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Finite multiple polylogarithms

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§. 13. *Hinc ergo nacti sumus sequentem aequationem maxime memorabilem*:

$$\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \frac{1}{4}\binom{n}{4} + \frac{1}{5}\binom{n}{5} \; etc. \; \ldots\ldots \; \pm \frac{1}{n}\binom{n}{n} =$$
$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \; etc. \; \ldots\ldots \; \frac{1}{n}.$$

Leonhard Euler

The properties I propose to prove in this article, for any prime number n, > 3, are (1) *that the numerator of the fraction*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

when reduced to its lowest terms is divisible by n^2, (2) *the numerator of the fraction*

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-1)^2}$$

is divisible by n, and (3) *that the number of combinations of $2n-1$ things, taken $n-1$ together, diminished by 1, is divisible by n^3.*

Joseph Wolstenholme

Contents

1 Introduction

In this thesis, we introduce *finite multiple polylogarithms* and investigate their fundamental relations. First, we provide some historical background.

Multiple zeta values

The study of zeta values goes back to *the Basel problem* proposed in the 17th century asking the value of $\zeta(2)$. In 1734, this famous problem was solved by Euler as

$$\zeta(2) = \frac{\pi^2}{6}$$

and he studied the values of the Riemann zeta function at positive integers. *The multiple zeta values (= MZVs)* are defined as certain nested series which are generalizations of the Riemann zeta values and surprisingly the history of MZVs also goes back to Euler. In the seminal paper [9], Euler introduced *the double zeta-star value* $\zeta^\star(k_1, k_2)$ which is an MZV of depth two and found some relations such as

$$\zeta^\star(k_1, k_2) + \zeta^\star(k_2, k_1) = \zeta(k_1)\zeta(k_2) + \zeta(k_1 + k_2). \tag{1}$$

After a long interval, the study of MZVs became active in late 1980s. Moen proved a certain relation among MZVs of depth three, which is a generalization of one of the relations among MZVs of depth two proved by Euler. Moen also conjectured that a similar relation holds for MZVs of general depth. In 1988, Hoffman who heard of Moen's conjecture has started the study of MZVs. He could not solve Moen's conjecture but he discovered a conjecture called *the duality formula* as a bi-product ([12]). MZVs have iterated integral expressions due to

Drinfel'd, Kontsevich, Le–Murakami and the duality formula is an immediate consequence of this expression ([59]). Moen's conjecture was proved by Granville [11] and Zagier later and it is now called *the sum formula*. In [59], Zagier conjectured that the dimension of the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ spanned by MZVs of weght k is equal to d_k defined by the recurrence $d_k = d_{k-2} + d_{k-3}, d_0 = 1, d_1 = 0, d_2 = 1$ and Terasoma [55] and Deligne–Goncharov [5] proved that $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$ by a motivic approach. This inequality implies that there exist many $\mathbb{Q}$-linear relations among MZVs of the same weight. For example, there are $2^{12-2} = 1024$ MZVs of weight 12 and the dimension of the $\mathbb{Q}$-vecter space spanned by these values is at most $d_{12} = 12$. On the other hand, it is believed that there exist no relations among MZVs of different weights. By the harmonic product formula which is a generalization of the relation (1), $\mathcal{Z} := \sum_{k\geq 0} \mathcal{Z}_k$ becomes a $\mathbb{Q}$-subalgebra of $\mathbb{R}$. One of the main purpose of the study of MZVs is to understand the algebraic structure of $\mathcal{Z}$. In particular, we want to find explicit relations among MZVs. Until now, there appeared huge amounts of explicit relations after the sum formula and the duality formula such as Ohno's relation [31], the derivation relation [17], the extended double shuffle relation [17], and so on. MZVs appear not only in number theory, but also in various fields of mathematics and physics. Nowadays, the study of MZVs is quite active.

Finite multiple zeta values

According to Hoffman [14], the research on multiple harmonic sums modulo a prime number by Moen and Hoffman was already going on before the article *"What divisibility properties do generalized harmonic sums have?"* appeared in American Mathematical Monthly in 1992 [25]. Moen and Hoffman proved conjectures of Matiyasevich stated in the above article but the proofs was not published.

Let $H_n(k) := \sum_{j=1}^{n} 1/j^k$ and $H_n(k_1, k_2) := \sum_{j=1}^{n} H_{j-1}(k_2)/j^{k_1}$ for positive integers n, k, k_1, and k_2. Then we have $H_{p-1}(k) \equiv 0 \mod p$ for any prime number p satisfying $p > k+1$. A more non-trivial congruence is

$$H_{p-1}(k_1,k_2) \equiv (-1)^{k_1}\binom{k_1+k_2}{k_1}\frac{B_{p-k_1-k_2}}{k_1+k_2} \pmod{p} \quad (p>k_1+k_2+1),$$

where B_n denotes the nth Seki–Bernoulli number. These are typical examples of congruences of multiple harmonic sums. Moen and Hoffman thought of these objects as a sort of "toy model" for MZVs. Recently, Zagier suggested that we should consider a collection of mod p multiple harmonic sums for all prime numbers as an element of a $\mathbb{Q}$-algebra

$$\mathcal{A} := \left(\prod_{p\colon \text{primes}} \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_{p\colon \text{primes}} \mathbb{Z}/p\mathbb{Z}\right).$$

Kaneko–Zagier [20] calls these elements *finite multiple zeta values (= FMZVs)*. For example, $\zeta_{\mathcal{A}}(k) := (H_{p-1}(k) \bmod p)_p$ and $\zeta_{\mathcal{A}}(k_1,k_2) := (H_{p-1}(k_1,k_2) \bmod p)_p$ in $\mathcal{A}$. Then we have $\zeta_{\mathcal{A}}(k)=0$ and

$$\zeta_{\mathcal{A}}(k_1,k_2) = (-1)^{k_1}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}$$

by the above examples. Here, $B_{\boldsymbol{p}-k} := (B_{p-k} \bmod p)_p$ in $\mathcal{A}$ for a positive integer k. Thanks to the excellent framework of Zagier, we can consider the $\mathbb{Q}$-vector space $\mathcal{Z}_{\mathcal{A},k}$ spanned by FMZVs of weight k. Then Zagier [20] conjectured the dimension formula $\dim_{\mathbb{Q}} \mathcal{Z}_{\mathcal{A},k} = d_{k-3}$ for $k \geq 3$ and Akagi–Hirose–Yasuda announced that they proved $\dim_{\mathbb{Q}} Z_{\mathcal{A},k} \leq d_{k-3}$ by using Jarossay's recent work. By this inequality, there also exist many $\mathbb{Q}$-linear relations among FMZVs. Hoffman [15] gave an explicit conjectural basis of $\mathcal{Z}_{\mathcal{A},k}$ for $k \leq 9$ by combining results of Pilehrood–Pilehrood–Tauraso [37]. Since the harmonic product formula also holds for FMZVs, we see that $\mathcal{Z}_{\mathcal{A}} := \sum_{k\geq 0} \mathcal{Z}_{\mathcal{A},k}$ becomes a $\mathbb{Q}$-subalgebra of $\mathcal{A}$. We believe that FMZVs are not a "toy model" of MZVs, but $\mathcal{Z}_{\mathcal{A}}$ has a very rich algebraic structure as much as an algebraic structure of $\mathcal{Z}$.

In [15], Hoffman proved some fundamental relations for FMZVs such as *the reversal relation, the duality formula for FMZSVs*, and *the relation between FMZVs and FMZSVs*. Here, FMZSV which is an abbreviation of *"finite multiple zeta-star value"* is a variant of

FMZV and can be written as a $\mathbb{Q}$-linear combination of FMZVs. It is remarkable that the duality formula holds not for non-star values but for star values. This shows a different phenomenon from the case of MZVs where the duality formula holds with non-star values compared to the case of FMZVs. After Hoffman's work, various analogous relations for FMZVs were proved such as the sum formula [43], Ohno's relation [36], the derivation relation [30], the shuffle relation [19], and so on. Since we have $\zeta_{\mathcal{A}}(k) = 0$ for every positive integer k, we do not regard $\zeta_{\mathcal{A}}(k)$ as a suitable finite analogue of the Riemann zeta value $\zeta(k)$. According to some observations the true counterpart of the Riemann zeta value $\zeta(k)$ is considered to be $B_{\boldsymbol{p}-k}/k$. Kaneko–Zagier [20] conjectures that there exists a mysterious isomorphism as algebra between $\mathcal{Z}_{\mathcal{A}}$ and $\mathcal{Z}/\zeta(2)\mathcal{Z}$.

On the other hand, the most classical result for congruence properties of multiple harmonic sums is

$$H_{p-1}(1) \equiv 0 \pmod{p^2} \quad (p > 3),$$

which is famous as *Wolstenholme's theorem* [56]. J. Zhao [60] studied mod p multiple harmonic sums independently of Moen and Hoffman. Zhao calculated not only mod p but also many mod p^2 multiple harmonic sums inspired by Wolstenholme's theorem. We want to extend Zagier's framework to investigate the algebraic structures of mod p^2 multiple harmonic sums or, more generally, mod p^n multiple harmonic sums for a positive integer n. Based on this idea, we can define *the $\mathcal{A}_n$-finite multiple zeta values* as elements of

$$\mathcal{A}_n := \left(\prod_{p\colon \text{primes}} \mathbb{Z}/p^n\mathbb{Z} \right) \Big/ \left(\bigoplus_{p\colon \text{primes}} \mathbb{Z}/p^n\mathbb{Z} \right).$$

Moreover, Rosen [41] defined a $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}$ to be the projective limit of $\{\mathcal{A}_n\}$. The algebra $\widehat{\mathcal{A}}$ is $\boldsymbol{p}$-adically complete and we call this *the $\boldsymbol{p}$-adic number ring*. Here, $\boldsymbol{p} \in \widehat{\mathcal{A}}$ is defined by collecting all prime numbers, that is, $\boldsymbol{p} = (\boldsymbol{p}_n)_n$ where $\boldsymbol{p}_n \in \mathcal{A}_n$ is defined to be $(p \mod p^n)_p$ for each n. We call $\boldsymbol{p}$ *the infinitely large prime*. Rosen suggested to study multiple harmonic sums as elements in this new algebra. He calls these elements *weighted finite multiple zeta values*, but we call *$\widehat{\mathcal{A}}$-finite multiple zeta values (= $\widehat{\mathcal{A}}$-FMZVs)* here. We can also define

$\widehat{\mathcal{A}}$*-finite multiple zeta-star values (=* $\widehat{\mathcal{A}}$*-FMZSVs).* In [41], Rosen only considered $\widehat{\mathcal{A}}$-FMZVs and proved $\boldsymbol{p}$-adic generalizations of the reversal relation and the ψ-duality formula. Here, the ψ-duality formula is a relation for $\mathcal{A}$-FMZVs and is equivalent to Hoffman's duality formula for $\mathcal{A}$-FMZSVs. We prove a direct generalization of the duality for $\mathcal{A}$-FMZSVs and the relation between $\widehat{\mathcal{A}}$-FMZVs and $\widehat{\mathcal{A}}$-FMZSVs. We do not know whether Rosen's duality is equivalent to our duality. We can also prove a $\boldsymbol{p}$-adic generalization of Kaneko's shuffle relation.

Finite multiple polylogarithms

There are also finite analogues of polylogarithms which are important objects in number theory. In the unpublished note [22], Kontsevich introduced so called *the* $1\frac{1}{2}$*-logarithm* which is a finite analogue of the logarithmic function and observed that it satisfies some functional equations such as *the four-term relation.* The reason why he called this function $1\frac{1}{2}$*-logarithm* is because the logarithmic function satisfies the three-term relation and the dilogarithm satisfies the five-term relation. Kontsevich asked what are functional equations satisfied by the finite analogue of dilogarithm. In [8], Elbaz-Vincent–Gangl introduced *finite polylogarithms* and they answered to Kontsevich's question. In fact, they proved that the finite dilogarithm satisfies *the* 22*-term relation* and developed a general theory of finite polylogarithms.

Motivated by these results, Sakugawa and the author introduced *finite multiple polylogarithms (= FMPs)* in [44]. Kontsevich and Elbaz-Vincent–Gangl considered finite polylogarithms for a fixed prime. But we generalized finite polylogarithms to multiple-cases with multi-variables in the framework of Zagier, that is $\mathcal{A}_n$-FMPs. There are four types of FMPs: *finite harmonic multiple polylogarithm (= FHMP)*, *finite shuffle multiple polylogarithm (= FSMP)*, *finite harmonic star-multiple polylogarithm (= FHSMP)*, and *finite shuffle star-multiple polylogarithms (= FSSMP).* These are also generalizations of FMZVs in the sense that our FMPs give FMZVs by substituting 1 into all variables. Our main purpose is to establish fundamental relations for FMPs which generalize fundamental relations for FMZVs obtained by Hoffman and some of functional equations for finite polylogarithms obtained by Elbaz-Vincent–Gangl. In [44], we proved *the reversal relation for $\mathcal{A}_2$-FH(S)MPs* ([44, Proposition 3.11]), *the functional equation for $\mathcal{A}_2$-FSSMPs* ([44, Theorem 3.12]), and *the relation between $\mathcal{A}_n$-FHMPs and $\mathcal{A}_n$-FHSMPs* for any n ([44, Theorem 3.15]). After this work, motivated by Rosen's work, the author defined $\widehat{\mathcal{A}}$-FMPs and extended the fundamental relations to $\boldsymbol{p}$-adic relations [46, Theorem 3.1 and Theorem 3.4].

This thesis is based on [44] and [46]. The main result of this thesis is as follows:

Main Theorem. *Let $\mathbf{k} = (k_1, \dots, k_r), \mathbf{k}_1, \dots, \mathbf{k}_r$ be indices. Let $t_1, \dots, t_r$ be indeterminates and $\bullet \in \{\emptyset, \star\}$. Let $\overline{\mathbf{k}} = (k_r, \dots, k_1)$ be the reverse index of $\mathbf{k}$. Put $l_i := \mathrm{dep}(\mathbf{k}_i)$ and $l'_i := \mathrm{dep}(\mathbf{k}_i^\vee)$ for $i = 1, \dots, r$. Here, $\mathbf{k}_i^\vee$ is the Hoffman dual of $\mathbf{k}_i$. Furthermore, we define a $\boldsymbol{p}$-adic series $\mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}}(t_1, \dots, t_r)$ by*

$$\mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}}(t_1, \dots, t_r) := \sum_{i=0}^{\infty} \left(\pounds^{\text{Ш},\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i, t_1, \dots, t_r) - \frac{1}{2} \pounds^{\text{Ш},\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i, t_1, \dots, t_{r-1}, 1) \right) \boldsymbol{p}^i.$$

Then we have the following three fundamental relations for $\widehat{\mathcal{A}}$-finite multiple polylogarithms:

(i) ***Reversal relation for $\widehat{\mathcal{A}}$-FH(S)MPs (= Theorem 12.1)***

$$\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\overline{\mathbf{k}}}(t_1, \dots, t_r) =$$

$$(-1)^{\mathrm{wt}(\mathbf{k})} (t_1 \cdots t_r)^{\boldsymbol{p}} \sum_{i=0}^{\infty} \sum_{\substack{(l_1,\dots,l_r) \in \mathbb{Z}^r_{\geq 0} \\ l_1 + \cdots + l_r = i}} \left[\prod_{j=1}^{r} \binom{k_j + l_j - 1}{l_j} \right] \pounds^{*,\bullet}_{\widehat{\mathcal{A}},(k_1+l_1,\dots,k_r+l_r)}(t_r^{-1}, \dots, t_1^{-1}) \boldsymbol{p}^i,$$

(ii) ***Functional equation for $\widehat{\mathcal{A}}$-FSSMPs (= Theorem 12.2)***

$$\mathcal{L}^\star_{\widehat{\mathcal{A}},(\mathbf{k}_1,\dots,\mathbf{k}_r)}(\{1\}^{l_1-1}, t_1, \dots, \{1\}^{l_r-1}, t_r) = \mathcal{L}^\star_{\widehat{\mathcal{A}},(\mathbf{k}_1^\vee,\dots,\mathbf{k}_r^\vee)}(\{1\}^{l'_1-1}, 1-t_1, \dots, \{1\}^{l'_r-1}, 1-t_r),$$

(iii) ***Relation between $\widehat{\mathcal{A}}$-FHMPs and $\widehat{\mathcal{A}}$-FHSMPs (= Theorem 12.11)***

$$\sum_{j=0}^{r} (-1)^j \pounds^{*}_{\widehat{\mathcal{A}},(k_1,\dots,k_j)}(t_1, \dots, t_j) \pounds^{*,\star}_{\widehat{\mathcal{A}},(k_r,\dots,k_{j+1})}(t_r, \dots, t_{j+1}) = 0.$$

The key ingredients of proofs of (ii) and (iii) are generalizations of the following identity by Euler

$$\sum_{n=1}^{N}(-1)^{n-1}\binom{N}{n}\frac{1}{n}=\sum_{n=1}^{N}\frac{1}{n}$$

to multiple summations with variables. We carry out this generalizations by introducing *the truncated integral operators*.

Independently of us, Ono–Yamamoto [35] defined another type of finite multiple polylogarithms and proved a shuffle relation of them. We give an explicit relation between Ono–Yamamoto's FMPs and our $\mathcal{A}$-FMPs.

Special values of finite multiple polylogarithms

By applying our main theorem, we can calculate special values at 2 or 1/2 of FMPs. Before stating our result, we recall a conjecture by Z. W. Sun. In [50], Z. W. Sun proved

$$\sum_{n=1}^{p-1}\frac{H_n(1)}{n2^n}\equiv 0 \pmod{p} \quad (p>3)$$

and conjectured

$$\sum_{n=1}^{p-1}\frac{H_n(1)}{n2^n}\equiv \frac{7}{24}B_{p-3}p \pmod{p^2} \quad (p>3), \quad \text{if } n \text{ is even.}$$

This conjecture was already proved by Z. W. Sun–L. L. Zhao [51] and Meštrović [27]. We noticed that these congruences can be regarded as explicit formulas of special values of FMPs. Namely, we can rewrite the above results as $\pounds^{\star}_{\mathcal{A},(1,1)}(1/2)=0$ and $\pounds^{\star}_{\mathcal{A}_2,(1,1)}(1/2)=\frac{7}{24}B_{\boldsymbol{p}-3}\boldsymbol{p}$. Once we interpret the congruences through FMPs, we have quite natural generalizations of the result and the conjecture by Z. W. Sun:

Theorem 1.1 (= Theorem 16.2 (143) and Theorem 16.5 (151))**.** *Let k be a positive integer. Then we have*

$$\pounds^{\star}_{\mathcal{A},\{1\}^k}(1/2)=\frac{2^{k-1}-1}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k},$$

$$\pounds^{\star}_{\mathcal{A}_2,\{1\}^k}(1/2) = \frac{2^{k+1}-1}{2^{k+1}} \frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} \quad (k : \mathit{even}).$$

The first equality is a finite analogue of the following Zlobin's result for the star-multiple polylogarithms:

Theorem 1.2 (Zlobin [65, Theorem 8])**.** *Let k be a positive integer. Then*

$$\mathrm{Li}^{\star}_{\{1\}^k}(1/2) = \frac{2^{k-1}-1}{2^{k-1}}\zeta(k).$$

By combining our main results and explicit evaluations of *finite alternating multiple zeta values* calculated by Z. H. Sun, Tauraso–J. Zhao, Pilehrood–Pilehrood–Tauraso, we can calculate not only the above theorem, but also many other special values obtained by substituting 1, 2, or 1/2 into variables of $\mathcal{A}$-FMPs or $\mathcal{A}_2$-FMPs.

Outline of this thesis

This paper is organized as follows:
In Part I, we review multiple zeta values and multiple polylogarithms. In Part II, we review finite multiple zeta values. There are some new results on $\widehat{\mathcal{A}}$-FMZVs. Part III is the main part of this thesis. After generalizing Euler's identity for binomial coefficients in Section 8 to prove our main results, we define the finite multiple polylogarithms in Section 11. We prove our main results in Section 12. Using our main results, we calculate special values at 2 and 1/2 of finite multiple polylogarithms in Section 16.

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1.1 Notations for indices

We use the following notations for indices throughout this thesis.

1.1.1 index

We define the set of indices I by

$$I := \coprod_{r\in\mathbb{Z}_{\geq 0}} (\underbrace{\mathbb{Z}_{>0} \times \cdots \times \mathbb{Z}_{>0}}_{r})$$

and we call an element of I *an index*.

1.1.2 weight, depth, and height

For an index $\mathbf{k} = (k_1, \ldots, k_r) \in I$, we define *the weight* (resp. *the depth*, resp. *the height*) of $\mathbf{k}$ to be $k_1 + \cdots + k_r$ (resp. r, resp. $\#\{i \mid k_i \geq 2\}$) and we denote it by $\mathrm{wt}(\mathbf{k})$ (resp. $\mathrm{dep}(\mathbf{k})$, resp. $\mathrm{ht}(\mathbf{k})$). We define by convention $\mathrm{wt}(\emptyset) = \mathrm{dep}(\emptyset) = \mathrm{ht}(\emptyset) := 0$.

1.1.3 admissible

An index $\mathbf{k} = (k_1, \ldots, k_r)$ is called *admissible* if $\mathbf{k} = \emptyset$ or $k_1 \geq 2$.

1.1.4 abbreviation for repetitions

For a non-negative integer k, the symbol $\{k\}^r$ denotes r repetitions of k. Namely, $\{k\}^r = \underbrace{k, \ldots, k}_{r}$. If $r = 0$, $\{k\}^r = \emptyset$. For $i = 1, \ldots, r$, put $\mathbf{e}_i := (\{0\}^{i-1}, 1, \{0\}^{r-i})$ when r is clear from the context.

1.1.5 reverse index

For an index $\mathbf{k} = (k_1, \dots, k_r)$, we define *the reverse index* $\overline{\mathbf{k}}$ to be $(k_r, \dots, k_1)$.

1.1.6 concatenation

For indices $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$, we define *the concatenation* $(\mathbf{k}, \mathbf{l})$ to be $(k_1, \dots, k_r, l_1, \dots, l_s)$. Furthermore, we use the notation $\mathbf{k} \uplus \mathbf{l} := (k_1, \dots, k_{r-1}, k_r + l_1, l_2, \dots, l_s)$.

1.1.7 componentwise addition

For indices $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_r)$, we define *the componentwise sum* $\mathbf{k} \oplus \mathbf{l}$ to be $(k_1 + l_1, \dots, k_r + l_r)$. We also use this notation when some components equal to zero.

1.1.8 eliminated index

For an index $\mathbf{k} = (k_1, \dots, k_r)$ and a positive integer i satisfying $0 \leq i \leq r$, we define $\mathbf{k}_{(i)}$ (resp. $\mathbf{k}^{(i)}$) to be $(k_1, \dots, k_i)$ (resp. $(k_{i+1}, \dots, k_r)$). $\mathbf{k}_{(0)} = \mathbf{k}^{(r)} = \emptyset$.

1.1.9 contraction index

Let $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$ be indices. If there exist $i_1, \dots, i_r$ such that $k_1 = l_1 + \cdots + l_{i_1}, k_2 = l_{i_1+1} + \cdots + l_{i_2}, \dots, k_r = l_{i_{r-1}+1} + \cdots + l_{r_r}$, then we say that $\mathbf{k}$ is *a contraction index of* $\mathbf{l}$ and we use the notation $\mathbf{k} \preceq \mathbf{l}$.

1.1.10 dual index

Any non-empty admissible index $\mathbf{k}$ can be written in the form

$$\mathbf{k} = (a_1 + 1, \{1\}^{b_1 - 1}, a_2 + 1, \{1\}^{b_2 - 1}, \dots, a_s + 1, \{1\}^{b_s - 1})$$

with positive integers $a_1, b_1, \dots, a_s, b_s$. Then we define *the dual index* $\mathbf{k}'$ by

$$\mathbf{k}' := (b_s + 1, \{1\}^{a_s - 1}, b_{s-1} + 1, \{1\}^{a_{s-1} - 1}, \dots, b_1 + 1, \{1\}^{a_1 - 1}).$$

By the definition of the dual index, we see that $\mathbf{k}'$ is admissible and that $\mathbf{k}''=\mathbf{k}$, $\mathrm{wt}(\mathbf{k}')=\mathrm{wt}(\mathbf{k})$, and $\mathrm{dep}(\mathbf{k})+\mathrm{dep}(\mathbf{k}')=\mathrm{wt}(\mathbf{k})$ hold for any non-empty admissible index $\mathbf{k}$.

Example 1.3. Let k_1 and k_2 be positive integers. Then

$$(k_1+1,\{1\}^{k_2-1})'=(k_2+1,\{1\}^{k_1-1})$$

holds. In particular, $(3)'=(2,1)$.

1.1.11 Hoffman dual

Let $\mathbf{k}$ be an index. Put $k:=\mathrm{wt}(\mathbf{k})$ and $r:=\mathrm{dep}(\mathbf{k})$. Then we define a subset $A(\mathbf{k})$ of $\{1,2,\ldots,k-1\}$ by

$$A(\mathbf{k}):=\{\mathrm{wt}(\mathbf{k}_{(1)}),\mathrm{wt}(\mathbf{k}_{(2)}),\ldots,\mathrm{wt}(\mathbf{k}_{(r-1)})\}.$$

We define *the Hoffman dual* $\mathbf{k}^\vee$ of $\mathbf{k}$ as the following equality holds:

$$A(\mathbf{k})\sqcup A(\mathbf{k}^\vee)=\{1,2,\ldots,k-1\}.$$

By the definition of the Hoffman dual, we see that $\mathbf{k}^{\vee\vee}=\mathbf{k}$, $\mathrm{wt}(\mathbf{k}^\vee)=\mathrm{wt}(\mathbf{k})$, and $\mathrm{dep}(\mathbf{k})+\mathrm{dep}(\mathbf{k}^\vee)=\mathrm{wt}(\mathbf{k})+1$ hold for any index $\mathbf{k}$. We can use the notation $\overline{\mathbf{k}}^\vee$ since the operator taking the Hoffman dual and the reversal operator $\mathbf{k}\mapsto\overline{\mathbf{k}}$ commute.

Example 1.4. We have the following equalities:

$$\begin{aligned}&r^\vee=\{1\}^r,\ (k_1,k_2)^\vee=(\{1\}^{k_1-1},2,\{1\}^{k_2-1}),\\&(k_1,k_2,k_3)^\vee=(\{1\}^{k_1-1},2,\{1\}^{k_2-2},2,\{1\}^{k_3-1}),\\&(k_1,\{1\}^{k_2-1})^\vee=(\{1\}^{k_1-1},k_2).\end{aligned}$$

Here, r,k_1,k_2, and k_3 are positive integers and the third equality holds only when k_2 is greater than or equal to 2.

1.2 Notations for indeterminates

Let $\boldsymbol{t} = (t_1, \ldots, t_r)$ be a tuple of indeterminates. For tuples of indeterminates, we also use notations of *abbreviation for repetitions*, *reverse tuple*, *concatenation*, *componentwise addition*, *eliminated tuple* in the same manner as indices. Furthermore, we use the notations $\boldsymbol{t}_1 := (\boldsymbol{t}_{(r-1)}, 1)$, $1 - \boldsymbol{t} := (1 - t_1, \ldots, 1 - t_r)$, and $\boldsymbol{t}^{-1} := (t_1^{-1}, \ldots, t_r^{-1})$. For a commutative ring R, we denote a multi-variable polynomial ring $R[t_1, \ldots, t_r]$ by $R[\boldsymbol{t}]$.

Part I
Review of Multiple Zeta Values and Multiple Polylogarithms

2 Review of multiple zeta values

2.1 Definition of multiple zeta values

Definition 2.1. Let $\mathbf{k} = (k_1, \ldots, k_r)$ be an admissible index. Then we define *the multiple zeta value* $\zeta(\mathbf{k})$ by

$$\zeta(\mathbf{k}) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

and define *the multiple zeta-star value* $\zeta^{\star}(\mathbf{k})$ by

$$\zeta^{\star}(\mathbf{k}) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

These series converge since $\mathbf{k}$ is admissible. $\zeta(\emptyset) = \zeta^{\star}(\emptyset) := 1$ by convention.

Multiple zeta values are defined by series expressions as above, but these values also have integral expressions. By this fact, multiple zeta values become periods in the sense of Kontsevich–Zagier [24]. This perspective is very important.

2.2 Integral expression of multiple zeta value

Theorem 2.2 (Drinfel'd, Kontsevich, Le–Murakami [59]). *Let* $\omega_0(t) = \frac{dt}{t}$ *and* $\omega_1(t) = \frac{dt}{1-t}$. *Let* $\mathbf{k}$ *be an admissible index. Put* $k := \mathrm{wt}(\mathbf{k})$. *For* $j = 1, \ldots, k$, *we define* $d(j) \in \{0,1\}$ *by*

$$d(j) := \begin{cases} 0 & j \notin A(\mathbf{k}) \cup \{k\} \\ 1 & j \in A(\mathbf{k}) \cup \{k\} \end{cases}.$$

Then we have

$$\zeta(\mathbf{k}) = \int_{1>t_1>\cdots>t_k>0} \omega_{d(1)}(t_1) \cdots \omega_{d(k)}(t_k). \tag{2}$$

2.3 Integral expression of multiple zeta-star value

Let $\omega_0(t) = \frac{dt}{t}$ and $\omega_1(t) = \frac{dt}{1-t}$ as in Theorem 2.2. Let $\mathbf{k}$ be an index and $k := \mathrm{wt}(\mathbf{k})$. For $j = 1, \ldots, k$, we define $\delta(j) \in \{0,1\}$ by

$$\delta(j) := \begin{cases} 0 & j-1 \notin A(\mathbf{k}) \cup \{0\} \\ 1 & j-1 \in A(\mathbf{k}) \cup \{0\} \end{cases}$$

and define a domain $\Delta(\mathbf{k})$ in $\mathbb{R}^k$ by

$$\Delta(\mathbf{k}) := \left\{ (t_1, \ldots, t_k) \in [0,1]^k \;\middle|\; \begin{array}{ll} t_j < t_{j+1} & j \notin A(\mathbf{k}) \\ t_j > t_{j+1} & j \in A(\mathbf{k}) \end{array} \right\}.$$

Theorem 2.3 (Yamamoto [58, Theorem 1.2]). *Let* n *be a positive integer and* $\mathbf{k} = (k_1, \ldots, k_r)$ *an index. We define* $s_{\mathbf{k}}(n)$ *by*

$$s_{\mathbf{k}}(n) := \sum_{n=n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Then we have

$$s_{\mathbf{k}}(n) = \int_{\Delta(\mathbf{k})} t_1^{n-1} \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k). \tag{3}$$

Corollary 2.4. *Let* $\mathbf{k}$ *be an admissible index. Then*

$$\zeta^\star(\mathbf{k}) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1)\cdots\omega_{\delta(k)}(t_k).$$

Proof. This is obtained by the equality $\zeta^\star(\mathbf{k}) = \sum_{n=1}^{\infty} s_{\mathbf{k}}(n)$. □

2.4 Relations among multiple zeta values

Theorem 2.5 (Duality formula)**.** *Let* $\mathbf{k}$ *be an admissible index. Then we have*

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}').$$

Proof. This equality is obtained by a change of variables $(t_1,\ldots,t_k)\mapsto(1-t_k,\ldots,1-t_1)$ in the integral expression (2). □

Euler's famous identity $\zeta(3)=\zeta(2,1)$ is a special case of the duality formula.

