''^t_k(F_p)=1+\sum_{a\geq 1}(an-1)j''^t_{0,a}(F_p)-\iota^t(F_p)-\kappa^t(F_p).$$

Summing up for $t=1,\dots,\eta_p$, we have (3.1).

Let $r^{t,k}=RC'^{t,k}$, $m^{t,k}$ the multiplicity of R at the point to which $C''^{t,k}$ is contracted and $d^{t,k}=[m^{t,k}/n]$. Let $x_1'^{t,k},\dots,x_{c'}'^{t,k}$ (resp. $x_1''^{t,k},\dots,x_{c''}''^{t,k}$) be all the singular points on $C'^{t,k}$ (resp. on $C''^{t,k}$), including infinitely near ones. Put $m_i'^{t,k}=\mathrm{mult}_{x_i'^{t,k}}(R)$, $d_i'^{t,k}=[m_i'^{t,k}/n]$, $m_i''^{t,k}=\mathrm{mult}_{x_i''^{t,k}}(R)$ and $d_i''^{t,k}=[m_i''^{t,k}/n]$. Applying Lemma 1.10 to

$C'^{t,k}$ and $C''^{t,k}$, we get that $r^{t,k}/n + a'^{t,k} = \sum_i d_i'^{t,k}$ if $C'^{t,k}$ is smooth, $r^{t,1}/n + a'^{t,1} = d_1'^{t,1} + \sum_i d_i'^{t,1}$ if $C'^{t,1}$ is singular rational, and $d^{t,k} + a''^{t,k} = \sum_i d_i''^{t,k}$. If $D'^t(p) \neq 0$ and every $C'^{t,k}$ is smooth, then

$$\begin{aligned}
\sum_k \frac{r^{t,k}}{n} + \sum_{a \geq 1} a j^t_{\bullet,a}(F_p) + \sum_{k \geq 1} k \alpha_k^{t,n\mathbb{Z}+1}(F_p) &= \sum_k \left(\frac{r^{t,k}}{n} + a'^{t,k} \right) + \sum_k \left(d^{t,k} + a''^{t,k} \right) \\
&= \sum_k \sum_i d_i'^{t,k} + \sum_k \sum_i d_i''^{t,k} \\
&= \sum_{k \geq 1} k \left(\alpha_k''^t(F_p) + \iota_k^t(F_p) + \kappa_k^t(F_p) \right).
\end{aligned}$$

Similarly, if $D'^t(p) \neq 0$ and $C'^{t,1}$ is singular rational, we have

$$\frac{r}{nm} - d_1'^{t,1} + \sum_{a \geq 1} a j^t_{\bullet,a}(F_p) + \sum_{k \geq 1} k \alpha_k^{t,n\mathbb{Z}+1}(F_p) = \sum_{k \geq 1} k \left(\alpha_k''^t(F_p) + \iota_k^t(F_p) + \kappa_k^t(F_p) \right).$$

If $D'^t(p) = 0$, then

$$\begin{aligned}
\sum_{a \geq 1} a j^t_{\bullet,a}(F_p) + \sum_{k \geq 1} k \alpha_k^{t,n\mathbb{Z}+1}(F_p) &= \sum_k \left(d^{t,k} + a''^{t,k} \right) \\
&= \sum_k \sum_i d_i''^{t,k} \\
&= \sum_{k \geq 1} k \left(\alpha_k''^t(F_p) + \iota_k^t(F_p) + \kappa_k^t(F_p) \right) - d^{t,1}.
\end{aligned}$$

Summing up for $t = 1, \ldots, \eta_p$, we get (3.2). □

Lemma 3.7. *The following hold.*

$$\gamma_p \leq \left(\frac{r}{n} - j'_{0,1}(F_p)\delta_{n=2} - \delta_{{}_m\mathrm{I}_1,\mathrm{II}} \right) \delta_{\eta'_p \neq 0} + \left(\frac{r}{n} - 1 \right) \eta''_p.$$
$$\iota(F_p) = j(F_p) - \eta_p + \delta_{\mathrm{cyc}},$$

where δ_{cyc} is defined to be 1 if the following (1), (2), (3) and (4) hold and $\delta_{\mathrm{cyc}} = 0$ otherwise.

(1) *Γ_p is a singular fiber of type $({}_m\mathrm{I}_k)_{k \geq 1}$, (II), (III) or (IV).*

(2) *Any irreducible component of Γ_p is contained in R.*

(3) $\iota^{(3)}(F_p) = \kappa^{(3)}(F_p) = 0$.

(4) *The multiplicity of the singular point of R which is singular for $(\Gamma_p)_{\mathrm{red}}$ belongs to $n\mathbb{Z} + 1$ if Γ_p is a singular fiber of type $({}_m\mathrm{I}_1)$ or (II).*

Proof. By the definition of γ_p, the first inequality is clear. We consider the following graph $\mathbf{G}^t$: The vertex set $V(\mathbf{G}^t)$ is defined by the symbol set $\{v^{t,k}\}_{k=1}^{j^t(F_p)}$. The edge set $E(\mathbf{G}^t)$ is defined by the symbol set $\{e_x\}_x \cup \{e_y\}_y \cup \{e'_y\}_y$, where x, y respectively move among $(t,2)$, $(t,3)$-vertical $n\mathbb{Z}$ type singularities. If the proper transform of $C^{t,k}$ meets that of $C^{t,k'}$ at a $(t,2)$-vertical $n\mathbb{Z}$ type singularity x, the edge e_x connects $v^{t,k}$ and $v^{t,k'}$. If the proper transforms of $C^{t,k}$, $C^{t,k'}$ and $C^{t,k''}$ $(k < k' < k'')$ intersects in a $(t,3)$-vertical $n\mathbb{Z}$ type singularity y, the edge e_y connects $v^{t,k}$ and $v^{t,k'}$, and e'_y connects $v^{t,k'}$ and $v^{t,k''}$. By the definition of the decomposition $\widetilde{R}_v(p) = D^1(p) + \cdots + D^{\eta_p}(p)$, the graph $\mathbf{G}^t$ is connected for any $t = 1, \ldots, \eta_p$. Clearly, $\iota(F_p)$ is the cardinality of $E(\mathbf{G}^t)$. Thus, the number of cycles in $\mathbf{G}^t$ is $\iota^t(F_p) - j^t(F_p) + 1$. One sees that $\mathbf{G}^t$ has at most one cycle, and it has one cycle only if $\{C^{t,k}\}_k$ contains all irreducible components of Γ_p. Hence at most one $\mathbf{G}^t$ has one cycle. We can see that $\mathbf{G}^t$ has one cycle for some t if and only if $\delta_{\mathrm{cyc}} = 1$. Thus, we get $\iota(F_p) = j(F_p) - \eta_p + \delta_{\mathrm{cyc}}$. □

For any singular point x of R, the multiplicity $\mathrm{mult}_x(R)$ at x does not exceed $r/m + 1$ since $R(\Gamma_p)_{\mathrm{red}} = r/m$. Thus we have $\alpha_k = 0$ for $k \geq r/nm + 1$. Moreover, the following lemma holds.

Lemma 3.8. *If $n \geq 3$, then we have $\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p) = 0$. If $n = 2$, then we have $\kappa_{\frac{r}{2m}}(F_p) = 0$.*

Proof. If $\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p) \neq 0$, then there exists an irreducible component C of Γ_p contained in R and a singular point x of R on C with multiplicity $r/m + 1$ such that any local horizontal branch of R around x is not tangential to C since $RD_p = r/m$. Then, the exceptional curve E for the blow-up at x and the proper transform of C form a singular point of multiplicity 2. Hence we have $n = 2$ from Lemma 1.2. It is clear that all singular points with multiplicity $r/m + 1$ are infinitely near to x and the exceptional curves for blow-ups of these singularities form a chain. In particular, any singular point with multiplicity $r/m + 1$ is a 1-vertical type singularity. □

To prove Theorem 3.1, we need some inequalities among several indices.

Lemma 3.9. (1) *The following holds.*

$$\sum_{k\geq 1} k\left(\alpha^{n\mathbb{Z}+1}_k(F_p) - \kappa_k(F_p)\right) \leq \left(\frac{r}{n} - 1\right)\left(j''(F_p) - \kappa(F_p)\right) + \left(\frac{r}{n} - 2\right)\kappa^{(3)}(F_p) + \alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p).$$

(2) *If $n = 2$, then the following holds more strongly.*

$$\sum_{k\geq 1} k\left(\alpha_k^{2\mathbb{Z}+1}(F_p)-\kappa_k(F_p)\right) \leq \sum_{k\geq 1} k\alpha_{(2k+1\to 2k+1)}(F_p) + \left(\frac{r}{2}-1\right)\left(\sum_{a\geq 2} j''_{0,a}(F_p)-\kappa(F_p)\right) + \left(\frac{r}{2}-2\right)\kappa^{(3)}(F_p) + \alpha^{2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p).$$

Proof. From Lemma 3.8, we have

$$\sum_{k\geq 1} k\left(\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa_k(F_p)\right) = \sum_{k=1}^{\frac{r}{nm}-1} k\left(\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa_k(F_p)\right) + \frac{r}{nm}\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p)$$
$$= \sum_{k=1}^{\frac{r}{nm}-1} k\left(\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa_k^{(2)}(F_p)-\kappa_k^{(3)}(F_p)\right) - \sum_{k=1}^{\frac{r}{nm}-1} k\kappa_k^{(3)}(F_p) + \frac{r}{nm}\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p)$$

Since $\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa_k^{(2)}(F_p)-\kappa_k^{(3)}(F_p)\geq 0$ and $\sum_{k=1}^{\frac{r}{nm}-1} k\kappa_k^{(3)}(F_p)=k_0\kappa^{(3)}(F_p)$ for some $1\leq k_0\leq r/n-1$ from Lemma 3.4, we have

$$\sum_{k=1}^{\frac{r}{nm}-1} k\left(\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa_k^{(2)}(F_p)-\kappa_k^{(3)}(F_p)\right) - \sum_{k=1}^{\frac{r}{nm}-1} k\kappa_k^{(3)}(F_p) + \frac{r}{nm}\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p)$$
$$\leq \left(\frac{r}{n}-1\right)\left(\sum_{k=1}^{\frac{r}{nm}-1}\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa^{(2)}(F_p)-\kappa^{(3)}(F_p)\right) - k_0\kappa^{(3)}(F_p) + \frac{r}{n}\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p)$$
$$= \left(\frac{r}{n}-1\right)\left(\sum_{k=1}^{\frac{r}{nm}-1}\alpha_k^{n\mathbb{Z}+1}(F_p)-\kappa(F_p)\right) + \left(\frac{r}{n}-1-k_0\right)\kappa^{(3)}(F_p) + \frac{r}{n}\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p).$$

Combining the above inequality with $j''(F_p)=\sum_{k=1}^{\frac{r}{nm}}\alpha_k^{n\mathbb{Z}+1}(F_p)$ and $r/n-1-k_0\leq r/n-2$, the assertion (1) follows.

Assume $n=2$. Note that any $(2k+1\to 2k+1)$ singularity is not involved in $\kappa(F_p)$. Then we have

$$\sum_{k\geq 1} k\left(\alpha_k^{2\mathbb{Z}+1}(F_p)-\kappa_k(F_p)\right) = \sum_{k=1}^{\frac{r}{2m}-1} k\left(\alpha_k^{2\mathbb{Z}+1}(F_p)-\alpha_{(2k+1\to 2k+1)}(F_p)-\kappa_k(F_p)\right) + \sum_{k\geq 1} k\alpha_{(2k+1\to 2k+1)}(F_p) + \frac{r}{2m}\alpha^{2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p).$$

Similarly as in (1), the assertion (2) follows. □

Lemma 3.10. *If $n = 2$, then we have*

$$\kappa(F_p) \leq \frac{2}{3}\sum_{a\geq 2}(a-1)j_{\bullet,a}(F_p) - \frac{2}{3}\alpha^{2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p).$$

Proof. It is sufficient to show that

$$\kappa^t(F_p) \leq \frac{2}{3}\sum_{a\geq 2}(a-1)j^t_{0,a}(F_p) - \frac{2}{3}\alpha^{t,2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p) \tag{3.3}$$

for any t. If $\kappa^t(F_p) = 0$, then it is clear. Thus, we may assume $\kappa^t(F_p) > 0$. Then clearly we have

$$j^t(F_p) \geq \kappa^{t,(2)}(F_p) + \kappa^{t,(3)}(F_p) + \alpha^{t,2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p) + 2. \tag{3.4}$$

Since any blow-up at a (t,u)-vertical type singularity contributes $-u$ to the number

$$\sum_{k\geq 1}(L^{t,k})^2 = -\sum_{a\geq 1} 2aj^t_{\bullet,a}(F_p)$$

and Γ_p contains no u-vertical type singularity for $u \geq 2$ if Γ_p is of type $({}_m\mathrm{I}_0)$, we get

$$\begin{aligned}\sum_{a\geq 1} 2aj^t_{\bullet,a}(F_p) \geq & j^t_{1,\bullet}(F_p) + (2+\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'^t_{0,\bullet}(F_p) + j''^t_{0,\bullet}(F_p) \\ & + \alpha^{t,2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p) + \sum_{u=2,3} u\left(\iota^{t,(u)}(F_p) + \kappa^{t,(u)}(F_p)\right).\end{aligned}$$

Combining this inequality with $\iota^t(F_p) \geq j^t(F_p) - 1$ and (3.4), we have

$$\begin{aligned}\sum_{a\geq 1} 2aj^t_{\bullet,a}(F_p) \geq & 3j^t(F_p) + (1+\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'^t_{0,\bullet}(F_p) + \alpha^{t,2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p) \\ & + 2\kappa^t(F_p) - 2 - \left(\iota^{t,(3)}(F_p) + \kappa^{t,(3)}(F_p)\right) \\ \geq & 2j^t(F_p) + (1+\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'^t_{0,\bullet}(F_p) + 2\alpha^{t,2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p) \\ & + 3\kappa^t(F_p) - \iota^{t,(3)}(F_p) - 2\kappa^{t,(3)}(F_p).\end{aligned}$$

On the other hand, it is easily seen that

$$(1+\delta_{{}_m\mathrm{I}_1,\mathrm{II}})j'^t_{0,\bullet}(F_p) - \iota^{t,(3)}(F_p) - 2\kappa^{t,(3)}(F_p) \geq 0.$$

Hence we get (3.3), as desired. □

Lemma 3.11. (1) *If $n = 3$, then the following hold.*

$(1,\mathrm{i})$ *If $j'^t_{0,1}(F_p) \leq 2$ for any t, then*

$$\frac{1}{2}j'_{0,1}(F_p) \leq \eta'_p - \delta_{\mathrm{cyc}}.$$

(1, ii) *If* $j'^{t}_{0,1}(F_p) = 3$ *for some* t, *then* Γ_p *is a singular fiber of type* (IV), (I_k^*), (II^*), (III^*) *or* (IV^*) *and*

$$\frac{1}{3}j'_{0,1}(F_p) \le \eta'_p, \quad \delta_{\mathrm{cyc}} = 0.$$

(1, iii) *If* $j'^{t}_{0,1}(F_p) = 4$ *for some* t, *then* Γ_p *is a singular fiber of type* (I_k^*) *and any component of* Γ_p *is contained in* R. *Moreover, we have* $\eta'_p = 1$, $j'_{0,1}(F_p) = 4$ *and* $\delta_{\mathrm{cyc}} = 0$.

(2) *If* $n = 2$, *then the following hold.*

(2, i) *If* $j'^{t}_{0,2,\mathrm{odd}}(F_p) \le 2$ *for any* t, *then*

$$j'_{0,1}(F_p) + \frac{1}{2}j'_{0,2,\mathrm{odd}}(F_p) \le \eta'_p - \delta_{\mathrm{cyc}},$$

where $j'_{0,2,\mathrm{odd}}(F_p)$ *denotes the number of irreducible components* C *of* Γ_p *involved in* $j'_{0,2}(F_p)$ *which has a singular point of* R *of odd multiplicity.*

(2, ii) *If* $j'^{t}_{0,2,\mathrm{odd}}(F_p) = 3$ *for some* t, *then* Γ_p *is a singular fiber of type* (IV), (I_k^*), (II^*), (III^*) *or* (IV^*) *and*

$$j'_{0,1}(F_p) + \frac{1}{3}j'_{0,2,\mathrm{odd}}(F_p) \le \eta'_p, \quad \delta_{\mathrm{cyc}} = 0.$$

(2, iii) *If* $j'^{t}_{0,2,\mathrm{odd}}(F_p) = 4$ *for some* t, *then* Γ_p *is a singular fiber of type* (I_k^*) *and any component of* Γ_p *is contained in* R. *Moreover, we have* $\eta'_p = 1$, $j'_{0,1}(F_p) = 0$, $j'_{0,2,\mathrm{odd}}(F_p) = 4$ *and* $\delta_{\mathrm{cyc}} = 0$.

Proof. If $n = 3$, then any curve C in Γ_p contributing to $j'_{0,1}(F_p)$ intersects at most one component of Γ_p contained in R, since C is blown up just once. Thus, considering the classification of singular fibers of elliptic surfaces, we can show easily the assertion (1).

Suppose that $n = 2$. Any curve in Γ_p contributing to $j'_{0,1}(F_p)$ is not blown up and any curve in Γ_p contributing to $j'_{0,2,\mathrm{odd}}(F_p)$ intersects at most one component of Γ_p contained in R. Hence we can show the assertion (2) similarly. □

Lemma 3.12. (1) *If* $n = 3$, *then we have*

$$j''_{0,1}(F_p) \le 2\eta''_p + \sum_{a\ge 1} 2a j'_{1,a}(F_p) + \sum_{a \ge 2}(2a-2)j'_{0,a}(F_p) + \sum_{a\ge 2}(2a-1)j''_{0,a}(F_p).$$

(2) *If* $n = 2$, *then we have*

$$\sum_{k\ge 1}\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p) \le j'_{0,2,\mathrm{odd}}(F_p) + \sum_{a\ge 3}(a-1)j'_{0,a}(F_p) + \sum_{a\ge 2}(a-1)j'_{1,a}(F_p) + \sum_{a\ge 3}(a-2)j''_{0,a}(F_p) + \widehat{\eta}_p.$$

Proof. Suppose that $n = 3$. Let $C_1, \ldots, C_{j''^t_{0,1}(F_p)}$ be all (-1)-curves in $\{C''^{t,k}\}_k$ contributing to $j''^t_{0,1}(F_p)$ and x_i the point to which C_i contracts for $i = 1, \ldots, j''^t_{0,1}(F_p)$. If $C_i \neq C^{t,1}$, then x_i is contained in some $C^{t,k}$. If $C^{t,k}$ contributes to $j''^t_{0,1}(F_p)$ and $k = 1$, then $j''^t(F_p) = j''^t_{0,1}(F_p) = 2$ from Example 1.15 (2). If $C^{t,k}$ contributes to $j''^t_{0,1}(F_p)$ and $k \neq 1$, then the point $x^{t,k}$ to which $C^{t,k}$ contracts is contained in another $C^{t,k'}$ which does not contribute to $j''^t_{0,1}(F_p)$ from the argument of Example 1.15 (2). Moreover, x_i is also contained in $C^{t,k'}$ since the singularity diagram of $C^{t,k}$ is type (iii) in Example 1.15 (2) and Lemma 1.9. For a curve $C^{t,k}$ which does not contribute to $j''^t_{0,1}(F_p)$, we consider how many points among $x_1, \ldots, x_{j''^t_{0,1}(F_p)}$ it contains.