Theorem 2.6 (Relation between MZVs and MZSVs [64])**.** *Let* $\mathbf{k}$ *be an admissible index such that* $\overline{\mathbf{k}}$ *is also admissible. Then*

$$\sum_{j=0}^{\mathrm{dep}(\mathbf{k})} (-1)^j \zeta(\mathbf{k}_{(j)})\zeta^\star(\overline{\mathbf{k}^{(j)}}) = 0.$$

Theorem 2.7 (Sum formula [11])**.** *Let* k *and* r *be positive integers satisfying* $r<k$*. Then we have the following two relations*:

$$\sum_{\substack{\mathbf{k}\colon \textit{admissible}\\ \mathrm{wt}(\mathbf{k})=k,\,\mathrm{dep}(\mathbf{k})=r}} \zeta(\mathbf{k}) = \zeta(k), \qquad \sum_{\substack{\mathbf{k}\colon \textit{admissible}\\ \mathrm{wt}(\mathbf{k})=k,\mathrm{dep}(\mathbf{k})=r}} \zeta^\star(\mathbf{k}) = \binom{k-1}{r-1}\zeta(k).$$

Theorem 2.8 (Aoki–Ohno [1]). *Let k and s be positive integers satisfying $k \geq 2s$. Then*

$$\sum_{\substack{\mathbf{k}\colon \textit{admissible} \\ \mathrm{wt}(\mathbf{k})=k, \mathrm{ht}(\mathbf{k})=s}} \zeta^{\star}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\zeta(k).$$

Theorem 2.9 (Ohno's relation [31]). *Let l be a non-negative integer and $\mathbf{k}$ an admissible index. Then*

$$\sum_{\substack{\mathbf{e}\in\mathbb{Z}_{\geq 0}^{\mathrm{dep}(\mathbf{k})} \\ \mathrm{wt}(\mathbf{e})=l}} \zeta(\mathbf{k}\oplus\mathbf{e}) = \sum_{\substack{\mathbf{e}'\in\mathbb{Z}_{\geq 0}^{\mathrm{dep}(\mathbf{k}')} \\ \mathrm{wt}(\mathbf{e}')=l}} \zeta(\mathbf{k}'\oplus\mathbf{e}'),$$

where $\mathbf{k}'$ is the dual index of $\mathbf{k}$ (See 1.1.10).

Many other relations are also known.

2.5 Hoffman's algebras

Let $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ be a non-commutative polynomial algebra in two variables and $\mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y \supset \mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y$ its subalgebras. For a word $w \in \mathfrak{H}$, we define the weight $\mathrm{wt}(w)$ of w as the total degree of w. For a positive integer k, we define $z_k \in \mathfrak{H}^1$ to be $x^{k-1}y$. Then $\mathfrak{H}^1$ is generated by z_k ($k = 1, 2, \dots$) as a non-commutative algebra. For an index $\mathbf{k} = (k_1, \dots, k_r)$, we define $z_{\mathbf{k}}$ to be $z_{k_1} \cdots z_{k_r} \in \mathfrak{H}^1$. If $\mathbf{k}$ is admissible, then $z_{\mathbf{k}}$ is an element of $\mathfrak{H}^0$.

Definition 2.10. We define *the harmonic product* $*$ on $\mathfrak{H}^1$ by the $\mathbb{Q}$-bilinearity and the following rules:

1. $w * 1 = 1 * w = w$ for any $w \in \mathfrak{H}^1$,

2. $z_{k_1}w_1 * z_{k_2}w_2 = z_{k_1}(w_1 * z_{k_2}w_2) + z_{k_2}(z_{k_1}w_1 * w_2) + z_{k_1+k_2}(w_1 * w_2)$
   for any $w_1, w_2 \in \mathfrak{H}^1$ and positive integers k_1, k_2.

We also define *the shuffle product* $ш$ on $\mathfrak{H}$ by the $\mathbb{Q}$-bilinearity and the following rules:

1. $w ш 1 = 1 ш w = 1$ for any $w \in \mathfrak{H}$,

2. $u_1w_1 ш\, u_2w_2 = u_1(w_1 ш\, u_2w_2) + u_2(u_1w_1 ш\, w_2)$

   for any $w_1, w_2 \in \mathfrak{H}$ and $u_1, u_2 \in \{x, y\}$.

Proposition 2.11 (Hoffman [13], Reutenauer [39])**.** *The harmonic product and the shuffle product are commutative and associative.*

Definition 2.12. We define a $\mathbb{Q}$-linear mapping $Z\colon \mathfrak{H}^0 \to \mathbb{R}$ by $Z(1) := 1$ and $Z(z_{\mathbf{k}}) := \zeta(\mathbf{k})$ for each admissible index $\mathbf{k}$.

Two expressions of multiple zeta value (Definition 2.1 and Theorem 2.2) leads to the following relation:

Proposition 2.13 (Double shuffle relation)**.** *For any elements w_1 and w_2 of $\mathfrak{H}^0$, we have*

$$Z(w_1 * w_2) = Z(w_1 ш\, w_2) = Z(w_1)Z(w_2).$$

Definition 2.14. Let n be a positive integer. Then we define a derivation ∂_n on $\mathfrak{H}$ by

$$\partial_n(x) = x(x+y)^{n-1}y, \quad \partial_n(y) = -x(x+y)^{n-1}y.$$

The extended double shuffle relation which is a generalization of the double shuffle relation onto $\mathfrak{H}^1$ implies the following relation:

Theorem 2.15 (Derivation relation [17])**.** *Let n be a positive integer. Then*

$$Z(\partial_n(w)) = 0$$

holds for any $w \in \mathfrak{H}^0$.

2.6 Interpolated multiple zeta values

Let t be an indeterminate. In this subsection, we review the t-multiple zeta values defined by Yamamoto.

Definition 2.16 (Yamamoto [57])**.** Let $\mathbf{k}$ be an admissible index and t an indeterminate. Then we define *the t-multiple zeta value* $\zeta^t(\mathbf{k})$ by

$$\zeta^t(\mathbf{k}) := \sum_{\mathbf{l} \preceq \mathbf{k}} t^{\mathrm{dep}(\mathbf{k})-\mathrm{dep}(\mathbf{l})} \zeta(\mathbf{l}).$$

Definition 2.17. Let $\mathfrak{z}$ be a $\mathbb{Q}$-submodule of $\mathfrak{H}^1$ generated by $\{z_k \mid k = 1, 2, \dots\}$. We define a $\mathbb{Q}$-linear action $\circ$ of $\mathfrak{z}$ on $\mathfrak{H}^1$ by

$$z_k \circ 1 = 0, \quad z_k \circ (z_l w) = z_{k+l} w$$

for any positive integers k, l and $w \in \mathfrak{H}^1$.

Definition 2.18. We define a $\mathbb{Q}$-linear operator S^t on $\mathfrak{H}^1[t]$ by

$$S^t(1) = 1, \quad S^t(z_k w) = z_k S^t(w) + t z_k \circ S^t(w).$$

for any positive integer k and $w \in \mathfrak{H}^1$.

Definition 2.19. We define a $\mathbb{Q}$-linear mapping $Z^t \colon \mathfrak{H}^0[t] \to \mathbb{R}[t]$ as a composition $Z^t := Z \circ S^t$. Here, we naturally extend Z to a mapping $\mathfrak{H}^0[t] \to \mathbb{R}[t]$.

$Z^t(z_{\mathbf{k}}) = \zeta^t(\mathbf{k})$ holds for any admissible index. We denote $S = S^1$ and $Z^\star = Z^1$. Then $Z^\star(z_{\mathbf{k}}) = \zeta^\star(\mathbf{k})$. The following proposition is a t-analogue of Theorem 2.6:

Proposition 2.20 (Relation between t-MZVs and $(1-t)$-MZVs [57, Proposition 3.7])**.** *Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index. Then we have*

$$\sum_{j=0}^{r} (-1)^j S^t(z_{k_1} \cdots z_{k_j}) * S^{1-t}(z_{k_r} \cdots z_{k_{j+1}}).$$

In particular, if $\mathbf{k}$ and $\overline{\mathbf{k}}$ are admissible, we have

$$\sum_{j=0}^{r} (-1)^j \zeta^t(\mathbf{k}_{(j)}) \zeta^{1-t}(\overline{\mathbf{k}^{(j)}}) = 0.$$

Yamamoto also proved a t-analogue of the sum formula (Theorem 2.7). Here, we recall his proof briefly.

Definition 2.21. Let N be a $\mathbb{Q}[t]$-submodule of $\mathfrak{H}^1[t]$. Then we call N *a differential submodule* if it is closed under the differentiation with respect to t.

Lemma 2.22 ([57, Lemma 5.1])**.** *Let N be a differential submodule, R a $\mathbb{Q}$-algebra, and $\alpha \in R$. Then the $R[t]$-module $R \otimes_{\mathbb{Q}} N$ is generated by an R-submodule $N_\alpha = \{f(\alpha) \mid f(t) \in R \otimes_{\mathbb{Q}} N\}$ of $R \otimes_{\mathbb{Q}} \mathfrak{H}^1$.*

For $k > r \geq 1$, we put

$$x_{k,r} := \sum_{\substack{\mathbf{k}\colon \text{admissible} \\ \mathrm{wt}(\mathbf{k})=k, \mathrm{dep}(\mathbf{k})=r}} z_{\mathbf{k}} \in \mathfrak{H}^1, \quad P_{k,r}(t) = \sum_{j=0}^{r-1} \binom{k-1}{j} t^j (1-t)^{r-1-j} \in \mathbb{Q}[t].$$

Lemma 2.23 ([57, Lemma 5.2])**.** *Let $k > r \geq 2$. Then*

$$\frac{d}{dt} S^t(x_{k,r}) = (k-r) S^t(x_{k,r-1}), \quad \frac{d}{dt} P_{k,r}(t) = (k-r) P_{k,r-1}(t).$$

Theorem 2.24 (Sum formula for t-MZVs [57, Theorem 1.1])**.** *Let k and r be positive integers satisfying $r < k$. Then we have the following relation*:

$$\sum_{\substack{\mathbf{k}\colon \textit{admissible} \\ \mathrm{wt}(\mathbf{k})=k, \mathrm{dep}(\mathbf{k})=r}} \zeta^t(\mathbf{k}) = \left(\sum_{j=0}^{r-1} \binom{k-1}{j} t^j (1-t)^{r-1-j} \right) \zeta(k).$$

Proof. Let N_k^{SF} be a $\mathbb{Q}[t]$-submodule of $\mathfrak{H}^1[t]$ generated by $\{S^t(x_{k,r}) - P_{k,r}(t) z_k \mid r < k\}$. Then N_k^{SF} is a differential submodule by Lemma 2.23. Therefore, it is sufficient to show that $(N_k^{\mathrm{SF}})_0$ is contained in $\mathrm{Ker}(Z)$. This follows from the sum formula (Theorem 2.7). $\square$

3 Review of multiple polylogarithms

Definition 3.1. Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index and $\boldsymbol{z} = (z_1, \dots, z_r)$ a tuple of complex numbers satisfying at least one of the following conditions for absolute convergence:

(i) $|z_1| < 1$ and $|z_i| \leq 1$ $(2 \leq i \leq r)$, (ii) $|z_i| \leq 1$ $(1 \leq i \leq r)$ and $\mathbf{k}$ is admissible.

Then we define *the multiple polylogarithms* by

$$\mathrm{Li}_{\mathbf{k}}(\boldsymbol{z}) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\mathrm{Li}^{\star}_{\mathbf{k}}(\boldsymbol{z}) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.$$

We define *the one-variable multiple polylogarithms* by

$$\mathrm{Li}_{\mathbf{k}}(z) := \mathrm{Li}_{\mathbf{k}}(z, \{1\}^{r-1}), \quad \mathrm{Li}^{\star}_{\mathbf{k}}(z) := \mathrm{Li}^{\star}_{\mathbf{k}}(z, \{1\}^{r-1}).$$

If $\mathbf{k}$ is admissible, then we have $\mathrm{Li}_{\mathbf{k}}(1) = \zeta(\mathbf{k})$ and $\mathrm{Li}^{\star}_{\mathbf{k}}(1) = \zeta^{\star}(\mathbf{k})$.

Theorem 3.2 (Landen connection formula, cf. [32]). *Let $\mathbf{k}$ be an index and z a complex number satisfying $|z| < 1$ and $\mathrm{Re}(z) < 1/2$. Then*

$$\mathrm{Li}_{\mathbf{k}}(z) = (-1)^{\mathrm{dep}(\mathbf{k})} \sum_{\mathbf{k} \preceq \mathbf{l}} \mathrm{Li}_{\mathbf{l}}\left(\frac{z}{z-1}\right).$$

Theorem 3.3 (Duality formula, Zlobin [63, Lemma 12], Imatomi [18, Proof of Theorem 3.2]). *Let $\mathbf{k}$ be an index and z a complex number satisfying $|z| < 1$ and $\mathrm{Re}(z) < 1/2$. Then*

$$\mathrm{Li}^{\star}_{\mathbf{k}}(z) = -\mathrm{Li}^{\star}_{\mathbf{k}^{\vee}}\left(\frac{z}{z-1}\right). \tag{4}$$

These two theorems are proved by the following easy lemma and by induction.

Lemma 3.4. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index and* z *a complex number satisfying* $|z| < 1$. *Then*

$$\frac{d}{dz}\operatorname{Li}_{\mathbf{k}}(z) = \begin{cases} \dfrac{1}{z}\operatorname{Li}_{(k_1-1,\mathbf{k}^{(1)})}(z) & \text{if } \mathbf{k} \text{ is admissible} \\ \dfrac{1}{1-z}\operatorname{Li}_{\mathbf{k}^{(1)}}(z) & \text{if } \mathbf{k} \text{ is not admissible} \end{cases}$$

and

$$\frac{d}{dz}\operatorname{Li}^{\star}_{\mathbf{k}}(z) = \begin{cases} \dfrac{1}{z}\operatorname{Li}^{\star}_{(k_1-1,\mathbf{k}^{(1)})}(z) & \text{if } \mathbf{k} \text{ is admissible} \\ \dfrac{1}{z(1-z)}\operatorname{Li}^{\star}_{\mathbf{k}^{(1)}}(z) & \text{if } \mathbf{k} \text{ is not admissible} \end{cases}$$

hold.

The following theorem which is a generalization of Theorem 2.6 is obtained by the result in Section 8 (= Theorem 8.11).

Theorem 3.5. *Let* $\mathbf{k}$ *be an index of* $\operatorname{dep}(\mathbf{k}) = r$ *and* $\boldsymbol{z} = (z_1, \ldots, z_r)$ *a tuple of complex numbers such that* $\operatorname{Li}_{\mathbf{k}}(\boldsymbol{z})$ *and* $\operatorname{Li}^{\star}_{\overline{\mathbf{k}}}(\overline{\boldsymbol{z}})$ *converge. Then we have*

$$\sum_{j=0}^{r}(-1)^j \operatorname{Li}_{\mathbf{k}_{(j)}}(\boldsymbol{z}_{(j)}) \operatorname{Li}^{\star}_{\overline{\mathbf{k}^{(j)}}}(\overline{\boldsymbol{z}^{(j)}}) = 0.$$

Part II
Finite Multiple Zeta Values

4 Definition of finite multiple zeta values

4.1 Multiple Harmonic Sums

Before defining finite multiple zeta values, we define multiple harmonic sums which are generalizations of *the harmonic number* $H_n = \sum_{j=1}^{n} 1/j$.

Definition 4.1. Let n be a positive integer and $\mathbf{k} = (k_1, \ldots, k_r)$ an index. Then we define *the multiple harmonic sum* $H_n(\mathbf{k})$ by

$$H_n(\mathbf{k}) := \sum_{n \geq n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

and we define *the star-multiple harmonic sum* $S_n(\mathbf{k})$ by

$$S_n(\mathbf{k}) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

These are rational numbers.

Lemma 4.2. *Let* $\mathbf{k}$ *be an index and* n *a positive integer. Then*

$$S_n(\mathbf{k}) = \sum_{\mathbf{l} \preceq \mathbf{k}} H_n(\mathbf{l}), \tag{5}$$

$$H_n(\mathbf{k}) = \sum_{\mathbf{l} \preceq \mathbf{k}} (-1)^{\mathrm{dep}(\mathbf{k}) - \mathrm{dep}(\mathbf{l})} S_n(\mathbf{l}). \tag{6}$$

Proof. The equality (5) is clear by definition. The equality (6) is obtained by the Möbius inversion formula for compositions (See [15]). □

4.2 The ring of integers modulo infinitely large primes and the p-adic number ring

Zagier [20] proposed to define the finite analogue of multiple zeta value as an element of *the ring of integers modulo infinitely large primes*

$$\mathcal{A} := \left(\prod_p \mathbb{F}_p \right) \Big/ \left(\bigoplus_p \mathbb{F}_p \right) = \left(\prod_p \mathbb{F}_p \right) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where p runs over all prime numbers, that is, $\zeta_{\mathcal{A}}(\mathbf{k}) := ((H_{p-1}(\mathbf{k}) \bmod p)_p) \in \mathcal{A}$ for an index $\mathbf{k}$. This ring was defined by Kontsevich in a different context ([23, 2.2 Infinitely large prime]). Rosen generalized this ring as follows in [41]:

Definition 4.3. Let n be a positive integer. Then we define a $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \left(\prod_p \mathbb{Z}/p^n\mathbb{Z}\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\right),$$

where p runs over all prime numbers.

Definition 4.4 (The $\boldsymbol{p}$-adic number ring)**.** A system of rings $\{\mathcal{A}_n\}$ becomes a projective system by natural projections and we define $\widehat{\mathcal{A}}$ to be the projective limit $\varprojlim_n \mathcal{A}_n$. We equip $\mathcal{A}_n$ with the discrete topology for each n and $\widehat{\mathcal{A}}$ with the projective limit topology.

The topological $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}$ is not locally compact. There exist natural projections $\pi\colon \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \twoheadrightarrow \widehat{\mathcal{A}}$ and $\pi_n\colon \widehat{\mathcal{A}} \twoheadrightarrow \mathcal{A}_n$ for any n, where $\mathbb{Z}_p$ is the p-adic integer ring.

Definition 4.5. We define *the infinitely large prime* $\boldsymbol{p}$ to be $\pi((p)_p) \in \widehat{\mathcal{A}}$.

For any positive integer n, π_n induces an isomorphism $\widehat{\mathcal{A}}/\boldsymbol{p}^n\widehat{\mathcal{A}} \simeq \mathcal{A}_n$ and the topology of $\widehat{\mathcal{A}}$ coincides with the $\boldsymbol{p}$-adic topology. So $\widehat{\mathcal{A}}$ is complete with respect to the $\boldsymbol{p}$-adic topology (See Lemma 9.4).

Definition 4.6 ($\boldsymbol{p}$-notation rule)**.** We assume that an element a_p of $\mathbb{Z}_p$ is given for all but finitely many prime number p. For the exceptional p's, we put $a_p = 0$. Then we denote $\pi((a_p)_p) \in \widehat{\mathcal{A}}$ by $a_{\boldsymbol{p}}$. By abuse of notation, we often use the same notation $a_{\boldsymbol{p}}$ for $\pi_n(a_{\boldsymbol{p}}) \in \mathcal{A}_n$.

We recall the definitions of *the Seki-Bernoulli numbers* and *the Fermat quotient.*

Definition 4.7. The numbers $B_1, B_2, \ldots$ are defined by the generating function

$$\frac{te^t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

We call these numbers *the Seki-Bernoulli numbers.* We define *the modified Seki-Bernoulli number* $\widehat{B}_k$ to be B_k/k for each k.

Definition 4.8. Let p be a prime number and a an element of $\mathbb{Z}_{(p)}^{\times}$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p. Then we define *the Fermat quotient* $q_p(a)$ by

$$q_p(a) := \frac{a^{p-1}-1}{p}.$$

$q_p(a)$ is also an element of $\mathbb{Z}_{(p)}$ by Fermat's little theorem.

Example 4.9. The following two elements are often used.

1. Let k be an integer greater than or equal to 2. If p is a prime number greater than $k+1$, the $(p-k)$-th Seki-Bernoulli number B_{p-k} is an element of $\mathbb{Z}_{(p)}$ by the von Staudt–Clausen theorem. Hence, $B_{\boldsymbol{p}-k}$ is well-defined by the $\boldsymbol{p}$-notation rule. If k is even, then $B_{\boldsymbol{p}-k} = 0$. We also use the notation $\widehat{B}_{\boldsymbol{p}-k}$.

2. Let a be a non-zero rational number. Then $q_{\boldsymbol{p}}(a)$ is defined by the $\boldsymbol{p}$-notation rule. We also denote $\pi_1(q_{\boldsymbol{p}}(a))$ by $\log_{\mathcal{A}}(a)$ in $\mathcal{A}$. Then $\log_{\mathcal{A}}(ab) = \log_{\mathcal{A}}(a) + \log_{\mathcal{A}}(b)$ holds for any non-zero rational number a and b.

Conjecture 4.10. *Let k be an odd integer greater than 1. Then $B_{\boldsymbol{p}-k}$ is non-zero in $\mathcal{A}$.*

Of course, if there exist infinitely many *regular primes*, the above conjecture is true. On the other hand, the following theorem is known:

Theorem 4.11 (Silverman [47])**.** *Let a be a non-zero rational number satisfying $a \neq \pm 1$. We assume that the abc-conjecture is true. Then $\log_{\mathcal{A}}(a)$ is non-zero.*

4.3 Definition of finite multiple zeta values

Definition 4.12. Let $\mathbf{k}$ be an index. Then we define the two kinds of *the $\widehat{\mathcal{A}}$-finite multiple zeta value* which are elements of $\widehat{\mathcal{A}}$ as follows:

$$\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}) := \pi((H_{p-1}(\mathbf{k}))_p) \quad (\textit{$\widehat{\mathcal{A}}$-finite multiple zeta value ($\widehat{\mathcal{A}}$-FMZV)}),$$

$$\zeta^{\star}_{\widehat{\mathcal{A}}}(\mathbf{k}) := \pi((S_{p-1}(\mathbf{k}))_p) \quad (\textit{$\widehat{\mathcal{A}}$-finite multiple zeta-star value ($\widehat{\mathcal{A}}$-FMZSV)}).$$

For a positive integer n and $\bullet \in \{\emptyset, \star\}$, we define *the $\mathcal{A}_n$-finite multiple zeta(-star) value* ($\mathcal{A}_n$-*FMZ(S)V*) $\zeta^{\bullet}_{\mathcal{A}_n}(\mathbf{k})$ as an element of $\mathcal{A}_n$ by

$$\zeta^{\bullet}_{\mathcal{A}_n}(\mathbf{k}) := \pi_n(\zeta^{\bullet}_{\widehat{\mathcal{A}}}(\mathbf{k})).$$

This definition is well-defined since $H_{p-1}(\mathbf{k})$ and $S_{p-1}(\mathbf{k})$ are elements of $\mathbb{Z}_{(p)}$ for each prime number. We denote $\zeta^{\bullet}_{\mathcal{A}_1}(\mathbf{k})$ by $\zeta^{\bullet}_{\mathcal{A}}(\mathbf{k})$ for $\bullet \in \{\emptyset, \star\}$ and call it *the $\mathcal{A}$-finite multiple zeta(-star) value* ($\mathcal{A}$-*FMZ(S)V*).

5 Relations among $\mathcal{A}$-finite multiple zeta values

5.1 Hoffman's fundamental relations for $\mathcal{A}$-FMZVs

Proposition 5.1 ([15, Theorem 4.4])**.** *Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index and $\bullet \in \{\emptyset, \star\}$. Then we have*

$$\sum_{\sigma \in \mathfrak{S}_r} \zeta^{\bullet}_{\mathcal{A}}(\sigma(\mathbf{k})) = 0. \tag{7}$$

Here, $\mathfrak{S}_r$ denotes the r-th symmetric group and $\sigma(\mathbf{k}) := (k_{\sigma(1)}, \dots, k_{\sigma(r)})$.

Proposition 5.2 (Reversal relation [15, Theorem 4.5])**.** *Let $\mathbf{k}$ be an index and $\bullet \in \{\emptyset, \star\}$. Then*

$$\zeta^{\bullet}_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})} \zeta^{\bullet}_{\mathcal{A}}(\overline{\mathbf{k}}).$$

Theorem 5.3 (Hoffman's duality formula [15, Theorem 4.6])**.** *Let $\mathbf{k}$ be an index and $\mathbf{k}^{\vee}$ its Hoffman dual. Then*

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = -\zeta^{\star}_{\mathcal{A}}(\mathbf{k}^{\vee}). \tag{8}$$

We give some comments for Hoffman's duality formula since one of the main themes of this thesis is to generalize this formula. Compared with the duality formula for multiple zeta values (= Theorem 2.5), I am very curious that the duality formula for finite multiple zeta values holds for *the star case* not for *the non-star case*. According to [14], Hoffman first observed that the equality

$$\zeta_{\mathcal{A}}(k_1, \{1\}^{k_2-1}) = \zeta_{\mathcal{A}}(k_2, \{1\}^{k_1-1}) \tag{9}$$

holds for positive integers k_1 and k_2. This seems to be a counterpart of the height 1 duality

$$\zeta(k_1+1, \{1\}^{k_2-1}) = \zeta(k_2+1, \{1\}^{k_1-1})$$

and he defined the corresponding dual $\overline{\mathbf{k}}^{\vee}$ of each index $\mathbf{k}$ from the equality (9). Unfortunately, the expected formula

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}(\overline{\mathbf{k}}^{\vee})$$

seems not to be true in general. He failed to find the true duality for five years. We should have seen the equality (9) as the following equivalent formula:

$$\zeta^{\star}_{\mathcal{A}}(k_1, \{1\}^{k_2-1}) = -\zeta^{\star}_{\mathcal{A}}(\{1\}^{k_1-1}, k_2).$$

The equivalence can be proved by Proposition 5.2, Proposition 5.5, and the equality (17) below. The true duality holds for FMZSVs! He proved his duality formula prior to September 2000.

Hoffman's original proof is based on Hoffman's identity (= Theorem 8.4). His proof of the identity is an induction on the weight of the index. We can also prove the identity by comparing the coefficients of powers of z in the equality (4). The proof presented here is essentially due to Yamamoto [58]:

Proof. Let p be a prime number. By the equality (3), we have

$$\begin{aligned}\zeta^{\star}_{p-1}(\mathbf{k}) = \sum_{n=1}^{p-1} s_{\mathbf{k}}(n) &= \sum_{n=1}^{p-1} \int_{\Delta(\mathbf{k})} t_1^{n-1} \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k) \\ &= \int_{\Delta(\mathbf{k})} \frac{1-t_1^{p-1}}{1-t_1} \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k).\end{aligned}$$

We put

$$\delta^*(j) = \begin{cases} 0 & j-1 \notin A(\mathbf{k}^\vee) \cup \{0\} \\ 1 & j-1 \in A(\mathbf{k}^\vee) \cup \{0\} \end{cases}.$$

Then by a change of variables $(t_1, \ldots, t_k) \mapsto (1-t_1, \ldots, 1-t_k)$, we have

$$\zeta^\star_{p-1}(\mathbf{k}) = \int_{\Delta(\mathbf{k}^\vee)} \frac{1-(1-t_1)^{p-1}}{1-(1-t_1)} \omega_{\delta^*(2)}(t_2) \cdots \omega_{\delta^*(k)}(t_k).$$

Here, we note that the following congruence holds:

$$\frac{1-(1-t_1)^{p-1}}{1-(1-t_1)} = \sum_{j=1}^{p-1} (-1)^{j-1} \binom{p-1}{j} t_1^{j-1} \equiv -\sum_{j=1}^{p-1} t_1^{j-1} = -\frac{1-t_1^{p-1}}{1-t_1} \pmod{p}.$$

Therefore, we have

$$\zeta^\star_{p-1}(\mathbf{k}) \equiv -\int_{\Delta(\mathbf{k}^\vee)} \frac{1-t_1^{p-1}}{1-t_1} \omega_{\delta^*(2)}(t_2) \cdots \omega_{\delta^*(k)}(t_k) = -\zeta^\star_{p-1}(\mathbf{k}^\vee) \pmod{p}.$$

This completes the proof of Hoffman's duality formula. □

Remark 5.4. If we use the $\boldsymbol{p}$ notation rule, the above proof symbolically can be written as

$$\begin{aligned} \zeta^\star_{\mathcal{A}}(\mathbf{k}) &= \int_{\Delta(\mathbf{k})} (1-t_1^{\boldsymbol{p}-1}) \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k)}(t_k) \\ &= -\int_{\Delta(\mathbf{k}^\vee)} (1-t_1^{\boldsymbol{p}-1}) \omega_{\delta^*(1)}(t_1) \cdots \omega_{\delta(k)^*}(t_k) = -\zeta^\star_{\mathcal{A}}(\mathbf{k}^\vee) \end{aligned}$$

by a change of variables $(t_1, \ldots, t_k) \mapsto (1-t_1, \ldots, 1-t_k)$. Compare with the proof of Theorem 2.5.

Proposition 5.5 (Relation between $\mathcal{A}$-FMZVs and $\mathcal{A}$-FMZSVs [15, Theorem 3.1], [16, Proposition 6], [42, Proposition 2.9])**.** *Let* $\mathbf{k}$ *be an index and* $r = \mathrm{dep}(\mathbf{k})$. *Then*

$$\sum_{j=0}^{r} (-1)^r \zeta_{\mathcal{A}}(\mathbf{k}_{(j)}) \zeta^\star_{\mathcal{A}}(\overline{\mathbf{k}^{(j)}}) = 0.$$

5.2 ψ-duality

Definition 5.6. We define a $\mathbb{Q}$-linear mapping $Z_{\mathcal{A}}\colon \mathfrak{H}^1 \to \mathcal{A}$ by $Z_{\mathcal{A}}(1) := 1$ and $Z_{\mathcal{A}}(z_{\mathbf{k}}) := \zeta_{\mathcal{A}}(\mathbf{k})$ for each index $\mathbf{k}$. We also define $Z^{\star}_{\mathcal{A}}\colon \mathfrak{H}^1 \to \mathcal{A}$ to be a composition $Z_{\mathcal{A}} \circ S$.

Definition 5.7. We define an algebra automorphism $\tau\colon \mathfrak{H} \to \mathfrak{H}$ by $x \mapsto y$ and $y \mapsto x$ and define a $\mathbb{Q}$-linear mapping $T\colon \mathfrak{H}^1 \to \mathfrak{H}^1$ by $T(1) := 1$ and $T(wy) := \tau(w)y$ for any $w \in \mathfrak{H}$. We also define an algebra automorphism $\psi\colon \mathfrak{H} \to \mathfrak{H}$ by $x \mapsto x+y$ and $y \mapsto -y$.

Using these terminology, Hoffman's duality formula can be rewritten as follows:

$$Z^{\star}_{\mathcal{A}}(w) = Z^{\star}_{\mathcal{A}}(T(w)) \tag{10}$$

for any $w \in \mathfrak{H}^1$.

Lemma 5.8. *We see that $STS^{-1} = -\psi$ holds on $\mathfrak{H}^1$.*

Proof. It is sufficient to calculate images of both sides at x and y. □

By this lemma, we see that Hoffman's duality is equivalent to the following relation for $\mathcal{A}$-FMZVs:

Theorem 5.9 (ψ-duality [15, Theorem 4.7])**.** *For any $w \in \mathfrak{H}^1$, we have*

$$Z_{\mathcal{A}}(w) = Z_{\mathcal{A}}(\psi(w)).$$

Remark 5.10. We can regard the above relation as a duality *for an element of* $\mathfrak{H}^1$ since $\psi^2 = \mathrm{id}$. But this is not a duality *for an index*. In fact, we can rewrite the ψ-duality as

$$\zeta_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{dep}(\mathbf{k})} \sum_{\mathbf{k} \preceq \mathbf{l}} \zeta_{\mathcal{A}}(\mathbf{l})$$

for an index $\mathbf{k}$. This is the reason why we call the relation (8) the *true duality* for finite multiple zeta values.

5.3 Recent results and some conjectures

Theorem 5.11 (Saito–Wakabayashi [43]). *Let k, r, and i be positive integers satisfying $1 \le i \le r \le k$. Then the following two relations hold*:

$$\sum_{\substack{\mathbf{k} \textit{ s.t. } \mathbf{k}^{(i-1)}:\textit{ adm.} \\ \mathrm{wt}(\mathbf{k})=k,\ \mathrm{dep}(\mathbf{k})=r}} \zeta_{\mathcal{A}}(\mathbf{k}) = (-1)^{i-1}\left\{\binom{k-1}{i-1} + (-1)^r \binom{k-1}{r-i}\right\}\frac{B_{\boldsymbol{p}-k}}{k},$$

$$\sum_{\substack{\mathbf{k} \textit{ s.t. } \mathbf{k}^{(i-1)}:\textit{ adm.} \\ \mathrm{wt}(\mathbf{k})=k,\ \mathrm{dep}(\mathbf{k})=r}} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = (-1)^{i-1}\left\{(-1)^r\binom{k-1}{i-1} + \binom{k-1}{r-i}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.$$

Theorem 5.12 (Oyama [36]). *Let l be a non-negative integer and $\mathbf{k}$ an index. Then*

$$\sum_{\substack{\mathbf{e}\in\mathbb{Z}_{\ge 0}^{\mathrm{dep}(\mathbf{k})} \\ \mathrm{wt}(\mathbf{e})=l}} \zeta_{\mathcal{A}}(\mathbf{k}\oplus\mathbf{e}) = \sum_{\substack{\mathbf{e}'\in\mathbb{Z}_{\ge 0}^{\mathrm{dep}(\mathbf{k}^{\vee})} \\ \mathrm{wt}(\mathbf{e}')=l}} \zeta_{\mathcal{A}}((\mathbf{k}^{\vee}\oplus\mathbf{e}')^{\vee}).$$

Theorem 5.13 (Murahara [30]). *Let n be a positive integer. Then*

$$Z_{\mathcal{A}}(\partial_n(w)) = -Z_{\mathcal{A}}((x+y)^{n-1}yw)$$

holds for any $w \in \mathfrak{H}^1$.

Similarly to the case of MZVs, FMZVs satisfy *the harmonic product formula* since multiple harmonic sums also satisfy the same formula. On the other hand, $Z_{\mathcal{A}}\colon (\mathfrak{H}^1, ш) \to \mathcal{A}$ is not a homomorphism as algebra.

Theorem 5.14 (Shuffle relation [19], [33, Corollary 4.1]). *Let $\mathbf{k}_1$ and $\mathbf{k}_2$ be indices. Then*

$$\zeta_{\mathcal{A}}(\mathbf{k}_1 ш\, \mathbf{k}_2) = (-1)^{\mathrm{wt}(\mathbf{k}_1)}\zeta_{\mathcal{A}}(\overline{\mathbf{k}_1}, \mathbf{k}_2), \tag{11}$$

where $\zeta_{\mathcal{A}}(\mathbf{k}_1 ш\, \mathbf{k}_2)$ is defined to be $Z_{\mathcal{A}}(z_{\mathbf{k}_1} ш\, z_{\mathbf{k}_2})$.

Conjecture 5.15 (Kaneko [19]). *Let k and s be positive integers satisfying $k \geq 2s$. Then*

$$\sum_{\substack{\mathbf{k}\colon \text{ admissible} \\ \mathrm{wt}(\mathbf{k})=k, \mathrm{ht}(\mathbf{k})=s}} (-1)^{\mathrm{dep}(\mathbf{k})} \zeta_{\mathcal{A}}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\frac{B_{\boldsymbol{p}-k}}{k}.$$

Conjecture 5.16 (Kaneko [19]). *Let k and s be positive integers satisfying $k \geq 2s$. Then*

$$\sum_{\substack{\mathbf{k}\colon \text{ admissible} \\ \mathrm{wt}(\mathbf{k})=k, \mathrm{ht}(\mathbf{k})=s}} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}) = 2\binom{k-1}{2s-1}(1-2^{1-k})\frac{B_{\boldsymbol{p}-k}}{k}.$$

Theorem 5.11, Theorem 5.12, Theorem 5.13, Theorem 5.14, and Conjecture 5.16 are finite analogues of Theorem 2.7, Theorem 2.9, Theorem 2.15, a part of Proposition 2.13, and Theorem 2.8, respectively.

5.4 Interpolated finite multiple zeta values

In this subsection, we discuss *the interpolated finite multiple zeta values* inspired by Yamamoto's t-MZVs.

Definition 5.17. Let $\mathbf{k}$ be an index and t an indeterminate. Then we define *the t-finite multiple zeta value (t-FMZV) $\zeta^t_{\mathcal{A}}(\mathbf{k})$* as an element of $\mathcal{A}[t]$ by

$$\zeta^t_{\mathcal{A}}(\mathbf{k}) := \sum_{\mathbf{l} \preceq \mathbf{k}} t^{\mathrm{dep}(\mathbf{k})-\mathrm{dep}(\mathbf{l})} \zeta_{\mathcal{A}}(\mathbf{l}).$$

Definition 5.18. We define a $\mathbb{Q}$-linear mapping $Z^t_{\mathcal{A}}\colon \mathfrak{H}^1[t] \to \mathcal{A}[t]$ as a composition $Z^t_{\mathcal{A}} := Z^t_{\mathcal{A}} \circ S^t$. Here we naturally extend $Z_{\mathcal{A}}$ to a mapping $\mathfrak{H}^1[t] \to \mathcal{A}[t]$.

Here, we observe that the following two interpolated relations for t-FMZVs hold.

Proposition 5.19 (Relation between t-FMZVs and $(1-t)$-FMZVs). *Let $\mathbf{k}$ be an index. Then*

$$\sum_{j=0}^{\mathrm{dep}(\mathbf{k})} (-1)^j \zeta^t_{\mathcal{A}}(\mathbf{k}_{(j)}) \zeta^{1-t}_{\mathcal{A}}(\overline{\mathbf{k}^{(j)}}) = 0.$$

Proof. This is an immediate consequence of Proposition 2.20. □

Proposition 5.20 (Sum formula for t-FMZVs)**.** *Let k and r be positive integers satisfying $r<k$. Then we have the following relation*:

$$\sum_{\substack{\mathbf{k}\colon \textit{admissible}\\ \mathrm{wt}(\mathbf{k})=k,\mathrm{dep}(\mathbf{k})=r}} \zeta_{\mathcal{A}}^{t}(\mathbf{k}) = \left(\sum_{j=0}^{r-1}\left\{\binom{k-1}{j}+(-1)^r\binom{k-1}{r-1-j}\right\}t^j(1-t)^{r-1-j}\right)\frac{B_{\boldsymbol{p}-k}}{k}.$$

Proof. Let b_k be an element of $\mathfrak{H}^1$ such that $Z_{\mathcal{A}}(b_k)=B_{\boldsymbol{p}-k}/k$. Let $N_{\mathcal{A},k}^{\mathrm{SF}}$ be a $\mathbb{Q}$-submodule of $\mathfrak{H}^1[t]$ generated by $\{S^t(x_{k,r})-P_{k,r}(t)b_k-(-1)^rP_{k,r}(1-t)b_k \mid r<k\}$. Then we can easily see that $N_{\mathcal{A},k}^{\mathrm{SF}}$ is a differential submodule by Lemma 2.23. Therefore, it is sufficient to show that $(N_{\mathcal{A},k}^{\mathrm{SF}})_0$ is contained in $\mathrm{Ker}(Z_{\mathcal{A}})$ by Lemma 2.22. This follows from the sum formula for $\mathcal{A}$-FMZVs (Theorem 5.11). □

Question 5.21. *What is an interpolated relation of Saito–Wakabayashi's sum formula* (*Theorem* 5.11) *of the case $i=2,\dots,r$?*

6 $\boldsymbol{p}$-adic relations among finite multiple zeta values

We call a relation among $\widehat{\mathcal{A}}$-FMZVs *a $\boldsymbol{p}$-adic relation.* We review Rosen's $\boldsymbol{p}$-adic relations and give some new $\boldsymbol{p}$-adic relations.

6.1 Some $\boldsymbol{p}$-adic relations

Proposition 6.1 (Rosen [40, Proposition 2.1])**.**

$$\sum_{i=1}^{\infty}(-1)^i\zeta_{\widehat{\mathcal{A}}}(\{1\}^i)\boldsymbol{p}^{i-1}=0. \tag{12}$$

Proposition 6.2 ($\boldsymbol{p}$-adic reversal relation [41, Theorem 4.1])**.** *Let $\mathbf{k}=(k_1,\dots,k_r)$ be an index and $\bullet\in\{\emptyset,\star\}$. Then*

$$\zeta_{\widehat{\mathcal{A}}}^{\bullet}(\overline{\mathbf{k}})=(-1)^{\mathrm{wt}(\mathbf{k})}\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}_{\geq 0}^r\\ l_1+\cdots+l_r=i}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\zeta_{\widehat{\mathcal{A}}}^{\bullet}(\mathbf{k}\oplus\mathbf{l})\boldsymbol{p}^i. \tag{13}$$

Proof. This is a corollary of Theorem 6.4, Theorem 6.12, and Theorem 12.1. $\square$

Proposition 6.3 (Relation between $\widehat{\mathcal{A}}$-FMZVs and $\widehat{\mathcal{A}}$-FMZSVs)**.** *Let* $\mathbf{k}$ *be an index. Then*

$$\sum_{j=0}^{\mathrm{dep}(\mathbf{k})}(-1)^j\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}_{(j)})\zeta^{\star}_{\widehat{\mathcal{A}}}(\overline{\mathbf{k}^{(j)}})=0.$$

Proof. This is a corollary of Theorem 12.11 below. $\square$

Theorem 6.4 (***p***-adic shuffle relation)**.** *Let* $\mathbf{k}_1=(k_1,\dots,k_r)$ *and* $\mathbf{k}_2$ *are indices. Then*

$$\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}_1 ш\, \mathbf{k}_2)=(-1)^{\mathrm{wt}(\mathbf{k}_1)}\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}^r_{\geq 0}\\ l_1+\cdots+l_r=i}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\zeta_{\widehat{\mathcal{A}}}(\overline{\mathbf{k}_1\oplus\mathbf{l}},\mathbf{k}_2)\boldsymbol{p}^i.$$

Here, we define $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}_1 ш\, \mathbf{k}_2)$ *to be* $\sum a_i\zeta_{\widehat{\mathcal{A}}}(\mathbf{l}_i)$ *when* $z_{\mathbf{k}_1} ш\, z_{\mathbf{k}_2}=\sum a_i z_{\mathbf{l}_i}$.

Proof. Let $\mathbf{k}_2=(k'_1,\dots,k'_s)$ and p a prime number. We use the notation $C(f(z);z^i)$ as the coefficient of z^i in a power series $f(z)$. For a positive integer n and an index $\mathbf{k}$, we have $H_n(\mathbf{k})=\sum_{i=1}^{n}C(\mathrm{Li}_{\mathbf{k}}(z);z^i)$. Since one-variable multiple polylogarithms satisfy the shuffle product formula and a p-adically convergent identity

$$\frac{1}{(p-n)^j}=(-1)^j\sum_{l=0}^{\infty}\binom{j+l-1}{l}\frac{p^l}{n^{j+l}} \tag{14}$$

holds for positive integers $n < p$ and j, we see that a p-component of $\zeta_{\widehat{\mathcal{A}}}(\mathbf{k}_1 ш\, \mathbf{k}_2)$ is equal to

$$\sum_{i=1}^{p-1} C(\mathrm{Li}_{\mathbf{k}_1}(z)\,\mathrm{Li}_{\mathbf{k}_2}(z); z^i) = \sum_{\substack{p-1\geq n, m\geq 1\\ p-1\geq n+m}} C(\mathrm{Li}_{\mathbf{k}_1}(z); z^n) C(\mathrm{Li}_{\mathbf{k}_2}(z); z^m)$$

$$= \sum_{\substack{p-1\geq n, m\geq 1\\ p-1\geq n+m}} \left(\sum_{n>n_2>\cdots>n_r\geq 1} \frac{1}{n^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \right) \left(\sum_{m>m_2>\cdots>m_s\geq 1} \frac{1}{m^{k'_1} m_2^{k'_2} \cdots m_s^{k'_s}} \right)$$

$$= \sum_{\substack{p-1\geq p-n, m\geq 1\\ p-1\geq (p-n)+m}} \left(\sum_{p-n>p-n_2>\cdots>p-n_r\geq 1} \frac{1}{(p-n)^{k_1} (p-n_2)^{k_2} \cdots (p-n_r)^{k_r}} \right)$$

$$\times \left(\sum_{m>m_2>\cdots>m_s\geq 1} \frac{1}{m^{k'_1} m_2^{k'_2} \cdots m_s^{k'_s}} \right)$$

$$= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{p-1\geq n_r>\cdots>n_2>n>m>m_2>\cdots>m_s\geq 1}$$

$$\times \sum_{(l_1,\ldots,l_r)\in\mathbb{Z}_{\geq 0}^r} \left[\prod_{r=1}^{r} \binom{k_j+l_j-1}{l_j} \right] \frac{1}{n_r^{k_r+l_r} \cdots n_2^{k_2+l_2} n^{k_1+l_1} m^{k'_1} m_2^{k'_2} \cdots m_s^{k'_s}} p^{l_1+\cdots+l_r}$$

$$= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\ldots,l_r)\\ l_1+\cdots+l_r=i}} \left[\prod_{r=1}^{r} \binom{k_j+l_j-1}{l_j} \right] H_{p-1}(\overline{\mathbf{k}_1 \oplus \mathbf{l}}, \mathbf{k}_2) p^i.$$

This completes the proof. □

Proposition 6.2, Proposition 6.3, and Theorem 6.4 are $\boldsymbol{p}$-adic generalizations of Proposition 5.2, Proposition 5.5, and Theorem 5.14, respectively.

6.2 Rosen's asymptotic duality theorem

Let $\widehat{\mathfrak{H}}^1$ be the completion of $\mathfrak{H}^1$. Namely, $\widehat{\mathfrak{H}}$ is defined as the non-commutative formal power series ring $\mathbb{Q}\langle\langle x, y\rangle\rangle$ and $\widehat{\mathfrak{H}}^1 := \mathbb{Q} + \widehat{\mathfrak{H}}y$. Then each element of $\widehat{\mathfrak{H}}^1$ is written as $\sum_{\mathbf{k}:\ \mathrm{index}} a_{\mathbf{k}} z_{\mathbf{k}}$, where $a_{\mathbf{k}}$ is a rational number.

Definition 6.5. We define *the weighted finite multiple zeta function* $Z_{\widehat{\mathcal{A}}}\colon \widehat{\mathfrak{H}}^1 \to \widehat{\mathcal{A}}$ by

$$\sum_{\mathbf{k}} a_{\mathbf{k}} z_{\mathbf{k}} \mapsto \sum_{\mathbf{k}} a_{\mathbf{k}} \zeta_{\widehat{\mathcal{A}}}(\mathbf{k}) \boldsymbol{p}^{\mathrm{wt}(\mathbf{k})}.$$

Since $\mathbb{Q}$ acts on $\boldsymbol{p}^i\widehat{\mathcal{A}}$ for each i, the above definition is well-defined. The algebra automorphism ψ on $\mathfrak{H}^1$ (resp. the harmonic product $*$) is extended continuously to the mapping on $\widehat{\mathfrak{H}}^1$ (resp. $\widehat{\mathfrak{H}}^1 \times \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$), respectively and we define a continuous algebra automorphism $\Phi\colon \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ by

$$w \mapsto (1+y)\left(\frac{1}{1+y} * w\right).$$

Then Rosen generalized Theorem 5.9 as follows:

Theorem 6.6 (Asymptotic duality theorem [41, Theorem 4.5])**.** *For any $w \in \widehat{\mathfrak{H}}^1$, we have*

$$Z_{\widehat{\mathcal{A}}}(\psi(w)) = Z_{\widehat{\mathcal{A}}}(\Phi(w)).$$

6.3 The $\boldsymbol{p}$-adic duality theorem for $\widehat{\mathcal{A}}$-FMZSVs

In this subsection, we investigate a $\boldsymbol{p}$-adic generalization of Hoffman's duality formula for $\mathcal{A}$-FMZVs. Zhao proved a $\mathcal{A}_2$-generalization of Hoffman's duality formula:

Theorem 6.7 (Zhao [60, Theorem 2.11])**.** *Let $\mathbf{k}$ be an index. Then the following $\mathcal{A}_2$-relation holds*:

$$-\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}^{\vee}) = \zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}) + \sum_{\mathbf{l} \preceq \mathbf{k}} \zeta_{\mathcal{A}_2}(1, \mathbf{l})\boldsymbol{p}.$$

We easily see that the above Zhao's relation can be rewritten as the following symmetric relation:

$$\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}) + \zeta^{\star}_{\mathcal{A}_2}(1, \mathbf{k})\boldsymbol{p} = -\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}^{\vee}) - \zeta^{\star}_{\mathcal{A}_2}(1, \mathbf{k}^{\vee})\boldsymbol{p}. \tag{15}$$

The simplest case of the duality (15) leads to a proof of Wolstenholme's theorem. This is essentially the same as Wolstenholme's original proof.

Theorem 6.8 (Wolstenholme's theorem [56])**.**

$$\zeta_{\mathcal{A}_2}(1) = 0.$$

Proof. We can reduce the equality "$\zeta_{\mathcal{A}_2}(1) = 0$" to the equality "$\zeta_{\mathcal{A}}(2) = 0$" as follows and to deduce "$\zeta_{\mathcal{A}}(2) = 0$" is quite easy. By the duality (15), we have

$$\zeta_{\mathcal{A}_2}(1) + \zeta^{\star}_{\mathcal{A}_2}(1,1)\boldsymbol{p} = 0.$$

Hence, the equality "$\zeta_{\mathcal{A}_2}(1) = 0$" is equivalent to the equality "$\zeta^{\star}_{\mathcal{A}}(1,1) = 0$". Furthermore, by Hoffman's duality formula (8), we have

$$\zeta_{\mathcal{A}}(2) = -\zeta^{\star}_{\mathcal{A}}(1,1).$$

This means that "$\zeta^{\star}_{\mathcal{A}}(1,1) = 0$" is equivalent to the equality "$\zeta_{\mathcal{A}}(2) = 0$". We are done. □

We generalize the duality (15) to a $\boldsymbol{p}$-adic duality:

Theorem 6.9. *Let* $\mathbf{k}$ *be an index. Then we have*

$$\sum_{i=0}^{\infty} \zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k})\boldsymbol{p}^i = -\sum_{i=0}^{\infty} \zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i, \mathbf{k}^{\vee})\boldsymbol{p}^i \tag{16}$$

in the ring $\widehat{\mathcal{A}}$.

Proof. This is a corollary of the functional equation for $\widehat{\mathcal{A}}$-finite shuffle star-multiple polylogarithms (Theorem 12.2) which is one of our main results in this thesis. □

We can naturally extend S and T to mappings on $\widehat{\mathfrak{H}}^1$. We define $Z^{\star}_{\widehat{\mathcal{A}}}\colon \widehat{\mathfrak{H}}^1 \to \widehat{\mathcal{A}}$ by $Z^{\star}_{\widehat{\mathcal{A}}} := Z_{\widehat{\mathcal{A}}} \circ S$. Then we can rewrite the $\widehat{\mathcal{A}}$-duality as

$$Z^{\star}_{\widehat{\mathcal{A}}}\left(\frac{1}{1-y}w\right) = Z^{\star}_{\widehat{\mathcal{A}}}\left(\frac{1}{1-y}T(w)\right).$$

for any $w \in \widehat{\mathfrak{H}}^1$.

6.4 The $\boldsymbol{p}$-adic shuffle relation for $\widehat{\mathcal{A}}$-FMZSVs.

Muneta [29] defined *the star-harmonic product* and *the star-shuffle product.* He established *the double shuffle relation* for multiple zeta star-values. Here, we only recall the star-shuffle product.

Definition 6.10 (Muneta [29])**.** We define *the star-shuffle product* $\bar{ш}$ on $\mathfrak{H}$ by the $\mathbb{Q}$-bilinearity and the following rules:

1. $w\bar{ш}1 = 1\bar{ш}w = 1$ for any $w \in \mathfrak{H}$,
2. $u_1w_1\bar{ш}u_2w_2 = u_1(w_1\bar{ш}u_2w_2) + u_2(u_1w_1\bar{ш}w_2) - \delta(w_1)\tau(u_1)u_2w_2 - \delta(w_2)\tau(u_2)u_1w_1$ for any $w_1, w_2 \in \mathfrak{H}$ and $u_1, u_2 \in \{x, y\}$.

For a word $w \in \mathfrak{H}^1$, the $\mathbb{Q}$-linear mapping δ is defined by

$$\delta(w) = \begin{cases} 1 & (w = 1) \\ 0 & (w \neq 1) \end{cases}.$$

Proposition 6.11 (Muneta [29, Proposition 2.6 and Proposition 2.7])**.** *The star-shuffle product is commutative and associative. Furthermore,*

$$S(w_1\bar{ш}w_2) = S(w_1)шS(w_2)$$

holds for any $w_1, w_2 \in \mathfrak{H}^1$.

A star-analogue of the shuffle relation (Theorem 5.14) is

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k}_1\bar{ш}\mathbf{k}_2) = (-1)^{\mathrm{wt}(\mathbf{k}_1)}\left(\zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}_1}, \mathbf{k}_2) - \zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}_1} \uplus \mathbf{k}_2)\right).$$

Moreover, the following $\boldsymbol{p}$-adic relation holds:

Theorem 6.12 ($\boldsymbol{p}$-adic star-shuffle relation)**.** *Let* $\mathbf{k}_1 = (k_1, \ldots, k_r)$ *and* $\mathbf{k}_2$ *are indices. Then*

$$\zeta^{\star}_{\widehat{\mathcal{A}}}(\mathbf{k}_1\bar{ш}\mathbf{k}_2)$$
$$= (-1)^{\mathrm{wt}(\mathbf{k}_1)}\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\ldots,l_r)\in\mathbb{Z}^r_{\geq 0}\\ l_1+\cdots+l_r=i}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\left(\zeta^{\star}_{\widehat{\mathcal{A}}}(\overline{\mathbf{k}_1 \oplus \mathbf{l}}, \mathbf{k}_2) - \zeta^{\star}_{\widehat{\mathcal{A}}}((\overline{\mathbf{k}_1 \oplus \mathbf{l}}) \uplus \mathbf{k}_2)\right)\boldsymbol{p}^i.$$

Here, we define $\zeta^\star_{\widehat{\mathcal{A}}}(\mathbf{k}_1 \overline{ш} \mathbf{k}_2)$ *to be* $\sum a_i \zeta^\star_{\widehat{\mathcal{A}}}(\mathbf{l}_i)$ *when* $z_{\mathbf{k}_1} \overline{ш} z_{\mathbf{k}_2} = \sum a_i z_{\mathbf{l}_i}$.

Lemma 6.13. *Let* $\mathbf{k}_1$ *and* $\mathbf{k}_2$ *be indices. Then the equality*

$$\sum_{\substack{\mathbf{l}_1 \preceq \mathbf{k}_1 \\ \mathbf{l}_2 \preceq \mathbf{k}_2}} z_{\mathbf{l}_1} z_{\mathbf{l}_2} = \sum_{\mathbf{m}_1 \preceq (\mathbf{k}_1, \mathbf{k}_2)} z_{\mathbf{m}_1} - \sum_{\mathbf{m}_2 \preceq \mathbf{k}_1 \uplus \mathbf{k}_2} z_{\mathbf{m}_2}$$

holds in $\mathfrak{H}^1$.

Proof. This is clear by the notations for indices. □

Lemma 6.14 (A variant of Vandermonde's identity)**.** *Let* m*,* i*, and* r *be positive integers. Let* $k_1, k_2, \ldots, k_r$ *be positive integers satisfying* $k_1 + \cdots + k_r = m$*. Then we have*

$$\binom{m+i-1}{i} = \sum_{\substack{(l_1,\ldots,l_r) \in \mathbb{Z}^r_{\geq 0} \\ l_1 + \cdots + l_r = i}} \prod_{j=1}^{r} \binom{k_j + l_j - 1}{l_j}.$$

Proof. This is obtained by the generating function

$$\frac{1}{(1-t)^m} = \sum_{i=0}^{\infty} \binom{m+i-1}{i} t^i$$

and the identity

$$\frac{1}{(1-t)^m} = \prod_{j=1}^{r} \frac{1}{(1-t)^{k_j}}.$$ □

Lemma 6.15. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index and* i *a non-negative integer. Then*

$$\sum_{\mathbf{m}=(m_1,\ldots,m_s) \preceq \mathbf{k}} \sum_{\substack{\mathbf{a}=(a_1,\ldots,a_s) \in \mathbb{Z}^s_{\geq 0} \\ a_1 + \cdots + a_s = i}} \left[\prod_{j=1}^{s} \binom{m_j + a_j - 1}{a_j} \right] z_{\mathbf{m} \oplus \mathbf{a}}$$

$$= \sum_{\substack{\mathbf{l}=(l_1,\ldots,l_r) \in \mathbb{Z}^r_{\geq 0} \\ l_1 + \cdots + l_r = i}} \left[\prod_{j=1}^{r} \binom{k_j + l_j - 1}{l_j} \right] \sum_{\mathbf{b} \preceq \mathbf{k} \oplus \mathbf{l}} z_{\mathbf{b}}$$

holds in $\mathfrak{H}^1$.

Proof. Let $\mathbf{l} \in \mathbb{Z}_{\geq 0}^r$ be a tuple of non-negative integers. Then there exists a natural bijection from the set of contraction indices of $\mathbf{k}$ to the set of contraction indices of $\mathbf{k} \oplus \mathbf{l}$. For a contraction index $\mathbf{b}$ of $\mathbf{k} \oplus \mathbf{l}$ and the corresponding contraction index $\mathbf{m}$ of $\mathbf{k}$, there exists a tuple of non-negative integers $\mathbf{a}_{\mathbf{m},\mathbf{l}} \preceq \mathbf{l}$ uniquely determined by $\mathbf{m}$ and $\mathbf{l}$ such that $\mathbf{b} = \mathbf{m} \oplus \mathbf{a}_{\mathbf{m},\mathbf{l}}$. Hence, we have

$$\begin{aligned}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^r \binom{k_j+l_j-1}{l_j}\right] \sum_{\mathbf{b}\preceq\mathbf{k}\oplus\mathbf{l}} z_{\mathbf{b}} &= \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^r \binom{k_j+l_j-1}{l_j}\right] \sum_{\mathbf{m}\preceq\mathbf{k}} z_{\mathbf{m}\oplus\mathbf{a}_{\mathbf{m},\mathbf{l}}} \\ &= \sum_{\mathbf{m}\preceq\mathbf{k}} \sum_{\substack{\mathbf{a}\in\mathbb{Z}_{\geq 0}^{\mathrm{dep}(\mathbf{m})} \\ \mathrm{wt}(\mathbf{a})=i}} \sum_{\mathbf{l}=(l_1,\dots,l_r)\succeq\mathbf{a}} \left[\prod_{j=1}^r \binom{k_j+l_j-1}{l_j}\right] z_{\mathbf{m}\oplus\mathbf{a}}.\end{aligned}$$

For an index $\mathbf{m} = (m_1, \dots, m_s) \preceq \mathbf{k}$ and a tuple of non-negative integers $\mathbf{a} = (a_1, \dots, a_s)$, the equality

$$\sum_{\mathbf{l}=(l_1,\dots,l_r)\succeq\mathbf{a}} \left[\prod_{j=1}^r \binom{k_j+l_j-1}{l_j}\right] = \prod_{j=1}^s \binom{m_j+a_j-1}{a_j}$$

holds by Lemma 6.14. Therefore, we have the conclusion. $\square$

Proof of Theorem 6.12. By Proposition 6.11, we have

$$\begin{aligned}Z_{\widehat{\mathcal{A}}}^{\star}(z_{\mathbf{k}_1}\overline{ш}z_{\mathbf{k}_2}) &= Z_{\widehat{\mathcal{A}}}(S(z_{\mathbf{k}_1}\overline{ш}z_{\mathbf{k}_2})) = Z_{\widehat{\mathcal{A}}}(S(z_{\mathbf{k}_1})шS(z_{\mathbf{k}_2})) \\ &= Z_{\widehat{\mathcal{A}}}\left(\left(\sum_{\mathbf{l}_1\preceq\mathbf{k}_1} z_{\mathbf{l}_1}\right)ш\left(\sum_{\mathbf{l}_2\preceq\mathbf{k}_2} z_{\mathbf{l}_2}\right)\right) = \sum_{\substack{\mathbf{l}_1\preceq\mathbf{k}_1 \\ \mathbf{l}_2\preceq\mathbf{k}_2}} Z_{\widehat{\mathcal{A}}}(z_{\mathbf{l}_1}шz_{\mathbf{l}_2}).\end{aligned}$$

For a contraction index $\mathbf{l}_2$ of $\mathbf{k}_2$, by Theorem 6.4, we have

$$\begin{aligned}\sum_{\mathbf{l}_1\preceq\mathbf{k}_1} Z_{\widehat{\mathcal{A}}}(z_{\mathbf{l}_1}шz_{\mathbf{l}_2}) &= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{\mathbf{l}_1=(m_1,\dots,m_s)\preceq\mathbf{k}_1} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{a}=(a_1,\dots,a_s)\in\mathbb{Z}_{\geq 0}^s \\ a_1+\cdots+a_s=i}} \left[\prod_{j=1}^s \binom{m_j+a_j-1}{a_j}\right] Z_{\widehat{\mathcal{A}}}(z_{\overline{\mathbf{l}_1\oplus\mathbf{a}}}z_{\mathbf{l}_2}) \\ &= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\mathbf{l}_1=(m_1,\dots,m_s)\preceq\mathbf{k}_1} \sum_{\substack{\mathbf{a}=(a_1,\dots,a_s)\in\mathbb{Z}_{\geq 0}^s \\ a_1+\cdots+a_s=i}} \left[\prod_{j=1}^s \binom{m_j+a_j-1}{a_j}\right] Z_{\widehat{\mathcal{A}}}(z_{\overline{\mathbf{l}_1\oplus\mathbf{a}}}z_{\mathbf{l}_2})\end{aligned}$$

and by Lemma 6.15, we have

$$\sum_{\mathbf{l}_1 \preceq \mathbf{k}_1} Z_{\widehat{\mathcal{A}}}(z_{\mathbf{l}_1} \text{ш} z_{\mathbf{l_2}}) = (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r) \in \mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^{r} \binom{k_j+l_j-1}{l_j} \right] \sum_{\mathbf{l}'_1 \preceq \overline{\mathbf{k}_1 \oplus \mathbf{l}}} Z_{\widehat{\mathcal{A}}}(z_{\mathbf{l}'_1} z_{\mathbf{l}_2}).$$

Therefore, by Lemma 6.13, we have

$$\begin{aligned}
&Z^{\star}_{\widehat{\mathcal{A}}}(z_{\mathbf{k}_1} \overline{\text{ш}} z_{\mathbf{k}_2}) \\
&= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r) \in \mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^{r} \binom{k_j+l_j-1}{l_j} \right] \sum_{\substack{\mathbf{l}'_1 \preceq \overline{\mathbf{k}_1 \oplus \mathbf{l}} \\ \mathbf{l}_2 \preceq \mathbf{k}_2}} Z_{\widehat{\mathcal{A}}}(z_{\mathbf{l}'_1} z_{\mathbf{l}_2}) \\
&= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r) \in \mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^{r} \binom{k_j+l_j-1}{l_j} \right] \\
&\quad \times Z_{\widehat{\mathcal{A}}} \left(\left(\sum_{\mathbf{m}_1 \preceq (\overline{\mathbf{k}_1 \oplus \mathbf{l}}, \mathbf{k}_2)} z_{\mathbf{m}_1} \right) - \left(\sum_{\mathbf{m}_2 \preceq (\overline{\mathbf{k}_1 \oplus \mathbf{l}}) \uplus \mathbf{k}_2} z_{\mathbf{m}_2} \right) \right) \\
&= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r) \in \mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^{r} \binom{k_j+l_j-1}{l_j} \right] Z_{\widehat{\mathcal{A}}}(S(z_{(\overline{\mathbf{k}_1 \oplus \mathbf{l}}, \mathbf{k}_2)}) - S(z_{(\overline{\mathbf{k}_1 \oplus \mathbf{l}}) \uplus \mathbf{k}_2})) \\
&= (-1)^{\mathrm{wt}(\mathbf{k}_1)} \sum_{i=0}^{\infty} \sum_{\substack{\mathbf{l}=(l_1,\dots,l_r) \in \mathbb{Z}_{\geq 0}^r \\ l_1+\cdots+l_r=i}} \left[\prod_{j=1}^{r} \binom{k_j+l_j-1}{l_j} \right] \left(Z^{\star}_{\widehat{\mathcal{A}}}(z_{(\overline{\mathbf{k}_1 \oplus \mathbf{l}}, \mathbf{k}_2)}) - Z^{\star}_{\widehat{\mathcal{A}}}(z_{(\overline{\mathbf{k}_1 \oplus \mathbf{l}}) \uplus \mathbf{k}_2}) \right).
\end{aligned}$$

This completes the proof. □

6.5 Lifting Conjecture

We equip $\widehat{\mathfrak{H}}^1$ a $\mathbb{Q}$-algebra structure by the harmonic product $*$. Then $Z_{\widehat{\mathcal{A}}}$ becomes an algebra homomorphism. For each n, we define an algebra homomorphism $Z_n \colon \widehat{\mathfrak{H}}^1 \to \mathcal{A}_n$ by a composition $Z_n := \pi_n \circ Z_{\widehat{\mathcal{A}}}$. We define an ideal $\mathbb{I}_n$ of $\widehat{\mathfrak{H}}^1$ by

$$\mathbb{I}_n := \left\{ \sum_{\mathbf{k}} a_{\mathbf{k}} z_{\mathbf{k}} \in \widehat{\mathfrak{H}}^1 \,\middle|\, a_{\mathbf{k}} = 0, \text{if } \mathrm{wt}(\mathbf{k}) < n \right\}.$$

Conjecture 6.16 (Lifting Conjecture [41, Conjecture A]). *Let n be a positive integer. Then the following equality of ideals holds*:

$$\mathrm{Ker}(Z_n) = \mathrm{Ker}(Z_{\widehat{\mathcal{A}}}) + \mathbb{I}_n.$$

This conjecture implies that every relation for $\mathcal{A}$-FMZVs lifts to a $\boldsymbol{p}$-adic relation for $\widehat{\mathcal{A}}$-FMZVs.