(i) Assume that $C^{t,k}$ contributes to $j''^t_{0,a}(F_p)$ for some $a \geq 2$. Then $C^{t,k}$ is blown up $3a - 1$ times. Let $(x_{i,j}, m_{i,j})$, $i = 1, \ldots, l$, $j = 1, \ldots, i_{\mathrm{bm}}$ be entries of the singularity diagram of $C^{t,k}$. We consider a subset of entries of the i-th column of its diagram $\{(x_{i,j}, m_{i,j})\}_{j=j_0+1,\ldots,j_0+N}$ satisfying that

$(*)$ $m_{i,j_0} \in 3\mathbb{Z}$ if $j_0 > 0$, $m_{i,j} \in 3\mathbb{Z} + 1$ for $j_0 < j < j_0 + N$ and $m_{i,j_0+N} \in 3\mathbb{Z}$.

Note that the set of all entries of the singularity diagram is the union of these subsets. Then we can see that the exceptional curve $C^{t,k'}$ obtained by the blow-up at $x^{i,j}$, $j_0 + 1 < j < j_0 + N$ does not contribute to $j''_{0,1}(F_p)$ from Lemma 1.14. Hence it contains at most $2a - 1$ points among $x_1, \ldots, x_{j''^t_{0,1}(F_p)}$.

(ii) Assume that $C^{t,k}$ contributes to $j'^t_{0,a}(F_p)$. Then $C^{t,k}$ is blown up $3a-2$ times when it is a (-2)-curve or $3a-3$ times when it is a singular rational curve. Hence it contains at most $2a - 2$ points among $x_1, \ldots, x_{j''^t_{0,1}(F_p)}$ by the same argument as in (i).

(iii) Assume that $C^{t,k}$ contributes to $j'^t_{1,a}(F_p)$. Then $C^{t,k}$ is blown up $3a$ times. Hence it contains at most $2a$ points among $x_1, \ldots, x_{j''^t_{0,1}(F_p)}$ by the same argument as in (i).

We estimate $j''^t_{0,1}(F_p)$ from (i), (ii), (iii) as follows.

(a) If $D'^t(p) = 0$ and $j''^t_{0,a}(F_p) = 0$ for any $a \geq 2$, then we have shown that $j''^t_{0,1}(F_p) \leq 2$ in Example 1.15 (2).

(b) If $D'^t(p) = 0$ and $j''^t_{0,a}(F_p) > 0$ for some $a \geq 2$, then x_i is the point to which $C^{t,1}$ contracts or contained in some $C^{t,k}$ which contributes to $j''^t_{0,a}(F_p)$ for some $a \geq 2$. Hence we have

$$j''^t_{0,1}(F_p) \leq 1 + \sum_{a \geq 2} (2a-1) j''^t_{0,a}(F_p).$$

(c) If $D'^t(p) \neq 0$, then x_i is contained in some $C^{t,k}$ which does not contribute to $j''^t_{0,1}(F_p)$. Hence we have

$$j''^t_{0,1}(F_p) \leq \sum_{a \geq 2} (2a-2) j'^t_{0,a}(F_p) + \sum_{a \geq 1} 2a j'^t_{1,a}(F_p) + \sum_{a \geq 2} (2a-1) j''^t_{0,a}(F_p).$$

From (a), (b) and (c), we have

$$j''_{0,1}(F_p) \le \overline{\eta}_p + \eta''_p + \sum_{a\ge 2}(2a-2)j'_{0,a}(F_p) + \sum_{a\ge 1} 2aj'_{1,a}(F_p) + \sum_{a\ge 2}(2a-1)j''_{0,a}(F_p)$$

by summing up for $t = 1, \dots, \eta_p$. Combining this with $\overline{\eta}_p \le \eta''_p$, the claim (1) follows.

Suppose $n = 2$. Let $x^{t,k}$ be the point to which $C''^{t,k}$ is contracted and $m^{t,k}$ the multiplicity of R at $x^{t,k}$. If $D'^t(p) = 0$, then $x^{t,k}$, $k \ge 2$ is contained in $C^{t,k'}$ for some $k' < k$. Otherwise, $x^{t,1}$ is also contained in $C^{t,k'}$ for some k'. If $C^{t,k}$ is smooth, a singularity with odd multiplicity which is not contained in $C^{t,k'}$ for any $k' > k$ corresponds to an entry $(x_{i,j}, m_{i,j})$ of the singularity diagram $\mathcal{D}^{t,k}$ of $C^{t,k}$ satisfying that $m_{i,j-1}$ is even if $j > 1$, and $m_{i,j}$ is odd and then corresponds to a subset of entries of the diagram satisfying $(*)$. For a curve $C^{t,k}$, we consider how many such subsets of entries of its singularity diagram there are.

(iv) If $C^{t,k}$ contributes to $j''^t_{0,a}(F_p)$, then $C^{t,k}$ is blown up $2a-1$ times. Then the singularity diagram of $C^{t,k}$ has at most $a-1$ subsets satisfying $(*)$.

(v) If $C^{t,k}$ contributes to $j'^t_{0,a}(F_p)$ and it is a (-2)-curve, then $C^{t,k}$ is blown up $2a-2$ times. Then the singularity diagram of $C^{t,k}$ has at most $a-1$ subsets satisfying $(*)$.

(vi) If $C^{t,k}$ contributes to $j'^t_{0,a}(F_p)$ and it is a singular rational curve, then $C^{t,k}$ is blown up $2a-3$ times. Considering the singularity diagram of the proper transform of $C^{t,k}$ by the blow-up at its singular point, $C^{t,k}$ has at most $a-1$ singularities with odd multiplicity which is not contained in $C^{t,k'}$ for any $k' > k$.

(vii) If $C^{t,k}$ contributes to $j'^t_{1,a}(F_p)$, then $C^{t,k}$ is blown up $2a$ times. Then the singularity diagram of $C^{t,k}$ has at most a subsets satisfying $(*)$.

We estimate $j''^t(F_p)$ using (iv), (v), (vi) and (vii) as follows.

(d) If $D'^t(p) = 0$, then the number of singularities with odd multiplicity appearing in $\{C^{t,k}\}_k$ is $j''^t(F_p) - 1$. Hence we have

$$j''^t(F_p) - 1 \le \sum_{a\ge 2}(a-1)j''^t_{0,a}(F_p).$$

(e) If $D'^t(p) \ne 0$, then the number of singularities with odd multiplicity appearing in $\{C^{t,k}\}_k$ is $j''^t(F_p)$. Hence we have

$$j''^t(F_p) \le j'^t_{0,1,\mathrm{odd}}(F_p) + \sum_{a\ge 3}(a-1)j'^t_{0,a}(F_p) + \sum_{a\ge 1} aj'^t_{1,a}(F_p) + \sum_{a\ge 2}(a-1)j''^t_{0,a}(F_p).$$

From (d) and (e), we have

$$j''(F_p) \le \eta''_p + j'_{0,1,\mathrm{odd}}(F_p) + \sum_{a\ge 2}(a-1)j'_{0,a}(F_p) + \sum_{a\ge 1} aj'_{1,a}(F_p) + \sum_{a\ge 2}(a-1)j''_{0,a}(F_p)$$

by summing up for $t = 1, \dots, \eta_p$. Combining this with (1.12) and (1.14), the claim (2) follows.

□

Lemma 3.13. (1) *If $n = 3$ and $j'_{0,1}(F_p) \neq 0$, then we have*

$$\chi_\varphi(F_p) \geq \frac{1}{12}(j'_{0,1}(F_p) + 1).$$

(2) *If $n = 2$, then the following hold.*

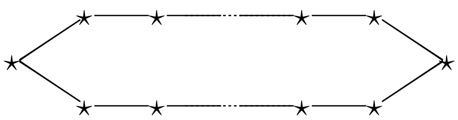
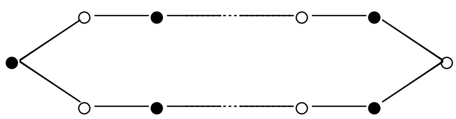
$(2, \mathrm{i})$ *If Γ_p is a singular fiber not of type $({}_m\mathrm{I}_k)$, (I_k^*), (II^*), (III^*), (IV^*), then we have*

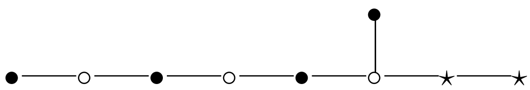
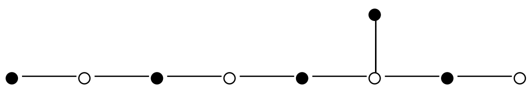
$$\chi_\varphi(F_p) \geq \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) + 1).$$

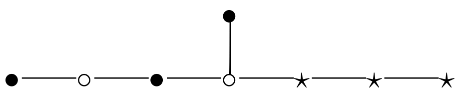
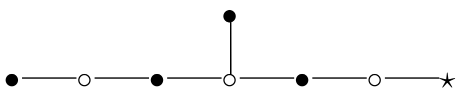
Moreover, all the cases where Γ_p is a singular fiber of type $({}_m\mathrm{I}_k)$, (II^*), (III^*), (IV^*) and

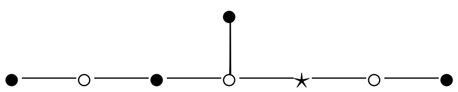
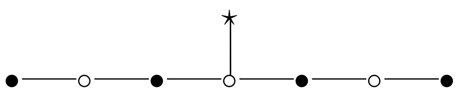
$$\chi_\varphi(F_p) < \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) + 1)$$

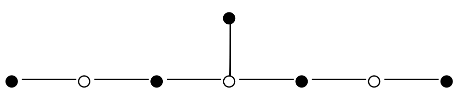
are as follows.

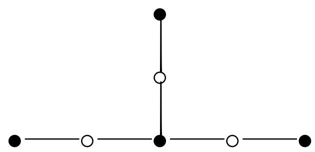
$({}_m\mathrm{I}_k)$  or 

(II^*)  or 

(III^*)  or 

or  or 

or 

(IV*) 

where, in the dual graphs of Γ_p*, the symbol* $\circ$*,* $\bullet$*,* $\star$ *respectively denotes a* (-2)*-curve not contained in* R*, contributing to* $j'_{0,1}(F_p)$*, contributing to* $j'_{0,2}(F_p)$ *or* $j'_{0,3}(F_p)$*. In these cases, we have*

$$\chi_\varphi(F_p) = \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) - 1)$$

when Γ_p *is of type* (III*) *and* $j'_{0,2}(F_p) = j'_{0,3}(F_p) = 0$ *and*

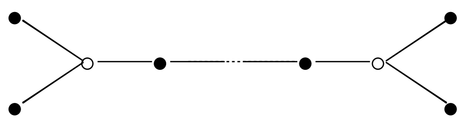
$$\chi_\varphi(F_p) = \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p))$$

otherwise.

(2, ii) *If* Γ_p *is a singular fiber of type* (I_k^*)*, then*

$$\chi_\varphi(F_p) \geq \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) - 2)$$

with equality holding if and only if Γ_p *and* R *satisfies the condition indicated in the following figure.*

(I_k^*) 

Proof. (1) Suppose that $n = 3$ and $j'_{0,1}(F_p) \neq 0$. If Γ_p is not of type $({}_m\mathrm{I}_k)$, the claim is clear. Thus we may assume that Γ_p is of type $({}_m\mathrm{I}_k)$. If $\chi_\varphi(F_p) = j'_{0,1}(F_p)/12$, then any component of Γ_p contributes to $j'_{0,1}(F_p)$ and contains at least 2 singular points of R, which is a contradiction.

(2) Suppose that $n = 2$. Any irreducible component C of Γ_p contributing to $j'_{0,1}(F_p)$ has no singular points of R. Thus any component of Γ_p intersecting with the curve C is not contained in R. From this observation and the classification of singular fibers of elliptic surfaces, the claims (2,i) and (2,ii) follow by an easy combinatorial argument. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $f\colon S\to B$ be a primitive cyclic covering fibration of type $(g,1,n)$. From Lemma 3.6, we have

$$
\begin{aligned}
e_f(F_p)-\mu\chi_f(F_p) &= A_n\alpha_0^+(F_p)+\sum_{k\geq 1}(n-\mu'k)\,\alpha_k'(F_p)+n\sum_{k\geq 1}\alpha_k''(F_p)\\
&\quad -\mu'\sum_{k\geq 1}k\alpha_k''(F_p)-\sum_{a\geq 1}(2A_n+\delta_{a=1})j_{0,a}(F_p)+C_n\chi_\varphi(F_p)-\frac{r\mu'}{n}\nu(F_p)\\
&= A_n\alpha_0^+(F_p)+\sum_{k\geq 1}(n-\mu'k)\,\alpha_k'(F_p)+C_n\chi_\varphi(F_p)\\
&\quad -\frac{r\mu'}{n}\left(1-\frac{1}{m}\right)-\mu'\gamma_p+n\eta_p''+\sum_{a\geq 1}\left(an^2-a\mu'\right)j_{1,a}'(F_p)\\
&\quad +\sum_{a\geq 1}\left(n(an-2-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})-a\mu'-2A_n-\delta_{a=1}\right)j_{0,a}'(F_p)\\
&\quad +\sum_{a\geq 1}\left(n(an-1)-a\mu'-2A_n-\delta_{a=1}\right)j_{0,a}''(F_p)\\
&\quad -\mu'\sum_{k\geq 1}k\alpha_k^{n\mathbb{Z}+1}(F_p)-\sum_{k\geq 1}(n-\mu'k)\,\iota_k(F_p)-\sum_{k\geq 1}(n-\mu'k)\,\kappa_k(F_p),
\end{aligned}
$$

where

$$
A_n:=n-1-\frac{(n-1)(2n-1)}{12n}\mu,\quad C_n:=12n-\left(\frac{r}{12n}(n-1)(n+1)+n\right)\mu
$$

and $\mu':=(n=1)(n+1)\mu/12$. Combining Lemma 3.7 with the above equality, we have

$$
\begin{aligned}
&e_f(F_p)-\mu\chi_f(F_p)\\
&\geq A_n\alpha_0^+(F_p)+\sum_{k\geq 1}(n-\mu'k)\,\alpha_k'(F_p)+C_n\chi_\varphi(F_p)-\frac{r\mu'}{n}\left(1-\frac{1}{m}\right)\\
&\quad -\mu'\left(\frac{r}{n}-j_{0,1}'(F_p)\delta_{n=2}-\delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta_p'\neq 0}+(n-\mu')\left(\eta_p'-\delta_{\mathrm{cyc}}\right)+\left(2n-\frac{r\mu'}{n}\right)\eta_p''\\
&\quad +\sum_{a\geq 1}\left(n(an-3-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})-(a-1)\mu'-2A_n-\delta_{a=1}\right)j_{0,a}'(F_p) \qquad (3.5)\\
&\quad +\sum_{a\geq 1}\left(n(an-2)-(a-1)\mu'-2A_n-\delta_{a=1}\right)j_{0,a}''(F_p)\\
&\quad +\sum_{a\geq 1}\left(n(an-1)-(a-1)\mu'\right)j_{1,a}'(F_p)-\mu'\sum_{k\geq 1}k\alpha_k^{n\mathbb{Z}+1}(F_p)-\sum_{k\geq 1}(n-\mu'k)\,\kappa_k(F_p).
\end{aligned}
$$

Assume that $A_n\geq 0$ and $C_n\geq 0$. We obtain by using Lemmas 3.5 and 3.9 (1) that

$$
\begin{aligned}
&e_f(F_p) - \mu\chi_f(F_p) \\
\geq &\sum_{k\geq 1} (n-\mu' k)\,\alpha'_k(F_p) + \frac{r(m-1)(n-1)(4-\mu)}{4m} + C_n\chi_\varphi(F_p) - \mu'\left(\frac{r}{n} - \delta_{m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta'_p\neq 0} \\
&+ A_n\beta_p + (n-\mu'-(n-2)A_n)\,(\eta'_p - \delta_{\mathrm{cyc}}) + \left(2n - \frac{r\mu'}{n} - (n-2)A_n\right)\eta''_p \\
&+ \sum_{a\geq 1}\left((n-4)A_n + n(an-3-\delta_{m\mathrm{I}_1,\mathrm{II}}) - (a-1)\mu' - (1-\delta_{n=2}\mu')\delta_{a=1}\right) j'_{0,a}(F_p) \\
&+ \sum_{a\geq 1}\left((n-4)A_n + n(an-2) - \mu'\left(a + \frac{r}{n} - 2\right) - \delta_{a=1}\right) j''_{0,a}(F_p) \\
&+ \sum_{a\geq 1}\left((n-2)A_n + n(an-1) - (a-1)\mu'\right) j'_{1,a}(F_p) \\
&+ \left(2(n-2)A_n - n + \left(\frac{r}{n} - 1\right)\mu'\right)\kappa(F_p) - \mu'\left(\frac{r}{n} - 2\right)\kappa^{(3)}(F_p) - \mu'\alpha^{n\mathbb{Z}+1}_{\frac{r}{nm}}(F_p).
\end{aligned}
\tag{3.6}
$$

We put

$$
\mu = \begin{cases} \dfrac{12n^2}{r(n-1)(n+1)}, & \text{if } n\geq 4, \\ \dfrac{24}{4r-13}, & \text{if } n=3, \\ \dfrac{4}{r-2}, & \text{if } n=2. \end{cases}
$$

(i) We assume $n \geq 4$. We write $r = kn$. The coefficient of η'_p is

$$
n - \frac{1}{12}(n-1)(n+1)\mu - (n-2)A_n = -(n^2-4n+2) + \frac{n^2-6n+2}{k(n+1)} < 0.
$$

The coefficient of η''_p is

$$
2n - \frac{(n-1)(n+1)r\mu}{12n} - (n-2)A_n = -(n^2-4n+2) + \frac{(n-2)(2n-1)}{k(n+1)}.
$$

It is negative if $n \geq 5$ or $k \geq 2$. Note that $\eta''_p = 0$ if $k < n-1$ since the multiplicity m' of a singular point of type $n\mathbb{Z}+1$ satisfies $(n-1)^2 \leq m' \leq r-n+1$. Thus, we may not consider the case where $n=4$ and $k=1$. Using $\eta'_p \leq j'(F_p)$ and $\eta''_p \leq j''(F_p)$, (3.6) is greater than or equal to

$$\sum_{k\geq 1}(n-\mu' k)\,\alpha'_k(F_p)+\frac{r(m-1)(n-1)(4-\mu)}{4m}+C_n\chi_\varphi(F_p)+A_n\beta_p$$
$$-\mu'\left(\frac{r}{n}-\delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta'_p\neq 0}+((n-2)A_n-n+\mu')\,\delta_{\mathrm{cyc}}+\sum_{a\geq 1}\left(an^2-a\mu'\right)j'_{1,a}(F_p)$$
$$+\sum_{a\geq 1}(n(an-2-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})-a\mu'-2A_n-\delta_{a=1})\,j'_{0,a}(F_p) \tag{3.7}$$
$$+\sum_{a\geq 1}\left(an^2-\mu'\left(a+\frac{2r}{n}-2\right)-2A_n-\delta_{a=1}\right)j''_{0,a}(F_p)$$
$$+\left(2(n-2)A_n-n+\left(\frac{r}{n}-1\right)\mu'\right)\kappa(F_p)-\mu'\left(\frac{r}{n}-2\right)\kappa^{(3)}(F_p).$$

Since $2(n-2)A_n-n+\mu'>0$, we have

$$\begin{aligned}&\left(2(n-2)A_n-n+\left(\frac{r}{n}-1\right)\mu'\right)\kappa(F_p)-\mu'\left(\frac{r}{n}-2\right)\kappa^{(3)}(F_p)\\&\geq(2(n-2)A_n-n+\mu')\,\kappa(F_p)\\&\geq 0.\end{aligned}$$

The coefficient of $j'_{0,1}(F_p)$ in (3.7) is

$$\begin{aligned}&n(n-2-\delta_{{}_m\mathrm{I}_1,\mathrm{II}})-\mu'-2A_n-\delta_{a=1}\\&=n^2-(4+\delta_{{}_m\mathrm{I}_1,\mathrm{II}})n+1-\frac{(n-1)(n-2)}{k(n+1)}.\end{aligned} \tag{3.8}$$

If $\delta_{{}_m\mathrm{I}_1,\mathrm{II}}=0$, (3.8) is negative if and only if $n=4$ and $k=1$. If $\delta_{{}_m\mathrm{I}_1,\mathrm{II}}=1$ and $n\geq 5$, (3.8) is non-negative. If $\delta_{{}_m\mathrm{I}_1,\mathrm{II}}=1$ and $n=4$, (3.8) is $-3-6/5k<0$. Note that $j'(F_p)=0$ if $k=1$ since $r=n$ and any singularity of R has the multiplicity n. Thus, we may not consider the case where $\delta_{{}_m\mathrm{I}_1,\mathrm{II}}=0$, $n=4$ and $k=1$. We can check that the coefficient of $j'_{0,a}(F_p)$ in (3.7) is positive for $a\geq 2$. Moreover, we also can check that the coefficient of $j''_{0,a}(F_p)$ in (3.7) is positive for $a\geq 1$.