Question 6.17. *We have generalized some relations for $\mathcal{A}$-FMZVs to $\boldsymbol{p}$-adic relations such as the reversal relation, the duality formula, and the shuffle relation. What are $\boldsymbol{p}$-adic liftings of other relations for $\mathcal{A}$-FMZVs, for example relations given in Subsection 5.3?*

7 Explicit evaluations of finite multiple zeta values

Proposition 7.1 (Zhou–Cai [62]). *Let k and r be positive integers and $\bullet \in \{\emptyset, \star\}$. Then*

$$\zeta^{\bullet}_{\mathcal{A}}(\{k\}^r) = 0, \tag{17}$$

$$\zeta_{\mathcal{A}_2}(\{k\}^r) = (-1)^{r-1} k \frac{B_{\boldsymbol{p}-rk-1}}{rk+1}\boldsymbol{p}, \tag{18}$$

$$\zeta^{\star}_{\mathcal{A}_2}(\{k\}^r) = k \frac{B_{\boldsymbol{p}-rk-1}}{rk+1}\boldsymbol{p}. \tag{19}$$

If rk is odd, then

$$\zeta_{\mathcal{A}_3}(\{k\}^r) = (-1)^r \frac{k(rk+1)}{2}\frac{B_{\boldsymbol{p}-rk-2}}{rk+2}\boldsymbol{p}^2, \tag{20}$$

$$\zeta^{\star}_{\mathcal{A}_3}(\{k\}^r) = -\frac{k(rk+1)}{2}\frac{B_{\boldsymbol{p}-rk-2}}{rk+2}\boldsymbol{p}^2. \tag{21}$$

Proposition 7.2 (Z. H. Sun [48, Theorem 5.1] and [48, Remark 5.1]). *Let k be a positive integer. Then*

$$\zeta_{\mathcal{A}_3}(k) = \begin{cases} \binom{k+1}{2}\widehat{B}_{\boldsymbol{p}-k-2}\boldsymbol{p}^2 & \text{if } k \text{ is odd,} \\ k\left(\widehat{B}_{2\boldsymbol{p}-k-2} - 2\widehat{B}_{\boldsymbol{p}-k-1}\right)\boldsymbol{p} & \text{if } k \text{ is even,} \end{cases} \tag{22}$$

$$\zeta_{\mathcal{A}_4}(k) = \begin{cases} -\binom{k+1}{2}\left(\widehat{B}_{2\boldsymbol{p}-k-3} - 2\widehat{B}_{\boldsymbol{p}-k-2}\right)\boldsymbol{p}^2 & \text{if } k \text{ is odd,} \\ -k\left(\widehat{B}_{3\boldsymbol{p}-k-3} - 3\widehat{B}_{2\boldsymbol{p}-k-2} + 3\widehat{B}_{\boldsymbol{p}-k-1}\right)\boldsymbol{p} - \binom{k+2}{3}\widehat{B}_{\boldsymbol{p}-k-3}\boldsymbol{p}^3 & \text{if } k \text{ is even.} \end{cases} \tag{23}$$

Theorem 7.3 (Tauraso [53, Theorem 2.1])**.**

$$\zeta_{\mathcal{A}_5}(1) = \left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right)\boldsymbol{p}^2 + \widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^4. \tag{24}$$

Proposition 7.4 (Hoffman [15, Theorem 6.1] and J. Zhao [60, Theorem 3.1, 3.2])**.** *Let k_1 and k_2 be positive integers and $\bullet \in \{\emptyset, \star\}$. Then*

$$\zeta^{\bullet}_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}. \tag{25}$$

If $k := k_1 + k_2$ is even, then

$$\zeta_{\mathcal{A}_2}(k_1, k_2) = \frac{1}{2}\left\{(-1)^{k_2}k_1\binom{k+1}{k_2} - (-1)^{k_1}k_2\binom{k+1}{k_1} - k\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} \tag{26}$$

and

$$\zeta^{\star}_{\mathcal{A}_2}(k_1, k_2) = \frac{1}{2}\left\{(-1)^{k_2}k_1\binom{k+1}{k_2} - (-1)^{k_1}k_2\binom{k+1}{k_1} + k\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} \tag{27}$$

hold.

Proposition 7.5 (Hoffman [15, Theorem 6.2] and J. Zhao [60, Theorem 3.5])**.** *Let k_1, k_2, and k_3 be positive integers. If $k := k_1 + k_2 + k_3$ is odd, then*

$$\zeta_{\mathcal{A}}(k_1, k_2, k_3) = \frac{1}{2}\left\{(-1)^{k_3}\binom{k}{k_3} - (-1)^{k_1}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k} \tag{28}$$

and

$$\zeta^{\star}_{\mathcal{A}}(k_1, k_2, k_3) = \frac{1}{2}\left\{(-1)^{k_1}\binom{k}{k_1} - (-1)^{k_3}\binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k} \tag{29}$$

hold.

Tauraso determined $\zeta_{\mathcal{A}_2}(1,2)$ and $\zeta_{\mathcal{A}_2}(2,1)$ ([53, Theorem 2.3]). Here, we calculate such values in $\mathcal{A}_3$:

Theorem 7.6.

$$\zeta_{\mathcal{A}_3}(1,2) = 3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right) - \frac{1}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2, \tag{30}$$

$$\zeta_{\mathcal{A}_3}(2,1) = -3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right) - \frac{11}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2, \tag{31}$$

$$\zeta^\star_{\mathcal{A}_3}(1,2) = 3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right) + \frac{11}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2, \tag{32}$$

$$\zeta^\star_{\mathcal{A}_3}(2,1) = -3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right) + \frac{1}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2. \tag{33}$$

Proof. By Theorem 6.9, we have

$$\begin{gathered}\zeta_{\mathcal{A}_5}(1) + \zeta^\star_{\mathcal{A}_5}(1,1)\boldsymbol{p} + \zeta^\star_{\mathcal{A}_5}(1,1,1)\boldsymbol{p}^2 + \zeta^\star_{\mathcal{A}_5}(1,1,1,1)\boldsymbol{p}^3 + \zeta^\star_{\mathcal{A}_5}(1,1,1,1,1)\boldsymbol{p}^4 = 0,\\ \zeta^\star_{\mathcal{A}_4}(1,1) + \zeta^\star_{\mathcal{A}_4}(1,1,1)\boldsymbol{p} + \zeta^\star_{\mathcal{A}_4}(1,1,1,1)\boldsymbol{p}^2 + \zeta^\star_{\mathcal{A}_4}(1,1,1,1,1)\boldsymbol{p}^3\\ = -\zeta_{\mathcal{A}_4}(2) - \zeta^\star_{\mathcal{A}_4}(1,2)\boldsymbol{p} - \zeta^\star_{\mathcal{A}_4}(1,1,2)\boldsymbol{p}^2 - \zeta^\star_{\mathcal{A}_4}(1,1,1,2)\boldsymbol{p}^3,\end{gathered}$$

and

$$-\zeta^\star_{\mathcal{A}_2}(1,1,2) - \zeta^\star_{\mathcal{A}_2}(1,1,1,2)\boldsymbol{p} = \zeta^\star_{\mathcal{A}_2}(3,1) + \zeta^\star_{\mathcal{A}_2}(1,3,1)\boldsymbol{p} = \frac{1}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}.$$

By combining these and the equalities (23), (24), we have

$$\zeta^\star_{\mathcal{A}_5}(1,2)\boldsymbol{p}^2 = \zeta_{\mathcal{A}_5}(1) - \zeta_{\mathcal{A}_4}(2)\boldsymbol{p} + \frac{1}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^3 = 3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right)\boldsymbol{p}^2 + \frac{11}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^4.$$

Dividing both sides by $\boldsymbol{p}^2$ gives the equality (32). By the duality, we have

$$\zeta^\star_{\mathcal{A}_3}(2,1) + \zeta^\star_{\mathcal{A}_3}(1,2,1)\boldsymbol{p} + \zeta^\star_{\mathcal{A}_3}(1,1,2,1)\boldsymbol{p}^2 = -\zeta^\star_{\mathcal{A}_3}(1,2) - \zeta^\star_{\mathcal{A}_3}(1,1,2)\boldsymbol{p} - \zeta^\star_{\mathcal{A}_3}(1,1,1,2)\boldsymbol{p}^2$$

and

$$\zeta^\star_{\mathcal{A}_2}(1,2,1) + \zeta^\star_{\mathcal{A}_2}(1,1,2,1)\boldsymbol{p} = -\zeta^\star_{\mathcal{A}_2}(2,2) - \zeta^\star_{\mathcal{A}_2}(1,2,2)\boldsymbol{p} = -\frac{11}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}.$$

Hence,

$$\zeta^\star_{\mathcal{A}_3}(2,1) = -\zeta^\star_{\mathcal{A}_3}(1,2) + 6\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2 = -3\left(\widehat{B}_{3\boldsymbol{p}-5} - 3\widehat{B}_{2\boldsymbol{p}-4} + 3\widehat{B}_{\boldsymbol{p}-3}\right) + \frac{1}{2}\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2.$$

The equalities (30) and (31) are obtained by

$$\zeta_{\mathcal{A}_3}(1,2) = \zeta^\star_{\mathcal{A}_3}(1,2) - \zeta_{\mathcal{A}_3}(3), \quad \zeta_{\mathcal{A}_3}(2,1) = \zeta^\star_{\mathcal{A}_3}(2,1) - \zeta_{\mathcal{A}_3}(3), \quad \zeta_{\mathcal{A}_3}(3) = 6\widehat{B}_{\boldsymbol{p}-5}\boldsymbol{p}^2. \quad \square$$

Remark 7.7. This is also obtained by Rosen's duality (Theorem 6.6). See [41, Paragraph 1.3.1 (1.3) and Paragraph 5.2.1].

Theorem 7.8 (Pilehrood–Pilehrood–Tauraso [37, Theorem 4.3])**.** *Let k_1 and k_2 be positive integers. Then*

$$\zeta_{\mathcal{A}}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = (-1)^{k_1-1}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}, \tag{34}$$

$$\zeta^\star_{\mathcal{A}}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = (-1)^{k_1-1}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}. \tag{35}$$

If k_1+k_2 is even, then

$$\zeta_{\mathcal{A}_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2}\left\{1 - (-1)^{k_1}\binom{k_1+k_2+1}{k_1+1}\right\}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}, \tag{36}$$

$$\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \frac{1}{2}\left\{1 - (-1)^{k_2}\binom{k_1+k_2+1}{k_2+1}\right\}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}. \tag{37}$$

Kh. Hessami Pilehrood, T. Hessami Pilehrood, and Tauraso calculated the value $\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})$ by using their new identity ([37, Theorem 2.2]). Here, we give another proof based on the duality formula.

Proof. By Hoffman's duality and the equality (25), we have

$$\zeta^\star_{\mathcal{A}}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = -\zeta^\star_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1-1}\binom{k_1+k_2}{k_1}\frac{B_{\boldsymbol{p}-k_1-k_2}}{k_1+k_2}.$$

This proves the equality (35). By Proposition 5.5, the equality (17), and Proposition 5.2, we have

$$(-1)^{k_1+k_2}\zeta_{\mathcal{A}}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}) = \zeta^\star_{\mathcal{A}}(\{1\}^{k_2-1}, 2, \{1\}^{k_1-1}) = (-1)^{k_1+k_2}\zeta^\star_{\mathcal{A}}(\{1\}^{k_1-1}, 2, \{1\}^{k_2-1}).$$

Therefore we obtain the equality (34).

Form now on, we assume that k_1+k_2 is even. By $\mathcal{A}_2$-duality (15) and the equalities (27), (29), (35), we can calculate as follows:

$$\begin{aligned}
&\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_1-1},2,\{1\}^{k_2-1})\\
&=-\zeta^\star_{\mathcal{A}_2}(k_1,k_2)-\zeta^\star_{\mathcal{A}_2}(1,k_1,k_2)\boldsymbol{p}-\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_1},2,\{1\}^{k_2-1})\boldsymbol{p}\\
&=-\frac{1}{2}\left\{(-1)^{k_2}k_1\binom{k_1+k_2+1}{k_2}-(-1)^{k_1}k_2\binom{k_1+k_2+1}{k_1}+k_1+k_2\right\}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}\\
&\quad-\frac{1}{2}\left\{-(k_1+k_2+1)-(-1)^{k_2}\binom{k_1+k_2+1}{k_2}\right\}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}\\
&\quad-(-1)^{k_1}\binom{k_1+k_2+1}{k_1+1}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}\\
&=\frac{1}{2}\left\{1-(-1)^{k_2}\binom{k_1+k_2+1}{k_2+1}\right\}\frac{B_{\boldsymbol{p}-k_1-k_2-1}}{k_1+k_2+1}\boldsymbol{p}.
\end{aligned}$$

By Proposition 6.3, we have

$$\begin{aligned}
&\zeta_{\mathcal{A}_2}(\{1\}^{k_1-1},2,\{1\}^{k_2-1})\\
&=\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-1},2,\{1\}^{k_1-1})+\sum_{j=1}^{k_1-1}(-1)^j\zeta_{\mathcal{A}_2}(\{1\}^j)\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-1},2,\{1\}^{k_1-1-j})\\
&\quad+\sum_{i=1}^{k_2-1}(-1)^{k_1-1+i}\zeta_{\mathcal{A}_2}(\{1\}^{k_1-1},2,\{1\}^{i-1})\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-i}).
\end{aligned}$$

If j is odd, $\zeta_{\mathcal{A}_2}(\{1\}^j)=0$. If j is even, $\zeta_{\mathcal{A}_2}(\{1\}^j)$ and $\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-1},2,\{1\}^{k_1-1-j})$ in $\boldsymbol{p}\mathcal{A}_2$ and hence $\zeta_{\mathcal{A}_2}(\{1\}^j)\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-1},2,\{1\}^{k_1-1-j})=0$ by the equality (18) and the equality (35). Similarly, $\zeta_{\mathcal{A}_2}(\{1\}^{k_1-1},2,\{1\}^{i-1})\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-i})=0$ for $i=1,\dots,k_2-1$. Therefore we have

$$\zeta_{\mathcal{A}_2}(\{1\}^{k_1-1},2,\{1\}^{k_2-1})=\zeta^\star_{\mathcal{A}_2}(\{1\}^{k_2-1},2,\{1\}^{k_1-1})$$

and the equality (36) is obtained by the equality (37). This completes the proof. □

Theorem 7.9 (Pilehrood–Pilehrood–Tauraso [37, Theorem 4.2]). *Let k_1 and k_2 be positive integers. Then*

$$\zeta_{\mathcal{A}}(\{2\}^{k_1-1},1,\{2\}^{k_2-1}) = (-1)^{k_1+k_2-1}\frac{(k_1-k_2)(4^{k_1+k_2-2}-1)}{4^{k_1+k_2-3}(2k_1-1)(2k_2-1)}\binom{2k_1+2k_2-4}{2k_1-2}B_{\boldsymbol{p}-2k_1-2k_2+3}, \tag{38}$$

$$\zeta^{\star}_{\mathcal{A}}(\{2\}^{k_1-1},1,\{2\}^{k_2-1}) = \frac{(k_1-k_2)(4^{k_1+k_2-2}-1)}{4^{k_1+k_2-3}(2k_1-1)(2k_2-1)}\binom{2k_1+2k_2-4}{2k_1-2}B_{\boldsymbol{p}-2k_1-2k_2+3}. \tag{39}$$

Theorem 7.10 (Pilehrood–Pilehrood–Tauraso [37, Theorem 4.1]). *Let k_1 and k_2 be positive integers. Then*

$$\zeta_{\mathcal{A}}(\{2\}^{k_1-1},3,\{2\}^{k_2-1}) = (-1)^{k_1+k_2-1}\frac{k_1-k_2}{k_1k_2}\binom{2k_1+2k_2-2}{2k_1-1}B_{\boldsymbol{p}-2k_1-2k_2-1}, \tag{40}$$

$$\zeta^{\star}_{\mathcal{A}}(\{2\}^{k_1-1},3,\{2\}^{k_2-1}) = \frac{k_1-k_2}{k_1k_2}\binom{2k_1+2k_2-2}{2k_1-1}B_{\boldsymbol{p}-2k_1-2k_2-1}. \tag{41}$$

Part III
Finite Multiple Polylogarithms

This part is the main part in this thesis.

8 Generalizations of Euler's identity

Through this section, let R be a commutative ring including the field of rational numbers and N a positive integer. The following identity with binomial coefficients is due to Euler:

Theorem 8.1 (Euler [10, §13]).

$$\sum_{n=1}^{N}(-1)^{n-1}\binom{N}{n}\frac{1}{n} = \sum_{n=1}^{N}\frac{1}{n}. \tag{42}$$

There are various generalizations of this identity as follows.

Theorem 8.2 (Dilcher's identity [6, Corollary 3]). *Let r be a positive integer. Then*

$$\sum_{n=1}^{N}(-1)^{n-1}\binom{N}{n}\frac{1}{n^r}=\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}\frac{1}{n_1\cdots n_r}.$$

Theorem 8.3 (Hernández' identity [2]). *Let r be a positive integer. Then*

$$\sum_{n=1}^{N}\frac{1}{n^r}=\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}(-1)^{n_1-1}\binom{N}{n_1}\frac{1}{n_1\cdots n_r}.$$

These two identities are special cases of the following identity due to Hoffman and we see that Hernández' identity is the dual of Dilcher's identity.

Theorem 8.4 (Hoffman's identity [15, Theorem 4.2]). *Let $\mathbf{k}=(k_1,\ldots,k_r)$ be an index and $\mathbf{k}^\vee=(k_1',\ldots,k_s')$ its Hoffman dual. Then*

$$\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}(-1)^{n_1-1}\binom{N}{n_1}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}=\sum_{N\geq n_1\geq\cdots\geq n_s\geq 1}\frac{1}{n_1^{k_1'}\cdots n_s^{k_s'}}.$$

The following theorem is a generalization of Dilcher's identity and Hernández' identity to a one-variable polynomial case:

Theorem 8.5 (Tauraso–Zhao's identities [54, Lemma 5.5 and 5.6]. cf. [2, Lossers' solution]). *Let r be a positive integer. Then we have the following polynomial identities:*

$$\sum_{n=1}^{N}(-1)^{n_1}\binom{N}{n}\frac{t^n}{n^r}=\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}\frac{(1-t)^{n_r}-1}{n_1\cdots n_r},$$

$$\sum_{n=1}^{N}\frac{t^n}{n^r}=\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}(-1)^{n_1}\binom{N}{n_1}\frac{(1-t)^{n_r}-1}{n_1\cdots n_r}.$$

In this section, we further generalize Euler's identity to obtain main results (Theorem 12.2 and Theorem 12.11).

Remark 8.6. In [6], Dilcher gave a q-analogue version of Dilcher's identity. A q-analogue version of Hernández' identity was proved by Prodinger [38].

8.1 Generalizations of Euler's identity

Theorem 8.7. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index of weight* k. *Then the following polynomial identities hold in* $R[t_1, \ldots, t_r]$:

$$\sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{r-1}^{n_{r-1}-n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{r-1})^{n_{l_{r-1}}-n_{l_{r-1}+1}} \{(1-t_r)^{n_{l_r}} - 1\}}{n_1 \cdots n_k} \tag{43}$$

and

$$\sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{t_1^{n_1-n_2} \cdots t_{r-1}^{n_{r-1}-n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} = \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{r-1})^{n_{l_{r-1}}-n_{l_{r-1}+1}} \{(1-t_r)^{n_{l_r}} - 1\}}{n_1 \cdots n_k}, \tag{44}$$

where $l_1 = k_1, l_2 = k_1 + k_2, \ldots, l_r = k_1 + \cdots + k_r (= k)$.

Corollary 8.8. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index and* $\mathbf{k}^\vee = (k'_1, \ldots, k'_s)$ *its Hoffman dual. Then we have polynomial identities*

$$\sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{t^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} = \sum_{N \geq n_1 \geq \cdots \geq n_s \geq 1} \frac{(1-t)^{n_s} - 1}{n_1^{k'_1} \cdots n_s^{k'_s}},$$

$$\sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{t^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} = \sum_{N \geq n_1 \geq \cdots \geq n_s \geq 1} (-1)^{n_1} \binom{N}{n_1} \frac{(1-t)^{n_s} - 1}{n_1^{k'_1} \cdots n_s^{k'_s}},$$

in $R[t]$. *By Hoffman's identity and* $t \mapsto 1-t$, *we see that these two are equivalent.*

Proof. See "Proof that Theorem 12.4 implies Theorem 12.2". □

Remark 8.9. Corollary 8.8 and hence Theorem 8.7 are clearly generalizations of Hoffman's identity and Tauraso–Zhao's identities. Theorem 8.7 is also deduced from Kawashima–Tanaka's formula [21, Theorem 2.6], which is a generalization of the identity

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \frac{1}{n+1} = \frac{1}{N+1}.$$

The above equality is equivalent to Euler's identity (42). See [2, Woord's solution] for a proof of the equivalence. Our proof of Theorem 8.7 given in Subsection 8.3 is quite different from the proof by Kawashima–Tanaka. Their proof is based on a generalization of Theorem 3.2 and Theorem 3.3.

We give the following other generalizations of Euler's identity:

Theorem 8.10. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index of weight* k. *Then the following identities hold in* $R[t_1, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$:

$$\begin{aligned}
&\sum_{N \geq n_1 > \cdots > n_r \geq 1} (-1)^{n_r} \binom{N}{n_r} \frac{(t_1/t_2)^{n_1} \cdots (t_{r-1}/t_r)^{n_{r-1}} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} \\
&= (-1)^{r-1} \sum_{N \geq n_1 \geq \cdots \geq n_k \geq 1} \frac{(1-t_r)^{n_{l_1} - n_{l_1+1}} \cdots (1-t_2)^{n_{l_{r-1}} - n_{l_{r-1}+1}} \{(1-t_1)^{n_{l_r}} - 1\}}{n_1 \cdots n_k} \\
&\quad + \sum_{j=1}^{r-1} (-1)^{r-j-1} \left(\sum_{N \geq n_1 > \cdots > n_j \geq 1} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \times \left(\sum_{N \geq n_1 \geq \cdots \geq n_{l_{r-j}} \geq 1} \right. \\
&\quad \left. \frac{(1-t_r)^{n_{l_1} - n_{l_1+1}} \cdots (1-t_{j+2})^{n_{l_{r-j-1}} - n_{l_{r-j-1}+1}} \{(1-t_{j+1})^{n_{l_{r-j}}} - 1\}}{n_1 \cdots n_{l_{r-j}}} \right)
\end{aligned} \tag{45}$$

and

$$\sum_{N\geq n_1>\cdots>n_r\geq 1}\frac{(t_1/t_2)^{n_1}\cdots(t_{r-1}/t_r)^{n_{r-1}}t_m^{n_m}}{n_1^{k_1}\cdots n_r^{k_r}}$$

$$=(-1)^{r-1}\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}(-1)^{n_1}\binom{N}{n_1}$$

$$\times\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_2)^{n_{l_{r-1}}-n_{l_{r-1}+1}}\{(1-t_1)^{n_{l_r}}-1\}}{n_1\cdots n_k}$$

$$+\sum_{j=1}^{r-1}(-1)^{r-j-1}\left(\sum_{N\geq n_1>\cdots>n_j\geq 1}\frac{(t_1/t_2)^{n_1}\cdots(t_j/t_{j+1})^{n_j}}{n_1^{k_1}\cdots n_j^{k_j}}\right) \tag{46}$$

$$\times\left(\sum_{N\geq n_1\geq\cdots\geq n_{l_{r-j}}\geq 1}(-1)^{n_1}\binom{N}{n_1}\right.$$

$$\left.\times\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_{j+2})^{n_{l_{r-j-1}}-n_{l_{r-j-1}+1}}\{(1-t_{j+1})^{n_{l_{r-j}}}-1\}}{n_1\cdots n_{l_{r-j}}}\right),$$

where $l_1=k_r, l_2=k_r+k_{r-1},\ldots,l_r=k_r+\cdots+k_1(=k)$.

By combining above results and eliminating the binomial coefficients, we have the following polynomial identity:

Theorem 8.11. *Let* $\mathbf{k}=(k_1,\ldots,k_r)$ *be an index. Then the following identity holds in* $R[t_1,\ldots,t_r]$ *:*

$$(-1)^{r-1}\sum_{N\geq n_1>\cdots>n_r\geq 1}\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}=\sum_{N\geq n_1\geq\cdots\geq n_r\geq 1}\frac{t_r^{n_1}\cdots t_1^{n_r}}{n_1^{k_r}\cdots n_r^{k_1}}$$

$$+\sum_{j=1}^{r-1}(-1)^j\left(\sum_{N\geq n_1>\cdots>n_j\geq 1}\frac{t_1^{n_1}\cdots t_j^{n_j}}{n_1^{k_1}\cdots n_j^{k_j}}\right)\left(\sum_{N\geq n_1\geq\cdots\geq n_{r-j}\geq 1}\frac{t_r^{n_1}\cdots t_{j+1}^{n_{r-j}}}{n_1^{k_r}\cdots n_{r-j}^{k_{j+1}}}\right).$$

Proof. By combining Theorem 8.7 (44) and Theorem 8.10 (46), we have

$$\begin{aligned}
&(-1)^{r-1} \sum_{N \geq n_1 > \cdots > n_r \geq 1} \frac{(t_1/t_2)^{n_1} \cdots (t_{r-1}/t_r)^{n_{r-1}} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}} \\
&= \sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{t_r^{n_1-n_2} \cdots t_2^{n_{r-1}-n_r} t_1^{n_r}}{n_1^{k_r} \cdots n_r^{k_1}} \\
&\quad + \sum_{j=1}^{r-1} (-1)^j \left(\sum_{N \geq n_1 > \cdots > n_j \geq 1} \frac{(t_1/t_2)^{n_1} \cdots (t_j/t_{j+1})^{n_j}}{n_1^{k_1} \cdots n_j^{k_j}} \right) \\
&\quad \times \left(\sum_{N \geq n_1 \geq \cdots \geq n_{r-j} \geq 1} \frac{t_r^{n_1-n_2} \cdots t_{j+2}^{n_{r-j-1}-n_{r-j}} t_{j+1}^{n_{r-j}}}{n_1^{k_r} \cdots n_{r-j}^{k_{j+1}}} \right)
\end{aligned}$$

in $R[t_1, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$. By replacing $t_1/t_2 \mapsto t_1, \ldots, t_{r-1}/t_r \mapsto t_{r-1}$, we obtain the desired identity. □

8.2 Truncated integral operators

In this subsection, we introduce *truncated integral operators* to prove the theorems in Subsection 8.1. Through this subsection, let t and s be indeterminates.

Let $\int * dt \colon R[\![t]\!] \to R[\![t]\!]$ be the formal indefinite integral operator satisfying the condition that the constant term with respect to t is equal to 0, that is,

$$\int \left(\sum_{n=0}^{\infty} a_n t^n \right) dt := \sum_{n=0}^{\infty} \frac{a_n}{n+1} t^{n+1}.$$

We prepare the following five R-linear operators:

(i) $I_{t,R} \colon tR[t] \longrightarrow tR[t], \quad f(t) \longmapsto \int \frac{f(t)}{t} dt.$

(ii) $I_{t,s;R} \colon R[t,s] \longrightarrow R[\![s/t]\!][t], \quad f(t,s) \longmapsto \int \frac{f(t,s)}{t-s} ds.$

(iii) $\tau_{t;R}^{\leq N} \colon R[\![t]\!] \longrightarrow R[t], \quad \sum_{n=0}^{\infty} a_n t^n \longmapsto \sum_{n=0}^{N} a_n t^n.$

(iv) $\mathrm{pr}_{t;R} \colon R(\!(t^{-1})\!) \longrightarrow R[t], \quad \sum_{n=-\infty}^{n_0} a_n t^n \longmapsto \sum_{n=0}^{n_0} a_n t^n.$

(v) $\mathrm{pr}^{-}_{t;R} \colon R(\!(t^{-1})\!) \longrightarrow t^{-1}R[\![t^{-1}]\!], \quad \sum_{n=n_0}^{\infty} a_n t^{-n} \longmapsto \sum_{n=1}^{\infty} a_n t^{-n}.$

For the definition of $\mathrm{pr}_{t,R}$, we define $\sum_{n=0}^{n_0} a_n t^n$ to be zero if n_0 is negative. We consider the formal integral operator in the definition of $I_{t,s;R}$ as an operator on $R(\!(t^{-1})\!)[\![s]\!]$. For instance, we have

$$I_{t,s;R}(s^n) = \sum_{j=1}^{\infty} \frac{s^{n+j}t^{-j}}{j} \tag{47}$$

for a non-negative integer n.

Definition 8.12. We define *the truncated integral operators* $J^{\star}_{t,s;R}$ and $J^{N}_{t,s;R}$ by

$$J^{\star}_{t,s;R} := \mathrm{pr}_{t;R[\![s]\!]} \circ I_{t,s;R} \colon R[t,s] \longrightarrow R[\![s]\!][t],$$
$$J^{N}_{t,s;R} := \tau^{\leq N}_{s;R[\![t^{-1}]\!]} \circ \mathrm{pr}^{-}_{t,R[\![s]\!]} \circ I_{t,s;R} \colon R[t,s] \longrightarrow t^{-1}R[\![t^{-1}]\!][s].$$

We can check easily that the image of $J^{\star}_{t,s;R}$ (resp. $J^{N}_{t,s;R}$) is included in $R[t,s]$ (resp. $t^{-1}R[t^{-1},s]$).

For simplicity, we omit the ring R from our notations. The following Lemma 8.13 and Lemma 8.16 are fundamental for the proofs of Theorem 8.7 and Theorem 8.10.

Lemma 8.13. *Let n be a positive integer. Then we have the following identities*:

$$I_t(t^n) = \frac{t^n}{n}, \tag{48}$$

$$I_t((1-t)^n - 1) = \sum_{j=1}^{n} \frac{(1-t)^j - 1}{j}, \tag{49}$$

$$J^{\star}_{t,s}(t^n) = \sum_{j=1}^{n} \frac{t^{n-j}s^j}{j}, \tag{50}$$

$$J^{\star}_{t,s}((1-t)^n - 1) = \sum_{j=1}^{n} \frac{(1-t)^{n-j}\{(1-s)^j - 1\}}{j}. \tag{51}$$

Proof. The equality (48) can be easily checked by definition. We show the equality (49). Set $T := 1 - t$. Then the left hand side of (49) equals to

$$-\int \frac{T^n - 1}{1 - T} dT = \int \sum_{j=0}^{n-1} T^j dT = \sum_{j=1}^{n} \frac{T^j - 1}{j}.$$

By the definition of T, we obtain the equality (49). Let us show the equality (50). By the equality (47), the following equalities hold:

$$J^{\star}_{t,s}(t^n) = \mathrm{pr}_{t,R[\![s]\!]}(t^n I_{t,s;R}(1)) = \mathrm{pr}_{t,R[\![s]\!]}\left(\sum_{j=1}^{\infty} \frac{s^j t^{n-j}}{j}\right) = \sum_{j=1}^{n} \frac{s^j t^{n-j}}{j}.$$

Finally, we show the equality (51). Note that the following equalities hold:

$$\begin{aligned}
\frac{(1-t)^n}{t-s} = -\frac{(1-t)^{n-1}}{1 - \dfrac{s-1}{t-1}} &= -(1-t)^{n-1}\left(\frac{1 - \left(\dfrac{s-1}{t-1}\right)^n}{1 - \dfrac{s-1}{t-1}} + \frac{\left(\dfrac{s-1}{t-1}\right)^n}{1 - \dfrac{s-1}{t-1}}\right) \\
&= -(1-t)^{n-1} \sum_{j=0}^{n-1} \left(\frac{s-1}{t-1}\right)^j + \frac{(1-s)^n}{t-s} \\
&= -\sum_{j=0}^{n-1} (1-s)^j (1-t)^{n-j-1} + \frac{(1-s)^n}{t-s}.
\end{aligned}$$

As $J^{\star}_{t,s}(f(s)) = \mathrm{pr}_{t,R[\![s]\!]}(\int \frac{f(s)ds}{t-s}) = 0$ for each $f(s) \in R[s]$ by the equality (47), we have

$$\begin{aligned}
J^{\star}_{t,s}((1-t)^n - 1) = J^{\star}_{t,s}((1-t)^n) &= \int \left(-\sum_{j=0}^{n-1} (1-s)^j (1-t)^{n-j-1}\right) ds \\
&= \sum_{j=1}^{n} \frac{(1-t)^{n-j}\{(1-s)^j - 1\}}{j}.
\end{aligned}$$

This completes the proof of the lemma. □

Before we give the lemma for $J^N_{t,s}$ (= Lemma 8.16), we prepare the following two auxiliary lemmas:

Lemma 8.14 (cf. [49, Proof of Lemma 4.1])**.** *We have the following polynomial identities in* $R[t]$:

$$\sum_{n=1}^{N}(-1)^n\binom{N}{n}\frac{t^n}{n}=\sum_{n=1}^{N}\frac{(1-t)^n-1}{n}, \tag{52}$$

$$\sum_{n=1}^{N}\frac{t^n}{n}=\sum_{n=1}^{N}(-1)^n\binom{N}{n}\frac{(1-t)^n-1}{n}. \tag{53}$$

Proof. By the binomial theorem, we have $(1-t)^N-1=\sum_{n=1}^{N}\binom{N}{n}(-t)^n$. Then by applying I_t on the both sides and using Lemma 8.13 (48) and (49), we obtain the identity (52). The identity (53) is obtained by the substitution $t\mapsto 1-t$ and the Euler's identity (42), which is a special case of the identity (52). $\square$

Lemma 8.15. *Let* j *and* n *be non-negative integers satisfying* $j\leq n$*. Then we have the following polynomial identity in* $R[t]$:

$$\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}t^k=\binom{n}{j}t^j(1+t)^{n-j}.$$

Proof. By the binomial theorem, we have

$$(t+s+ts)^n=\{t+(1+t)s\}^n=\sum_{j=0}^{n}\binom{n}{j}t^j(1+t)^{n-j}s^{n-j}.$$

On the other hand, we have

$$\begin{aligned}(t+s+ts)^n&=\{t(1+s)+s\}^n=\sum_{k=0}^{n}\binom{n}{k}t^k(1+s)^k s^{n-k}\\&=\sum_{k=0}^{n}\binom{n}{k}t^k\sum_{j=0}^{k}\binom{k}{j}s^{k-j}s^{n-k}=\sum_{j=0}^{n}\left\{\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j}t^k\right\}s^{n-j}.\end{aligned}$$

Compare the coefficient of s^{n-j}. $\square$

Lemma 8.16. *Let n be a positive integer. Then we have the following identities in $R[t^{\pm 1}, s]$:*

$$J_{t,s}^N(t^n) = \sum_{j=n+1}^{N} \frac{s^j t^{n-j}}{j}, \tag{54}$$

$$J_{t,s}^N((1-t)^n - 1) = -\sum_{j=1}^{n} \frac{(1-t)^{n-j}\{(1-s)^j - 1\}}{j} + \left(\sum_{j=1}^{N} \frac{(s/t)^j}{j}\right)\{(1-t)^n - 1\}. \tag{55}$$

Here, we understand the summation in the right hand side of the equality (54) *as* 0 *if* $n+1$ *is greater than* N.

Proof. The equality (54) is an immediate consequence of the equality (47). We show the equality (55). By the equality

$$I_{t,s;R}((1-t)^n) = (1-t)^n \sum_{j=1}^{\infty} \frac{s^j t^{-j}}{j} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\sum_{j=1}^{\infty} \frac{s^j t^{k-j}}{j}\right),$$

we have

$$\begin{aligned} J_{t,s}^N((1-t)^n) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\sum_{j=k+1}^{N} \frac{s^j t^{k-j}}{j}\right) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\sum_{j=1}^{N} \frac{s^j t^{k-j}}{j} - \sum_{j=1}^{k} \frac{s^j t^{k-j}}{j}\right) \\ &= \sum_{j=1}^{N} \frac{(s/t)^j}{j}(1-t)^n - \sum_{n \geq k \geq j \geq 1} (-1)^k \binom{n}{k} \frac{s^j t^{k-j}}{j}. \end{aligned} \tag{56}$$

Since the equality

$$\sum_{j=1}^{k} \frac{(s/t)^j}{j} = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{(1-s/t)^j - 1}{j}$$

holds by Lemma 8.14 (53), we have

$$\sum_{n \geq k \geq j \geq 1} (-1)^k \binom{n}{k} \frac{s^j t^{k-j}}{j} = \sum_{n \geq k \geq j \geq 1} (-1)^{j+k} \binom{n}{k} \binom{k}{j} \frac{(1-s/t)^j - 1}{j} t^k.$$

Furthermore, by Lemma 8.15, we have

$$\begin{aligned}
&\sum_{n\geq k\geq j\geq 1}(-1)^{j+k}\binom{n}{k}\binom{k}{j}\frac{(1-s/t)^j-1}{j}t^k\\
&=\sum_{j=1}^{n}\binom{n}{j}t^j(1-t)^{n-j}\frac{(1-s/t)^j-1}{j}\\
&=\sum_{j=1}^{n}\binom{n}{j}(1-t)^{n-j}\frac{(t-s)^j-t^j}{j}\\
&=(1-t)^n\sum_{j=1}^{n}\binom{n}{j}\frac{1}{j}\left\{\left(\frac{t-s}{1-t}\right)^j-\left(\frac{t}{1-t}\right)^j\right\}.
\end{aligned}$$

Therefore, according to Lemma 8.14 (52), we can delete the binomial coefficients completely as follows:

$$\begin{aligned}
&\sum_{n\geq k\geq j\geq 1}(-1)^{j+k}\binom{n}{k}\binom{k}{j}\frac{(1-s/t)^j-1}{j}t^k\\
&=(1-t)^n\sum_{j=1}^{n}\frac{1}{j}\left\{\left(1-\frac{s-t}{1-t}\right)^j-\left(1-\frac{-t}{1-t}\right)^j\right\}\\
&=(1-t)^n\sum_{j=1}^{n}\frac{1}{j}\left\{\left(\frac{1-s}{1-t}\right)^j-\left(\frac{1}{1-t}\right)^j\right\}\\
&=\sum_{j=1}^{n}\frac{(1-t)^{n-j}\left\{(1-s)^j-1\right\}}{j}.
\end{aligned}$$

Hence, we have the desired identity by the equality (56) and $J_{t,s}^N(1)=\sum_{j=1}^N\frac{(s/t)^j}{j}$. □

8.3 Proof of Theorem 8.7 and Theorem 8.10

Proof of Theorem 8.7. We show this theorem by the induction on the weight k of the index. If $k=1$, the assertion of the theorem is nothing but Lemma 8.14. We show only the equality (43) because the proof of the equality (44) is completely the same. Now, we assume that the assertion holds for an index $\mathbf{k} = (k_1, \dots, k_r)$. Then it is sufficient to show that the assertions also hold for the indices $\mathbf{k} \oplus \mathbf{e}_r = (k_1, \dots, k_r + 1)$ and $(\mathbf{k}, 1) = (k_1, \dots, k_r, 1)$.

First, we consider the case $\mathbf{k} \oplus \mathbf{e}_r$. By Lemma 8.13 (48), we have

$$I_{t_r}\left(\sum_{N \ge n_1 \ge \cdots \ge n_r \ge 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{r-1}^{n_{r-1}-n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}\right)$$

$$= \sum_{N \ge n_1 \ge \cdots \ge n_r \ge 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{r-1}^{n_{r-1}-n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r+1}},$$

where $I_{t_r} := I_{t_r; R[t_1, \dots, t_{r-1}]}$. On the other hand, by Lemma 8.13 (49), we have

$$I_{t_r}\left(\sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{r-1})^{n_{l_{r-1}}-n_{l_{r-1}+1}} \{(1-t_r)^{n_{l_r}} - 1\}}{n_1 \cdots n_k}\right)$$

$$= \sum_{N \ge n_1 \ge \cdots \ge n_{k+1} \ge 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{r-1})^{n_{l_{r-1}}-n_{l_{r-1}+1}} \{(1-t_r)^{n_{l_r+1}} - 1\}}{n_1 \cdots n_{k+1}},$$

where $l_1 = k_1, l_2 = k_1 + k_2, \dots, l_r = k_1 + \cdots + k_r (= k)$. Thus, the equality (43) in the theorem also holds for $\mathbf{k} \oplus \mathbf{e}_r$ by the induction hypothesis.

Next, we check the equality for the index $(\mathbf{k}, 1)$. By Lemma 8.13 (50), we have

$$J^\star_{t_r, t_{r+1}}\left(\sum_{N \ge n_1 \ge \cdots \ge n_r \ge 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{r-1}^{n_{r-1}-n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}\right)$$

$$= \sum_{N \ge n_1 \ge \cdots \ge n_{r+1} \ge 1} (-1)^{n_1} \binom{N}{n_1} \frac{t_1^{n_1-n_2} \cdots t_r^{n_r-n_{r+1}} t_{r+1}^{n_{r+1}}}{n_1^{k_1} \cdots n_r^{k_r} n_{r+1}},$$

where $J^\star_{t_r, t_{r+1}} := J^\star_{t_r, t_{r+1}; R[t_1, \dots, t_{r-1}]}$. On the other hand, by Lemma 8.13 (51), we have

$$J^\star_{t_r, t_{r+1}}\left(\sum_{N \ge n_1 \ge \cdots \ge n_k \ge 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_{r-1})^{n_{l_{r-1}}-n_{l_{r-1}+1}} \{(1-t_r)^{n_{l_r}} - 1\}}{n_1 \cdots n_k}\right)$$

$$= \sum_{N \ge n_1 \ge \cdots \ge n_{k+1} \ge 1} \frac{(1-t_1)^{n_{l_1}-n_{l_1+1}} \cdots (1-t_r)^{n_{l_r}-n_{l_r+1}} \{(1-t_{r+1})^{n_{l_r+1}} - 1\}}{n_1 \cdots n_{k+1}}.$$

Using the induction hypothesis, the assertion of the equality (43) holds for the index $(\mathbf{k}, 1)$. This completes the proof of Theorem 8.7. $\square$

Proof of Theorem 8.10. We show this theorem by the induction on the weight k of the index. If $k=1$, the assertion of the theorem is nothing but Lemma 8.14. We show only the equality (45) because the proof of the equality (46) is completely the same. Now, we assume that the assertion holds for an index $\mathbf{k}=(k_1,\dots,k_r)$. Then it is sufficient to show that the assertions also hold for the indices $\mathbf{k}\oplus\mathbf{e}_1=(k_1+1,\dots,k_r)$ and $(1,\mathbf{k})=(1,k_1,\dots,k_r)$.

First, we consider the case $\mathbf{k}\oplus\mathbf{e}_1$. By Lemma 8.13 (48), we have

$$\begin{aligned}
&I_{t_1}\left(\sum_{N\ge n_1>\cdots>n_r\ge 1}(-1)^{n_r}\binom{N}{n_r}\frac{(t_1/t_2)^{n_1}\cdots(t_{r-1}/t_r)^{n_{r-1}}t_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}\right)\\
&=\sum_{N\ge n_1>\cdots>n_r\ge 1}(-1)^{n_r}\binom{N}{n_r}\frac{(t_1/t_2)^{n_1}\cdots(t_{r-1}/t_r)^{n_{r-1}}t_r^{n_r}}{n_1^{k_1+1}\cdots n_r^{k_r}},
\end{aligned}$$

where $I_{t_1}:=I_{t_1;R[t_2^{\pm1},\dots,t_r^{\pm1}]}$. On the other hand, by Lemma 8.13 (49), we have

$$\begin{aligned}
&I_{t_1}(\text{R. H. S. of the equality (45)})=\\
&(-1)^{r-1}\sum_{N\ge n_1\ge\cdots\ge n_{k+1}\ge 1}\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_2)^{n_{l_{r-1}}-n_{l_{r-1}+1}}\{(1-t_1)^{n_{l_r+1}}-1\}}{n_1\cdots n_{k+1}}\\
&+\sum_{j=1}^{r-1}(-1)^{r-j-1}\left(\sum_{N\ge n_1>\cdots>n_j\ge 1}\frac{(t_1/t_2)^{n_1}\cdots(t_j/t_{j+1})^{n_j}}{n_1^{k_1+1}\cdots n_j^{k_j}}\right)\\
&\quad\times\left(\sum_{N\ge n_1\ge\cdots\ge n_{l_{r-j}}\ge 1}\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_{j+2})^{n_{l_{r-j-1}}-n_{l_{r-j-1}+1}}\{(1-t_{j+1})^{n_{l_{r-j}}}-1\}}{n_1\cdots n_{l_{r-j}}}\right)
\end{aligned}$$

where $l_1=k_r, l_2=k_r+k_{r-1},\dots,l_r=k_r+\cdots+k_1(=k)$. Thus the equality (45) in the theorem also holds for the index $\mathbf{k}\oplus\mathbf{e}_1$ by the induction hypothesis.

Next, we check the equality for the index $(1,\mathbf{k})$. By Lemma 8.16 (54), we have

$$\begin{aligned}
&J_{t_1,t_0}^N\left(\sum_{N\ge n_1>\cdots>n_r\ge 1}(-1)^{n_r}\binom{N}{n_r}\frac{(t_1/t_2)^{n_1}\cdots(t_{r-1}/t_r)^{n_{r-1}}t_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}\right)\\
&=\sum_{N\ge n_0>n_1>\cdots>n_r\ge 1}(-1)^{n_r}\binom{N}{n_r}\frac{(t_0/t_1)^{n_0}\cdots(t_{r-1}/t_r)^{n_{r-1}}t_r^{n_r}}{n_0n_1^{k_1}\cdots n_r^{k_r}},
\end{aligned}$$

where $J^N_{t_1,t_0} := J^N_{t_1,t_0;R[t_2^{\pm 1},\dots,t_r^{\pm 1}]}$. On the other hand, by Lemma 8.16 (55), we have

$$
\begin{aligned}
&(-1)^r J^N_{t_1,t_0}(\text{R. H. S. of the equality (45)}) = \\
&\sum_{N\geq n_1\geq\cdots\geq n_{k+1}\geq 1} \frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_1)^{n_{l_r}-n_{l_r+1}}\{(1-t_0)^{n_{l_r+1}}-1\}}{n_1\cdots n_{k+1}} \\
&-\left(\sum_{n_0=1}^{N}\frac{(t_0/t_1)^{n_0}}{n_0}\right) \\
&\times\left(\sum_{N\geq n_1\geq\cdots\geq n_k\geq 1}\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_2)^{n_{l_{r-1}}-n_{l_{r-1}+1}}\{(1-t_1)^{n_{l_r}}-1\}}{n_1\cdots n_k}\right) \\
&+\sum_{j=1}^{r-1}(-1)^{j-1}\left(\sum_{N\geq n_0>n_1>\cdots>n_j\geq 1}\frac{(t_0/t_1)^{n_0}\cdots(t_j/t_{j+1})^{n_j}}{n_0 n_1^{k_1}\cdots n_j^{k_j}}\right) \\
&\times\left(\sum_{N\geq n_1\geq\cdots\geq n_{l_{r-j}}\geq 1}\frac{(1-t_r)^{n_{l_1}-n_{l_1+1}}\cdots(1-t_{j+2})^{n_{l_{r-j-1}}-n_{l_{r-j-1}+1}}\{(1-t_{j+1})^{n_{l_{r-j}}}-1\}}{n_1\cdots n_{l_{r-j}}}\right).
\end{aligned}
$$

Using the induction hypothesis, the assertion of the equality (45) holds for the index $(1,\mathbf{k})$. This completes the proof of Theorem 8.10. $\square$

9 Adelic rings

In order to define the finite multiple polylogarithms, we introduce some adelic rings in a general setting.

Definition 9.1. Let R be a commutative ring and Σ an infinite family of ideals of R. We define a ring $\mathcal{A}^{\Sigma}_{n,R}$ for each positive integer n by

$$\mathcal{A}^{\Sigma}_{n,R} := \left(\prod_{I\in\Sigma} R/I^n\right)\Big/\left(\bigoplus_{I\in\Sigma} R/I^n\right).$$

Then $\{\mathcal{A}^{\Sigma}_{n,R}\}$ becomes a projective system by natural projections and we define a ring $\widehat{\mathcal{A}}^{\Sigma}_{R}$ by

$$\widehat{\mathcal{A}}^{\Sigma}_{R} := \varprojlim_{n} \mathcal{A}^{\Sigma}_{n,R}.$$

We put the discrete topology on $\mathcal{A}^{\Sigma}_{n,R}$ for each n and we define the topology of $\widehat{\mathcal{A}}^{\Sigma}_{R}$ to be the projective limit topology.

Lemma 9.2. *We use the same notations as Definition* 9.1 *and we define the I-adic completion $\widehat{R}_I$ of R to be $\varprojlim_n R/I^nR$. Then there exists the following natural surjective ring homomorphism*:

$$\pi\colon \prod_{I\in\Sigma} \widehat{R}_I \longrightarrow \widehat{\mathcal{A}}^{\Sigma}_{R}.$$

Proof. For a short exact sequence of projective systems of rings

$$0 \longrightarrow \left\{\bigoplus_{I\in\Sigma} R/I^n\right\} \longrightarrow \left\{\prod_{I\in\Sigma} R/I^n\right\} \longrightarrow \{\mathcal{A}^{\Sigma}_{n,R}\} \longrightarrow 0,$$

the system $\left\{\bigoplus_{I\in\Sigma} R/I^n\right\}$ satisfies the Mittag–Leffler condition. Therefore, there exists a natural surjection

$$\prod_{I\in\Sigma} \widehat{R}_I \simeq \varprojlim_n \prod_{I\in\Sigma} R/I^n \longrightarrow \widehat{\mathcal{A}}^{\Sigma}_{R}. \qquad \square$$

Remark 9.3. We assume that some topology of R/I^n is defined for any $I \in \Sigma$. If we put the product topology on $\prod_{I\in\Sigma} R/I^n$ and the quotient topology on $\mathcal{A}^{\Sigma}_{n,R}$ by $\prod_{I\in\Sigma} R/I^n \twoheadrightarrow \mathcal{A}^{\Sigma}_{n,R}$, then the topology becomes indiscrete. However, we put the discrete topology on $\mathcal{A}^{\Sigma}_{n,R}$ in this thesis.

Lemma 9.4. *We use the same notations as Definition* 9.1 *and Definition* 9.2. *We assume that $I\widehat{R}_I$ is a principal ideal of $\widehat{R}_I$ for any $I \in \Sigma$. Furthermore, we define an ideal $\boldsymbol{I}$ of $\widehat{\mathcal{A}}^{\Sigma}_{R}$ to be $\pi((I\widehat{R}_I)_{I\in\Sigma})$. Let π_n be the natural projection $\pi_n\colon \widehat{\mathcal{A}}^{\Sigma}_{R} \twoheadrightarrow \mathcal{A}^{\Sigma}_{n,R}$ for any positive integer n. Then we have $\ker(\pi_n) = \boldsymbol{I}^n$. In particular, the topology of $\widehat{\mathcal{A}}^{\Sigma}_{R}$ coincides with the $\boldsymbol{I}$-adic topology and $\widehat{\mathcal{A}}^{\Sigma}_{R}$ is complete with respect to the $\boldsymbol{I}$-adic topology.*

Proof. Let n be a positive integer. Take any element x of $\ker(\pi_n)$. Then there exists an element $\{x_I\}_{I\in\Sigma}$ of $\prod_{I\in\Sigma} \widehat{R}_I$ such that $x = \pi((x_I)_{I\in\Sigma})$ by Lemma 9.2. By the commutative diagram

$$\begin{array}{ccc} \prod_{I\in\Sigma} \widehat{R}_I & \xrightarrow{\pi} & \widehat{\mathcal{A}}_R^{\Sigma} \\ {\scriptstyle (\bmod I^n)_{I\in\Sigma}}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ \prod_{I\in\Sigma} R/I^n & \xrightarrow{\rho_n} & \mathcal{A}_{n,R}^{\Sigma} \end{array}$$

we have

$$\pi_n(x) = \pi_n \circ \pi((x_I)_{I\in\Sigma}) = \rho_n((x_I \bmod I^n)_{I\in\Sigma}) = 0.$$

Here, ρ_n is the canonical projection. Therefore, there exists a subset Σ' of Σ such that $\Sigma \setminus \Sigma'$ is finite and $x_I \in I^n\widehat{R}_I$ for every $I \in \Sigma'$. We can take a generator a_I of $I\widehat{R}_I$ for any $I \in \Sigma$ by the assumption. Then there exists an element $\{y_I\}_{I\in\Sigma'}$ of $\prod_{I\in\Sigma'} \widehat{R}_I$ such that $x_I = a_I^n y_I$ holds for any $I \in \Sigma'$. We define y_I to be zero for $I \in \Sigma \setminus \Sigma'$. Then we have

$$x = \pi((x_I)_{I\in\Sigma}) = \pi((a_I^n y_I)_{I\in\Sigma}) = (\pi((a_I)_{I\in\Sigma}))^n \cdot \pi((y_I)_{I\in\Sigma}) \in \boldsymbol{I}^n$$

and we obtain the inclusion $\ker(\pi_n) \subset \boldsymbol{I}^n$. The opposite inclusion is trivial and the last assertion follows from the fact that $\{\ker(\pi_n)\}$ is a neighborhood basis of zero. $\square$

In the rest of this thesis, we only use the case $\Sigma = \{pR \mid p \text{ is a prime number}\}$ and we omit the notation Σ. We will define the $\widehat{\mathcal{A}}$-finite multiple polylogarithms as elements of the $\mathbb{Q}$-algebra $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$ in Section 11. Let $\pi\colon \prod_p \widehat{\mathbb{Z}[\boldsymbol{t}]}_p \twoheadrightarrow \widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$ be the natural surjection obtained by Lemma 9.2. Here, $\widehat{\mathbb{Z}[\boldsymbol{t}]}_p := \varprojlim_n \mathbb{Z}[\boldsymbol{t}]/p^n\mathbb{Z}[\boldsymbol{t}]$ is the p-adic completion of $\mathbb{Z}[\boldsymbol{t}]$. Let $\pi_n\colon \widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]} \twoheadrightarrow \mathcal{A}_{n,\mathbb{Z}[\boldsymbol{t}]}$ be the natural projection for each n. The topology of $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$ coincides with the $\boldsymbol{p}$-adic topology and $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$ is complete with respect to the topology by Lemma 9.4.

Since an equality $\pi\left(\left(\sum_{i=0}^{\infty} a_i^{(p)} p^i\right)_p\right) = \sum_{i=0}^{\infty} (a_i^{(p)})_p \boldsymbol{p}^i$ holds, in order to obtain a $\boldsymbol{p}$-adic relation, it is sufficient to show the p-adic relations given by taking the p-components for all but finitely many prime numbers p. Here, $a_i^{(p)} \in \mathbb{Z}_{(p)}[\boldsymbol{t}]$. It seems that the opposite assertion does not hold in general.

10 Review of finite polylogarithms

After that Kontsevich defined "*the* $1\frac{1}{2}$*-logarithm*" = *the finite* 1*-logarithm* in [22], Elbaz-Vincent–Gangl defined *the finite polylogarithm* for fixed prime number p in [8]. In this section, we define the $\widehat{\mathcal{A}}$-finite (resp. $\mathcal{A}_n$-finite) polylogarithm as an element of $\widehat{\mathcal{A}}_{\mathbb{Z}[t]}$ (resp. $\mathcal{A}_{n,\mathbb{Z}[t]}$).

Definition 10.1. Let m and k be positive integers. *The truncated polylogarithm* $\pounds_{m,k}(t)$ is defined by

$$\pounds_{m,k}(t) := \sum_{n=1}^{m} \frac{t^n}{n^k}.$$

Then *the* $\widehat{\mathcal{A}}$*-finite polylogarithm of weight* k $\pounds_{\widehat{\mathcal{A}},k}(t)$ is defined by

$$\pounds_{\widehat{\mathcal{A}},k}(t) := \pi\left(\left(\pounds_{p-1,k}(t)\right)_p \right)$$

in $\widehat{\mathcal{A}}_{\mathbb{Z}[t]}$ and *the* $\mathcal{A}_n$*-finite polylogarithm of weight* k $\pounds_{\mathcal{A}_n,k}(t)$ is defined by

$$\pounds_{\mathcal{A}_n,k}(t) := \pi_n\left(\pounds_{\widehat{\mathcal{A}},k}(t)\right)$$

in $\mathcal{A}_{n,\mathbb{Z}[t]}$ for each positive number n.

Kontsevich observed that the following three functional equations for $\mathcal{A}$-finite polylogarithms of weight 1 hold:

Proposition 10.2 (Kontsevich [22])**.** *Let t and s be indeterminates. Then*

$$\pounds_{\mathcal{A},1}(t) = \pounds_{\mathcal{A},1}(1-t), \tag{57}$$

$$\pounds_{\mathcal{A},1}(t) = -t^{\boldsymbol{p}}\pounds_{\mathcal{A},1}(t^{-1}), \tag{58}$$

$$\pounds_{\mathcal{A},1}(t) - \pounds_{\mathcal{A},1}(s) + t^{\boldsymbol{p}}\pounds_{\mathcal{A},1}\left(\frac{s}{t}\right) + (1-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},1}\left(\frac{1-s}{1-t}\right) = 0. \tag{59}$$

Here, we consider the equality (58) *in* $\mathcal{A}_{\mathbb{Z}[t,t^{-1}]}$ *and the equality* (59) *in* $\mathcal{A}_{\mathbb{Z}[t^{\pm1},(1-t)^{-1},s]}$.

The following functional equation is a generalization of the equality (58):

Proposition 10.3 (Elbaz-Vincent–Gangl [8, Theorem 5.7 (1)])**.** *Let k be a positive integer. Then*

$$\pounds_{\mathcal{A},k}(t) = (-1)^k t^{\boldsymbol{p}} \pounds_{\mathcal{A},k}(t^{-1}) \tag{60}$$

in $\mathcal{A}_{\mathbb{Z}[t^{\pm 1}]}$.

The equalities (57) and (60) will be generalized to $\widehat{\mathcal{A}}$-finite multiple cases (Section 12). Elbaz-Vincent–Gangl proved the following distribution property for $\pounds_{\mathcal{A},k}$:

Proposition 10.4 (Elbaz-Vincent–Gangl [8, Proposition 5.7 (2)])**.** *Let m be a non-zero integer and k a positive integer. Let ζ_m be a primitive $|m|$-th root of unity. Then we have the following equality in* $\mathcal{A}_{\mathbb{Z}[\zeta_m, t^{\pm 1}]}$:

$$\pounds_{\mathcal{A},k}(t^m) = m^{k-1} \sum_{j=0}^{|m|-1} \frac{1-t^{m\boldsymbol{p}}}{1-(\zeta_m^j t)^{\boldsymbol{p}}} \pounds_{\mathcal{A},k}(\zeta_m^j t). \tag{61}$$

Proof. We assume that m is positive. Let p be a prime number not dividing m. Then

$$\begin{aligned}
\frac{1}{m}\sum_{j=0}^{m-1} \frac{1-t^{mp}}{1-(\zeta_m^j t)^p} \pounds_{p-1,k}(\zeta_m^j t) &= \frac{1}{m}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{m-1}(\zeta_m^j t)^{ip}\right)\left(\sum_{n=1}^{p-1}\frac{(\zeta_m^j t)^n}{n^k}\right) \\
&= \frac{1}{m}\sum_{n=1}^{p-1}\frac{1}{n^k}\sum_{j=0}^{m-1}\sum_{i=0}^{m-1}(\zeta_m^j t)^{ip+n} \\
&\equiv \frac{1}{m}\sum_{n=1}^{p-1}\sum_{j=0}^{m-1}\sum_{i=0}^{m-1}\frac{(\zeta_m^j t)^{ip+n}}{(ip+n)^k} \\
&= \frac{1}{m}\sum_{\substack{l=1 \\ p\nmid l}}^{pm-1}\left(\sum_{j=0}^{m-1}\zeta_m^{jl}\right)\frac{t^l}{l^k} \pmod{p}.
\end{aligned}$$

in $\mathbb{Z}_{(p)}[\zeta_m, t]$. By

$$\sum_{j=0}^{m-1}\zeta_m^{jl} = \begin{cases} m & \text{if } m \mid l \\ 0 & \text{otherwise} \end{cases},$$

we have

$$\frac{1}{m}\sum_{j=0}^{m-1}\frac{1-t^{mp}}{1-(\zeta_m^j t)^p}\pounds_{p-1,k}(\zeta_m^j t)\equiv\sum_{n=1}^{p-1}\frac{t^{nm}}{(nm)^k}=\frac{1}{m^k}\pounds_{p-1,k}(t^m)\pmod{p}.$$

The negative case is obtained by combining the positive case with Proposition 10.3. $\square$

Kontsevich proved the 4-term relation (57) for $\pounds_{\mathcal{A},1}$ and he raised a question to find functional equations for $\pounds_{\mathcal{A},2}$. Elbaz-Vincent–Gangl gave answers to the question in [8]. Especially, they proved the following 22-term relation for $\pounds_{\mathcal{A},2}$:

Theorem 10.5 (Elbaz-Vincent–Gangl [8, Theorem 5.12])**.** *Let s, t, and u be indeterminates. Then we have the following functional equation for $\mathcal{A}$-finite dilogarithm*:

$$\begin{aligned}
&u^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(s)-u^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(t)+(s-t+1)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(u)\\
&+(1-u)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(1-s)-(1-u)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(1-t)+(t-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}(1-u)\\
&-s^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{u}{s}\right)+t^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{u}{t}\right)+u^{\boldsymbol{p}}s^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{t}{s}\right)\\
&-(1-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{1-u}{1-s}\right)+(1-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{1-u}{1-t}\right)+u^{\boldsymbol{p}}(1-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{1-t}{1-s}\right)\\
&+u^{\boldsymbol{p}}(1-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{s(1-u)}{u(1-s)}\right)-u^{\boldsymbol{p}}(1-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{t(1-u)}{u(1-t)}\right)\\
&-t^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{us}{t}\right)-(1-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{u(1-s)}{1-t}\right)\\
&+(1-u)^{\boldsymbol{p}}s^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{s-t}{s}\right)+(1-u)^{\boldsymbol{p}}(1-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{t-s}{1-s}\right)\\
&-(s-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{(1-u)s}{s-t}\right)-(t-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{(1-u)(1-s)}{t-s}\right)\\
&+u^{\boldsymbol{p}}(s-t)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{(1-u)t}{u(s-t)}\right)+u^{\boldsymbol{p}}(t-s)^{\boldsymbol{p}}\pounds_{\mathcal{A},2}\left(\frac{(1-u)(1-t)}{u(t-s)}\right)\\
&=0
\end{aligned}$$

in $\mathcal{A}_{\mathbb{Z}[s^{\pm1},t^{\pm1},u^{\pm1},(1-s)^{-1},(1-t)^{-1},(s-t)^{-1}]}$.