(i,1) We assume that $\eta'_p=0$. Then, clearly (3.7) is non-negative.

(i,2) We assume that $\eta'_p\neq 0$, $\chi_\varphi(F_p)\geq 1/6$ (i.e. Γ_p is a singular fiber not of type $({}_{\mathrm{m}}\mathrm{I}_1)$) and $\iota^{(3)}(F_p)=0$. Then we have

$$C_n\chi_\varphi(F_p)\geq\frac{1}{6}C_n=\frac{n}{6}\left(11-\frac{12n}{k(n-1)(n+1)}\right).$$

The coefficient of $\delta_{\eta'_p\neq 0}$ is

$$\mu'\left(\frac{r}{n}-\delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)=-n+\frac{n}{k}\delta_{{}_m\mathrm{I}_1,\mathrm{II}}.$$

If $n \neq 4$ or $\delta_{{}_m\mathrm{I}_1,\mathrm{II}} = 0$ or $j'_{0,1}(F_p) = 0$, then (3.7) is positive since

$$\frac{n}{6}\left(11 - \frac{12n}{k(n-1)(n+1)}\right) - n = \frac{n}{6}\left(5 - \frac{12n}{k(n-1)(n+1)}\right) > 0.$$

If $n = 4$ and $\delta_{{}_m\mathrm{I}_1,\mathrm{II}} = 1$ and $j'_{0,1}(F_p) \neq 0$ (we denote this condition by (#)), then we have $j'_{0,1}(F_p) = 1$ and then (3.7) is positive since

$$\frac{n}{6}\left(11 - \frac{12n}{k(n-1)(n+1)}\right) - n + \frac{n}{k} - 3 - \frac{6}{5k} = \frac{1}{3} + \frac{26}{15k} > 0.$$

(i,3) We assume that $\eta'_p \neq 0$ and $\chi_\varphi(F_p) = 1/12$ (i.e. Γ_p is a singular fiber of type $({}_{\mathrm{m}}\mathrm{I}_1)$). Let m_1 be the multiplicity of the singular point x_1 of R which is singular for Γ_p. If $m_1 \in n\mathbb{Z}$, then x_1 contributes $m_1 - 2$ to $\alpha_0^+(F_p)$ (Note that x_1 is a 1-vertical type singularity). In particular, x_1 contributes at least $n-2$ to $\alpha_0^+(F_p)$. If the condition (#) does not hold, (3.7) is positive since

$$\frac{n}{12}\left(11 - \frac{12n}{k(n-1)(n+1)}\right) - n + (n-2)A_n = n^2 - \frac{37}{12}n + 2 - \frac{n^2 + (n-2)(n-1)(2n-1)}{k(n-1)(n+1)}$$

increases monotonically with respect to n and

$$\frac{17}{3} - \frac{58}{15k} > 0$$

when $n = 4$. If the condition (#) holds, (3.7) is also positive since

$$\frac{17}{3} - \frac{58}{15k} + \frac{4}{k} - 3 - \frac{6}{5k} = \frac{8}{3} - \frac{16}{15k} > 0.$$

(i,4) We assume that $\eta'_p \neq 0$ and $\chi_\varphi(F_p) = 0$ (i.e. Γ_p is a smooth elliptic curve). Then (3.7) is positive since $j'_{1,\bullet}(F_p) = 1$.

(i,5) We assume that $\iota^{(3)}(F_p) = 1$. From Lemma 3.4, we may consider the following 3 cases.

(i,5,II) If Γ_p is a singular fiber of type (II), then $\beta_p = n - 7$, $\chi_\varphi(F_p) = 1/6$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. From the argument in (i,2), it is sufficient to show that

$$A_n\beta_p + \left(2(n-2)A_n - n + \left(\frac{r}{n} - 1\right)\mu'\right)\kappa(F_p) > 0.$$

This inequality is true, since

$$\begin{aligned}&(n-7)A_n + 2(n-2)A_n - n + \left(\frac{r}{n} - 1\right)\mu' \\ &= (3n-11)\left(n - 1 - \frac{2n-1}{k(n+1)}\right) - \frac{n}{k} \\ &> 0.\end{aligned}$$

Note that $k \geq n-1$ since $j''(F_p) \neq 0$.

(i,5,Ⅲ) If Γ_p is a singular fiber of type (Ⅲ), then $\beta_p = -n-1$, $\chi_\varphi(F_p) = 1/4$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. Since $j''(F_p) \geq 1$ and the coefficient of $j''_{0,a}(F_p)$ is greater than 1 for any $a \geq 1$, it is sufficient to show that

$$1 + A_n\beta_p + \frac{1}{12}C_n + \left(2(n-2)A_n - n + \left(\frac{r}{n} - 1\right)\mu'\right)\kappa(F_p) \geq 0.$$

The left hand side of it is greater than or equal to

$$\begin{aligned} &1 - (n+1)A_n + \frac{1}{12}C_n + 2(n-2)A_n - n + \left(\frac{r}{n} - 1\right)\mu' \\ &= n^2 - \frac{61}{12}n + 6 - \frac{3n^3 - 12n^2 + 15n - 5}{k(n-1)(n+1)} \end{aligned} \tag{3.9}$$

and (3.9) increases monotonically with respect to n. If $n = 4$, (3.9) is $5/3 - 11/3k > 0$. Note that $k \geq n-1$ since $j''(F_p) \neq 0$.

(i,5,Ⅳ) If Γ_p is a singular fiber of type (Ⅳ), then $\beta_p = -2$, $\chi_\varphi(F_p) = 1/3$ and $j'_{0,\bullet}(F_p) \geq 3$. Thus, it is sufficient to show that $A_n\beta_p + C_n/6 + 3\cdot(3.8)$ is positive. By a computation, this is equal to

$$3n^2 - \frac{73}{6}n + 5 - \frac{3n^3 - 14n^2 + 21n - 8}{k(n-1)(n+1)}$$

and we can check that it is positive.

From (i,1) through (i,5), we have $e_f(F_p) - \mu\chi_f(F_p) \geq 0$ for $n \geq 4$. On the other hand, if $n = 3$ and $g = 4$, one can easily classify all singular fibers of primitive cyclic covering fibrations of type $(4,1,3)$ because R has no singularities of multiplicity greater than 3, and check $e_f(F_p) \geq (9/2)\chi_f(F_p)$ for any fiber germ F_p. Thus, Theorem 3.1 (1) follows.

(ii) We assume $n = 3$ and $g > 4$. The coefficient of $j''_{0,1}(F_p)$ in (3.6) is

$$-\left(\frac{2}{9}r - \frac{17}{18}\right)\mu < 0.$$

Applying Lemma 3.12 (1) to the term of $j''_{0,1}(F_p)$, (3.6) is greater than or equal to

$$\begin{aligned} &\sum_{k\geq 1}\left(3 - \frac{2}{3}\mu k\right)\alpha'_k(F_p) + \frac{r(m-1)(4-\mu)}{2m} + \left(36 - \left(\frac{2}{9}r + 3\right)\mu\right)\chi_\varphi(F_p) \\ &+ \left(2 - \frac{5}{18}\mu\right)\beta_p - \frac{2}{3}\mu\left(\frac{r}{3} - \delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta'_p\neq 0} + \left(1 - \frac{7}{18}\mu\right)(\eta'_p - \delta_{\mathrm{cyc}}) \\ &+ \sum_{a\geq 2}\left(9a - 11 - 3\delta_{{}_m\mathrm{I}_1,\mathrm{II}} - \left(\frac{4}{9}(a-1)r - \frac{11}{9}a + \frac{17}{18}\right)\mu\right)j'_{0,a}(F_p) \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&+\left(-3+\frac{5}{18}\mu\right) j'_{0,1}(F_p)+\sum_{a\geq 2}\left(9a-8-\left(\frac{4}{9}ar-\frac{11}{9}a-\frac{2}{3}\right)\mu\right) j''_{0,a}(F_p)\\
&+\sum_{a\geq 1}\left(9a-1-\left(\frac{4}{9}ar-\frac{11}{9}a-\frac{7}{18}\right)\mu\right) j'_{1,a}(F_p)\\
&+\left(1+\left(\frac{2}{9}r-\frac{11}{9}\right)\mu\right)\kappa(F_p)-\frac{2}{3}\mu\left(\frac{r}{3}-2\right)\kappa^{(3)}(F_p).
\end{aligned}
$$

We remark that the term of η''_p vanishes by the definition of μ and

$$
\begin{aligned}
&\left(1+\left(\frac{2}{9}r-\frac{11}{9}\right)\mu\right)\kappa(F_p)-\frac{2}{3}\mu\left(\frac{r}{3}-2\right)\kappa^{(3)}(F_p)\\
&\geq\left(1+\frac{1}{9}\mu\right)\kappa(F_p)\\
&\geq 0.
\end{aligned}
$$

We can check that the coefficient of $j'_{0,a}(F_p)$ (resp. $j''_{0,a}(F_p)$, $j'_{1,a}(F_p)$) in (3.10) are positive for $a\geq 2$ (resp. $a\geq 2$, $a\geq 1$).

(ii,1) Assume that $\eta'_p=0$. Then (3.10) is non-negative.

(ii,2) Assume that $\eta'_p\neq 0$ and $j'^{t}_{0,1}(F_p)=4$ for some t. From Lemma 3.11 (1,iii), it follows that Γ_p is a singular fiber of type (I_k^*) for some k, $\chi_\varphi(F_p)=(k+6)/12$, $r\geq 9$, $\Gamma_p\subset R$, $\eta'_p=1$, $j'_{0,1}(F_p)=4$ and $\delta_{\mathrm{cyc}}=0$. Considering the terms of $\chi_\varphi(F_p)$, $\delta_{\eta'_p\neq 0}$, η'_p and $j'_{0,1}(F_p)$, (3.10) is greater than or equal to

$$
\begin{aligned}
&\left(36-\left(\frac{2}{9}r+3\right)\mu\right)\frac{k+6}{12}-\frac{2}{9}r\mu+\left(1-\frac{7}{18}\mu\right)+4\left(-3+\frac{5}{18}\mu\right)\\
&=7-\frac{8(3r+2)}{3(4r-13)}.
\end{aligned}
$$

This is positive since $r\geq 9$.

(ii,3) Assume that $\eta'_p\neq 0$, $\iota^{(3)}(F_p)=0$, $r\geq 9$ and $j'^{t}_{0,1}(F_p)\geq 3$ for any t. Then

$$
\frac{1}{3}j'_{0,1}(F_p)\leq\eta'_p-\delta_{\mathrm{cyc}}
$$

from Lemma 3.11 (1,i), (1,ii). From this and Lemma 3.13 (1), (3.10) is greater than or equal to

$$
\left(36-\left(\frac{2}{9}r+3\right)\mu\right)\frac{j'_{0,1}(F_p)+1}{12}-\frac{2}{9}r\mu+\frac{1}{3}\left(1-\frac{7}{18}\mu\right)j'_{0,1}(F_p)+\left(-3+\frac{5}{18}\mu\right)j'_{0,1}(F_p).
$$

One can check by a computation that this is positive since $r\geq 9$.

(ii,4) Assume that $\eta'_p \neq 0$, $\iota^{(3)}(F_p) = 0$, $r = 6$ and $\chi_\varphi(F_p) \geq (j'_{0,1}(F_p) + 2)/12$. By the same argument as in (ii,3), (3.10) is greater than or equal to

$$\left(36 - \left(\frac{2}{9}r + 3\right)\mu\right)\frac{j'_{0,1}(F_p) + 2}{12} - \frac{2}{9}r\mu + \frac{1}{3}\left(1 - \frac{7}{18}\mu\right)j'_{0,1}(F_p) + \left(-3 + \frac{5}{18}\mu\right)j'_{0,1}(F_p)$$
$$= -\frac{13}{99}j'_{0,1}(F_p) + \frac{50}{33}.$$

On the other hand, one sees that $j'_{0,1}(F_p) \leq 6$ since $r = 6$. Then

$$-\frac{13}{99}j'_{0,1}(F_p) + \frac{50}{33} > 0.$$

(ii,5) Assume that $\eta'_p \neq 0$, $\iota^{(3)}(F_p) = 0$, $r = 6$ and $\chi_\varphi(F_p) < (j'_{0,1}(F_p) + 2)/12$. If $j'_{0,1}(F_p) = 0$, then (3.10) is positive since $j'(F_p) \neq 0$. Then there are the following two cases only.

(ii,5,I_2) Γ_p is a singular fiber of type $({}_m\mathrm{I}_2)$ and only one component of Γ_p is contained in R and brown up just once.

(ii,5,I_3) Γ_p is a singular fiber of type $({}_m\mathrm{I}_3)$ and only two component of Γ_p are contained in R and brown up just once at the intersection point of these.

In both cases, we can see that

$$\gamma_p \leq \left(\frac{r}{n} - 1\right)\delta_{\eta'_p \neq 0} + \left(\frac{r}{n} - 1\right)\eta''_p$$

from the proof of Lemma 3.7, since the component of Γ_p not contained in R intersects with R_h. Thus, it is sufficient to show that

$$\left(36 - \left(\frac{2}{9}r + 3\right)\mu\right)\frac{j'_{0,1}(F_p) + 1}{12} - \frac{2}{3}\mu\left(\frac{r}{3} - 1\right) + \left(1 - \frac{7}{18}\mu\right) + \left(-3 + \frac{5}{18}\mu\right)j'_{0,1}(F_p)$$

is positive. This is equal to $(-2j'_{0,1}(F_p) + 10)/11 > 0$ by a computation.

(ii,6) Assume that $\iota^{(3)}(F_p) = 1$. From Lemma 3.4, we may consider the following 3 cases.

(ii,6,II) If Γ_p is a singular fiber of type (II), then we have $\chi_\varphi(F_p) = 1/6$, $\beta_p = -4$, $\delta_{\mathrm{cyc}} = 0$, $j'_{0,1}(F_p) = 0$, $j'_{0,\bullet}(F_p) = 1$, $j''_{0,\bullet}(F_p) \geq 2$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. One can see easily that (3.10) is positive by a computation.

(ii,6,III) If Γ_p is a singular fiber of type (III), then we have $\chi_\varphi(F_p) = 1/4$, $\beta_p = -4$, $\delta_{\mathrm{cyc}} = 0$, $j'_{0,1}(F_p) = 0$, $j'_{0,\bullet}(F_p) = 2$, $j''_{0,\bullet}(F_p) \geq 1$ and $\kappa(F_p) = \kappa^{(2)}(F_p) \geq 1$. One can see easily that (3.10) is positive by a computation.

(ii,6,IV) If Γ_p is a singular fiber of type (IV), then we have $\chi_\varphi(F_p) = 1/3$, $\beta_p = -2$, $\delta_{\mathrm{cyc}} = 0$, $j'_{0,\bullet}(F_p) = 3$ and $\kappa^{(3)}(F_p) = 0$. If $j'_{0,1}(F_p) \leq 2$, then we can check that (3.10) is positive. Suppose that $j'_{0,1}(F_p)=3$. Then any component of Γ_p is brown up only once. Thus, the multiplicity of the singularity of R on Γ_p is $r/3+3\geq 6$. In particular, $r\geq 9$. Since this singularity is a 3-vertical $3\mathbb{Z}$ type singularity, we have

$$\alpha_0^+(F_p)\geq \iota(F_p)+2\kappa(F_p)+\beta_p+3$$

from the proof of Lemma 3.5. Then it suffices to show that

$$\left(2-\frac{5}{18}\mu\right)+\left(36-\left(\frac{2}{9}r+3\right)\mu\right)\frac{1}{3}-\frac{2}{9}r\mu+\left(1-\frac{7}{18}\mu\right)+3\left(-3+\frac{5}{18}\mu\right)$$

is positive. This is equal to

$$6-\frac{4(16r+45)}{9(4r-13)}>0.$$

From (ii,1) through (ii,6), we have $e_f(F_p)-\mu\chi_f(F_p)\geq 0$. Thus, Theorem 3.1 (2) follows.