11 Definition of finite multiple polylogarithms

Before defining finite multiple polylogarithms, we define truncated multiple polylogarithms which are generalizations of multiple harmonic sums (Definition 4.1).

Definition 11.1. Let n be a positive integer, $\mathbf{k} = (k_1, \dots, k_r)$ an index, and $\boldsymbol{t} = (t_1, \dots, t_r)$ a tuple of indeterminates. Then we define the four kinds of *the truncated multiple polylogarithms* which are elements of $\mathbb{Q}[\boldsymbol{t}]$ as follows:

$$\pounds^{*}_{n,\mathbf{k}}(\boldsymbol{t}) := \sum_{n \geq n_1 > \cdots > n_r \geq 1} \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\pounds^{*,\star}_{n,\mathbf{k}}(\boldsymbol{t}) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{t_1^{n_1} \cdots t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\pounds^{ш}_{n,\mathbf{k}}(\boldsymbol{t}) := \sum_{n \geq n_1 > \cdots > n_r \geq 1} \frac{t_1^{n_1 - n_2} \cdots t_{r-1}^{n_{r-1} - n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\pounds^{ш,\star}_{n,\mathbf{k}}(\boldsymbol{t}) := \sum_{n \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{t_1^{n_1 - n_2} \cdots t_{r-1}^{n_{r-1} - n_r} t_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}.$$

If $\mathbf{k} = \emptyset$, we consider the truncated multiple polylogarithms as 1.

Definition 11.2. Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index and $\boldsymbol{t} = (t_1, \dots, t_r)$ a tuple of indeterminates. Then we define the four kinds of *the $\widehat{\mathcal{A}}$-finite multiple polylogarithms* which are elements of $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$ as follows:

$\pounds^{*}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) := \pi((\pounds^{*}_{p-1,\mathbf{k}}(\boldsymbol{t}))_p)$ (*$\widehat{\mathcal{A}}$-finite harmonic multiple polylogarithm = $\widehat{\mathcal{A}}$-FHMP*),

$\pounds^{*,\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) := \pi((\pounds^{*,\star}_{p-1,\mathbf{k}}(\boldsymbol{t}))_p)$ (*$\widehat{\mathcal{A}}$-finite harmonic star-multiple polylogarithm = $\widehat{\mathcal{A}}$-FHSMP*),

$\pounds^{ш}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) := \pi((\pounds^{ш}_{p-1,\mathbf{k}}(\boldsymbol{t}))_p)$ (*$\widehat{\mathcal{A}}$-finite shuffle multiple polylogarithm = $\widehat{\mathcal{A}}$-FSMP*),

$\pounds^{ш,\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) := \pi((\pounds^{ш,\star}_{p-1,\mathbf{k}}(\boldsymbol{t}))_p)$ (*$\widehat{\mathcal{A}}$-finite shuffle star-multiple polylogarithm = $\widehat{\mathcal{A}}$-FSSMP*).

This definition is well-defined since $\pounds^{\circ,\bullet}_{p-1,\mathbf{k}}(\boldsymbol{t})$ is an element of $\mathbb{Z}_{(p)}[\boldsymbol{t}]$ for each prime number p, $\circ \in \{*, ш\}$, and $\bullet \in \{\emptyset, \star\}$. We also define *the $\mathcal{A}_n$-finite multiple polylogarithm* (*$\mathcal{A}_n$-FMP*)

$\mathcal{L}^{\circ;\bullet}_{\mathcal{A}_n,\mathbf{k}}(\boldsymbol{t})$ as an element of $\mathcal{A}_{n,\mathbb{Z}[\boldsymbol{t}]}$ by

$$\mathcal{L}^{\circ;\bullet}_{\mathcal{A}_n,\mathbf{k}}(\boldsymbol{t}) := \pi_n(\mathcal{L}^{\circ;\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}))$$

for each positive integer n, $\circ \in \{*, ш\}$, and $\bullet \in \{\emptyset, \star\}$. We define 1-variable $\widehat{\mathcal{A}}$-F(S)MPs as follows:

$$\mathcal{L}^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t) := \mathcal{L}^{*;\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t, \{1\}^{r-1}) = \mathcal{L}^{ш;\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{t\}^{r}) \in \widehat{\mathcal{A}}_{\mathbb{Z}[t]},$$

$$\widetilde{\mathcal{L}}^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(t) := \mathcal{L}^{*;\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{r-1}, t) = \mathcal{L}^{ш;\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{r-1}, t) \in \widehat{\mathcal{A}}_{\mathbb{Z}[t]},$$

where t is an indeterminate and $\bullet \in \{\emptyset, \star\}$. In the same way, we can define 1-variable $\mathcal{A}_n$-F(S)MPs $\mathcal{L}^{\bullet}_{\mathcal{A}_n,\mathbf{k}}(t)$ and $\widetilde{\mathcal{L}}^{\bullet}_{\mathcal{A}_n,\mathbf{k}}(t)$ for each n.

Remark 11.3. Let R be a commutative ring. For any subset $\{i_1, \ldots, i_h\}$ of $\{1, \ldots, r\}$ and $a_1, \ldots, a_h \in R$, the substitution mapping

$$\widehat{\mathcal{A}}_{\mathbb{Z}[t_1,\ldots,t_r]} \longrightarrow \widehat{\mathcal{A}}_{R[t_{j_1},\ldots,t_{j_{h'}}]}$$

defined by

$$(f_p(t_1, \cdots, t_r))_p \mapsto (f_p(t_1, \ldots, t_r)|_{t_{i_1}=a_1,\ldots,t_{i_h}=a_h})_p$$

where $\{j_1, \ldots, j_{h'}\}$ is the complement of $\{i_1, \ldots, i_h\}$ with respect to $\{1, \ldots, r\}$. For example, we have

$$\mathcal{L}^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(1) = \widetilde{\mathcal{L}}^{\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(1) = \zeta^{\bullet}_{\widehat{\mathcal{A}}}(\mathbf{k}) \in \widehat{\mathcal{A}}$$

for $\bullet \in \{\emptyset, \star\}$. Our definition of FMPs is natural in this sense.

12 Fundamental relations of finite multiple polylogarithms

We prove three fundamental formulas as follows:

- **Reversal relation for $\widehat{\mathcal{A}}$-FH(S)MPs** (= Theorem 12.1),

- **Functional equation for $\widehat{\mathcal{A}}$-FSSMPs** (= Theorem 12.2),
- **Relation between $\widehat{\mathcal{A}}$-FHMPs and $\widehat{\mathcal{A}}$-FHSMPs** (= Theorem 12.11).

These are main results of this thesis.

12.1 Reversal relation for $\widehat{\mathcal{A}}$-finite harmonic (star-)multiple polylogarithms

Theorem 12.1. *Let* $\mathbf{k}=(k_1,\dots,k_r)$ *be an index,* $\boldsymbol{t}=(t_1,\dots,t_r)$ *a tuple of indeterminates, and* $\bullet\in\{\emptyset,\star\}$*. Then we have the following* $\boldsymbol{p}$*-adic relation in* $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t},\boldsymbol{t}^{-1}]}$:

$$\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\overline{\mathbf{k}}}(\boldsymbol{t})=(-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_r)^{\boldsymbol{p}}\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}^r_{\geq 0}\\ l_1+\cdots+l_r=i}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}\oplus\mathbf{l}}(\overline{\boldsymbol{t}^{-1}})\boldsymbol{p}^i,$$

where $\pounds^{*,\bullet}_{\widehat{\mathcal{A}},\mathbf{k}\oplus\mathbf{l}}(\overline{\boldsymbol{t}^{-1}})$ *is an element of* $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}^{-1}]}$.

Proof. Let p be a prime number. By the substitutions $n_i\mapsto p-n_{r+1-i}$ and a p-adically convergent identity (14), we have

$$\begin{aligned}
\pounds^{*}_{p-1,\overline{\mathbf{k}}}(\boldsymbol{t})&=\sum_{p-1\geq n_1>\cdots>n_r\geq 1}\frac{t_1^{n_1}\cdots t_r^{n_r}}{n_1^{k_r}\cdots n_r^{k_1}}\\
&=\sum_{p-1\geq p-n_r>\cdots>p-n_1\geq 1}\frac{t_1^{p-n_r}\cdots t_r^{p-n_1}}{(p-n_r)^{k_r}\cdots(p-n_1)^{k_1}}\\
&=(-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_r)^p\\
&\quad\times\sum_{p-1\geq n_1>\cdots>n_r\geq 1}\ \sum_{(l_1,\dots,l_r)\in\mathbb{Z}^r_{\geq 0}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\frac{t_r^{-n_1}\cdots t_1^{-n_r}}{n_1^{k_1+l_1}\cdots n_r^{k_r+l_r}}p^{l_1+\cdots+l_r}\\
&=(-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_r)^p\sum_{i=0}^{\infty}\sum_{\substack{\mathbf{l}=(l_1,\dots,l_r)\in\mathbb{Z}^r_{\geq 0}\\ l_1+\cdots+l_r=i}}\left[\prod_{j=1}^{r}\binom{k_j+l_j-1}{l_j}\right]\pounds^{*}_{p-1,\mathbf{k}\oplus\mathbf{l}}(\overline{\boldsymbol{t}^{-1}})p^i
\end{aligned}$$

in the ring $\widehat{\mathbb{Z}[\boldsymbol{t}]}_p$. Therefore, we have the conclusion for non-star case. The star case is similar. $\square$

12.2 Functional equation for $\widehat{\mathcal{A}}$-finite shuffle star-multiple polylogarithms

To state the functional equation for $\widehat{\mathcal{A}}$-FSSMP, we define a $\boldsymbol{p}$-adically convergent series $\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t})$ with $\widehat{\mathcal{A}}$-FSSMP-coefficients for an index $\mathbf{k}$ and a tuple of indeterminates $\boldsymbol{t}$ by

$$\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) := \sum_{i=0}^{\infty}\left(\pounds^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i,\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(\{1\}^i,\boldsymbol{t}_1)\right)\boldsymbol{p}^i. \tag{62}$$

We have the following functional equation for the series (62):

Theorem 12.2. *Let r be a positive integer, $\mathbf{k}_1,\dots,\mathbf{k}_r$ indices, and $\boldsymbol{t}=(t_1,\dots,t_r)$ a tuple of indeterminates. We define an index $\mathbf{k}$ to be $(\mathbf{k}_1,\dots,\mathbf{k}_r)$ and $\mathbf{k}^*$ to be $(\mathbf{k}_1^{\vee},\dots,\mathbf{k}_r^{\vee})$. Furthermore, we define l_i and l'_i by $l_i := \mathrm{dep}(\mathbf{k}_i)$ and $l'_i := \mathrm{dep}(\mathbf{k}_i^{\vee})$ respectively for $i=1,\dots,r$. Then we have a multi-variable functional equation*

$$\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{l_1-1},t_1,\dots,\{1\}^{l_r-1},t_r) = \mathcal{L}^{\star}_{\widehat{\mathcal{A}},\mathbf{k}^*}(\{1\}^{l'_1-1},1-t_1,\dots,\{1\}^{l'_r-1},1-t_r)$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$.

The one-variable case of Theorem 12.2 is as follows:

Corollary 12.3. *Let $\mathbf{k}$ be an index. Then we have*

$$\sum_{i=0}^{\infty}\left(\widetilde{\pounds}^{\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k})}(t) - \frac{1}{2}\zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i,\mathbf{k})\right)\boldsymbol{p}^i = \sum_{i=0}^{\infty}\left(\widetilde{\pounds}^{\star}_{\widehat{\mathcal{A}},(\{1\}^i,\mathbf{k}^{\vee})}(1-t) - \frac{1}{2}\zeta^{\star}_{\widehat{\mathcal{A}}}(\{1\}^i,\mathbf{k}^{\vee})\right)\boldsymbol{p}^i \tag{63}$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[t]}$.

In particular, we obtain Theorem 6.9 by substituting $t=1$. In order to prove Theorem 12.2, the case that $\mathbf{k}$ is k repetitions of 1 for a positive integer k is essential:

Theorem 12.4. *Let k be a positive integer and $\boldsymbol{t}$ a tuple of k indeterminates. Then we have*

$$\mathcal{L}^{\star}_{\widehat{\mathcal{A}},\{1\}^k}(\boldsymbol{t}) = \mathcal{L}^{\star}_{\widehat{\mathcal{A}},\{1\}^k}(1-\boldsymbol{t})$$

in the ring $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$.

In fact, the following lemma is the reason why k repetitions of 1 case is important for shuffle type FMPs.

Lemma 12.5. *Let* $\mathbf{k} = (k_1, \dots, k_r)$ *be an index,* $k = \mathrm{wt}(\mathbf{k})$*,* $\boldsymbol{t} = (t_1, \dots, t_r)$ *a tuple of indeterminates, and* $\bullet \in \{\emptyset, \star\}$*. Then we have*

$$\pounds^{\text{ш},\bullet}_{\widehat{\mathcal{A}},\mathbf{k}}(\boldsymbol{t}) = \pounds^{\text{ш},\bullet}_{\widehat{\mathcal{A}},\{1\}^k}(\{0\}^{k_1-1}, t_1, \dots, \{0\}^{k_r-1}, t_r).$$

Proof. We can easily check it by the definition of the finite shuffle multiple polylogarithms.

□

Proof that Theorem 12.4 *implies Theorem* 12.2. Let $\mathbf{k}_i = (k_1^{(i)}, \dots, k_{l_i}^{(i)})$ and $\mathbf{k}_i^\vee = ({k'}_1^{(i)}, \dots, {k'}_{l'_i}^{(i)})$ for $i = 1, \dots, r$. Put $k := \mathrm{wt}(\mathbf{k})$. Then

$$\begin{aligned}
&\mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}}(\{1\}^{l_1-1}, t_1, \dots, \{1\}^{l_r-1}, t_r) \\
&\overset{\text{Lem 12.5}}{=} \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^k}(\dots, \{0\}^{k_1^{(i)}-1}, 1, \dots, \{0\}^{k_{l_i-1}^{(i)}-1}, 1, \{0\}^{k_{l_i}^{(i)}-1}, t_i, \dots) \\
&\overset{\text{Thm 12.4}}{=} \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^k}(\dots, \{1\}^{k_1^{(i)}-1}, 0, \dots, \{1\}^{k_{l_i-1}^{(i)}-1}, 0, \{1\}^{k_{l_i}^{(i)}-1}, 1-t_i, \dots) \\
&\overset{(1.1.11)}{=} \mathcal{L}^\star_{\widehat{\mathcal{A}},\{1\}^k}(\dots, \{0\}^{{k'}_1^{(i)}-1}, 1, \dots, \{0\}^{{k'}_{l'_i-1}^{(i)}-1}, 1, \{0\}^{{k'}_{l'_i}^{(i)}-1}, 1-t_i, \dots) \\
&\overset{\text{Lem 12.5}}{=} \mathcal{L}^\star_{\widehat{\mathcal{A}},\mathbf{k}^*}(\{1\}^{l'_1-1}, 1-t_1, \dots, \{1\}^{l'_r-1}, 1-t_r).
\end{aligned}$$

Therefore, we have the conclusion. □

We prove Theorem 12.4. The following proposition is the key ingredient:

Proposition 12.6. *Let* p *be an odd prime number and* $\boldsymbol{t} = (t_1, \dots, t_k)$ *a tuple of indeterminates. Then we have the following* p*-adic expansion*:

$$\begin{aligned}
&\sum_{p-1 \geq n_1 \geq \cdots \geq n_k \geq 1} (-1)^{n_1} \binom{p-1}{n_1} \frac{t_1^{n_1-n_2} \cdots t_{k-1}^{n_{k-1}-n_k} t_k^{n_k}}{n_1 \cdots n_k} \\
&= \pounds^{\text{ш},\star}_{p-1,\{1\}^k}(\boldsymbol{t}) + \sum_{i=1}^\infty \left(\pounds^{\text{ш},\star}_{p-1,\{1\}^{k+i}}(\{1\}^i, \boldsymbol{t}) - \pounds^{\text{ш},\star}_{p-1,(\{1\}^{i-1},2,\{1\}^{k-1})}(\{1\}^{i-1}, \boldsymbol{t}) \right) p^i,
\end{aligned}$$

in the ring $\widehat{\mathbb{Z}[\boldsymbol{t}]}_p$.

We give two proofs of Proposition 12.6. The first one is author's original proof. The second one is by an anonymous referee of [46].

The first proof of Proposition 12.6

Lemma 12.7. *Let p be a prime number and n a positive integer less than p. Then the following equality holds*:

$$(-1)^n \binom{p-1}{n} = \sum_{i=0}^{\infty} (-1)^i H_n(\{1\}^i) p^i. \tag{64}$$

Proof. By the definition of a binomial coefficient, we can calculate as follows:

$$(-1)^n \binom{p-1}{n} = (-1)^n \frac{(p-1)(p-2)\cdots(p-n)}{1\cdot 2\cdots n} = \prod_{j=1}^{n}\left(1-\frac{p}{j}\right) = \sum_{i=0}^{n}(-1)^i H_n(\{1\}^i)p^i.$$

Since $H_n(\{1\}^i)$ is zero if i is greater than n, we obtain the equality (64). $\square$

The first proof of Proposition 12.6. By the substitution $n_i \mapsto p - n_i$ for every i satisfying $1 \le i \le k$ and the p-adic expansion formula (14), we have

$$\begin{aligned}
&\sum_{p-1\ge n_1\ge\cdots\ge n_k\ge 1} (-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}\\
&= \sum_{p-1\ge p-n_1\ge\cdots\ge p-n_k\ge 1} (-1)^{p-n_1}\binom{p-1}{p-n_1}\frac{t_1^{(p-n_1)-(p-n_2)}\cdots t_{k-1}^{(p-n_{k-1})-(p-n_k)}t_k^{p-n_k}}{(p-n_1)\cdots(p-n_k)}\\
&= (-1)^k \sum_{p-1\ge n_k\ge\cdots\ge n_1\ge 1} (-1)^{n_1-1}\binom{p-1}{n_1-1}\sum_{l_1,\dots,l_k\ge 0}\frac{t_1^{n_2-n_1}\cdots t_{k-1}^{n_k-n_{k-1}}t_k^{p-n_k}}{n_k^{l_k+1}\cdots n_1^{l_1+1}}p^{l_1+\cdots+l_k}\\
&= (-1)^k \sum_{p-1\ge n_k\ge\cdots\ge n_1\ge 1} (-1)^{n_1-1}\frac{n_1}{p-n_1}\binom{p-1}{n_1}\sum_{l_1,\dots,l_k\ge 0}\frac{t_1^{n_2-n_1}\cdots t_{k-1}^{n_k-n_{k-1}}t_k^{p-n_k}}{n_k^{l_k+1}\cdots n_1^{l_1+1}}p^{l_1+\cdots+l_k}.
\end{aligned}$$

By Lemma 12.7 and Lemma 4.2 (6),

$$\begin{aligned}
&\sum_{p-1\geq n_1\geq\cdots\geq n_k\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}\\
&=(-1)^k\sum_{p-1\geq n_k\geq\cdots\geq n_1\geq 1}\left(\sum_{j=0}^{\infty}\frac{p^j}{n_1^j}\right)\left(\sum_{i=0}^{\infty}(-1)^i H_{n_1}(\{1\}^i)p^i\right)\\
&\qquad\times\sum_{l_1,\ldots,l_k\geq 0}\frac{t_1^{n_2-n_1}\cdots t_{k-1}^{n_k-n_{k-1}}t_k^{p-n_k}}{n_k^{l_k+1}\cdots n_1^{l_1+1}}p^{l_1+\cdots+l_k}\\
&=(-1)^k\sum_{p-1\geq n_k\geq\cdots\geq n_1\geq 1}\left(\sum_{j=0}^{\infty}\frac{p^j}{n_1^j}\right)\left(\sum_{i=0}^{\infty}(-1)^i p^i\sum_{\mathrm{wt}(\mathbf{k})=i}(-1)^{i-\mathrm{dep}(\mathbf{k})}S_{n_1}(\mathbf{k})\right)\\
&\qquad\times\sum_{l_1,\ldots,l_k\geq 0}\frac{t_1^{n_2-n_1}\cdots t_{k-1}^{n_k-n_{k-1}}t_k^{p-n_k}}{n_k^{l_k+1}\cdots n_1^{l_1+1}}p^{l_1+\cdots+l_k}\\
&=(-1)^k\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{k_1,\ldots,k_l\geq 1\\ k_1+\cdots+k_l=i}}\sum_{l_1,\ldots,l_k\geq 0}(-1)^l p^{j+i+l_1+\cdots+l_k}\\
&\qquad\times\sum_{p-1\geq n_k\geq\cdots\geq n_1\geq m_l\geq\cdots\geq m_1\geq 1}\frac{t_1^{n_2-n_1}\cdots t_{k-1}^{n_k-n_{k-1}}t_k^{p-n_k}}{n_k^{l_k+1}\cdots n_2^{l_2+1}n_1^{l_1+j+1}m_l^{k_l}\cdots m_1^{k_1}}.
\end{aligned}$$

Further, by the substitution $n_k\mapsto p-n_k,\ldots,n_1\mapsto p-n_1, m_l\mapsto p-m_l,\ldots,m_1\mapsto p-m_1$

and the p-adic expansion formula (14),

$$\sum_{p-1\geq n_1\geq\cdots\geq n_k\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}$$

$$=(-1)^k\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{k_1,\dots,k_l\geq 1\\ k_1+\cdots+k_l=i}}\sum_{l_1,\dots,l_k\geq 0}(-1)^l p^{j+i+l_1+\cdots+l_k}\sum_{p-1\geq p-n_k\geq\cdots\geq p-n_1\geq p-m_l\geq\cdots\geq p-m_1\geq 1}$$

$$\times\frac{t_1^{(p-n_2)-(p-n_1)}\cdots t_{k-1}^{(p-n_k)-(p-n_{k-1})}t_k^{p-(p-n_k)}}{(p-n_k)^{l_k+1}\cdots(p-n_2)^{l_2+1}(p-n_1)^{l_1+j+1}(p-m_l)^{k_l}\cdots(p-m_1)^{k_1}}$$

$$=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{\substack{k_1,\dots,k_l\geq 1\\ k_1+\cdots+k_l=i}}\sum_{l_1,\dots,l_k\geq 0}(-1)^{l+j+i+l_1+\cdots+l_k}p^{j+i+l_1+\cdots+l_k}$$

$$\times\sum_{p-1\geq m_1\geq\cdots\geq m_l\geq n_1\geq\cdots\geq n_k\geq 1}t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}$$

$$\times\sum_{\substack{r_1,\dots,r_l\geq 0\\ s_1,\dots,s_k\geq 0}}\left[\prod_{a=1}^{l}\binom{k_a+r_a-1}{r_a}\binom{l_1+j+s_1}{s_1}\prod_{b=2}^{k}\binom{l_b+s_b}{s_b}\right]$$

$$\times\frac{p^{r_1+\cdots+r_l+s_1+\cdots+s_k}}{m_1^{k_1+r_1}\cdots m_l^{k_l+r_l}n_1^{l_1+j+s_1+1}n_2^{l_2+s_2+1}\cdots n_k^{l_k+s_k+1}}.$$

By collecting terms of the same indices, we have

$$\sum_{p-1\geq n_1\geq\cdots\geq n_k\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}$$

$$=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{l+n}p^n$$

(65)
$$\times\sum_{\substack{e_1,\dots,e_l,f_1,\dots,f_k\geq 1\\ e_1+\cdots+e_l+f_1+\cdots+f_k=n+k}}\prod_{a=1}^{l}\left[\sum_{r_a=0}^{e_a-1}(-1)^{r_a}\binom{e_a-1}{r_a}\right]\cdot\left[\sum_{s_1=0}^{f_1-1}\sum_{j=0}^{f_1-1-s_1}(-1)^{s_1}\binom{f_1-1}{s_1}\right]$$

$$\times\prod_{b=2}^{k}\left[\sum_{s_b=0}^{f_b-1}(-1)^{s_b}\binom{f_b-1}{s_b}\right]\sum_{p-1\geq m_1\geq\cdots\geq m_l\geq n_1\geq\cdots\geq n_k\geq 1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{m_1^{e_1}\cdots m_l^{e_l}n_1^{f_1}\cdots n_k^{f_k}}.$$

For $a = 1, \ldots, l$,

$$\sum_{r_a=0}^{e_a-1} (-1)^{r_a} \binom{e_a - 1}{r_a} = \begin{cases} 1 & \text{if } e_a = 1, \\ 0 & \text{if } e_a \geq 2, \end{cases}$$

holds and for $b = 2, \ldots, k$,

$$\sum_{s_b=0}^{f_b-1} (-1)^{s_b} \binom{f_b - 1}{s_b} = \begin{cases} 1 & \text{if } f_b = 1, \\ 0 & \text{if } f_b \geq 2, \end{cases}$$

holds. Furthermore, if f_1 is greater than or equal to 3, we have

$$\sum_{s_1=0}^{f_1-1} \sum_{j=1}^{f_1-1-s_1} (-1)^{s_1} \binom{f_1 - 1}{s_1} = \sum_{s_1=0}^{f_1-1} (-1)^{s_1+1} s_1 \binom{f_1 - 1}{s_1} = (f_1 - 1)(1 - 1)^{f_1-2} = 0.$$

If $f_1 = 1$ or 2, this summation is equal to 1. Hence, we see that all terms of the right hand side of (65) vanish except the cases $(e_1, \ldots, e_l, f_1, \ldots, f_k) = (\{1\}^{l+k})$ (then $l = n$) or $(e_1, \ldots, e_l, f_1, \ldots, f_k) = (\{1\}^l, 2, \{1\}^{k-1})$ (then $l = n - 1$). Therefore, we see that the desired identity holds. □

The second proof of Proposition 12.6

Lemma 12.8. *Let p be a prime number and n a positive integer satisfying $n < p$. Then we have the following p-adic expansion*:

$$(-1)^n \binom{p-1}{n} = (-1)^{p-1} \left(1 - \frac{p}{n}\right) \sum_{i=0}^{\infty} \sum_{p-1 \geq m_1 \geq \cdots \geq m_i \geq n} \frac{p^i}{m_1 \cdots m_i}.$$

Proof. We can calculate as follows:

$$\begin{aligned}
(-1)^n\binom{p-1}{n} &= (-1)^n\binom{p-1}{p-1-n} = (-1)^n\frac{p-n}{n}\binom{p-1}{p-n}\\
&= (-1)^{n-1}\left(1-\frac{p}{n}\right)\frac{(p-1)(p-2)\cdots n}{1\cdot 2\cdots(p-n)}\\
&= (-1)^{n-1}\left(1-\frac{p}{n}\right)\prod_{m=n}^{p-1}\frac{m}{p-m}\\
&= (-1)^{p-1}\left(1-\frac{p}{n}\right)\prod_{m=n}^{p-1}\left(1-\frac{p}{m}\right)^{-1}\\
&= (-1)^{p-1}\left(1-\frac{p}{n}\right)\prod_{m=n}^{p-1}\left(1+\frac{p}{m}+\frac{p^2}{m^2}+\cdots\right)\\
&= (-1)^{p-1}\left(1-\frac{p}{n}\right)\sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n}\frac{p^i}{m_1\cdots m_i}.
\end{aligned}$$

This completes the proof of the lemma. ∎

The second proof of Proposition 12.6. By Lemma 12.8, we have

$$\begin{aligned}
&\sum_{p-1\geq n_1\geq\cdots\geq n_k\geq 1}(-1)^{n_1}\binom{p-1}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}\\
&= \sum_{p-1\geq n_1\geq\cdots\geq n_k\geq 1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}\left(1-\frac{p}{n_1}\right)\sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n_1}\frac{p^i}{m_1\cdots m_i}\\
&= \sum_{i=0}^{\infty}\sum_{p-1\geq m_1\geq\cdots\geq m_i\geq n_1\geq\cdots n_k\geq 1}\left(1-\frac{p}{n_1}\right)\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{m_1\cdots m_i n_1\cdots n_k}p^i\\
&= \pounds^{\text{ш},\star}_{p-1,\{1\}^k}(\boldsymbol{t}) + \sum_{i=1}^{\infty}\left(\pounds^{\text{ш},\star}_{p-1,\{1\}^{k+i}}(\{1\}^i,\boldsymbol{t}) - \pounds^{\text{ш},\star}_{p-1,(\{1\}^{i-1},2,\{1\}^{k-1})}(\{1\}^{i-1},\boldsymbol{t})\right)p^i.
\end{aligned}$$

This completes the proof of the proposition. ∎

Proposition 12.9. *Let N and k be positive integers. Then the following polynomial identity holds in $\mathbb{Q}[t_1,\dots,t_k]$:*

$$\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}(t_k^{n_k}-\frac{1}{2})}{n_1\cdots n_k}$$
$$=\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{k-1})^{n_{k-1}-n_k}\{(1-t_k)^{n_k}-\frac{1}{2}\}}{n_1\cdots n_k}.$$

Proof. By Theorem 8.7 (43), we have

$$\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}t_k^{n_k}}{n_1\cdots n_k}$$
$$=\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{k-1})^{n_{k-1}-n_k}\{(1-t_k)^{n_k}-1\}}{n_1\cdots n_k},$$

and by substituting $t_k=1$, we have

$$\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}(-1)^{n_1}\binom{N}{n_1}\frac{t_1^{n_1-n_2}\cdots t_{k-1}^{n_{k-1}-n_k}}{n_1\cdots n_k}$$
$$=-\sum_{N\ge n_1\ge\cdots\ge n_k\ge 1}\frac{(1-t_1)^{n_1-n_2}\cdots(1-t_{k-1})^{n_{k-1}-n_k}}{n_1\cdots n_k}.$$

By combining these two identities, we obtain the desired identity. □

In order to prove Theorem 12.4, it is sufficient to show the following theorem:

Theorem 12.10. *Let n and k be positive integers and $\boldsymbol{t}$ a tuple of k indeterminates. We define $\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(\boldsymbol{t})$ to be*

$$\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(\boldsymbol{t}):=\sum_{i=0}^{n-1}\left(\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^{k+i}}(\{1\}^i,\boldsymbol{t})-\frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^{k+i}}(\{1\}^i,\boldsymbol{t}_1)\right)\boldsymbol{p}^i.$$

Then we have

$$\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(\boldsymbol{t})=\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}) \tag{66}$$

in $\mathcal{A}_{n,\mathbb{Z}[\boldsymbol{t}]}$.