(iii) We assume $n=2$ and $g\geq 3$. From Lemmas 3.5 and 3.9 (2), (3.5) is greater than or equal to

$$\begin{aligned}
&\sum_{k\geq 1}\left(2-\frac{1}{4}\mu k\right)\alpha'_k(F_p)+\frac{r(m-1)(4-\mu)}{4m}+\left(24-\left(\frac{r}{8}+2\right)\mu\right)\chi_\varphi(F_p)\\
&-\frac{1}{4}\mu\left(\frac{r}{2}-j'_{0,1}(F_p)-\delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta'_p\neq 0}+\left(2-\frac{1}{4}\mu\right)(\eta'_p-\delta_{\mathrm{cyc}})+\left(4-\frac{1}{8}r\mu\right)\widehat{\eta}_p\\
&+\sum_{a\geq 1}\left(4a-8-2\delta_{{}_m\mathrm{I}_1,\mathrm{II}}-\delta_{a=1}-\frac{1}{4}(a-2)\mu\right)j'_{0,a}(F_p)\\
&+\sum_{a\geq 2}\left(4a-6-\frac{1}{4}\left(a+\frac{r}{2}-3\right)\mu\right)j''_{0,a}(F_p)+\sum_{a\geq 1}\left(4a-2-\frac{1}{4}\left(a-1\right)\mu\right)j'_{1,a}(F_p)\\
&+\sum_{k\geq 1}\left(1-\frac{1}{8}(r-2)\mu-\frac{1}{4}\mu k\right)\alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p)-\frac{1}{4}\mu\alpha^{2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p)\\
&+\sum_{k\geq 1}\left(1-\frac{1}{8}(r-2)\mu+\left(2-\frac{1}{2}\mu\right)k\right)\alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p)-\left(2-\frac{1}{8}(r-2)\mu\right)\kappa(F_p)\\
&+\sum_{k\geq 1}\left(-3+\frac{1}{4}\mu+\left(2-\frac{1}{2}\mu\right)k\right)\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)-\frac{1}{8}(r-4)\mu\kappa^{(3)}(F_p).
\end{aligned}\tag{3.11}$$

The coefficients of $\alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p)$ and $\alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p)$ in (3.11) are non-negative and that of $\alpha^{\mathrm{tr}}_{(r-1\to r-1)}(F_p)$ is 0 by the definition of μ. The coefficient of

$\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)$ in (3.11) is positive except for $k=1$ and that of $\alpha^{\mathrm{co},1}_{(3\to 3)}(F_p)$ is $-1-\mu/4<0$. The coefficient of $\kappa(F_p)$ in (3.11) is $-3/2$. Applying Lemmas 3.10 and 3.12 (2) to $(3/2)\kappa(F_p)$ and $(1+\mu/4)\sum_{k\ge 1}\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)$, we see that (3.11) is greater than or equal to

$$
\begin{aligned}
&\sum_{k\ge 1}\left(2-\frac{1}{4}\mu k\right)\alpha'_k(F_p)+\frac{r(m-1)(4-\mu)}{4m}+\left(24-\left(\frac{r}{8}+2\right)\mu\right)\chi_\varphi(F_p)\\
&-\frac{1}{4}\mu\left(\frac{r}{2}-\delta_{{}_m\mathrm{I}_1,\mathrm{II}}\right)\delta_{\eta'_p\neq 0}+\left(2-\frac{1}{4}\mu\right)(\eta'_p-\delta_{\mathrm{cyc}})+\left(3-\frac{1}{8}(r+2)\mu\right)\widehat{\eta}_p\\
&-\left(5-\frac{1}{2}\mu\right)j'_{0,1}(F_p)-(1+2\delta_{{}_m\mathrm{I}_1,\mathrm{II}})\,j'_{0,2,\mathrm{even}}(F_p)-\left(2+\frac{1}{4}\mu\right)j'_{0,2,\mathrm{odd}}(F_p)\\
&-\left(2\delta_{{}_m\mathrm{I}_1,\mathrm{II}}+\frac{3}{4}\mu\right)j'_{0,3}(F_p)+\sum_{a\ge 4}\left(2a-6-2\delta_{{}_m\mathrm{I}_1,\mathrm{II}}-\frac{1}{4}(2a-3)\mu\right)j'_{0,a}(F_p)\\
&+\sum_{a\ge 2}\left(2a-3-\frac{1}{4}\left(2a+\frac{r}{2}-5\right)\mu\right)j''_{0,a}(F_p)+\sum_{a\ge 1}\left(2a-\frac{1}{2}\left(a-1\right)\mu\right)j'_{1,a}(F_p) \quad (3.12)\\
&+\sum_{k\ge 1}\left(1-\frac{1}{8}(r-2)\mu-\frac{1}{4}\mu k\right)\alpha^{\mathrm{tr}}_{(2k+1\to 2k+1)}(F_p)+\left(1-\frac{1}{4}\mu\right)\alpha^{2\mathbb{Z}+1}_{\frac{r}{2m}}(F_p)\\
&+\sum_{k\ge 1}\left(1-\frac{1}{8}(r-2)\mu+\left(2-\frac{1}{2}\mu\right)k\right)\alpha^{\mathrm{co},0}_{(2k+1\to 2k+1)}(F_p)\\
&+\sum_{k\ge 1}\left(2-\frac{1}{2}\mu\right)(k-1)\alpha^{\mathrm{co},1}_{(2k+1\to 2k+1)}(F_p)-\frac{1}{8}\left(r-4\right)\mu\kappa^{(3)}(F_p),
\end{aligned}
$$

where $j'_{0,2,\mathrm{even}}(F_p)=j'_{0,2}(F_p)-j'_{0,2,\mathrm{odd}}(F_p)$.

(iii,1) Assume that $\eta'_p=0$. Then (3.12) is clearly non-negative.

(iii,2) Assume that $\eta'_p\neq 0$ and $j'^{t}_{0,2,\mathrm{odd}}(F_p)=4$ for some t. From Lemma 3.11 (2,iii), Γ_p is a singular fiber of type (I^*_k), $\Gamma_p\subset R$, $\kappa^{(3)}(F_p)=0$, $\eta'_p=1$, $\delta_{\mathrm{cyc}}=0$, $j'_{0,1}(F_p)=0$ and $j'_{0,2,\mathrm{odd}}(F_p)=4$. Clearly we have $\chi_\varphi(F_p)=(j'_{0,2}(F_p)+j'_{0,3}(F_p)+1)/12$. Considering the terms of $\chi_\varphi(F_p)$, $\delta_{\eta'_p\neq 0}$, η'_p, $j'_{0,2,\mathrm{even}}(F_p)$, $j'_{0,2,\mathrm{odd}}(F_p)$ and $j'_{0,3}(F_p)$, (3.12) is greater than or equal to

$$
\begin{aligned}
&\left(24-\left(\frac{r}{8}+2\right)\mu\right)\frac{1}{12}(j'_{0,2,\mathrm{even}}(F_p)+j'_{0,3}(F_p)+5)-\frac{1}{8}r\mu+\left(2-\frac{1}{4}\mu\right)\\
&-j'_{0,2,\mathrm{even}}(F_p)-4\left(2+\frac{1}{4}\mu\right)-\frac{3}{4}\mu j'_{0,3}(F_p)\\
&=\left(1-\left(\frac{r}{96}+\frac{1}{6}\right)\mu\right)j'_{0,2,\mathrm{even}}(F_p)+\left(2-\left(\frac{r}{96}+\frac{11}{12}\right)\mu\right)j'_{0,3}(F_p)+4-\left(\frac{17}{96}r+\frac{25}{12}\right)\mu,
\end{aligned}
$$

which is positive.

(iii,3) Assume that $\eta'_p \neq 0$, $\kappa^{(3)}(F_p) = 0$, $\delta_{{}_m\mathrm{I}_1,\mathrm{II}} = 0$, $j'^t_{0,2,\mathrm{odd}}(F_p) \leq 3$ for any t and

$$\chi_\varphi(F_p) \geq \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) + 1).$$

From Lemma 3.11 (2,i), (2,ii), we have $j'_{0,1}(F_p) + j'_{0,2,\mathrm{odd}}(F_p)/3 \leq \eta'_p - \delta_{\mathrm{cyc}}$. Then (3.12) is greater than or equal to

$$\begin{aligned}
&\left(24 - \left(\frac{r}{8} + 2\right)\mu\right)\frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) + 1) - \frac{1}{8}r\mu \\
&+ \left(2 - \frac{1}{4}\mu\right)\left(j'_{0,1}(F_p) + \frac{1}{3}j'_{0,2,\mathrm{odd}}(F_p)\right) - \left(5 - \frac{1}{2}\mu\right)j'_{0,1}(F_p) - j'_{0,2,\mathrm{even}}(F_p) \\
&- \left(2 + \frac{1}{4}\mu\right)j'_{0,2,\mathrm{odd}}(F_p) - \frac{3}{4}\mu j'_{0,3}(F_p) \\
&= \left(1 - \left(\frac{r}{48} + \frac{1}{12}\right)\mu\right)j'_{0,1}(F_p) + \left(\frac{2}{3} - \left(\frac{r}{96} + \frac{1}{2}\right)\mu\right)j'_{0,2,\mathrm{odd}}(F_p) \\
&+ \left(1 - \left(\frac{r}{96} + \frac{1}{6}\right)\mu\right)j'_{0,2,\mathrm{even}}(F_p) + \left(2 - \left(\frac{r}{96} + \frac{11}{12}\right)\mu\right)j'_{0,3}(F_p) \\
&+ 2 - \left(\frac{13}{96}r + \frac{1}{6}\right)\mu.
\end{aligned}$$

The coefficients of $j'_{0,1}(F_p)$, $j'_{0,2,\mathrm{even}}(F_p)$, $j'_{0,3}(F_p)$ and the constant term are positive since $r \geq 4$, and $j_{0,2,\mathrm{odd}}(F_p)$ is also positive for $r \geq 6$. Thus the above equation is positive when $r \geq 6$. If $r = 4$, then one can check by an easy computation that (3.12) is also positive, since $j'_{0,2,\mathrm{odd}}(F_p) \leq 2$.

(iii,4) Assume that $\eta'_p \neq 0$, $\kappa^{(3)}(F_p) = 0$, $\delta_{{}_m\mathrm{I}_1,\mathrm{II}} = 0$, $j'^t_{0,2,\mathrm{odd}}(F_p) \leq 3$ for any t and

$$\chi_\varphi(F_p) < \frac{1}{12}(2j'_{0,1}(F_p) + j'_{0,2}(F_p) + j'_{0,3}(F_p) + 1).$$

From Lemma 3.13 (2), Γ_p is of type $({}_m\mathrm{I}_k)$, (I_k^*), (II^*), (III^*) or (IV^*). Considering the numbers $j'_{0,1}(F_p)$, $\chi_\phi(F_p)$ and Lemma 3.13 (2), we can see that (3.12) is positive by the same argument as in (iii,3) except Γ_p is of type (I_0^*) and $j'_{0,1}(F_p) = 4$.

Suppose that Γ_p is of type (I_0^*) and $j'_{0,1}(F_p) = 4$. Then the component not contributing to $j'_{0,1}(F_p)$ is a double component in Γ_p and intersects with R_h. Thus we have

$$\alpha_0^+(F_p) \geq \frac{r}{2} + \sum_{k \geq 1} 2k\alpha^{\mathrm{co}}_{(2k+1 \to 2k+1)}(F_p)$$

from the proof of Lemma 3.5. Then one can see that $e_f(F_p) - \mu\chi_\varphi(F_p)$ is positive by a computation.

(iii,5) Assume that $\eta'_p \neq 0$, $\kappa^{(3)}(F_p) = 0$ and $\delta_{{}_m\mathrm{I}_1,\mathrm{II}} = 1$. Then $j'_{0,1}(F_p) = 0$, $j'(F_p) = 1$, $\eta'_p = 1$ and $\chi_\varphi(F_p) = 1/12$ or $1/6$. If $j'_{0,2,\mathrm{even}}(F_p) = 1$, then $(\Gamma_p)_{\mathrm{red}}$ is blown up just once. Thus, the multiplicity of the singularity of R which is singular for $(\Gamma_p)_{\mathrm{red}}$ is even. Hence $\delta_{\mathrm{cyc}} = 0$. Then we can check by a computation that (3.12) is positive. If $j'_{0,2,\mathrm{even}}(F_p) = 0$, then we can also check by a computation that (3.12) is positive.

(iii,6) Assume that $\kappa^{(3)}(F_p) = 1$. From Lemma 3.4, we may consider the following 3 cases.

(iii,6,II) If Γ_p is a singular fiber of type (II), then $\eta'_p = 1$, $j'(F_p) = 1$, $j'_{0,a}(F_p) = 0$ for $a \le 3$, $j''(F_p) - j''_{0,1}(F_p) \ge 3$ and $\chi_\varphi(F_p) = 1/6$. Considering the terms of $\delta_{\eta'_p \neq 0}$, η'_p, $\kappa^{(3)}(F_p)$, $j'_{0,a}(F_p)$, $j''_{0,a}(F_p)$ and $\chi_\varphi(F_p)$ in (3.12), we can check that (3.12) is positive.

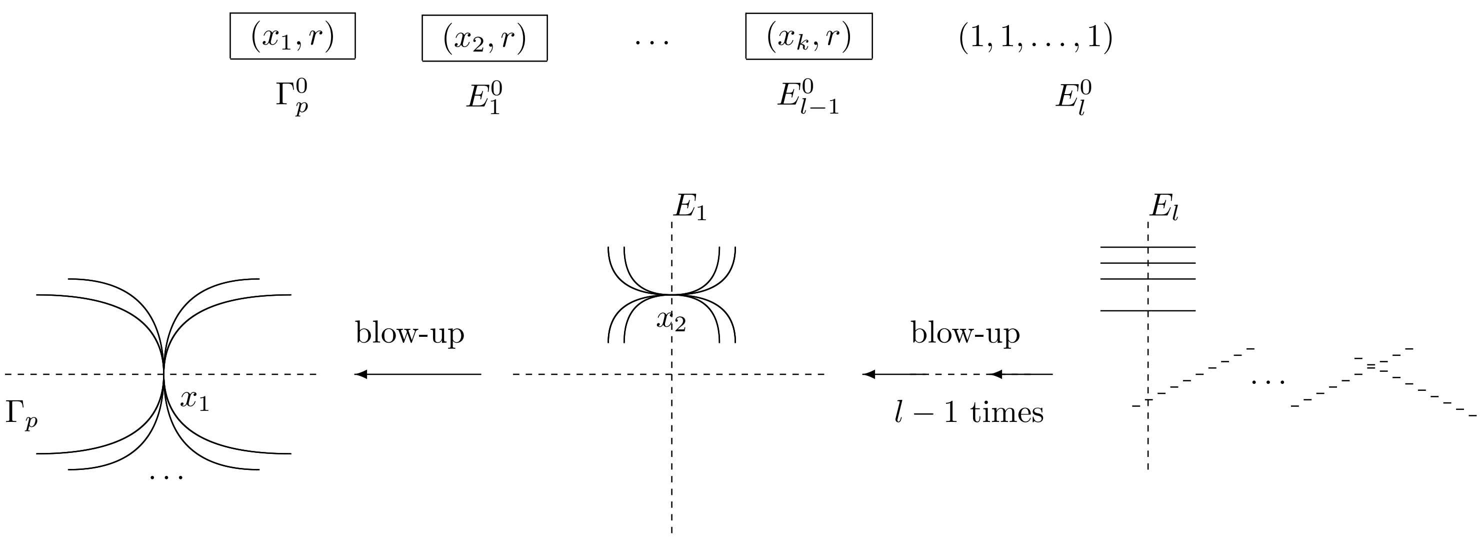
(iii,6,III) If Γ_p is a singular fiber of type (III), then $\eta'_p = 1$, $j'(F_p) = 2$, $j'_{0,a}(F_p) = 0$ for $a \le 2$, $j''(F_p) - j''_{0,1}(F_p) \ge 2$ and $\chi_\varphi(F_p) = 1/4$. Then we can also check that (3.12) is positive.

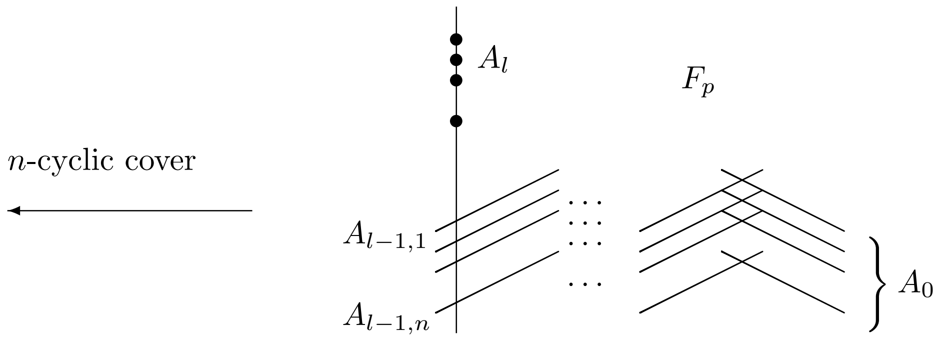
(iii,6,IV) If Γ_p is a singular fiber of type (IV), then $\eta'_p = 1$, $j'(F_p) = 3$, $j'_{0,1}(F_p) = j'_{0,2,\mathrm{even}}(F_p) = 0$, $j''(F_p) - j''_{0,1}(F_p) \ge 1$ and $\chi_\varphi(F_p) = 1/3$. Similarly, we can check that (3.12) is positive.

From (iii,1) through (iii,6), we have $e_f(F_p) - \mu\chi_f(F_p) \ge 0$. Thus, Theorem 3.1 (3) follows. □

Example 3.14. There exist singular fiber germs F_p such that $K_f^2(F_p) = (12 - \mu)\chi_f(F_p)$ and we can classify them.

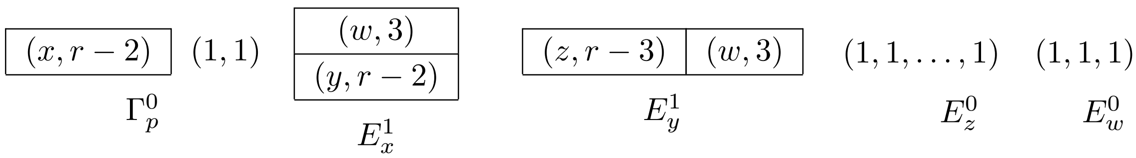
(i) Assume that $n \ge 4$, or $n = 3$ and $g = 4$. Consider the situation that Γ_p is smooth and F_p is obtained the following sequence of singularity diagrams associated with Γ_p (cf. [18]):

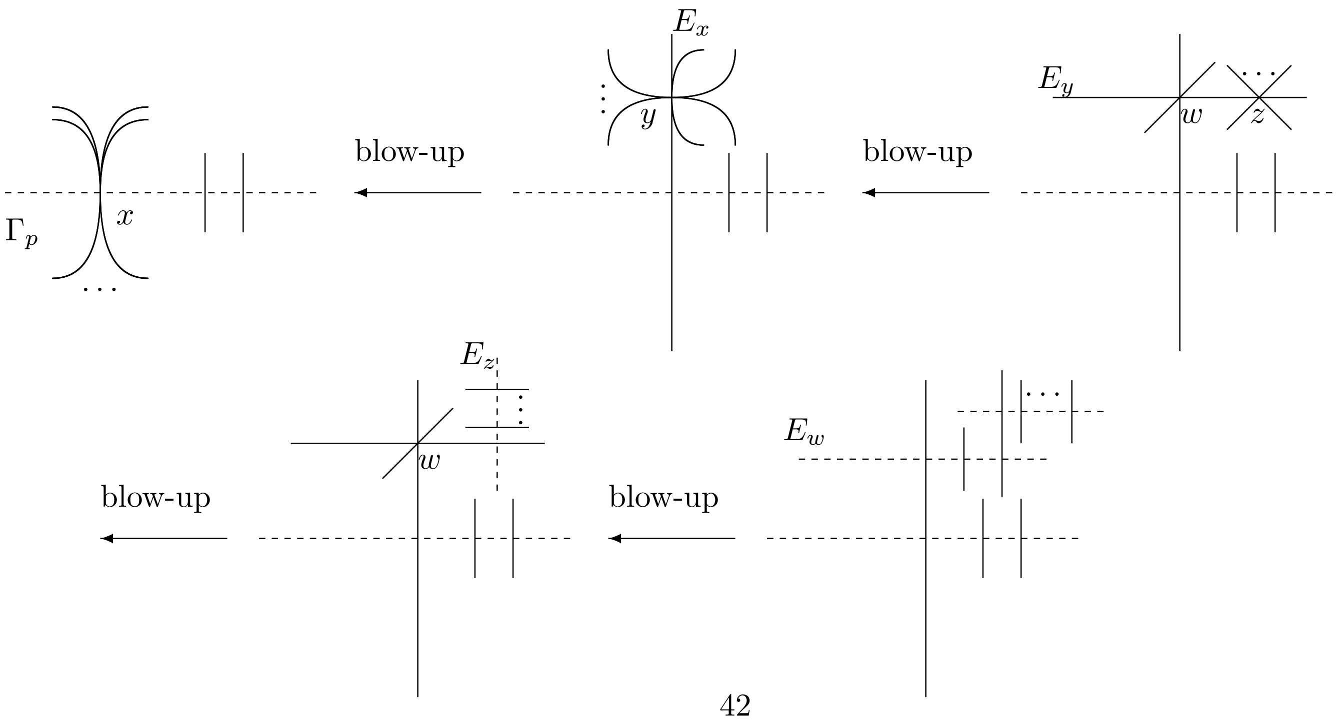


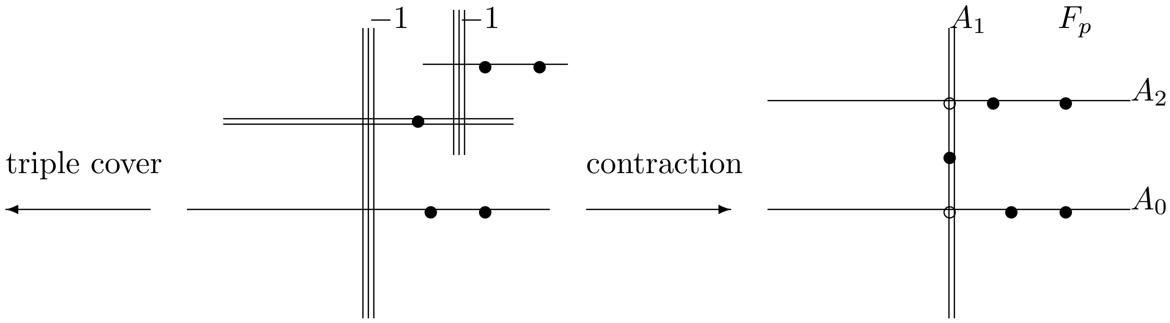


Then we can write $F_p = A_0 + \sum_{i=1}^{n}(A_{1,i} + \cdots + A_{l-1,i}) + A_l$, $p_a(A_0) = 0$, $g(A_{k,i}) = 0$ for $k = 1, \ldots, l-1$ and $g(A_l) = (r/2-1)(n-1)$ (note that A_0 may not be irreducible). This singular fiber satisfies $K_f^2(F_p) = (12-\mu)\chi_f(F_p)$. Indeed, $\alpha_k(F_p) = 0$ for $k = 0, 1, \ldots, r/n-1$, $\alpha_{r/n}(F_p) = l$, $\varepsilon(F_p) = 0$ and $\chi_\varphi(F_p) = 0$. Thus $\chi_f(F_p) = r(n-1)(n+1)l/12n$, $e_f(F_p) = nl$ and then $e_f(F_p)/\chi_f(F_p) = 12n^2/r(n-1)(n+1) = \mu$. We can see from the proof of Theorem 3.1 that any singular fiber F_p satisfying $K_f^2(F_p) = (12-\mu)\chi_f(F_p)$ is obtained in this way.

(ii) Assume that $n = 3$ and $g > 4$. Consider the situation that Γ_p is smooth and F_p is obtained the following sequence of singularity diagrams associated with Γ_p:



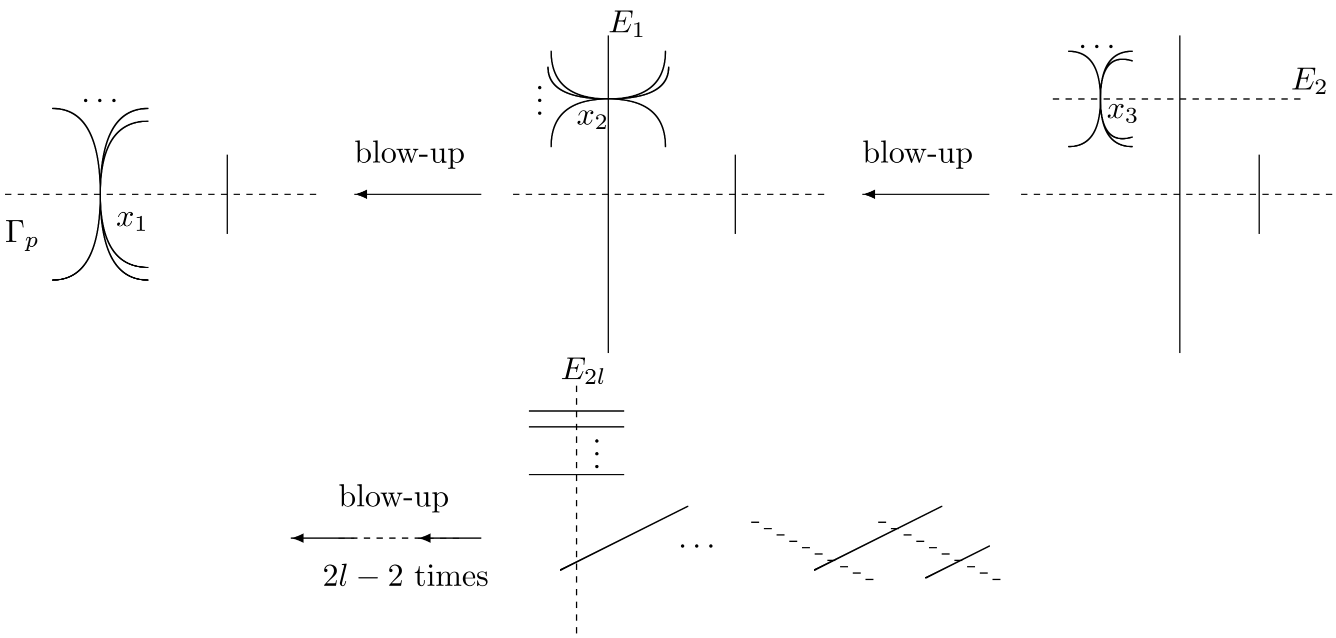


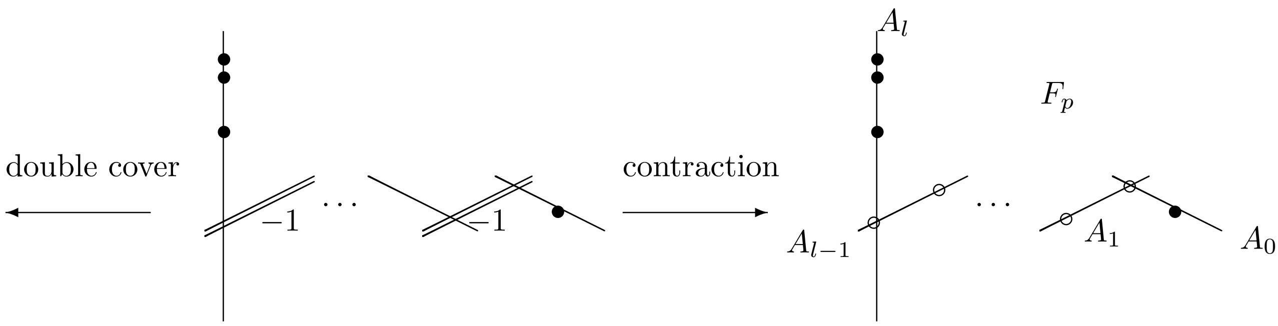


Then we can write $F_p = A_0 + 2A_1 + A_3$, $p_a(A_0) = 4$, $g(A_1) = 1$ and $g(A_2) = r - 5$. This singular fiber satisfies $K_f^2(F_p) = (12 - \mu)\chi_f(F_p)$. Indeed, $\alpha_0(F_p) = 1$, $\alpha_1(F_p) = l$, $\alpha_{r/3-1}(F_p) = 3$, $\alpha_k(F_p) = 0$ for $k \neq 1, r/3 - 1$, $\varepsilon(F_p) = 2$ and $\chi_\varphi(F_p) = 0$. Thus $\chi_f(F_p) = (4r - 13)/6$, $e_f(F_p) = 4$ and then $e_f(F_p)/\chi_f(F_p) = 24/(4r - 13) = \mu$. We can see from the proof of Theorem 3.1 that any singular fiber F_p satisfying $K_f^2(F_p) = (12 - \mu)\chi_f(F_p)$ is obtained in this way.

(iii) Assume that $n = 2$ and $g \geq 3$. Consider the situation that Γ_p is smooth and F_p is obtained the following sequence of singularity diagrams associated with Γ_p:

$(x_1, r-1)$	(1)	(x_2, r)	$\dots$	$(x_{2l-1}, r-1)$	(1)	(x_{2l}, r)	$(1, 1, \dots, 1)$
Γ_p^0		E_1^1		E_{2l-2}^0		E_{2l-1}^1	E_{2l}^0





Then we can write $F_p = A_0 + A_1 + \cdots + A_l$, $p_a(A_0) = 2$, $g(A_k) = 0$ for $k = 1, \dots, l-1$ and $g(A_l) = r/2 - 1$. This singular fiber satisfies $K_f^2(F_p) = (12-\mu)\chi_f(F_p)$. Indeed, $\alpha_{r/2-1}(F_p) = \alpha_{r/2}(F_p) = l$, $\alpha_k(F_p) = 0$ for $k \neq r/2-1, r/2$, $\varepsilon(F_p) = l$ and $\chi_\varphi(F_p) = 0$. Thus $\chi_f(F_p) = (r-2)l/4$, $e_f(F_p) = l$ and then $e_f(F_p)/\chi_f(F_p) = 4/(r-2) = \mu$. We can see from the proof of Theorem 3.1 that any singular fiber F_p satisfying $K_f^2(F_p) = (12-\mu)\chi_f(F_p)$ is obtained in this way.

From Theorem 3.1 and Example 3.14, we can characterize primitive cyclic covering fibrations of type $(g, 1, n)$ whose slope attains the upper bound in Theorem 3.1.

Corollary 3.15. *Let $f\colon S \to B$ be a primitive cyclic covering fibration of type $(g,1,n)$. Then the slope λ_f attains the upper bound in Theorem* 3.1 *if and only if any singular fiber of f is as in Example* 3.14.

4 Glueing linear series

For a smooth projective curve C (resp. a family of smooth projective curves $f\colon X \to B$), let $\mathcal{G}_d^r(C)$ (resp. $\mathcal{G}_d^r(f)$) be the (resp. relative) Brill-Noether variety parametrizing $\mathfrak{g}_d^r$'s on C (resp. on fibers of f), where we denote by $\mathfrak{g}_d^r$ a linear system of degree d and of dimension r (cf. [2] Chapter XXI).

In this section, we prove the following theorem for the later use, which is a slight improvement of Theorem 3.1 in [8].

Theorem 4.1. *Let X, B be normal algebraic varieties (resp. normal analytic varieties) and $f\colon X \to B$ a proper flat morphism whose general fiber is a non-singular projective curve. Let $B_0 \subset B$ be the Zariski open subset consisting of smooth points p of B such that $F_p = f^{-1}(p)$ is non-singular and $f_0\colon X_0 = f^{-1}(B_0) \to B_0$ the restriction of f to B_0. Let r, d be positive integers. Assume that there exists a rational section $\eta\colon B_0 \cdots \to \mathcal{G}_d^r(f_0)$. Then there exist a divisorial sheaf $\mathcal{L}$ on X and a subsheaf $\mathcal{G} \subset f_*\mathcal{L}$ such that the linear subspace $\mathcal{G} \otimes \mathbb{C}(p) \subset H^0(F_p, \mathcal{L}|_{F_p})$ defines $\eta(p)$ for any general $p \in B_0$.*

Proof. We may assume that $\eta(p)$ is base point free for any general $p \in B_0$ by removing the locus of all base points of $\eta(p)$, $p \in B_0$. Shrinking B_0 if necessary, we may assume that η is a section. For $p \in B_0$, we can write $\eta(p) = \{D(p)_\lambda\}_{\lambda \in \mathbb{P}^r}$, where $D(p)_\lambda$ is an effective divisor of degree d on F_p. Let $\mathcal{E}_0$ be a locally free sheaf on B_0 such that X_0 is embedded in $\mathbb{P}_{B_0}(\mathcal{E}_0)$ over B_0 (such $\mathcal{E}_0$ exists, e.g., take the direct image sheaf of a sufficiently f_0-ample invertible sheaf on X_0). We regard each fiber F_p as a subvariety of $\mathbb{P}(\mathcal{E}_0 \otimes \mathbb{C}(p)) \simeq \mathbb{P}^{\mathrm{rank}(\mathcal{E}_0)-1}$ via the inclusion $X_0 \subset \mathbb{P}_{B_0}(\mathcal{E}_0)$. Let $\overline{D(p)}_\lambda$ denote the plane in $\mathbb{P}(\mathcal{E}_0 \otimes \mathbb{C}(p))$ spanned by $D(p)_\lambda$. Then the dimension $k := \dim \overline{D(p)}_\lambda$ does not depend on the choices of p and λ from the Riemann-Roch theorem. Now, we consider the subvariety P of the relative Grassmannian $Gr_{B_0}(k, \mathbb{P}(\mathcal{E}_0)) = \cup_{p \in B_0} Gr(k, \mathbb{P}(\mathcal{E}_0 \otimes \mathbb{C}(p)))$ defined by

$$P := \{[\overline{D(p)}_\lambda] \in Gr(k, \mathbb{P}(\mathcal{E}_0 \otimes \mathbb{C}(p))) | \lambda \in \mathbb{P}^r, p \in B_0\}.$$

It is a holomorphic $\mathbb{P}^r$-bundle over B_0 via the natural projection. We can define a morphism Φ from X_0 to $P^* := Gr_{B_0}(r-1, P)$ by mapping x to $\{[\overline{D(p)}_\lambda] | x \in D(p)_\lambda\}$, the restriction of which to the fiber F_p is nothing but the morphism associated with $\eta(p)$. Let $\mathcal{G}_0$, $\mathcal{L}_0$ respectively be the direct image sheaf of the tautological line bundle $\mathcal{O}_{P^*}(1)$ via the natural projection $P^* \to B_0$, the pull-back of $\mathcal{O}_{P^*}(1)$ via Φ. It follows that $P^* = \mathcal{P}_{B_0}(\mathcal{G}_0)$ and $\mathcal{G}_0 \subset f_{0*}\mathcal{L}_0$. Let $i_{B_0} \colon B_0 \to B$ and $i_{X_0} \colon X_0 \to X$ be the natural inclusions. We put $\mathcal{G} := i_{B_0*}\mathcal{G}_0$ and $\mathcal{L} := (i_{X_0*}\mathcal{L})^{**}$, which are the desired sheaves. Indeed, we have $\mathcal{G} \subset i_{B_0*} f_{0*}\mathcal{L}_0 = f_* i_{X_0*}\mathcal{L}_0 \subset f_*\mathcal{L}$. □

Remark 4.2. If $f \colon S \to B$ has a section, Theorem 4.1 follows directly from the existence of the relative Poincare line bundle (cf. [2]).

Corollary 4.3. *Let X and B be normal algebraic varieties and $f \colon X \to B$ and d, r as in Theorem 4.1. Assume that the fiber F_p has a base point free $\mathfrak{g}^r_d$ for general $p \in B$. Then, after a suitable finite base change $B' \to B$, there exist a $\mathbb{P}^r$-bundle P' over B' and a rational map $\varphi \colon X' \cdots \to P'$ over B' of degree d, where $f' \colon X' \to B'$ is a base change fibration of f.*

Proof. By assumption, the general fiber of $\mathcal{G}^r_d(f_0) \to B_0$ is non-empty. Since $\mathcal{G}^r_d(f_0)$ is algebraic, we can take a subvariety B'_0 of $\mathcal{G}^r_d(f_0)$ such that the natural map $B'_0 \to B_0$ is finite (after shrinking B_0 if necessary). We take a compactification $B' \to B$ of it and perform base change via this map. Let $f' \colon S' \to B'$ be the base change fibration of f and $f'_0 \colon X'_0 \to B'_0$ the restriction of f' to $X'_0 = f'^{-1}(B'_0)$. Since $\mathcal{G}^r_d(f'_0) = \mathcal{G}^r_d(f_0) \times_{B_0} B'_0$, we can take a section $B'_0 \to \mathcal{G}^r_d(f'_0)$ by $p \mapsto (p, p)$. From Theorem 4.1, there exist a line bundle $\mathcal{L}$ on B' and a subbundle $\mathcal{G} \subset f'_*\mathcal{L}$ such that the rational map $X' \cdots \to \mathbb{P}_{B'}(\mathcal{G})$ associated to $f'^*\mathcal{G} \to \mathcal{L}$ is of degree d.