Proof. We prove the equality (66) by induction on n. By combining Proposition 12.6 with Proposition 12.9, we have

$$\begin{aligned}
&\pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^k}(\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^k}(\boldsymbol{t}_1) \\
&\quad + \sum_{i=1}^{\infty}\Bigg\{\left(\pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^{k+i}}(\{1\}^i,\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^{k+i}}(\{1\}^i,\boldsymbol{t}_1)\right) \\
&\qquad\qquad - \left(\pounds^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^{i-1},2,\{1\}^{k-1})}(\{1\}^{i-1},\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\widehat{\mathcal{A}},(\{1\}^{i-1},2,\{1\}^{k-1})}(\{1\}^{i-1},\boldsymbol{t}_1)\right)\Bigg\}\boldsymbol{p}^i \\
&= \pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^k}(1-\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\widehat{\mathcal{A}},\{1\}^k}((1-\boldsymbol{t})_1).
\end{aligned} \tag{67}$$

We see that the equality (66) for $n=1$ holds by the projection $\pi_1\colon \widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]} \twoheadrightarrow \mathcal{A}_{\mathbb{Z}[\boldsymbol{t}]}$. We assume that the equality (66) for $n-1$ holds for any tuple of indeterminates with any depth. By the equality (67) and the projection $\pi_n\colon \widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]} \twoheadrightarrow \mathcal{A}_{n,\mathbb{Z}[\boldsymbol{t}]}$, we have

$$\begin{aligned}
\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(\boldsymbol{t}) &= \pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}((1-\boldsymbol{t})_1) \\
&\qquad + \sum_{i=1}^{n-1}\left(\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^{k+i}}(\{1\}^{i-1},t_0,\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^{k+i}}(\{1\}^{i-1},t_0,\boldsymbol{t}_1)\right)\boldsymbol{p}^i\Bigg|_{t_0=0} \\
&= \pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}((1-\boldsymbol{t})_1) + \mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^{k+1}}(t_0,\boldsymbol{t})\boldsymbol{p}\Big|_{t_0=0}.
\end{aligned} \tag{68}$$

On the other hand, by the induction hypothesis, we have

$$\mathcal{L}^{\star}_{\mathcal{A}_{n-1},\{1\}^{k+1}}(t_0,\boldsymbol{t}) = \mathcal{L}^{\star}_{\mathcal{A}_{n-1},\{1\}^{k+1}}(1-t_0,1-\boldsymbol{t}).$$

Therefore, by the equality (68) and the canonical isomorphism $\mathcal{A}_{n-1,\mathbb{Z}[\boldsymbol{t}]} \simeq \boldsymbol{p}\mathcal{A}_{n,\mathbb{Z}[\boldsymbol{t}]}$, we have

$$\begin{aligned}
\mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(\boldsymbol{t}) &= \pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}((1-\boldsymbol{t})_1) + \mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^{k+1}}(t_0,\boldsymbol{t})\boldsymbol{p}\Big|_{t_0=0} \\
&= \pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}) - \frac{1}{2}\pounds^{ш,\star}_{\mathcal{A}_n,\{1\}^k}((1-\boldsymbol{t})_1) + \mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^{k+1}}(1-t_0,1-\boldsymbol{t})\boldsymbol{p}\Big|_{t_0=0} \\
&= \mathcal{L}^{\star}_{\mathcal{A}_n,\{1\}^k}(1-\boldsymbol{t}).
\end{aligned}$$

Hence, the equality (66) for n also holds. □

Thus, we have finished the proof of Theorem 12.2.

12.3 Relation between $\widehat{\mathcal{A}}$-finite harmonic multiple and star-multiple polylogarithms

Theorem 12.11. *Let* $\mathbf{k} = (k_1, \ldots, k_r)$ *be an index. Then we have the following* $\boldsymbol{p}$*-adic relation in* $\widehat{\mathcal{A}}_{\mathbb{Z}[\boldsymbol{t}]}$:

$$\sum_{j=0}^{r} (-1)^j \pounds^{*}_{\widehat{\mathcal{A}},\mathbf{k}_{(j)}}(\boldsymbol{t}_{(j)}) \pounds^{*,\star}_{\widehat{\mathcal{A}},\overline{\mathbf{k}^{(j)}}}(\overline{\boldsymbol{t}^{(j)}}) = 0.$$

Proof. This is an immediate consequence of Theorem 8.11. □

12.4 Summary of fundamental relations of $\mathcal{A}$-FMPs and $\mathcal{A}_2$-FMPs

We summarize the fundamental relations of $\mathcal{A}$-FMPs and $\mathcal{A}_2$-FMPs in non-symmetrical forms in order to refer them when we calculate special values of $\mathcal{A}$-FMPs and $\mathcal{A}_2$-FMPs.

Corollary 12.12 (Fundamental relations for one-variable $\mathcal{A}$-FMPs)**.** *Let* $k, r,$ *and* i *be positive integers satisfying* $1 \le i \le r$, $\mathbf{k}$ *an index, and* $\bullet \in \{\emptyset, \star\}$*. Then we have*

$$\pounds^{\bullet}_{\mathcal{A},\mathbf{k}}(t) = (-1)^{\mathrm{wt}(\mathbf{k})} t^{\boldsymbol{p}} \widetilde{\pounds}^{\bullet}_{\mathcal{A},\overline{\mathbf{k}}}(t^{-1}), \tag{69}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\mathbf{k}}(t) = \widetilde{\pounds}^{\star}_{\mathcal{A},\mathbf{k}^{\vee}}(1-t) - \zeta^{\star}_{\mathcal{A}}(\mathbf{k}^{\vee}), \tag{70}$$

$$(-1)^{\mathrm{dep}(\mathbf{k})-1} \pounds_{\mathcal{A},\mathbf{k}}(t) = \widetilde{\pounds}^{\star}_{\mathcal{A},\overline{\mathbf{k}}}(t) + \sum_{j=1}^{\mathrm{dep}(\mathbf{k})-1} (-1)^j \pounds_{\mathcal{A},\mathbf{k}_{(j)}}(t) \zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}^{(j)}}), \tag{71}$$

$$(-1)^{\mathrm{dep}(\mathbf{k})-1} \widetilde{\pounds}_{\mathcal{A},\mathbf{k}}(t) = \pounds^{\star}_{\mathcal{A},\overline{\mathbf{k}}}(t) + \sum_{j=1}^{\mathrm{dep}(\mathbf{k})-1} (-1)^j \zeta_{\mathcal{A}}(\mathbf{k}_{(j)}) \pounds^{\star}_{\mathcal{A},\overline{\mathbf{k}^{(j)}}}(t), \tag{72}$$

$$\pounds^{*}_{\mathcal{A},\{k\}^r}(\{1\}^{i-1}, t, \{1\}^{r-i}) + (-1)^r \pounds^{*,\star}_{\mathcal{A},\{k\}^r}(\{1\}^{r-i}, t, \{1\}^{i-1}) = 0. \tag{73}$$

Corollary 12.13 (Fundamental relations for multi-variable $\mathcal{A}$-FMPs)**.** *Let* $\mathbf{k}=(k_1,\dots,k_r)$ *be an index,* $\boldsymbol{t}=(t_1,\dots,t_r)$ *a tuple of indeterminates, and* $\bullet\in\{\emptyset,\star\}$*. Let* $\mathbf{k}_1,\dots,\mathbf{k}_r$ *be indices,* $\Bbbk:=(\mathbf{k}_1,\dots,\mathbf{k}_r)$*,* $\Bbbk^*:=(\mathbf{k}_1^\vee,\dots,\mathbf{k}_r^\vee)$*, and* $l_i:=\mathrm{dep}(\mathbf{k}_i), l'_i:=\mathrm{dep}(\mathbf{k}_i^\vee)$ *for* $i=1,\dots,r$*. Then we have*

$$\pounds^{*,\bullet}_{\mathcal{A},\overline{\mathbf{k}}}(\boldsymbol{t})=(-1)^{\mathrm{wt}(\mathbf{k})}(t_1\cdots t_r)^{\boldsymbol{p}}\pounds^{*,\bullet}_{\mathcal{A},\mathbf{k}}(\overline{\boldsymbol{t}^{-1}}),\tag{74}$$

$$\begin{aligned}&\pounds^{\text{ш},\star}_{\mathcal{A},\Bbbk}(\{1\}^{l_1-1},t_1,\dots,\{1\}^{l_r-1},t_r)\\&=\pounds^{\text{ш},\star}_{\mathcal{A},\Bbbk^*}(\{1\}^{l'_1-1},1-t_1,\dots,\{1\}^{l'_r-1},1-t_r)-\pounds^{\text{ш},\star}_{\mathcal{A},\Bbbk^*}(\{1\}^{l'_1-1},1-t_1,\dots,\{1\}^{l'_r}),\end{aligned}\tag{75}$$

$$(-1)^{r-1}\pounds^{*}_{\mathcal{A},\mathbf{k}}(\boldsymbol{t})=\pounds^{*,\star}_{\mathcal{A},\overline{\mathbf{k}}}(\overline{\boldsymbol{t}^{-1}})+\sum_{j=1}^{r-1}(-1)^j\pounds^{*}_{\mathcal{A},\mathbf{k}_{(j)}}(\boldsymbol{t}_{(j)})\pounds^{*,\star}_{\mathcal{A},\overline{\mathbf{k}^{(j)}}}(\overline{\boldsymbol{t}^{(j)}}).\tag{76}$$

Corollary 12.14 (Fundamental relations for one-variable $\mathcal{A}_2$-FMPs)**.** *Let* $\mathbf{k}=(k_1,\dots,k_r)$ *be an index,* t *an indeterminate, and* $\bullet\in\{\emptyset,\star\}$*. Then we have*

$$\pounds^{\bullet}_{\mathcal{A}_2,\overline{\mathbf{k}}}(t)=(-1)^{\mathrm{wt}(\mathbf{k})}t^{\boldsymbol{p}}\left(\widetilde{\pounds}^{\bullet}_{\mathcal{A}_2,\mathbf{k}}(t^{-1})+\sum_{j=1}^{r}k_j\widetilde{\pounds}^{\bullet}_{\mathcal{A}_2,\mathbf{k}\oplus\mathbf{e}_j}(t^{-1})\boldsymbol{p}\right),\tag{77}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\mathbf{k}}(t)+\left(\widetilde{\pounds}^{\star}_{\mathcal{A}_2,(1,\mathbf{k})}(t)-\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\mathbf{e}_1\oplus\mathbf{k}}(t)\right)\boldsymbol{p}=\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\mathbf{k}^\vee}(1-t)-\zeta^{\star}_{\mathcal{A}_2}(\mathbf{k}^\vee),\tag{78}$$

$$(-1)^{r-1}\pounds_{\mathcal{A}_2,\mathbf{k}}(t)=\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\overline{\mathbf{k}}}(t)+\sum_{j=1}^{r-1}(-1)^j\pounds_{\mathcal{A}_2,\mathbf{k}_{(j)}}(t)\zeta^{\star}_{\mathcal{A}_2}(\overline{\mathbf{k}^{(j)}}),\tag{79}$$

$$(-1)^{r-1}\widetilde{\pounds}_{\mathcal{A}_2,\mathbf{k}}(t)=\pounds^{\star}_{\mathcal{A}_2,\overline{\mathbf{k}}}(t)+\sum_{j=1}^{r-1}(-1)^j\zeta_{\mathcal{A}_2}(\mathbf{k}_{(j)})\pounds^{\star}_{\mathcal{A}_2,\overline{\mathbf{k}^{(j)}}}(t).\tag{80}$$

Corollary 12.15 (Fundamental relations for multi-variable $\mathcal{A}_2$-FMPs)**.** *Let* $\mathbf{k} = (k_1, \dots, k_r)$ *be an index,* $\boldsymbol{t} = (t_1, \dots, t_r)$ *a tuple of indeterminates, and* $\bullet \in \{\emptyset, \star\}$*. Let* $\mathbf{k}_1, \dots, \mathbf{k}_r$ *be indices,* $\Bbbk := (\mathbf{k}_1, \dots, \mathbf{k}_r)$*,* $\Bbbk^* := (\mathbf{k}_1^\vee, \dots, \mathbf{k}_r^\vee)$*, and* $l_i := \mathrm{dep}(\mathbf{k}_i), l_i' := \mathrm{dep}(\mathbf{k}_i^\vee)$ *for* $i = 1, \dots, r$*. Then we have*

$$\pounds^{*,\bullet}_{\mathcal{A}_2,\overline{\mathbf{k}}}(\boldsymbol{t}) = (-1)^{\mathrm{wt}(\mathbf{k})}(t_1 \cdots t_r)^{\boldsymbol{p}} \left(\pounds^{*,\bullet}_{\mathcal{A}_2,\mathbf{k}}(\overline{\boldsymbol{t}^{-1}}) + \sum_{j=1}^{r} k_j \pounds^{*,\bullet}_{\mathcal{A}_2,\mathbf{k}\oplus\mathbf{e}_j}(\overline{\boldsymbol{t}^{-1}})\boldsymbol{p} \right), \tag{81}$$

$$\begin{aligned} &\pounds^{\text{ш},\star}_{\mathcal{A}_2,\Bbbk}(\{1\}^{l_1-1}, t_1, \dots, \{1\}^{l_r-1}, t_r) \\ &+ \left(\pounds^{\text{ш},\star}_{\mathcal{A}_2,(1,\Bbbk)}(\{1\}^{l_1}, t_1, \dots, \{1\}^{l_r-1}, t_r) - \pounds^{\text{ш},\star}_{\mathcal{A}_2,(\mathbf{e}_1\oplus\Bbbk)}(\{1\}^{l_1-1}, t_1, \dots, \{1\}^{l_r-1}, t_r) \right) \boldsymbol{p} \\ &= \pounds^{\text{ш},\star}_{\mathcal{A}_2,\Bbbk^*}(\{1\}^{l_1'-1}, 1-t_1, \dots, \{1\}^{l_r'-1}, 1-t_r) - \pounds^{\text{ш},\star}_{\mathcal{A}_2,\Bbbk^*}(\{1\}^{l_1'-1}, 1-t_1, \dots, \{1\}^{l_r'}), \end{aligned} \tag{82}$$

$$(-1)^{r-1} \pounds^{*}_{\mathcal{A}_2,\mathbf{k}}(\boldsymbol{t}) = \pounds^{*,\star}_{\mathcal{A}_2,\overline{\mathbf{k}}}(\overline{\boldsymbol{t}^{-1}}) + \sum_{j=1}^{r-1} (-1)^j \pounds^{*}_{\mathcal{A}_2,\mathbf{k}_{(j)}}(\boldsymbol{t}_{(j)}) \pounds^{*,\star}_{\mathcal{A}_2,\overline{\mathbf{k}^{(j)}}}(\overline{\boldsymbol{t}^{(j)}}). \tag{83}$$

Proof. These corollaries are deduced from our main results immediately. Note that we use Proposition 7.1 (17) for a proof of the equality (73). The equality (73) was also proved by Tauraso–J. Zhao ([54, Lemma 5.9]). □

Theorem 12.16. *Let* $\mathbf{k}$ *be an index. Then we have*

$$\pounds^{\star}_{\mathcal{A},\mathbf{k}}(t) = (t^{\boldsymbol{p}} - 1)\pounds^{\star}_{\mathcal{A},\mathbf{k}^\vee}\left(\frac{t}{t-1}\right) - t^{\boldsymbol{p}} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}^\vee)$$

in $\mathcal{A}_{\mathbb{Z}[t,(t-1)^{-1}]}$.

Proof. By the reversal relation (69) and the functional equation (70), we can calculate as follows:

$$\begin{aligned} \pounds^{\star}_{\mathcal{A},\mathbf{k}}(t) &= (-1)^{\mathrm{wt}(\mathbf{k})} t^{\boldsymbol{p}} \widetilde{\pounds}^{\star}_{\mathcal{A},\overline{\mathbf{k}}}(t^{-1}) \\ &= (-1)^{\mathrm{wt}(\mathbf{k})} t^{\boldsymbol{p}} \left(\widetilde{\pounds}^{\star}_{\mathcal{A},\overline{\mathbf{k}}^\vee}(1 - t^{-1}) - \zeta^{\star}_{\mathcal{A}}(\overline{\mathbf{k}}^\vee) \right) \\ &= t^{\boldsymbol{p}}(1 - t^{-1})^{\boldsymbol{p}} \pounds^{\star}_{\mathcal{A},\mathbf{k}^\vee}((1 - t^{-1})^{-1}) - t^{\boldsymbol{p}} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}^\vee) \\ &= (t^{\boldsymbol{p}} - 1)\pounds^{\star}_{\mathcal{A},\mathbf{k}^\vee}\left(\frac{t}{t-1}\right) - t^{\boldsymbol{p}} \zeta^{\star}_{\mathcal{A}}(\mathbf{k}^\vee). \end{aligned}$$

□

Remark 12.17. This is a finite analogue of the duality formula for star-multiple polylogarithms (Theorem 3.3).

13 Functional equations of finite multiple polylogarithms of index $\{1\}^k$

In this subsection, we argue about functional equations of 1-variable $\mathcal{A}$-FMPs of the index $\{1\}^k$.

Lemma 13.1. *Let k be a positive integer. Then we have*

$$\pounds_{\mathcal{A},\{1\}^k}(t) = (-1)^{k-1}\pounds_{\mathcal{A},k}(1-t), \tag{84}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^k}(t) = \pounds_{\mathcal{A},k}(1-t), \tag{85}$$

$$\widetilde{\pounds}_{\mathcal{A},\{1\}^k}(t) = (-1)^{k-1}(t^{\boldsymbol{p}}-1)\pounds_{\mathcal{A},k}\left(\frac{t}{t-1}\right), \tag{86}$$

$$\pounds^{\star}_{\mathcal{A},\{1\}^k}(t) = (t^{\boldsymbol{p}}-1)\pounds_{\mathcal{A},k}\left(\frac{t}{t-1}\right) \tag{87}$$

in $\mathcal{A}_{\mathbb{Z}[t,(t-1)^{-1}]}$.

Proof. The equalities (85) and (87) are obtained as corollaries of the equality (70) and Theorem 12.16, respectively. The equality (84) is obtained by (71) (or (73)) and (85). The equality (86) is obtained by (72) (or (73)) and (87). □

This lemma says that the above four types of $\mathcal{A}$-finite multiple polylogarithms are essentially $\mathcal{A}$-finite polylogarithm. Therefore we have the following principle:

Principle 13.2. *We can obtain functional equations of $\mathcal{A}$-finite multiple polylogarithms $\pounds_{\mathcal{A},\{1\}^k}$, $\widetilde{\pounds}_{\mathcal{A},\{1\}^k}$, $\pounds^{\star}_{\mathcal{A},\{1\}^k}$, and $\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^k}$ from functional equations of $\mathcal{A}$-finite polylogarithm $\pounds_{\mathcal{A},k}$ via Lemma* 13.1.

Theorem 13.3 (Distribution properties for FMPs of the index $\{1\}^k$)**.** *Let m be a non-zero integer and k a positive integer. Let ζ_m be a primitive $|m|$-th root of unity. Then the following* *equalities hold in* $\mathcal{A}_{\mathbb{Z}[\zeta_m][\![t]\!]}$:

$$\pounds_{\mathcal{A},\{1\}^k}(1-t^m) = m^{k-1}\sum_{j=0}^{|m|-1}\frac{1-t^{m\boldsymbol{p}}}{1-(\zeta_m^j t)^{\boldsymbol{p}}}\pounds_{\mathcal{A},\{1\}^k}(1-\zeta_m^j t), \tag{88}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^k}(1-t^m) = m^{k-1}\sum_{j=0}^{|m|-1}\frac{1-t^{m\boldsymbol{p}}}{1-(\zeta_m^j t)^{\boldsymbol{p}}}\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^k}(1-\zeta_m^j t), \tag{89}$$

$$\widetilde{\pounds}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-t^m}\right) = m^{k-1}\sum_{j=0}^{|m|-1}\widetilde{\pounds}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-\zeta_m^j t}\right), \tag{90}$$

$$\pounds^{\star}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-t^m}\right) = m^{k-1}\sum_{j=0}^{|m|-1}\pounds^{\star}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-\zeta_m^j t}\right). \tag{91}$$

Proof. This theorem is obtained by Proposition 10.4 and Principle 13.2. Here, we only show the formula (91). By the equality (87) and Proposition 10.4, we can calculate as follows:

$$\begin{aligned}
\pounds^{\star}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-t^m}\right) &= \frac{t^{m\boldsymbol{p}}}{1-t^{m\boldsymbol{p}}}\pounds_{\mathcal{A},k}(t^{-m}) \\
&= \frac{t^{m\boldsymbol{p}}}{1-t^{m\boldsymbol{p}}}m^{k-1}\sum_{j=0}^{|m|-1}\frac{1-t^{-m\boldsymbol{p}}}{1-(\zeta_m^j t^{-1})^{\boldsymbol{p}}}\pounds_{\mathcal{A},k}(\zeta_m^j t^{-1}) \\
&= m^{k-1}\sum_{j=0}^{|m|-1}\frac{(\zeta_m^{-j}t)^{\boldsymbol{p}}}{1-(\zeta_m^{-j}t)^{\boldsymbol{p}}}\pounds_{\mathcal{A},k}((\zeta_m^{-j}t)^{-1}) \\
&= m^{k-1}\sum_{j=0}^{|m|-1}\pounds^{\star}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-\zeta_m^{-j}t}\right) \\
&= m^{k-1}\sum_{j=0}^{|m|-1}\pounds^{\star}_{\mathcal{A},\{1\}^k}\left(\frac{1}{1-\zeta_m^{j}t}\right).
\end{aligned}$$

□

Corollary 13.4. *Let k be a positive integer. Then the following functional equations hold in* $\mathcal{A}_{\mathbb{Z}[t]}$:

$$\widetilde{\pounds}_{\mathcal{A},\{1\}^k}(t) = (-1)^{k-1}\widetilde{\pounds}_{\mathcal{A},\{1\}^k}(1-t), \tag{92}$$

$$\pounds^{\star}_{\mathcal{A},\{1\}^k}(t) = (-1)^{k-1}\pounds^{\star}_{\mathcal{A},\{1\}^k}(1-t). \tag{93}$$

Proof. Let $m = -1$ in Theorem 13.3. Then we have the desired formulas by replacing $t/(1-t)$ with t. $\square$

Remark 13.5. Lemma 13.1 (84) has been proved by Mattarei and Tauraso ([52, The proof of Theorem 2.3], [26, Lemma 3.2]) and Lemma 13.4 (92) has been proved by L. L. Zhao and Z. W. Sun ([61, Theorem 1.2]).

By Theorem 10.5 and Principle 13.2, we see that each of $\mathcal{A}$-finite $(1,1)$-polylogarithms $\pounds_{\mathcal{A},(1,1)}$, $\widetilde{\pounds}_{\mathcal{A},(1,1)}$, $\pounds^{\star}_{\mathcal{A},(1,1)}$, and $\widetilde{\pounds}^{\star}_{\mathcal{A},(1,1)}$ satisfies a 22-term relation. Here, we only state a 22-term relation for $\pounds^{\star}_{\mathcal{A},(1,1)}$.

Theorem 13.6. *Let $s, t,$ and u be indeterminates. Then we have the following functional equation for $\pounds^{\star}_{\mathcal{A},(1,1)}$:*

$$\begin{aligned}
&u^{\boldsymbol{p}}(s-1)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{s}{s-1}\right)-u^{\boldsymbol{p}}(t-1)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{t}{t-1}\right)\\
&+(s-t+1)^{\boldsymbol{p}}(u-1)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{u}{u-1}\right)\\
&-s^{\boldsymbol{p}}(1-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{s-1}{s}\right)+t^{\boldsymbol{p}}(1-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{t-1}{t}\right)-u^{\boldsymbol{p}}(t-s)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{u-1}{u}\right)\\
&-(u-s)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{u}{u-s}\right)+(u-t)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{u}{u-t}\right)+u^{\boldsymbol{p}}(t-s)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{t}{t-s}\right)\\
&-(s-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{1-u}{s-u}\right)+(t-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{1-u}{t-u}\right)+u^{\boldsymbol{p}}(s-t)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{1-t}{s-t}\right)\\
&+(s-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{s(1-u)}{s-u}\right)-(t-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{t(1-u)}{t-u}\right)\\
&-(us-t)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{us}{us-t}\right)-(u(1-s)-(1-t))^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{u(1-s)}{u(1-s)-(1-t)}\right)\\
&-t^{\boldsymbol{p}}(1-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{t-s}{t}\right)-(1-t)^{\boldsymbol{p}}(1-u)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{s-t}{1-t}\right)\\
&-(t-us)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{(1-u)s}{t-us}\right)-((1-t)-u(1-s))^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{(1-u)(1-s)}{(1-t)-u(1-s)}\right)\\
&+(t-us)^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{(1-u)t}{t-us}\right)+((1-t)-u(1-s))^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A},(1,1)}\left(\frac{(1-u)(1-t)}{(1-t)-u(1-s)}\right)\\
&=0.
\end{aligned}$$

Remark 13.7. Besser proved a formula relating Coleman's p-adic polylogarithms and the finite polylogarithms in [3]. His formula plays a key role in Elbaz-Vincent–Gangl's theory. On the other hand, the author and Sakugawa proved a formula relating Wojtkowiak's étale polylogarithms and the finite polylogarithms in [45]. Surprisingly, in the proof of the main theorem of [45], we use the functional equation (87).

14 Ono–Yamamoto's finite multiple polylogarithms

Definition 14.1 (Ono–Yamamoto [35])**.** Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index. Then *Ono-Yamamoto's finite multiple polylogarithm* $\pounds^{\mathrm{OY}}_{\mathcal{A},\mathbf{k}}(t) \in \mathcal{A}_{\mathbb{Z}[t]}$ is defined by

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\mathbf{k}}(t) := \left({\sum_{0<l_1,\dots,l_r<p}}' \frac{t^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \bmod p \right)_p,$$

where the summation $\sum'$ runs over only fractions whose denominators are prime to p.

We prepare the following notations to discuss the relation between Ono–Yamamoto's FMPs and our $\mathcal{A}$-FMPs (cf. [35, Section 2]):

$$[l] := \{1, \dots, l\},$$

$$X_r^{(p)} := \{\mathbf{l} = (l_1, \dots, l_r) \in [p-1]^r \mid (\mathrm{wt}(\mathbf{l}_{(1)}), p) = \cdots = (\mathrm{wt}(\mathbf{l}_{(r)}), p) = 1\},$$

$$\alpha \colon X_r^{(p)} \to [r] \text{ is defined by } \alpha(l_1, \dots, l_r) = n \text{ such that } (n-1)p < l_1 + \cdots + l_r < np,$$

$$X_{r,i}^{(p)} := \alpha^{-1}(i),$$

$$\Phi_{r,l} := \{\phi \colon [r] \to [l] : \text{surjective} \mid \phi(a) \neq \phi(a+1) \text{ for all } a \in [r-1]\},$$

$$r_\phi := l \text{ when } \phi \in \Phi_{r,l},$$

$$\Phi_r := \bigsqcup_{l=1}^{r} \Phi_{r,l},\ \ \delta_\phi(i) := \#\{a \in [i-1] \mid \phi(a) > \phi(a+1)\} \text{ for } \phi \in \Phi_r,$$

$$\beta \colon \Phi_r \to [r] \text{ is defined by } \beta(\phi) := \delta_\phi(r) + 1,\ \ \Phi_r^i := \beta^{-1}(i),$$

$$Y_l^{(p)} := \{(n_1, \dots, n_l) \in [p-1]^l \mid 1 \le n_1 < \cdots < n_l \le p-1\},$$

where i, l, and r are positive integers satisfying $1 \le i, l \le r$ and p is a prime number.

Lemma 14.2 ([35, Lemma 2.3])**.** *For any $x = (l_1, \dots, l_r) \in X_r^{(p)}$, there exist unique $l \in [r]$, $\phi \in \Phi_{r,l}$, and $(n_1, \dots, n_l) \in Y_l^{(p)}$ such that $l_1 + \cdots + l_i = n_{\phi(i)} + \delta_\phi(i)p$ for any $i = 1, \dots, r$.*

In the above situation, we use the notation $\phi_x := \phi$. Further notations:

$$X_{\phi}^{(p)} := \{x \in X_r^{(p)} \mid \phi_x = \phi\} \text{ for } \phi \in \Phi_r,$$

$$\mathbf{k}_\phi := \Bigl(\sum_{\phi(j)=1} k_j, \ldots, \sum_{\phi(j)=r_\phi} k_j\Bigr) \text{ for an index } \mathbf{k} = (k_1, \ldots, k_r) \text{ and } \phi \in \Phi_r.$$

Note that $X_{r,i}^{(p)} = \bigsqcup_{\phi \in \Phi_r^i} X_\phi^{(p)}$ for $i = 1, \ldots, r$.