□

5 Localizations

Let $f\colon S \to B$ be a fibered surface of genus $g = (d-1)(d-2)/2 > 1$ whose general fiber has a $\mathfrak{g}^2_d$, that is, it is a smooth plane curve of degree d. Since the $\mathfrak{g}^2_d$ is unique, there exists a line bundle $\mathcal{L}$ on S (unique up to a multiple of a divisor consisting of components of fibers) such that $\mathcal{L}|_F$ is the $\mathfrak{g}^2_d$ on any general fiber F by Theorem 4.1. Then $\mathcal{L}^{\otimes d-3}$ is isomorphic to $\omega_f(\Gamma)$ for some divisor Γ consisting of components of fibers (it depends on a choice of $\mathcal{L}$) since $\mathcal{L}^{\otimes d-3}|_F$ is the canonical bundle K_F for a general fiber F. On the other hand, for $k = 1, \dots, d-1$, there exists a natural exact sequence

$$0 \to \mathrm{Sym}^k f_*\mathcal{L} \to f_*\mathcal{L}^{\otimes k} \to \mathcal{T}_k \to 0$$

induced from the multiplicative map $\mathrm{Sym}^k H^0(\mathcal{L}|_F) \to H^0(\mathcal{L}^{\otimes k}|_F)$ on fibers, where the cokernel $\mathcal{T}_k$ is a torsion sheaf. Thus, we get

$$\deg(f_*\mathcal{L}^{\otimes k}) = \deg(\mathrm{Sym}^k f_*\mathcal{L}) + \mathrm{length}(\mathcal{T}_k). \tag{5.1}$$

By the Grothendieck Riemann-Roch theorem, we have

$$\deg(f_*\mathcal{L}^{\otimes k}) - \deg(R^1 f_*\mathcal{L}^{\otimes k}) = \frac{k^2}{2}L^2 - \frac{k}{2}LK_f + \chi_f, \tag{5.2}$$

where $L = c_1(\mathcal{L})$. From (5.1) and (5.2), we obtain

$$\begin{aligned} &\frac{k^2}{2}L^2 - \frac{k}{2}LK_f + \chi_f + \deg(R^1 f_*\mathcal{L}^{\otimes k}) - \mathrm{length}(\mathcal{T}_k) \\ &= \binom{k+2}{3}\left(\frac{1}{2}L^2 - \frac{1}{2}LK_f + \chi_f + \deg(R^1 f_*\mathcal{L})\right). \end{aligned}$$

In particular, for $k = d-2, d-1$, we have

$$\begin{aligned} &\frac{(d-2)^2}{2}L^2 - \frac{d-2}{2}LK_f + \chi_f + \mathrm{length}(R^1 f_*\mathcal{L}^{\otimes d-2}) - \mathrm{length}(\mathcal{T}_{d-2}) \\ &= \binom{d}{3}\left(\frac{1}{2}L^2 - \frac{1}{2}LK_f + \chi_f + \deg(R^1 f_*\mathcal{L})\right), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\frac{(d-1)^2}{2}L^2 - \frac{d-1}{2}LK_f + \chi_f + \mathrm{length}(R^1 f_*\mathcal{L}^{\otimes d-1}) - \mathrm{length}(\mathcal{T}_{d-1}) \\ &= \binom{d+1}{3}\left(\frac{1}{2}L^2 - \frac{1}{2}LK_f + \chi_f + \deg(R^1 f_*\mathcal{L})\right), \end{aligned} \tag{5.4}$$

where two sheaves $R^1f_*\mathcal{L}^{\otimes d-2}$ and $R^1f_*\mathcal{L}^{\otimes d-1}$ are torsion sheaves. Since $\binom{d+1}{3}$ times the left hand side of (5.3) is equal to $\binom{d}{3}$ times the left hand side of (5.4), we obtain by a calculation that

$$K_f^2 = \frac{6(d-3)}{d-2}\chi_f + \sum_{p\in B} \mathrm{Ind}_d(F_p), \tag{5.5}$$

where we put

$$\begin{aligned}\mathrm{Ind}_d(F_p) :=& \Gamma_p^2 + 2(d-3)\left(\frac{d+1}{d-2}\mathrm{length}_p(R^1f_*\mathcal{L}^{\otimes d-2}) - \mathrm{length}_p(R^1f_*\mathcal{L}^{\otimes d-1})\right)\\ &+ 2(d-3)\left(\mathrm{length}_p(\mathcal{T}_{d-1}) - \frac{d+1}{d-2}\mathrm{length}_p(\mathcal{T}_{d-2})\right).\end{aligned}$$

From (5.5), the value $\mathrm{Ind}_d(F_p)$ is independent of a choice of the line bundle $\mathcal{L}$ since $\mathcal{L}$ is unique up to a multiple of an f-vertical divisor. But it seems hard to show that $\mathrm{Ind}_d(F_p)$ is non-negative directly from the definition.

6 Lower bound of the slope

In this section, we prove the following inequality for plane curve fibrations.

Theorem 6.1. *Let $f\colon S \to B$ be a relatively minimal plane curve fibration of degree $d \geq 4$. Then we have*

$$K_f^2 \geq \frac{6(d-3)}{d-2}\chi_f.$$

Let $f\colon S \to B$ be a relatively minimal plane curve fibration of degree d. Since a $\mathfrak{g}_d^2$ on the general fiber F is unique, there exists a line bundle $\mathcal{L}$ on S such that the restriction $\mathcal{L}|_F$ is the $\mathfrak{g}_d^2$ and it is unique up to a multiple of divisors consisting of components of fibers. Since $\mathcal{L}|_F^{\otimes d-3} = \omega_F$, we can write $\mathcal{L}^{\otimes d-3}(J) = \omega_f$ for some divisor J consisting of components of fibers. Tensoring components of fibers to $\mathcal{L}$, we may assume that J is effective. Then we have an injection $f_*\mathcal{L}^{\otimes d-3} \to f_*\omega_f$. The composite of it and the natural homomorphism $\mathrm{Sym}^{d-3}f_*\mathcal{L} \to f_*\mathcal{L}^{\otimes d-3}$ induces an injection $\mathrm{Sym}^{d-3}f_*\mathcal{L} \to f_*\omega_f$ whose cokernel is a torsion sheaf. Let $\mathfrak{c}$ be the maximal effective divisor on B such that the image of the homomorphism $\mathrm{Sym}^{d-3}f_*\mathcal{L} \to f_*\omega_f$ is contained in $f_*\omega_f(-\mathfrak{c})$. Then there is an exact sequence

$$0 \to \mathrm{Sym}^{d-3}f_*\mathcal{L} \to f_*\omega_f(-\mathfrak{c}) \to \mathcal{T} \to 0,$$

which induces an elementary transformation

$$P := \mathbb{P}_B(f_*\omega_f) = \mathbb{P}_B(f_*\omega_f(-\mathfrak{c})) \xleftarrow{\tau} \widetilde{P} \xrightarrow{\tau'} P' := \mathbb{P}_B(\mathrm{Sym}^{d-3}(f_*\mathcal{L}))$$

such that

$$\tau^*\mathcal{O}_{\mathbb{P}_B(f_*\omega_f(-\mathfrak{c}))}(1) - E_\tau = \tau'^*\mathcal{O}_{\mathbb{P}_B(\mathrm{Sym}^{d-3}(f_*\mathcal{L}))}(1)$$

holds, where $\mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ is the tautological line bundle associated with $\mathcal{E}$ and E_τ is an effective exceptional divisor of τ. On the other hand, we have

$$\mathcal{O}_{\mathbb{P}_B(f_*\omega_f(-\mathfrak{c}))}(1) = \mathcal{O}_{\mathbb{P}_B(f_*\omega_f)}(1) - \pi^*\mathfrak{c},$$

and then we get

$$\tau^*\mathcal{O}_{\mathbb{P}_B(f_*\omega_f)}(1) - \widetilde{\pi}^*\mathfrak{c} - E_\tau = \tau'^*\mathcal{O}_{\mathbb{P}_B(\mathrm{Sym}^{d-3}(f_*\mathcal{L}))}(1),$$

where $\pi\colon P \to B$, $\widetilde{\pi}\colon \widetilde{P} \to B$ are the natural projections. Now we consider the relative Veronese embedding $W' := \mathbb{P}_B(f_*\mathcal{L}) \to P'$ of degree $d-3$ corresponding to the surjective homomorphism $\phi'^*\mathrm{Sym}^{d-3}(f_*\mathcal{L}) \to \mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(d-3)$, where $\phi'\colon W' \to B$ is the natural projection. There is a rational map $S \cdots \to W'$ corresponding to the homomorphism $f^*f_*\mathcal{L} \to \mathcal{L}$. Let $X' \subset W'$ be (the closure of) its image. Let $\widetilde{W}$, $\widetilde{X}$ be the proper transforms of W', X' with respect to τ' and W, X the image of $\widetilde{W}$, $\widetilde{X}$ via τ, respectively. Note that X coincides with the image of the relative canonical map $S \cdots \to P$ and two birational maps $S \cdots \to X \cdots \to \widetilde{X}$ and $S \cdots \to X' \cdots \to \widetilde{X}$ coincide. Let $\rho\colon \widetilde{S} \to S$ be the resolution of indeterminacy of $S \cdots \to \widetilde{X}$ and $\widetilde{\Phi}\colon \widetilde{S} \to \widetilde{X}$ the induced birational morphism. We put $T := \mathcal{O}_{\mathbb{P}_B(f_*\omega_f)}(1)$, $T' := \mathcal{O}_{\mathbb{P}_B(\mathrm{Sym}^{d-3}(f_*\mathcal{L}))}(1)$ and denote also τ^*T, τ'^*T' by T, T' for simplicity. Let Γ, Γ' respectively be the numerical equivalence classes of fibers of $\pi\colon P \to B$, $\pi'\colon P' \to B$. Note that $\tau^*\Gamma = \tau'^*\Gamma'$ and we also denote it by Γ or Γ'. From the above arguments, we have

$$T - T' \equiv c\Gamma + E_\tau,$$

where c is the degree of $\mathfrak{c}$. Put $N := T|_{\widetilde{W}}$, $N' := T'|_{\widetilde{W}}$, $M := \widetilde{\Phi}^*T$ and $M' := \widetilde{\Phi}^*T'$. The numerical equivalence classes of W', X' in P' as cycles can be written by

$$W' \equiv (d-3)^2T'^{g-3} + \alpha'T'^{g-4}\Gamma', \quad X' \equiv d(d-3)T'^{g-2} + \beta'T'^{g-3}\Gamma'$$

for some α', β'. Then we have

$$N'^3 = T'^3W' = (d-3)^2(\chi_f - l) + \alpha', \quad M'^2 = T'^2X' = d(d-3)(\chi_f - l) + \beta',$$

where $l := \mathrm{length}(f_*\omega_f/\mathrm{Sym}^{d-3}f_*\mathcal{L}) \geq 0$. Note that $T'|_{W'} = \mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(d-3)$ and then we have

$$N'^3 = (d-3)^3\mathrm{deg}f_*\mathcal{L}.$$

Then the numerical class of the canonical divisor $K_{W'}$ of W' is

$$\begin{aligned} K_{W'} &\equiv -3\mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(1) + (\deg f_*\mathcal{L} + 2b - 2)\Gamma'|_{W'} \\ &= -3\mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(1) + \left(\frac{N'^3}{(d-3)^3} + 2b - 2\right)\Gamma'|_{W'}, \end{aligned}$$

where $b := g(B)$, the genus of B. The numerical class $[X']_{W'}$ of X' in W' can be denoted by

$$[X']_{W'} \equiv d\mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(1) + \beta''\Gamma'|_{W'}$$

for some β''. Since $\mathcal{O}_{\mathbb{P}_B(f_*\mathcal{L})}(d-3) = T'|_{W'}$, we have

$$\left(\frac{d}{d-3}T' + \beta''\Gamma\right)W' = X'$$

and thus we get

$$\beta' = (d-3)^2\beta'' + \frac{d}{d-3}\alpha'.$$

By the definition of M, we can write $\rho^*K_f = M + Z$ for some effective vertical divisor Z with respect to $\widetilde{f}\colon \widetilde{S} \to B$. Then we have

$$K_f^2 = (\rho^*K_f)^2 = (M+Z)^2 = M^2 + (\rho^*K_f + M)Z \geq M^2, \tag{6.1}$$

where the last inequality follows from the nefness of K_f.

Lemma 6.2.

$$M^2 \geq \frac{d-1}{d-3}N^3.$$

Proof. Take a sufficiently ample divisor $\mathfrak{a}$ such that $|M' + \widetilde{f}^*\mathfrak{a}|$ is free from base points. Then we can take a smooth general member $C \in |M' + \widetilde{f}^*\mathfrak{a}|$ by Bertini's theorem. Let $C' := (\tau' \circ \widetilde{\Phi})(C)$. Now we compare the genus $g(C)$ of C and the arithmetic genus $p_a(C')$ of C'.

First, we compute $g(C)$. The adjunction formula says that

$$\begin{aligned} 2g(C) - 2 &= (K_{\widetilde{S}} + C)C \\ &= (\rho^*K_f + E + (2b-2)\widetilde{F} + C)C \\ &= (M + Z + E + (2b-2)\widetilde{F} + M' + a\widetilde{F})(M' + a\widetilde{F}) \\ &= (2M + Z + E + (2b - 2 + a - c)\widetilde{F} - \widetilde{\Phi}^*E_\tau)(M + (a-c)\widetilde{F} - \widetilde{\Phi}^*E_\tau) \\ &= 2M^2 + (Z+E)(M - \widetilde{\Phi}^*E_\tau) + (2b - 2 + 3a - 3c)(2g-2) + (\widetilde{\Phi}^*E_\tau)^2, \end{aligned} \tag{6.2}$$

where $b := g(B)$, $\widetilde{F}$ is the numerical class of a fiber of $\widetilde{f}$, E is the exceptional divisor of ρ such that $K_{\widetilde{S}} = \rho^* K_S + E$ and $a := \deg \mathfrak{a}$.

Next, we compute $p_a(C')$. The adjunction formula also says that

$$
\begin{aligned}
2p_a(C') - 2 &= (K_{X'} + C')C' \\
&= ((K_{W'} + [X']_{W'})|_{X'} + C')C' \\
&= \left(\left(T' + \left(\frac{N'^3}{(d-3)^3} + 2b - 2 + \beta'' \right) \Gamma' \right) |_{X'} + (T' + a\Gamma')|_{X'} \right) (T' + a\Gamma')|_{X'} \\
&= \left(2T' + \left(\frac{N'^3}{(d-3)^3} + 2b - 2 + \beta'' + a \right) \Gamma' \right) (T' + a\Gamma')X' \\
&= \left(2T'^2 + \left(\frac{N'^3}{(d-3)^3} + 2b - 2 + \beta'' + 3a \right) T'\Gamma' \right) (d(d-3)T'^{g-2} + \beta' T'^{g-3}\Gamma') \\
&= 2d(d-3)(\chi_f - l) + d(d-3) \left(\frac{N'^3}{(d-3)^3} + 2b - 2 + \beta'' + 3a \right) + 2\beta' \\
&= -\frac{d(d-1)}{(d-3)^2} N'^3 + \frac{3d-6}{d-3} M'^2 + (2b - 2 + 3a)(2g-2) \\
&= -\frac{d(d-1)}{(d-3)^2} N^3 + \frac{3d-6}{d-3} M^2 + \frac{d(d-1)}{(d-3)^2} (E_\tau|_{\widetilde{W}})^3 + \frac{3d-6}{d-3} (\widetilde{\Phi}^* E_\tau)^2 \\
&\quad + (2b - 2 + 3a - 3c)(2g-2), \qquad (6.3)
\end{aligned}
$$

where the last equality follows from $N^3 - (E_\tau|_{\widetilde{W}})^3 = N'^3 + 3c(d-3)^2$ and $M^2 + (\widetilde{\Phi}^* E_\tau)^2 = M'^2 + 2cd(d-3)$. From (6.2) and (6.3), we get

$$
\begin{aligned}
2p_a(C') - 2g(C) = &- \frac{d(d-1)}{(d-3)^2} N^3 + \frac{d}{d-3} M^2 - (Z + E)(M - \widetilde{\Phi}^* E_\tau) \\
&+ \frac{d(d-1)}{(d-3)^2} (E_\tau|_{\widetilde{W}})^3 + \frac{2d-3}{d-3} (\widetilde{\Phi}^* E_\tau)^2 \qquad (6.4)
\end{aligned}
$$

and it is non-negative since $C \to C'$ is birational. On the other hand, we have

$$
\begin{aligned}
(E_\tau|_{\widetilde{W}})^3 &= (N - N' - c\Gamma|_{\widetilde{W}})^2 E_\tau|_{\widetilde{W}} = N'^2 E_\tau|_{\widetilde{W}} \\
&= T'^2 E_\tau \widetilde{W} = T'^2 E_\tau W' = (d-3)^2 T'^{g-1} E_\tau \qquad (6.5)
\end{aligned}
$$

and

$$
\begin{aligned}
(\widetilde{\Phi}^* E_\tau)^2 &= (M - M' - c\widetilde{F}) \widetilde{\Phi}^* E_\tau = -M' \widetilde{\Phi}^* E_\tau \\
&= -T' E_\tau \widetilde{X} = -T' E_\tau X' = -d(d-3) T'^{g-1} E_\tau. \qquad (6.6)
\end{aligned}
$$

Note that $T'^{g-1}E_\tau = \text{length}\mathcal{T} \geq 0$ by a simple computation. From (6.4), (6.5), (6.6) and $(Z+E)(M - \widetilde{\Phi}^*E_\tau) = (Z+E)C \geq 0$, we have

$$-\frac{d(d-1)}{(d-3)^2}N^3 + \frac{d}{d-3}M^2 \geq (Z+E)C + d(d-2)\text{length}\mathcal{T} \geq 0,$$

which is the desired inequality. □

Lemma 6.3.

$$N^3 \geq \frac{6(d-3)^2}{(d-1)(d-2)}\chi_f.$$

Proof. Since the linear system $\widetilde{\phi}_*N \otimes \mathbb{C}(p) = H^0(\widetilde{\phi}^{-1}(p), N|_{\widetilde{\phi}^{-1}(p)})$ on a general fiber $\widetilde{\phi}^{-1}(p) \simeq \mathbb{P}^2$ induces a Veronese embedding of degree $d-3$, the pair $(\widetilde{\phi}^{-1}(p), \widetilde{\phi}_*N \otimes \mathbb{C}(p))$ is Hilbert stable by Corollary 5.3 in [24]. Thus we can apply Theorem 2.2 in [10] to the pair $(N, \widetilde{\phi}_*N)$ and hence we get

$$\text{rank}(\widetilde{\phi}_*N)N^3 - \dim(\widetilde{W})\deg(\widetilde{\phi}_*N)(N|_{\widetilde{\phi}^{-1}(p)})^2 \geq 0,$$

which is the desired inequality since $\widetilde{\phi}_*N \simeq f_*\omega_f$. □

Proof of Theorem 6.1. From (6.1), Lemma 6.2 and Lemma 6.3, we have

$$K_f^2 \geq M^2 \geq \frac{d-1}{d-3}N^3 \geq \frac{6(d-3)}{d-2}\chi_f.$$

□

Proposition 6.4 (cf. [27])**.** *Let $f\colon S \to B$ be a relatively minimal plane curve fibration of degree d. Then the following are equivalent.*

(i) $M^2 = \dfrac{d-1}{d-3}N^3$.

(ii) $K_f^2 = \dfrac{6(d-3)}{d-2}\chi_f$.

(iii) *There exists a $\mathbb{P}^2$-bundle $\phi\colon W = \mathbb{P}(\mathcal{E}) \to B$ and a member $X \in |d\mathcal{O}_W(1) + \phi^*\mathfrak{k}|$ with at most rational double points as singularities such that S is the minimal resolution of X.*

Proof. ((i)⇒(iii)) From the proof of Lemma 6.2, (i) implies that $p_a(C') = g(C)$ for general $C \in |M' + \widetilde{f}^*\mathfrak{a}|$, $T'^{g-1}E_\tau = 0$ and $(Z+E)M' = 0$. The former implies that X' has at most isolated singularities. $T'^{g-1}E_\tau = 0$ implies that $P = P'$ and $E_\tau = 0$. Hence we have $M - M' = \widetilde{f}^*\mathfrak{c}$ and then $(Z+E)M = 0$. It follows from the nefness of M that $ZM = 0$ and $EM = 0$. Therefore, we have $Z^2 = MZ + Z^2 = \rho^*K_fZ \geq 0$. Thus, by the

Hodge index theorem, we get $Z = 0$. On the other hand, we have $\deg f_*\mathcal{L} + \beta'' = 0$ from the proof of Lemma 6.2 and the assumption (i). Thus $X = X'$ in $W = W'$ is linearly equivalent to $d\mathcal{O}_W(1) - \phi^*\mathfrak{d}$ for some divisor $\mathfrak{d}$ of degree $\deg f_*\mathcal{L}$. It follows that

$$\begin{aligned}
\chi(\mathcal{O}_X) &= \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-X)) \\
&= 1 - b + \chi(\mathcal{O}_W(K_W + X)) \\
&= 1 - b + \chi(\mathrm{Sym}^{d-3} f_*\mathcal{L} \otimes (\det f_*\mathcal{L} \otimes \omega_B \otimes \mathcal{O}_B(-\mathfrak{d}))) \\
&= 1 - b + g(1-b) + \chi_f - c + g(2b-2) \\
&= \chi(\mathcal{O}_S) - c \\
&\leq \chi(\mathcal{O}_S).
\end{aligned}$$

On the other hand, since $\Phi\colon \widetilde{S} \to X$ is a resolution of singularities of X, we have $\chi(\mathcal{O}_X) \geq \chi(\mathcal{O}_{\widetilde{S}}) = \chi(\mathcal{O}_S)$. Hence $c = 0$ and X has at most rational singularities. Since X is a hypersurface of W, any singularity of X is a rational double point. We can see that $\widetilde{S} = S$ and $\mathfrak{d} = \det f_*\mathcal{L}$.

((iii)⇒(ii)) By a simple computation, we have

$$K_f^2 = d(d-1)(d-3)\deg\mathcal{E} + 3(d-1)(d-3)k,$$

where $k := \deg\mathfrak{k}$. Moreover, by the similar computation as above, we have

$$\begin{aligned}
\chi(\mathcal{O}_X) &= \chi(\mathcal{O}_W) - \chi(\mathcal{O}_W(-X)) \\
&= (g-1)(b-1) + \frac{d(d-1)(d-2)}{6}\deg\mathcal{E} + \frac{(d-1)(d-2)}{2}k.
\end{aligned}$$

Since X has at most rational double points, we get

$$\begin{aligned}
\chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1) \\
&= \chi(\mathcal{O}_X) - (g-1)(b-1) \\
&= \frac{d(d-1)(d-2)}{6}\deg\mathcal{E} + \frac{(d-1)(d-2)}{2}k.
\end{aligned}$$

Hence (ii) holds.

((ii)⇒(i)) It is clear. □

7 Algebraization of fibers

We consider a proper surjective holomorphic map $f\colon S\to\Delta$ from a non-singular complex surface S to a small disk $\Delta=\{t\in\mathbb{C}||t|<\epsilon\}$ such that the general fiber $f^{-1}(t)$ over $t\neq 0$ is a non-singular curve of genus g and put $F:=f^{-1}(0)$. The pair (f,F) is called a *fiber germ of genus g*. A fiber germ (f,F) is *relatively minimal* if F contains no (-1)-curves. In the sequel, we always assume that any fiber germ is relatively minimal. Two relatively minimal fiber germs $(f\colon S\to\Delta,F)$ and $(f'\colon S'\to\Delta,F)$ are *holomorphically equivalent* if there exist biholomorphic maps $\phi\colon S\to S'$ and $\psi\colon\Delta\to\Delta$ such that $f'\circ\phi=\psi\circ f$ after shrinking Δ if necessary. Let $\mathcal{A}$ be a set of holomorphically equivalence classes of fiber germs of genus g and $\chi\colon\mathcal{A}\to\Sigma$ a map from $\mathcal{A}$ to a set Σ. The map χ is an *algebraic invariant* (cf. [35]) if for any fiber germ $(f\colon S\to\Delta,F)$ in $\mathcal{A}$, there exists a natural number n such that for any fiber germ $(f'\colon S'\to\Delta,F')$ in $\mathcal{A}$ which satisfies $S_n\simeq S'_n$ over $\mathrm{Spec}\mathbb{C}[t]/(t^n)$, we have $\chi(f,F)=\chi(f',F')$, where $S_n:=S\times_\Delta\mathrm{Spec}\mathbb{C}[t]/(t^n)$. For example, the map $\mu\colon\mathcal{A}\to\widehat{\Gamma}_g$ which sends a fiber germ (f,F) to its topological monodromy μ_f is an algebraic invariant, where Γ_g is the mapping class group of genus g and $\widehat{\Gamma}_g$ is the set of its conjugacy classes.

Let $\mathcal{A}_d$ denote the set of holomorphically equivalence classes of fiber germs whose general fiber is a smooth plane curve of degree d. The following is our main theorem:

Theorem 7.1. *There exists a non-negative algebraic invariant* $\mathrm{Ind}_d\colon\mathcal{A}_d\to\frac{1}{d-2}\mathbb{Z}_{\geq 0}$ *such that for any relatively minimal plane curve fibration $f\colon S\to B$ of degree d, the value* $\mathrm{Ind}_d(F)$ *equals to* 0 *for any general fiber F of f and*

$$K_f^2=\frac{6(d-3)}{d-2}\chi_f+\sum_{p\in B}\mathrm{Ind}_d(F_p)$$

holds.

Now, we define the function Ind_d. Let $(f\colon S\to\Delta,F)$ be a fiber germ in $\mathcal{A}_d$. Then, by Theorem 4.1, there exists a line bundle $\mathcal{L}$ on S such that the restriction $\mathcal{L}|_{F_t}$ is a $\mathfrak{g}^2_d$ on $F_t=f^{-1}(t)$ for any $t\neq 0$ and it is unique up to a multiple of a divisor consisting of components of $F=f^{-1}(0)$. It follows that $\mathcal{L}^{\otimes d-3}\simeq\omega_f(\Gamma)$ for some divisor Γ consisting of components of F. Using the line bundle $\mathcal{L}$, we define $\mathrm{Ind}_d(f,F)$ by

$$\begin{aligned}\mathrm{Ind}_d(f,F):=&\Gamma^2+2(d-3)\left(\frac{d+1}{d-2}\mathrm{length}(R^1f_*\mathcal{L}^{\otimes d-2})-\mathrm{length}(R^1f_*\mathcal{L}^{\otimes d-1})\right)\\&+2(d-3)\left(\mathrm{length}(\mathcal{T}_{d-1})-\frac{d+1}{d-2}\mathrm{length}(\mathcal{T}_{d-2})\right),\end{aligned}$$

where $\mathcal{T}_k$ is the torsion sheaf defined by the natural exact sequence

$$0 \to \mathrm{Sym}^k f_*\mathcal{L} \to f_*\mathcal{L}^{\otimes k} \to \mathcal{T}_k \to 0.$$

We have seen that the value $\mathrm{Ind}_d(f, F)$ is independent of a choice of the line bundle $\mathcal{L}$ when the fiber germ (f, F) is realized in a global fibration $S \to B$. From (5.5), in order to prove Theorem 7.1, we must show that for any fiber germ (f, F) in $\mathcal{A}_d$, $\mathrm{Ind}_d(f, F)$ is well-defined, that is, not depend on a choice of $\mathcal{L}$ and non-negative algebraic invariant. The following is a key lemma.

Lemma 7.2. *For any fiber germ $(f: S \to \Delta, F)$ in $\mathcal{A}_d$ and any natural number n, there exists a plane curve fibration $\overline{f}: \overline{S} \to \mathbb{P}^1$ of degree d such that S_n is isomorphic to $\overline{S}_n := \overline{S} \times_{\mathbb{P}^1} \mathrm{Spec}\mathcal{O}_{\mathbb{P}^1,0}/\mathfrak{m}^n$ over $\mathrm{Spec}\mathbb{C}[t]/(t^n) \simeq \mathrm{Spec}\mathcal{O}_{\mathbb{P}^1,0}/\mathfrak{m}^n$ and all the other singular fibers of $\overline{f}$ are irreducible Lefschetz plane curves of degree d with one node, where $\mathfrak{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{P}^1,0}$.*

Proof. We can take a line bundle $\mathcal{L}$ on S such that $\mathcal{L}|_{F_t}$ is the $\mathfrak{g}^2_d$ on F_t for any $t \neq 0$ from Theorem 4.1. Thus, we can take a rational map $S \cdots \to \Delta \times \mathbb{P}^2$ over Δ that embeds F_t to $\mathbb{P}^2 = \{t\} \times \mathbb{P}^2$ for any $t \neq 0$. Let $\varphi(t; X, Y, Z)$ be a defining equation of $F_t \subset \mathbb{P}^2_{(X:Y:Z)}$ for $t \neq 0$, which is a homogeneous polynomial of degree d with respect to X, Y, Z and determined uniquely up to a multiple of a constant. We may assume that $\varphi(t; X, Y, Z)$ is holomorphic in $t \neq 0$ after shrinking Δ if necessary. By Riemann's extension theorem, $\varphi(t; X, Y, Z)$ is holomorphic at $t = 0$. Thus the image of a rational map $S \cdots \to \Delta \times \mathbb{P}^2$ can be written as $X := \{(t, (X : Y : Z)) \in \Delta \times \mathbb{P}^2 | \varphi(t; X, Y, Z) = 0\}$. Let

$$\varphi(t; X, Y, Z) = \varphi(0; X, Y, Z) + t\frac{d\varphi}{dt}(0; X, Y, Z) + \cdots + \frac{t^m}{m!}\frac{d^m\varphi}{dt^m}(0; X, Y, Z) + \cdots$$

be the Taylor expansion near $0 \in \Delta$ and define

$$\varphi^{[n]}(t; X, Y, Z) := \varphi(0; X, Y, Z) + t\frac{d\varphi}{dt}(0; X, Y, Z) + \cdots + \frac{t^n}{n!}\frac{d^n\varphi}{dt^n}(0; X, Y, Z).$$

Take a sufficiently large $m \gg n$ and general homogeneous polynomials $\psi_{n+1}(X, Y, Z)$, $\ldots$, $\psi_m(X, Y, Z)$ of degree d. Let $\Phi(t_0, t_1; X, Y, Z)$ be the homogenization of the polynomial

$$\varphi^{[n]}(t; X, Y, Z) + t^{n+1}\psi_{n+1}(t; X, Y, Z) + \cdots + t^m\psi_m(t; X, Y, Z)$$

with respect to $t \in \mathbb{C}$ and put $\overline{X} := \{((t_0 : t_1), (X : Y : Z)) \in \mathbb{P}^1 \times \mathbb{P}^2 | \Phi(t_0, t_1; X, Y, Z) = 0\}$. Taking a resolution of singularities of $\overline{X}$ and its relatively minimal model over $\mathbb{P}^1$, we get a plane curve fibration $\overline{f}: \overline{S} \to \mathbb{P}^1$ of degree d such that S_n is isomorphic to $\overline{S}_n$. Since ψ_{n+1}, $\ldots$, ψ_m are general, any singular fiber of $\overline{f}$ over $\mathbb{P}^1 \setminus \{0\}$ is an irreducible Lefschetz plane curve of degree d with one node by Kuno's result [29]. □

Lemma 7.3. $\mathrm{Ind}_d\colon \mathcal{A}_d \to \mathbb{Q}$ *is a well-defined algebraic invariant.*

Proof. Fix a fiber germ (f, F) of $\mathcal{A}_d$ arbitrarily and denote by $\mathrm{Ind}_d^{\mathcal{L}}(f, F)$ the value $\mathrm{Ind}_d(f, F)$ defined by using a line bundle $\mathcal{L}$ as above. Note that the value $\mathrm{Ind}_d^{\mathcal{L}}(f, F)$ is completely determined by the restriction $\mathcal{L}_n := \mathcal{L}|_{S_n}$ for a sufficiently large n (depending on (f, F)). From Lemma 7.2, we can take a plane curve fibration $\overline{f}\colon \overline{S} \to \mathbb{P}^1$ of degree d such that S_n is isomorphic to $\overline{S}_n$. We will show that the line bundle $\mathcal{L}_n$ is the restriction of some line bundle $\overline{\mathcal{L}}$ on $\overline{S}$ to $\overline{S}_n$ via the isomorphism $S_n \simeq \overline{S}_n$. Note that the topological monodromies of (f, F) and $(\overline{f}, \overline{F})$ are the same and $F \simeq \overline{F}$. Take a subvariety $\mathcal{U}$ of the Kuranishi space of the stable model F' of (f, F) parametrizing smooth plane curves of degree d or its limit and consider the universal family $\mathcal{C} \to \mathcal{U}$. Then the cyclic group $G = \mathbb{Z}_N$ acts on $\mathcal{C}$ and $\mathcal{U}$ equivariantly and the quotient fibration $\mathcal{C}/G \to \mathcal{U}/G$ contains the two fiber germs (f, F) and $(\overline{f}, \overline{F})$, where the number N is the minimal pseudo-period of the topological monodromy of f. We may assume that $\mathcal{C}/G$ and $\mathcal{U}/G$ are normal by taking normalizations. Applying Theorem 4.1 to $\mathcal{C}/G \to \mathcal{U}/G$, we obtain a divisorial sheaf $\mathbb{L}$ on $\mathcal{C}/G$ such that the restriction of $\mathbb{L}$ to any general fiber is a $\mathfrak{g}^2_d$. We can write $\mathcal{L} \simeq \mathbb{L}|_S \otimes \mathcal{O}_S(D)$ for some divisor D consisting of components of F and then $\mathcal{L}_n \simeq \mathbb{L}|_{\overline{S}} \otimes \mathcal{O}_{\overline{S}}(D)|_{\overline{S}_n}$, where $\mathbb{L}|_{\overline{S}}$ is a line bundle on $\overline{S}$ obtained by glueing the restriction of $\mathbb{L}$ to a neighborhood of the fiber $\overline{F}$ with a line bundle on $\overline{S} \setminus \overline{F}$ obtained by Theorem 4.1. The line bundle $\overline{\mathcal{L}} := \mathbb{L}|_{\overline{S}} \otimes \mathcal{O}_{\overline{S}}(D)$ is the desired one. Since $\mathrm{Ind}_d^{\mathcal{L}}(f, F)$ and $\mathrm{Ind}_d^{\overline{\mathcal{L}}}(\overline{f}, \overline{F})$ are determined by $\mathcal{L}_n$, we have $\mathrm{Ind}_d^{\mathcal{L}}(f, F) = \mathrm{Ind}_d^{\overline{\mathcal{L}}}(\overline{f}, \overline{F})$. Since $\mathrm{Ind}_d^{\overline{\mathcal{L}}}(\overline{f}, \overline{F})$ is independent of the choice of the line bundle, we see that Ind_d is well-defined. In order to prove that Ind_d is an algebraic invariant, we apply the similar arguments as above to any fiber germ $(f'\colon S' \to \Delta, F')$ in $\mathcal{A}_d$ with $S_n \simeq S'_n$. Thus we have $\mathrm{Ind}_d(f, F) = \mathrm{Ind}_d(f', F')$ for a sufficiently large n. Such a number n depends only on (f, F) and $\mathbb{L}$. Thus Ind_d is an algebraic invariant. □

Definition 7.4. A fiber germ $(f\colon S \to \Delta, F)$ in $\mathcal{A}_d$ is called a *Lefschetz fiber germ of type* 0 if $S \subset \Delta \times \mathbb{P}^2$ and $F = f^{-1}(0)$ is an irreducible Lefschetz plane curve of degree d with one node.

Lemma 7.5. *For any Lefschetz fiber germ* (f, F) *of type* 0 *in* $\mathcal{A}_d$*, we have* $\mathrm{Ind}_d(f, F) = 0$.

Proof. We can take a line bundle $\mathcal{L}$ defining $\mathrm{Ind}_d(f, F)$ such that $\mathcal{L}^{\otimes d-3} \simeq \omega_f$ by restricting $\mathcal{O}(1)$ on $\Delta \times \mathbb{P}^2$ to S. Moreover, we can see that $R^1 f_* \mathcal{L}^{\otimes d-2} = R^1 f_* \mathcal{L}^{\otimes d-1} = \mathcal{T}_{d-2} = \mathcal{T}_{d-1} = 0$ since F is irreducible and $H^1(F, \mathcal{L}^{\otimes d-2}|_F) = H^1(F, \mathcal{L}^{\otimes d-1}|_F) = 0$. Thus we have $\mathrm{Ind}_d(f, F) = 0$. □

Lemma 7.6. *For any fiber germ* (f, F) *in* $\mathcal{A}_d$*, the value* $\mathrm{Ind}_d(f, F)$ *is non-negative.*

Proof. Fix a fiber germ $(f\colon S \to \Delta, F)$ in $\mathcal{A}_d$ arbitrarily. Since Ind_d is an algebraic invariant, we can take a natural number n such that for any fiber germ $(f'\colon S' \to \Delta, F')$ of $\mathcal{A}_d$ such that $S_n \simeq S'_n$, we have $\mathrm{Ind}_d(f, F) = \mathrm{Ind}_d(f', F')$. From Lemma 7.2, we can take a plane curve fibration $\overline{f}\colon \overline{S} \to \mathbb{P}^1$ of degree d such that $S_n \simeq \overline{S}_n$ and any other fiber germ of $\overline{f}$ is Lefschetz of type 0. Thus we get from (5.5), Theorem 6.1 and Lemma 7.5 that

$$\mathrm{Ind}_d(f, F) = \mathrm{Ind}_d(\overline{f}, \overline{F}) = K_f^2 - \frac{6(d-3)}{d-2}\chi_f \geq 0.$$

□

Combining (5.5) with Lemma 7.3 and Lemma 7.6, we get Theorem 7.1.

Proposition 7.7. *For a fiber germ $(f\colon S \to \Delta, F) \in \mathcal{A}_d$, $\mathrm{Ind}_d(f, F) = 0$ holds if and only if S is obtained by resolving singularities of some family $X \subset \Delta \times \mathbb{P}^2$ of plane curves of degree d with at most rational double points as singularities.*

Proof. From Proposition 6.4 and Theorem 7.2, we get the assertion. □

8 Local signature

For an oriented compact real 4-dimensional manifold X, the *signature* $\mathrm{Sign}(X)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form on $H^2(X)$. In this section, we consider the signature for complex surfaces with plane curve fibrations. For a given condition $\mathcal{P}$ on smooth curves, let $\mathcal{A}_{\mathcal{P}}$ be the set of holomorphically equivalence classes of fiber germs whose general fiber has the condition $\mathcal{P}$. Then a $\mathbb{Q}$-valued function $\sigma\colon \mathcal{A}_{\mathcal{P}} \to \mathbb{Q}$ is a *local signature* if for any relatively minimal fibered surface $f\colon S \to B$ whose general fiber F satisfies the condition $\mathcal{P}$, we have $\sigma(F) = 0$ and $\mathrm{Sign}(X) = \sum_{p \in B} \sigma(F_p)$. In this section, we treat relatively minimal plane curve fibrations $f\colon S \to B$ of degree d.

Definition 8.1. We define $\sigma_d^{\mathrm{alg}}\colon \mathcal{A}_d \to \mathbb{Q}$ by

$$\sigma_d^{\mathrm{alg}} = \frac{4}{12-\lambda_d}\mathrm{Ind}_d - \frac{8-\lambda_d}{12-\lambda_d}e,$$

where $\lambda_d := 6(d-3)/(d-2)$ and $e\colon \mathcal{A}_d \to \mathbb{Q}$ is defined by $e(f, F) := e_{\mathrm{top}}(F) - 2 + 2g$, which is clearly an algebraic invariant.

The function σ_d^{alg} is in fact a local signature, that is, the following holds:

Proposition 8.2 (cf. [5])**.** *For a relatively minimal plane curve fibration $f\colon S \to B$ of* *degree d, we have*

$$\mathrm{Sign}(S) = \sum_{p \in B} \sigma_d^{\mathrm{alg}}(F_p).$$

Proof. The claim holds from Hirzebruch's signature theorem $\mathrm{Sign}(S) = K_f^2 - 8\chi_f$, Theorem 7.1 and Noether's formula $12\chi_f = K_f^2 + e_f$. □

Recall that Kuno [29] defines a local signature σ_d^{top} for (C^∞-)fibrations of plane curves of degree d over a closed surface by using Meyer's signature cocycle from the topological point of view. In fact, two local signatures σ_d^{alg} and σ_d^{top} coincide on $\mathcal{A}_d$:

Theorem 8.3 (cf. [35])**.** *We have $\sigma_d^{\mathrm{alg}}(f, F) = \sigma_d^{\mathrm{top}}(f, F)$ for any fiber germ (f, F) in $\mathcal{A}_d$.*

Proof. We see that two functions σ_d^{alg} and σ_d^{top} are algebraic invariants. Moreover, we have

$$\sigma_d^{\mathrm{alg}}(f, F) = \sigma_d^{\mathrm{top}}(f, F) = -\frac{d+1}{3(d-1)}$$

for any Lefschetz fiber germ of type 0. Thus the claim holds from Lemma 7.2. □

9 Local signature associated with an effective divisor on $\mathcal{M}_g$

Let $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ respectively denote the moduli space of smooth curves of genus g and the moduli space of stable curves of genus g. The rational Picard group of $\overline{\mathcal{M}}_g$ is generated freely by the Hodge bundle λ and the boundary divisors $\delta_0, \delta_1, \ldots, \delta_{[g/2]}$ for $g \geq 3$, where we use the notation in [21]. Let D be an effective divisor on $\mathcal{M}_g$ and $\overline{D}$ the compactification of D in $\overline{\mathcal{M}}_g$. Then we can write $\overline{D} \sim_{\mathbb{Q}} a\lambda - \sum_{i=0}^{[g/2]} b_i \delta_i$ for some rational numbers $a, b_i > 0$, where the symbol $\sim_{\mathbb{Q}}$ means the $\mathbb{Q}$-linear equivalence.

Let $f \colon S \to \Delta$ be a relatively minimal degeneration of curves of genus g, that is, f is a surjective proper flat morphism from a complex smooth surface S to a small open disk Δ such that $f^{-1}(t)$ is a smooth curve of genus g for any $t \neq 0$ and the central fiber $F := f^{-1}(0)$ is relatively minimal. We take the stable reduction $\widetilde{f} \colon \widetilde{S} \to \widetilde{\Delta}$ of f via $\widetilde{\Delta} \to \Delta$; $z \mapsto z^N$. Resolving singularities of $\widetilde{S}$, we obtain a semi-stable reduction $\widehat{f} \colon \widehat{S} \to \widetilde{\Delta}$. Note that N can be taken as the pseudo-period of the topological monodromy μ_f of f as a pseudo-periodic class (cf. [3]). Put $F := f^{-1}(0)$ and $\widehat{F} := \widehat{f}^{-1}(0)$. Let

$$\mathrm{Lsd}(F) := \sigma(f, F; h_{\partial S}) - \frac{1}{N}\sigma(\widehat{f}, \widehat{F}; h_{\partial \widehat{S}})$$

be the local signature defect of (f, F) (more precisely, see [3]) and

$$e_F := (e_{\mathrm{top}}(F) - (2-2g)) - \frac{1}{N}\left(e_{\mathrm{top}}(\widehat{F}) - (2-2g)\right).$$

On the other hand, the local invariants $c_1^2(F)$, $c_2(F)$ and χ_F were defined in [34] for a fiber germ F of a global fibration $f\colon S \to B$. Indeed,

Proposition 9.1. *We have $e_F = c_2(F)$ and*

$$\mathrm{Lsd}(F) = \frac{1}{3}(c_1^2(F) - 2e_F) = 4\chi_F - e_F.$$

Proof. These invariants satisfy the following properties: Let $f\colon S \to B$ be a fibered surface of genus g and $\widehat{f}\colon \widehat{S} \to \widetilde{B}$ be the semi-stable reduction of f via a cyclic covering $\widetilde{B} \to B$ of degree N. Then we have

$$\begin{aligned}
\mathrm{Sign}(S) - \frac{1}{N}\mathrm{Sign}(\widehat{S}) &= \sum_{p\in B} \mathrm{Lsd}(F_p),\\
K_f^2 - \frac{1}{N}K_{\widehat{f}}^2 &= \sum_{p\in B} c_1^2(F_p),\\
e_f - \frac{1}{N}e_{\widehat{f}} &= \sum_{p\in B} c_2(F_p) = \sum_{p\in B} e_{F_p}, \qquad (9.1)\\
\chi_f - \frac{1}{N}\chi_{\widehat{f}} &= \sum_{p\in B} \chi_{F_p}.
\end{aligned}$$

Let F be an arbitrary fiber germ in a global fibration $f\colon S \to B$. Taking base change, we may assume that any fiber of f other than F is semi-stable. Thus we get the assertion from Hirzebruch's signature formula $\mathrm{Sign}(S) = K_f^2 - 8\chi_f$, Noether's formula $12\chi_f = K_f^2 + e_f$ and (9.1) since $\mathrm{Lsd}(\widehat{F}) = c_1^2(\widehat{F}) = c_2(\widehat{F}) = e_{\widehat{F}} = \chi_{\widehat{F}} = 0$ for any semi-stable fiber germ $\widehat{F}$. □

Let $\rho_{\widehat{f}}\colon \widetilde{\Delta} \to \overline{\mathcal{M}}_g$ be the moduli map of the semi-stable reduction $\widehat{f}\colon \widehat{S} \to \widetilde{\Delta}$. For an effective divisor E on $\overline{\mathcal{M}}_g$ not containing the image $\rho_{\widehat{f}}(\widetilde{\Delta})$, we can define the pull-back $\rho_{\widehat{f}}^*E$. Let $E(\widehat{F}) := \deg(\rho_{\widehat{f}}^*E)$. Note that even when $E \sim E'$ holds for two effective divisors E and E', it is not always true that $E(\widehat{F}) = E'(\widehat{F})$ because we treat local fibrations here. Given an effective divisor D on $\mathcal{M}_g$ such that $\overline{D}$ does not contain $\rho_{\widehat{f}}(\widetilde{\Delta})$ with $\overline{D} \sim_{\mathbb{Q}} a\lambda - \sum_{i=0}^{[g/2]} b_i\delta_i$, we put

$$\lambda_D(\widehat{F}) := \frac{1}{a}\left(\overline{D}(\widehat{F}) + \sum_i b_i\delta_i(\widehat{F})\right).$$

In general, for a relatively minimal fiber germ F, we define

$$\lambda_D(F) := \chi_F + \frac{\lambda_D(\widehat{F})}{N}$$

and

$$\delta(F) := e_F + \frac{\delta(\widehat{F})}{N} = e_{\mathrm{top}}(F) - (2-2g),$$

which are independent of the choice of N.

Now we consider a global fibration $f\colon S \to B$, that is, a surjective morphism from a smooth projective surface S to a smooth projective curve B with connected fibers. Assume that the moduli point of the general fiber of f is not contained in D. From (9.1), we have

$$\chi_f = \sum_{p\in B} \lambda_D(F_p), \quad e_f = \sum_{p\in B} \delta(F_p).$$

From Hirzebruch's signature formula $\mathrm{Sign}(S) = 4\chi_f - e_f$, we can write

$$\mathrm{Sign}(S) = \sum_{p\in B} (4\lambda_D(F_p) - \delta(F_p)).$$

We call $\sigma_D(F) := 4\lambda_D(F) - \delta(F)$ the *local signature of a fiber germ F associated with D*. Note that the divisor $4\lambda - \delta$ is called the signature divisor in [6].

10 Examples

Now we consider two effective divisors $E_{g,-1}$ and $E_{g,1}$ on $\mathcal{M}_g$, which parameterize curves C of genus g having a special Weierstrass point. Let C be a smooth curve of genus g. Let p be a Weierstrass point of C, i.e., a point on C satisfying $h^0(gp) \geq 2$. Then p is said to be *exceptional of type* $g-1$ (resp. *of type* $g+1$) if $h^0((g-1)p) \geq 2$ (resp. $h^0((g+1)p) \geq 3$). The locus $E_{g,-1}$ (resp. $E_{g,1}$) on $\mathcal{M}_g$ is (roughly) defined by the set of curves of genus g with an exceptional Weierstrass point of type $g-1$ (resp. of type $g+1$) with the natural scheme structure, which is of codimension 1 for $g \geq 3$. For more details, see [16]. For $g = 2$, the loci $\overline{E}_{2,-1}$ and $\overline{E}_{2,1}$ are empty. For $g = 3$, $\overline{E}_{3,-1}$ is coincide with the hyperelliptic locus $\overline{\mathcal{H}}_3$ as a set, but as a divisor, we have $\overline{E}_{3,-1} = 8\overline{\mathcal{H}}_3$. Indeed, once a genus 3 curve has one exceptional Weierstrass point of type 2, it becomes hyperelliptic and hence has 8 Weierstrass points of type 2 automatically. Since the hyperelliptic Weierstrass point is exceptional of type $g-1$ and $g+1$, the hyperelliptic locus $\overline{\mathcal{H}}_g$ is contained in both $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$. In particular, $\overline{E}_{3,-1} = 8\overline{\mathcal{H}}_3$ is a subdivisor of $\overline{E}_{3,1}$. Thus we can define an effective divisor $\overline{\mathcal{HF}} := \overline{E}_{3,1} - \overline{E}_{3,-1}$. As a different definition, let $\mathcal{HF}$ be the locus on the moduli space $\mathcal{M}_3 \setminus \mathcal{H}_3$ of smooth plane quartics parameterizing plane quartic curves with a hyperflex, i.e., 4-fold tangent point. Then the above $\overline{\mathcal{HF}}$ is just the closure of $\mathcal{HF}$ in $\overline{\mathcal{M}}_3$. The locus $\overline{\mathcal{HF}}$ has multiplicity 1 around general points.

For $g \geq 4$, $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ also have multiplicity 1 around general points. It is known that the rational divisor classes of $\overline{E}_{g,-1}$ and $\overline{E}_{g,1}$ are given by

$$\overline{E}_{g,-1} = \frac{g^2(g-1)(3g-1)}{2}\lambda - \frac{(g-1)^2 g(g+1)}{6}\delta_0 - \sum_{i=1}^{[g/2]} \frac{i(g-i)g(g^2+g-4)}{2}\delta_i,$$

$$\overline{E}_{g,1} = \frac{(g+1)(g+2)(3g^2+3g+2)}{2}\lambda - \frac{g(g+1)^2(g+2)}{6}\delta_0 - \sum_{i=1}^{[g/2]} \frac{i(g-i)(g+1)(g+2)^2}{2}\delta_i$$

(cf. [16], [13], [14]). In particular, we have

$$\overline{E}_{3,-1} = 72\lambda - 8\delta_0 - 24\delta_1, \quad \overline{E}_{3,1} = 380\lambda - 40\delta_0 - 100\delta_1,$$
$$\overline{\mathcal{H}}_3 = 9\lambda - \delta_0 - 3\delta_1, \quad \overline{\mathcal{HF}} = 308\lambda - 32\delta_0 - 76\delta_1.$$

Now, we will check using the simplest example of fibered surface of genus 3 that two local signatures $\sigma_{\mathcal{H}_3}$ and $\sigma_{\mathcal{HF}}$ associated with $\overline{\mathcal{H}}_3$ and $\overline{\mathcal{HF}}$ give different localizations.

Example 10.1. Let $\{C_\lambda\}_\lambda \subset |4H_{\mathbb{P}^2}|$ be a general Lefschetz pencil of quartics. The base locus of $\{C_\lambda\}_\lambda$ consists of 16 points and they are on smooth members. Blowing up at these 16 points, we obtain a non-hyperelliptic fibration $f\colon S \to \mathbb{P}^1$ of genus 3. By a simple computation, we get $\chi_f = 3$, $e_f = 27$, $K_f^2 = 9$ and $\mathrm{Sign}(S) = -15$. Note that all singular fibers of f are irreducible curves with one node and the number of them is 27. Thus we have $\overline{\mathcal{H}}_3(f) = 0$, $\lambda(f) = 3$, $\delta_0(f) = 27$ and $\delta_1(f) = 0$. Hence we have $\overline{\mathcal{HF}}(f) = 60$. This implies that the number of smooth curves in a general Lefschetz pencil of quartic curves with a hyperflex is 60. Let F_{hf} and F_0 respectively be a smooth quartic fiber germ of f with one hyperflex and an irreducible fiber germ of f with one node. Then clearly we have

$$\delta_0(F_{\mathrm{hf}}) = 0, \quad \delta_1(F_{\mathrm{hf}}) = 0, \quad \overline{\mathcal{H}}_3(F_{\mathrm{hf}}) = 0, \quad \overline{\mathcal{HF}}(F_{\mathrm{hf}}) = 1$$

and

$$\delta_0(F_0) = 1, \quad \delta_1(F_0) = 0, \quad \overline{\mathcal{H}}_3(F_0) = 0, \quad \overline{\mathcal{HF}}(F_0) = 0.$$

Thus we get

$$\lambda_{\mathcal{H}_3}(F_{\mathrm{hf}}) = 0, \quad \lambda_{\mathcal{H}_3}(F_0) = \frac{1}{9}, \quad \sigma_{\mathcal{H}_3}(F_{\mathrm{hf}}) = 0, \quad \sigma_{\mathcal{H}_3}(F_0) = -\frac{5}{9}$$

and

$$\lambda_{\mathcal{HF}}(F_{\mathrm{hf}}) = \frac{1}{308}, \quad \lambda_{\mathcal{HF}}(F_0) = \frac{8}{77}, \quad \sigma_{\mathcal{HF}}(F_{\mathrm{hf}}) = \frac{1}{77}, \quad \sigma_{\mathcal{HF}}(F_0) = -\frac{45}{77}.$$

Thus two local signatures $\sigma_{\mathcal{H}_3}$ and $\sigma_{\mathcal{HF}}$ are different.

Next, let us consider the genus 2 case. The rational Picard group of $\overline{\mathcal{M}}_2$ is generated by λ, δ_0 and δ_1 with one relation $10\lambda = \delta_0 + 2\delta_1$. For a semi-stable fiber germ $\widehat{F}$ of genus 2, we put $\lambda(\widehat{F}) := (\delta_0(\widehat{F}) + 2\delta_1(\widehat{F}))/10$. For a not necessarily semi-stable fiber germ F, we define $\lambda(F)$ by using the semi-stable reduction similarly as in the previous section. We also define a (pre-)Horikawa index $\mathrm{Ind}(F) := 10\lambda(F) - \delta(F)$ for a relatively minimal genus 2 fiber germ F. It coincides with the original Horikawa index defined by using the double covering data (cf. [35], [22], [37]) and hence it is non-negative. A local signature can be defined by $\sigma(F) := 4\lambda(F) - \delta(F)$ for any fiber germ F of genus 2.

Now, we define another local signature for non-bielliptic genus 2 fiber germs. Let $\mathcal{B}_2$ be the bielliptic locus on $\mathcal{M}_2$ and $\overline{\mathcal{B}}_2$ its closure in $\overline{\mathcal{M}}_2$. They are irreducible codimension 1 loci. From [19], the rational linearly equivalence class of $\overline{\mathcal{B}}_2$ is

$$\overline{\mathcal{B}}_2 = \frac{3}{2}\delta_0 + 6\delta_1 = 30\lambda - \frac{3}{2}\delta_0 = 15\lambda + 3\delta_1.$$

Thus, for non-bielliptic genus 2 fiber germs, two localizations of the Hodge bundle λ can be realized as follows. We put

$$\lambda_{\mathcal{B}_2,0}(\widehat{F}) := \frac{1}{30}\overline{\mathcal{B}}_2(\widehat{F}) + \frac{1}{20}\delta_0(\widehat{F})$$

and

$$\lambda_{\mathcal{B}_2,1}(\widehat{F}) := \frac{1}{15}\overline{\mathcal{B}}_2(\widehat{F}) - \frac{1}{5}\delta_1(\widehat{F})$$

for a semi-stable non-bielliptic fiber germ $\widehat{F}$ of genus 2. By using semi-stable reduction, we define $\lambda_{\mathcal{B}_2,0}(F)$, $\lambda_{\mathcal{B}_2,1}(F)$ for any non-bielliptic fiber germ F of genus 2. Then $\sigma_{\mathcal{B}_2,i}(F) := 4\lambda_{\mathcal{B}_2,i}(F) - \delta(F)$, $i = 1, 2$ are local signatures for genus 2 non-bielliptic fibrations.

Example 10.2. Let F_0, F_1 and F_b respectively be non-bielliptic genus 2 fiber germs the image of whose moduli map meets Δ_0, Δ_1 and $\mathcal{B}_2$ transversally (and does not meet other loci among them) at the moduli point of the central fiber. Then we have

$$\sigma(F_0) = -\frac{3}{5}, \quad \sigma(F_1) = -\frac{1}{5}, \quad \sigma(F_b) = 0,$$

$$\sigma_{\mathcal{B}_2,0}(F_0) = -\frac{4}{5}, \quad \sigma_{\mathcal{B}_2,0}(F_1) = -1, \quad \sigma_{\mathcal{B}_2,0}(F_b) = \frac{2}{5},$$

$$\sigma_{\mathcal{B}_2,1}(F_0) = -1, \quad \sigma_{\mathcal{B}_2,1}(F_1) = -\frac{9}{5}, \quad \sigma_{\mathcal{B}_2,0}(F_b) = \frac{4}{15}.$$

For example, take a general member R in the complete linear system $|pr_1^*\mathcal{O}_{\mathbb{P}^1}(N) \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(6)|$, $N \in 2\mathbb{Z}_{>0}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and construct the double covering $S \to \mathbb{P}^1 \times \mathbb{P}^1$

branched over R. Then the composite $f\colon S \to \mathbb{P}^1$ of the double covering and the first projection pr_1 is a non-bielliptic fibration of genus 2. By a simple computation, we have

$$\chi_f = N, \quad K_f^2 = 2N, \quad e_f = 10N, \quad \mathrm{Sign}(S) = -6N.$$

Since R is general, we may assume that any singular fiber germ of f is of type F_0 as above. Thus the number of fiber germs of type F_0, F_1 and F_b is $10N$, 0 and $15N$, respectively.

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