Proposition 14.3. *Let* $\mathbf{k}$ *be an index of* $\operatorname{dep}(\mathbf{k}) = r$. *Then we have*

$$\pounds_{\mathcal{A},\mathbf{k}}^{\mathrm{OY}}(t) = \sum_{i=1}^{r} t^{(i-1)\boldsymbol{p}} \sum_{\phi \in \Phi_r^i} \pounds^*_{\mathcal{A},\overline{\mathbf{k}_\phi}}(\{1\}^{r_\phi - \phi(r)}, t, \{1\}^{\phi(r)-1}). \tag{94}$$

Proof. Let p be a prime number. By the above notations and Lemma 14.2, we have

$$\begin{aligned}
&\sideset{}{'}\sum_{0<l_1,\ldots,l_r<p} \frac{t^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \\
&= \sum_{i=1}^{r} \sum_{(l_1,\ldots,l_r)\in X_{r,i}^{(p)}} \frac{t^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \\
&= \sum_{i=1}^{r} \sum_{\phi \in \Phi_r^i} \sum_{(l_1,\ldots,l_r)\in X_{\phi}^{(p)}} \frac{t^{l_1+\cdots+l_r}}{l_1^{k_1}(l_1+l_2)^{k_2}\cdots(l_1+\cdots+l_r)^{k_r}} \\
&\equiv \sum_{i=1}^{r} t^{(i-1)p} \sum_{\phi \in \Phi_r^i} \sum_{1\le n_1<\cdots<n_{r_\phi}\le p-1} \frac{t^{n_{\phi(r)}}}{n_{\phi(1)}^{k_1}\cdots n_{\phi(r)}^{k_r}} \\
&= \sum_{i=1}^{r} t^{(i-1)p} \sum_{\phi \in \Phi_r^i} \sum_{1\le n_1<\cdots<n_{r_\phi}\le p-1} \frac{t^{n_{\phi(r)}}}{n_1^{\sum_{\phi(j)=1} k_j}\cdots n_{r_\phi}^{\sum_{\phi(j)=r_\phi} k_j}} \\
&= \sum_{i=1}^{r} t^{(i-1)p} \sum_{\phi \in \Phi_r^i} \pounds^*_{p-1,\overline{\mathbf{k}_\phi}}(\{1\}^{r_\phi-\phi(r)}, t, \{1\}^{\phi(r)-1}) \pmod{p}.
\end{aligned}$$

Therefore, we complete the proof. $\square$

Corollary 14.4. *Let $k_1, k_2,$ and k_3 be positive integers. Then we have*

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(k_1,k_2)}(t) = \pounds_{\mathcal{A},(k_2,k_1)}(t) + t^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A},(k_1,k_2)}(t), \tag{95}$$

$$\begin{aligned} \pounds^{\mathrm{OY}}_{\mathcal{A},(k_1,k_2,k_3)}(t) &= \pounds_{\mathcal{A},(k_3,k_2,k_1)}(t) + t^{\boldsymbol{p}}\pounds_{\mathcal{A},(k_3,k_1,k_2)}(t) + t^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A},(k_2,k_1,k_3)}(t) \\ &\quad + t^{\boldsymbol{p}}\pounds^{*}_{\mathcal{A},(k_2,k_3,k_1)}(1,t,1) + t^{\boldsymbol{p}}\pounds^{*}_{\mathcal{A},(k_1,k_3,k_2)}(1,t,1) \\ &\quad + t^{\boldsymbol{p}}\pounds_{\mathcal{A},(k_1+k_3,k_2)}(t) + t^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A},(k_2,k_1+k_3)}(t) + t^{2\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A},(k_1,k_2,k_3)}(t). \end{aligned} \tag{96}$$

Proposition 14.5. *Let $\mathbf{k} = (k_1, \dots, k_r)$ be an index. Then we have*

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\mathbf{k}}(1) = 0.$$

Proof. If $r = 1$, we have $\pounds^{\mathrm{OY}}_{\mathcal{A},\mathbf{k}}(1) = \zeta_{\mathcal{A}}(k_1) = 0$. We assume that r is greater than 1. Let l be one of $2, \dots, r$ and $\mathfrak{S}_l$ the l-th symmetric group. We define an equivalence relation on $\Phi_{r,l}$ as follows: $\phi \sim \phi'$ holds for $\phi, \phi' \in \Phi_{r,l}$ if and only if there exists $\sigma \in \mathfrak{S}_l$ such that $\phi = \sigma \circ \phi'$ holds. We take and fix a system of representatives $\{\phi_{l,1}, \dots \phi_{l,i_l}\}$ of the quotient set $\Phi_{r,l}/\mathfrak{S}_l$ where i_l is the cardinality of $\Phi_{r,l}/\mathfrak{S}_l$. Then, by Proposition 14.3, we have

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\mathbf{k}}(1) = \sum_{\phi\in\Phi_r} \zeta_{\mathcal{A}}(\overline{\mathbf{k}_\phi}) = \sum_{l=2}^{r} \sum_{\phi\in\Phi_{r,l}} \zeta_{\mathcal{A}}(\overline{\mathbf{k}_\phi}) = \sum_{l=2}^{r} \sum_{s=1}^{i_l} \left(\sum_{\sigma\in\mathfrak{S}_l} \zeta_{\mathcal{A}}(\sigma(\overline{\mathbf{k}_{\phi_{l,s}}})) \right).$$

We see that this is zero by Proposition 5.1 (7). $\square$

We prove the following functional equations for Ono–Yamamoto's finite multiple polylogarithms:

Theorem 14.6.

$$\pounds^{\mathrm{OY}}_{\mathcal{A},1}(t) = \pounds^{\mathrm{OY}}_{\mathcal{A},1}(1-t), \quad \pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(t) = \pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(1-t).$$

Proof. Since $\pounds^{\mathrm{OY}}_{\mathcal{A},1}(t) = \pounds_{\mathcal{A},1}(t)$, the first one is exactly the functional equation (57). So, we prove the second one. By the equality (95), we have

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(t) = \pounds_{\mathcal{A},(1,1)}(t) + t^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A},(1,1)}(t).$$

On the other hand, by (84) and (92), we have

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(1-t) = \pounds_{\mathcal{A},(1,1)}(1-t) + (1-t^{\boldsymbol{p}})\widetilde{\pounds}_{\mathcal{A},(1,1)}(1-t) = -\pounds_{\mathcal{A},2}(t) - (1-t^{\boldsymbol{p}})\widetilde{\pounds}_{\mathcal{A},(1,1)}(t).$$

Hence,

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(t) - \pounds^{\mathrm{OY}}_{\mathcal{A},(1,1)}(1-t) = \pounds_{\mathcal{A},(1,1)}(t) + \pounds_{\mathcal{A},2}(t) + \widetilde{\pounds}_{\mathcal{A},(1,1)}(t) = \zeta_{\mathcal{A}}(1)\pounds_{\mathcal{A},1}(t) = 0.$$

This completes the proof. $\square$

Remark 14.7. Recently, Ono proved the functional equation $\pounds^{\mathrm{OY}}_{\mathcal{A},(1,1,1)}(t) = \pounds^{\mathrm{OY}}_{\mathcal{A},(1,1,1)}(1-t)$ and more general functional equations in [34]. His result suggests that $\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^k}(t) = \pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^k}(1-t)$ does not hold in general. In fact, he proved $\pounds^{\mathrm{OY}}_{\mathcal{A},(1,1,1,1)}(t) \neq \pounds^{\mathrm{OY}}_{\mathcal{A},(1,1,1,1)}(1-t)$ under the hypothesis that $B_{\boldsymbol{p}-5} \neq 0$ in $\mathcal{A}$.

15 Explicit evaluations of finite alternating multiple zeta values

We call values obtained by substituting ± 1 into the variables of finite multiple polylogarithms *the finite alternating multiple zeta values.* In order to calculate special values of finite multiple polylogarithms in the next section, we summarize known results on finite alternating multiple zeta values.

15.1 Calculations in general weights

Lemma 15.1 (Chamberland–Dilcher [4], Tauraso–J. Zhao [54, Corollary 2.3])**.** *Let k be a positive integer greater than* 1*. Then*

$$\pounds_{\mathcal{A},k}(-1) = \frac{1-2^{k-1}}{2^{k-2}}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{97}$$

Lemma 15.2 (Chamberland–Dilcher [4], Tauraso–J. Zhao [54, Theorem 3.1 (17), Theorem 3.1 (18)])**.** *Let k_1 and k_2 be positive integers such that $k := k_1 + k_2$ is odd. Then we have the following equalities*:

$$\pounds_{\mathcal{A},(k_1,k_2)}(-1) = \frac{2^{k-1}-1}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{98}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(k_1,k_2)}(-1) = \frac{1-2^{k-1}}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{99}$$

$$\widetilde{\pounds}_{\mathcal{A},(k_1,k_2)}(-1) = \frac{2^{k-1}-1}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{100}$$

$$\pounds^{\star}_{\mathcal{A},(k_1,k_2)}(-1) = \frac{1-2^{k-1}}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{101}$$

Lemma 15.3 (Z. H. Sun)**.** *Let k be an integer greater than 1. If k is even, then we have*

$$\pounds_{\mathcal{A}_2,k}(-1) = \frac{k(2^k-1)}{2^k}\widehat{B}_{\boldsymbol{p}-k-1}\boldsymbol{p}, \tag{102}$$

and if k is odd, then we have

$$\pounds_{\mathcal{A}_2,k}(-1) = \frac{2^{k-1}-1}{2^{k-2}}(2\widehat{B}_{\boldsymbol{p}-k} - \widehat{B}_{2\boldsymbol{p}-k-1}). \tag{103}$$

Proof. This is obtained by Z. H. Sun's results ([48, Theorem 5.2 (b), Corollary 5.2 (a)]) and the relation

$$\sum_{n=1}^{p-1}\frac{(-1)^n}{n^k} = -\sum_{n=1}^{p-1}\frac{1}{n^k} + \frac{1}{2^{k-1}}\sum_{n=1}^{\frac{p-1}{2}}\frac{1}{n^k}. \tag{104}$$

Here, p is any odd number. $\square$

Remark 15.4. Tauraso and J. Zhao also proved the even case of the equality (102) ([54, Corollary 2.3]).

Proposition 15.5. *Let k_1 and k_2 be positive integers such that $k := k_1 + k_2$ is odd. Then we have the following equalities*:

$$\mathscr{L}_{\mathcal{A}_2,(k_1,k_2)}(-1) = \frac{1-2^{k-1}}{2^{k-1}}(2\widehat{B}_{\boldsymbol{p}-k} - \widehat{B}_{2\boldsymbol{p}-k-1}) + \frac{k_2(1-2^{k_1-1})}{2^{k_1-1}}\widehat{B}_{\boldsymbol{p}-k_1}\widehat{B}_{\boldsymbol{p}-k_2-1}\boldsymbol{p}, \tag{105}$$

$$\widetilde{\mathscr{L}}^{\star}_{\mathcal{A}_2,(k_1,k_2)}(-1) = \frac{2^{k-1}-1}{2^{k-1}}(2\widehat{B}_{\boldsymbol{p}-k} - \widehat{B}_{2\boldsymbol{p}-k-1}) + \frac{k_1(1-2^{k_2-1})}{2^{k_2-1}}\widehat{B}_{\boldsymbol{p}-k_1-1}\widehat{B}_{\boldsymbol{p}-k_2}\boldsymbol{p}, \tag{106}$$

$$\widetilde{\mathscr{L}}_{\mathcal{A}_2,(k_1,k_2)}(-1) = \frac{1-2^{k-1}}{2^{k-1}}(2\widehat{B}_{\boldsymbol{p}-k} - \widehat{B}_{2\boldsymbol{p}-k-1}) + \frac{k_1(1-2^{k_2-1})}{2^{k_2-1}}\widehat{B}_{\boldsymbol{p}-k_1-1}\widehat{B}_{\boldsymbol{p}-k_2}\boldsymbol{p}, \tag{107}$$

$$\mathscr{L}^{\star}_{\mathcal{A}_2,(k_1,k_2)}(-1) = \frac{2^{k-1}-1}{2^{k-1}}(2\widehat{B}_{\boldsymbol{p}-k} - \widehat{B}_{2\boldsymbol{p}-k-1}) + \frac{k_2(1-2^{k_1-1})}{2^{k_1-1}}\widehat{B}_{\boldsymbol{p}-k_1}\widehat{B}_{\boldsymbol{p}-k_2-1}\boldsymbol{p}, \tag{108}$$

where we assume that k_1 (resp. k_2) is greater than 1 *in the equalities* (105) *and* (108) (*resp.* (106) *and* (107)).

Proof. We consider the following relation:

$$\mathscr{L}_{\mathcal{A}_2,k_1}(-1)\zeta_{\mathcal{A}_2}(k_2) = \mathscr{L}_{\mathcal{A}_2,(k_1,k_2)}(-1) + \widetilde{\mathscr{L}}_{\mathcal{A}_2,(k_2,k_1)}(-1) + \mathscr{L}_{\mathcal{A}_2,k_1+k_2}(-1).$$

Since $k_1 + k_2$ is odd, we have $\mathscr{L}_{\mathcal{A}_2,(k_1,k_2)}(-1) = \widetilde{\mathscr{L}}_{\mathcal{A}_2,(k_2,k_1)}(-1)$ by Corollary 12.14 (77) and Lemma 15.1 (97). Therefore, we obtain the equalities (105) and (107) by Proposition 7.1 (18), Lemma 15.1 (97), and Lemma 15.3 (103). The proof of the equalities (106) and (108) is similar. Namely, we can use the relation

$$\mathscr{L}_{\mathcal{A}_2,k_1}(-1)\zeta_{\mathcal{A}_2}(k_2) = \mathscr{L}^{\star}_{\mathcal{A}_2,(k_1,k_2)}(-1) + \widetilde{\mathscr{L}}^{\star}_{\mathcal{A}_2,(k_2,k_1)}(-1) - \mathscr{L}_{\mathcal{A}_2,k_1+k_2}(-1). \qquad \square$$

Lemma 15.6 (Chamberland–Dilcher [4], Tauraso–Zhao [54])**.** *Let k, k_1, k_2, and k_3 be positive integers and $\bullet \in \{\emptyset, \star\}$. If $k = k_1 + k_2$ and k is odd, then we have*

$$\mathscr{L}^{*,\bullet}_{\mathcal{A},(k_1,k_2)}(-1,-1) = (-1)^{k_1}\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{109}$$

If $k = k_1 + k_2$, k is even, and k_1, k_2 are greater than or equal to 2*, then we have*

$$\mathscr{L}^{*,\bullet}_{\mathcal{A},(k_1,k_2)}(-1,-1) = \frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{k-3}k_1k_2}B_{\boldsymbol{p}-k_1}B_{\boldsymbol{p}-k_2}. \tag{110}$$

If k is odd and greater than or equal to 3, then we have

$$\pounds^{*,\bullet}_{\mathcal{A},(k,1)}(-1,-1)=\frac{2^{k-1}-1}{2^{k-2}k}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-k}. \tag{111}$$

If $k=k_1+k_2+k_3$, k_1 is even, and k_2+k_3 is odd, then we have

$$\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,-1,1)=\frac{1}{2}\left\{(-1)^{k_3}\binom{k}{k_3}-\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{112}$$

If $k=k_1+k_2+k_3$, k_1 is even, and k_2+k_3 is odd, then we have

$$\pounds^{*,\star}_{\mathcal{A},(k_1,k_2,k_3)}(-1,-1,1)=\frac{1}{2}\left\{(-1)^{k_3-1}\binom{k}{k_3}+\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{113}$$

If $k=k_1+k_2+k_3$, k_1+k_2 is odd, and k_3 is even, then we have

$$\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(1,-1,-1)=\frac{1}{2}\left\{(-1)^{k_1-1}\binom{k}{k_1}+\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k} \tag{114}$$

and

$$\pounds^{*,\star}_{\mathcal{A},(k_1,k_2,k_3)}(1,-1,-1)=\frac{1}{2}\left\{(-1)^{k_1}\binom{k}{k_1}-\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{115}$$

If $k=k_1+k_2+k_3$, k_1 is even, k_2 is odd, and k_3 is even, then we have

$$\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,1,-1)=\frac{1-2^{k-1}}{2^{k}}\left\{\binom{k}{k_3}-\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k} \tag{116}$$

and

$$\pounds^{*,\star}_{\mathcal{A},(k_1,k_2,k_3)}(-1,1,-1)=\frac{2^{k-1}-1}{2^{k}}\left\{\binom{k}{k_3}-\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{117}$$

Proof. The non-star case of the equality (109) is [54, Theorem 3.1 (15)]. The equality (110) and (111) are [54, Theorem 3.1 (20)]. Now, suppose that $k:=k_1+k_2+k_3$ is odd. By [54, Theorem 4.1], we have

$$\begin{aligned}&2\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,-1,1)\\&=\zeta_{\mathcal{A}}(k_3,k_1+k_2)+\pounds^{*}_{\mathcal{A},(k_2+k_3,k_1)}(-1,-1)-\pounds_{\mathcal{A},k_1}(-1)\widetilde{\pounds}_{\mathcal{A},(k_3,k_2)}(-1)\end{aligned} \tag{118}$$

and

$$
\begin{aligned}
(119)\qquad 2\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,1,-1) = -\pounds_{\mathcal{A},k_1}(-1)\pounds_{\mathcal{A},(k_3,k_2)}(-1) - \widetilde{\pounds}_{\mathcal{A},(k_2,k_1)}(-1)\pounds_{\mathcal{A},k_3}(-1) \\
+ \pounds^{*}_{\mathcal{A},(k_3,k_1+k_2)}(-1,-1) + \pounds^{*}_{\mathcal{A},(k_2+k_3,k_1)}(-1,-1).
\end{aligned}
$$

If k_1 is even, then we have $\pounds_{\mathcal{A},k_1}(-1) = 0$ by Lemma 15.1 (97). Therefore, we can calculate $\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,-1,1)$ by the equality (118), Proposition 7.4 (25), and the equality (109). This proves the equality (112). If k_1 and k_3 are even, then we have $\pounds_{\mathcal{A},k_1}(-1) = \pounds_{\mathcal{A},k_3}(-1) = 0$ by Lemma 15.1 (97). Therefore, we can calculate $\pounds^{*}_{\mathcal{A},(k_1,k_2,k_3)}(-1,1,-1)$ by the equalities (109) and (119). This proves the equality (116). The equality (114) is obtained by Corollary 12.13 (74) and the equality (112). All star cases are obtained by Corollary 12.13 (76). Note that [54, Theorem 3.1 (16)] which is the corresponding formula to the star case of the equality (109) is incorrect. □

Lemma 15.7 (Pilehrood–Pilehrood–Tauraso [37])**.** *Let k_1 and k_2 be positive even integers. Let $k := k_1 + k_2$. Then we have*

$$
(120)\qquad \pounds^{*}_{\mathcal{A}_2,(k_1,k_2)}(-1,-1) = \left\{ \frac{(k_2-k_1)(2^k-1)}{2^{k+1}(k+2)} \binom{k+2}{k_1+1} - \frac{k}{2} \right\} \frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}
$$

and

$$
(121)\qquad \pounds^{*,\star}_{\mathcal{A}_2,(k_1,k_2)}(-1,-1) = \left\{ \frac{(k_2-k_1)(2^k-1)}{2^{k+1}(k+2)} \binom{k+2}{k_1+1} + \frac{k}{2} \right\} \frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}.
$$

Proof. The equality (120) is [37, Lemma 3.1]. The equality (121) is obtained by the relation

$$
\pounds^{*,\star}_{\mathcal{A}_2,(k_1,k_2)}(-1,-1) = \pounds^{*}_{\mathcal{A}_2,(k_1,k_2)}(-1,-1) + \zeta_{\mathcal{A}_2}(k_1+k_2)
$$

and Proposition 7.1 (18). □

15.2 Calculations in low weights

Lemma 15.8 (Tauraso–Zhao [54]). *Let $\bullet \in \{\emptyset, \star\}$. Then we have*

$$\pounds^{*}_{\mathcal{A},\{1\}^3}(1,-1,-1) = q_{\boldsymbol{p}}(2)^3 + \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{122}$$

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^3}(1,-1,-1) = q_{\boldsymbol{p}}(2)^3 - \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{123}$$

$$\pounds^{*}_{\mathcal{A},\{1\}^3}(-1,-1,1) = -q_{\boldsymbol{p}}(2)^3 - \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{124}$$

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^3}(-1,-1,1) = -q_{\boldsymbol{p}}(2)^3 + \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{125}$$

$$\pounds^{*,\bullet}_{\mathcal{A},\{1\}^3}(-1,1,-1) = 0, \tag{126}$$

$$\pounds^{*,\bullet}_{\mathcal{A},\{1\}^3}(-1,-1,-1) = -\frac{4}{3}q_{\boldsymbol{p}}(2)^3 - \frac{1}{6}B_{\boldsymbol{p}-3}, \tag{127}$$

$$\pounds^{*,\bullet}_{\mathcal{A},\{1\}^3}(1,-1,1) = \frac{2}{3}q_{\boldsymbol{p}}(2)^3 + \frac{1}{12}B_{\boldsymbol{p}-3}. \tag{128}$$

Proof. All non-star cases of these values are obtained by [54, Proposition 7.6]. All star cases can be calculated by Corollary 12.13 (76). $\square$

Lemma 15.9 (Tauraso–Zhao [54]).

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^2}(-1,-1) = 2q_{\boldsymbol{p}}(2)^2 - \left(2q_{\boldsymbol{p}}(2)^3 + \frac{1}{3}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{129}$$

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(-1,-1) = 2q_{\boldsymbol{p}}(2)^2 - \left(2q_{\boldsymbol{p}}(2)^3 - \frac{1}{3}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{130}$$

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^3}(-1,-1,-1) = -\frac{4}{3}q_{\boldsymbol{p}}(2)^3 + \widehat{B}_{\boldsymbol{p}-3} - \frac{1}{2}\widehat{B}_{2\boldsymbol{p}-4} + 2\left(q_{\boldsymbol{p}}(2)^4 - q_{\boldsymbol{p}}(2)\widehat{B}_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{131}$$

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^3}(-1,-1,-1) = -\frac{4}{3}q_{\boldsymbol{p}}(2)^3 + \widehat{B}_{\boldsymbol{p}-3} - \frac{1}{2}\widehat{B}_{2\boldsymbol{p}-4} + 2\left(q_{\boldsymbol{p}}(2)^4 + q_{\boldsymbol{p}}(2)\widehat{B}_{\boldsymbol{p}-3}\right)\boldsymbol{p}. \tag{132}$$

Proof. The equalities (129), and (131) are [54, Proposition 7.3 (100)] and [54, Proposition

7.6 (117)], respectively. The equalities (130) and (132) are obtained by the relations

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(-1,-1) = \pounds^{*}_{\mathcal{A}_2,\{1\}^2}(-1,-1) + \zeta_{\mathcal{A}_2}(2),$$

and

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^3}(-1,-1,-1) = \pounds^{*}_{\mathcal{A}_2,\{1\}^3}(-1,-1,-1) + \widetilde{\pounds}_{\mathcal{A}_2,(2,1)}(-1) + \pounds_{\mathcal{A}_2,(1,2)}(-1) + \pounds_{\mathcal{A}_2,3}(-1),$$

respectively. Here, note that

$$\widetilde{\pounds}_{\mathcal{A}_2,(2,1)}(-1) + \pounds_{\mathcal{A}_2,(1,2)}(-1) = -\frac{3}{2}\left(2\widehat{B}_{\boldsymbol{p}-3} - \widehat{B}_{2\boldsymbol{p}-4}\right) - \frac{4}{3}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3}\boldsymbol{p}$$

holds by [54, Proposition 7.3 (105) and (106)]. □

16 Special values of finite multiple polylogarithms

16.1 Special values of finite polylogarithms

Now, we recall the following results for finite polylogarithms obtained by Z. H. Sun [48, 49], Dilcher–Skula [7], and Meštrović [28]:

Lemma 16.1. *The following equalities hold*:

$$\pounds_{\mathcal{A}_3,1}(-1) = -2q_{\boldsymbol{p}}(2) + q_{\boldsymbol{p}}(2)^2\boldsymbol{p} - \left(\frac{2}{3}q_{\boldsymbol{p}}(2)^3 + \frac{1}{4}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}^2, \tag{133}$$

$$\pounds_{\mathcal{A}_3,1}(2) = -2q_{\boldsymbol{p}}(2) - \frac{7}{12}B_{\boldsymbol{p}-3}\boldsymbol{p}^2, \tag{134}$$

$$\pounds_{\mathcal{A}_2,2}(2) = -q_{\boldsymbol{p}}(2)^2 + \left(\frac{2}{3}q_{\boldsymbol{p}}(2)^3 + \frac{7}{6}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{135}$$

$$\pounds_{\mathcal{A},3}(2) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{24}B_{\boldsymbol{p}-3}, \tag{136}$$

$$\pounds_{\mathcal{A}_3,1}(1/2) = q_{\boldsymbol{p}}(2) - \frac{1}{2}q_{\boldsymbol{p}}(2)^2\boldsymbol{p} + \left(\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{48}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}^2, \tag{137}$$

$$\pounds_{\mathcal{A}_2,2}(1/2) = -\frac{1}{2}q_{\boldsymbol{p}}(2)^2 + \left(\frac{1}{2}q_{\boldsymbol{p}}(2)^3 + \frac{7}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{138}$$

$$\pounds_{\mathcal{A},3}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{7}{48}B_{\boldsymbol{p}-3}. \tag{139}$$

Proof. The equality (133) is obtained by [48, Theorem 5.2 (c)] and the equality (104). The equalities (134) and (135) are [49, Theorem 4.1 (i)] and [49, Theorem 4.1 (ii)], respectively. The equalities (136), (137), and (138) are essentially due to Dilcher–Skula [7] (see [49, Remark 4.1]). The equality (137) is also shown by Meštrović [28]. The equality (139) is obtained by the equality (136) and Proposition 10.3 (60). □

16.2 Calculations in general weights

Proposition 16.2. *Let k be an integer greater than* 1. *Then*

$$\pounds_{\mathcal{A},\{1\}^k}(2) = \frac{1-2^{k-1}}{2^{k-2}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{140}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^k}(2) = \frac{1-2^{k-1}}{2^{k-2}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{141}$$

$$\widetilde{\pounds}_{\mathcal{A},\{1\}^k}(1/2) = \frac{2^{k-1}-1}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{142}$$

$$\pounds^{\star}_{\mathcal{A},\{1\}^k}(1/2) = \frac{2^{k-1}-1}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{143}$$

Proof. These values can be calculated by Lemma 13.1 and Lemma 15.1. □

Proposition 16.3. *Let k_1 and k_2 be positive integers such that $k := k_1 + k_2$ is odd. Then*

$$\pounds_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(2) = \left\{\frac{2^{k-1}-1}{2^{k-1}} - (-1)^{k_1}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{144}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(2) = \left\{\frac{1-2^{k-1}}{2^{k-1}} - (-1)^{k_1}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{145}$$

$$\widetilde{\pounds}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(1/2) = \frac{1}{2}\left\{\frac{1-2^{k-1}}{2^{k-1}} - (-1)^{k_1}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}, \tag{146}$$

$$\pounds^{\star}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(1/2) = \frac{1}{2}\left\{\frac{2^{k-1}-1}{2^{k-1}} - (-1)^{k_1}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{147}$$

Proof. By substituting $t = 2$ and $\mathbf{k} = (\{1\}^{k_1-1}, 2, \{1\}^{k_2-1})$ in Corollary 12.12 (70), we have

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(2) = \widetilde{\pounds}^{\star}_{\mathcal{A},(k_1,k_2)}(-1) - \zeta^{\star}_{\mathcal{A}}(k_1, k_2).$$

Hence the equality (145) can be calculated by Proposition 7.4 (25) and Lemma 15.2 (99). The equality (147) is obtained by Corollary 12.12 (69). We can also calculate this directly by Theorem 12.16. By Corollary 12.12 (71), we have

$$\begin{aligned}-\pounds_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(2) = {} & \widetilde{\pounds}^{\star}_{\mathcal{A},(\{1\}^{k_2-1},2,\{1\}^{k_1-1})}(2) \\ & + \sum_{j=1}^{k_1-1}(-1)^j \pounds_{\mathcal{A},\{1\}^j}(2)\zeta^{\star}_{\mathcal{A}}(\{1\}^{k_2-1}, 2, \{1\}^{k_1-1-j}) \\ & + \sum_{i=1}^{k_2-1}(-1)^{k_1-1+i} \pounds_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{i-1})}(2)\zeta^{\star}_{\mathcal{A}}(\{1\}^{k_2-i}).\end{aligned}$$

The last summation vanishes by Proposition 7.1 (17). Let $j \in \{1, 2, \ldots, k_1 - 1\}$. If j is odd, we have $\zeta^{\star}_{\mathcal{A}}(\{1\}^{k_2-1}, 2, \{1\}^{k_1-1-j}) = 0$ by Theorem 7.8 (35) and if j is even, we have $\pounds_{\mathcal{A},\{1\}^j}(2) = 0$ by Proposition 16.2 (140). Therefore, we have

$$\pounds_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(2) = -\widetilde{\pounds}^{\star}_{\mathcal{A},(\{1\}^{k_2-1},2,\{1\}^{k_1-1})}(2).$$

This proves the equality (144). The equality (146) is obtained by Corollary 12.12 (69). $\square$

Remark 16.4. Z. W. Sun proved that $\pounds^{\star}_{\mathcal{A},\{1\}^2}(1/2) = 0$ (see [50, Theorem 1.1]). The proof is based on some technical calculations. The case $(k_1, k_2) = (1, 2)$ or $(k_1, k_2) = (2, 1)$ of Proposition 16.3 have already been obtained by Meštrović [27, Theorem 1.1, Corollary 1.2] and by Tauraso–J. Zhao [54, Proposition 7.1].

Theorem 16.5. *Let k be a positive even number. Then we have the following in $\mathcal{A}_2$:*

$$\pounds_{\mathcal{A}_2,\{1\}^k}(2) = \left(\frac{k+1}{2^k} - k - 2\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \tag{148}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) = \left(k + 2 - \frac{k+1}{2^k}\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \tag{149}$$

$$\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^k}(1/2) = \frac{1-2^{k+1}}{2^{k+1}}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \tag{150}$$

$$\pounds^{\star}_{\mathcal{A}_2,\{1\}^k}(1/2) = \frac{2^{k+1}-1}{2^{k+1}}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \tag{151}$$

Proof. First, we prove the star cases. By the functional equation Corollary 12.14 (78), we have

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) - \zeta^{\star}_{\mathcal{A}_2}(\{1\}^k) = \pounds_{\mathcal{A}_2,k}(-1) + \big(\widetilde{\pounds}^{\star}_{\mathcal{A}_2,(1,k)}(-1) - \pounds_{\mathcal{A}_2,k+1}(-1)\big)\boldsymbol{p}.$$

Therefore, by combining Proposition 7.1 (19), Lemma 15.3, Lemma 15.2 (99), and Lemma 15.1, we have

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) - \frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} = \frac{k(2^k-1)}{2^k}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} + \left(\frac{1-2^k}{2^k}\frac{B_{\boldsymbol{p}-k-1}}{k+1} - \frac{1-2^k}{2^{k-1}}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\right)\boldsymbol{p}$$

or

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) = \left(k + 2 - \frac{k+1}{2^k}\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \tag{152}$$

By Corollary 12.14 (77), we have

$$2^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A}_2,\{1\}^k}(1/2) = \widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) + \sum_{i=1}^{k}\widetilde{\pounds}^{\star}_{\mathcal{A}_2,(\{1\}^{i-1},2,\{1\}^{k-i})}(2)\boldsymbol{p}.$$

Hence, by combining the equality (152) and Proposition 16.3 (145), we have

$$\begin{aligned}
&2^{\boldsymbol{p}}\pounds^{\star}_{\mathcal{A}_2,\{1\}^k}(1/2)\\
&= \left(k + 2 - \frac{k+1}{2^k}\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} + \sum_{i=1}^{k}\left\{\frac{1-2^k}{2^k} - (-1)^i\binom{k+1}{i}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}\\
&= \frac{2^{k+1}-1}{2^k}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p},
\end{aligned}$$

since k is even and $\sum_{i=1}^{k}(-1)^i\binom{k+1}{i} = 0$. Since the equality $2^{\boldsymbol{p}} = 2(1+q_{\boldsymbol{p}}(2)\boldsymbol{p})$ holds in $\mathcal{A}_2$ and $\pounds^{\star}_{\mathcal{A}_2,\{1\}^k}(1/2)\boldsymbol{p} = 0$ by Proposition 16.2 (143), we obtain the equality (151).

Next, we prove the non-star cases. By Corollary 12.14 (79), we have

$$\pounds_{\mathcal{A}_2,\{1\}^k}(2) = -\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) + \sum_{j=1}^{k-1}(-1)^{j-1}\pounds_{\mathcal{A}_2,\{1\}^j}(2)\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k-j}). \tag{153}$$

Since $\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k-j})$ is contained in $\boldsymbol{p}\mathcal{A}_2$, we have

$$\pounds_{\mathcal{A}_2,\{1\}^j}(2)\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k-j}) = (\text{a certain rational number}) \times B_{\boldsymbol{p}-j}B_{\boldsymbol{p}-k+j-1}\boldsymbol{p}$$

for any $j = 1, \ldots, k-1$ by Proposition 7.1 (19) and Proposition 16.2 (140). If j is even, we have $B_{\boldsymbol{p}-j} = 0$ and if j is odd, we have $B_{\boldsymbol{p}-k+j-1} = 0$. Therefore, the summation in the right hand side of (153) vanishes and we have

$$\pounds_{\mathcal{A}_2,\{1\}^k}(2) = -\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^k}(2) = \left(\frac{k+1}{2^k} - k - 2\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \tag{154}$$

By Corollary 12.14 (80), we have

$$2^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^k}(1/2) = \pounds_{\mathcal{A}_2,\{1\}^k}(2) + \sum_{i=1}^{k}\pounds_{\mathcal{A}_2,(\{1\}^{i-1},2,\{1\}^{k-i})}(2)\boldsymbol{p}.$$

Hence, by the equality (154) and Proposition 16.3 (144), we have

$$\begin{aligned}
&2^{\boldsymbol{p}}\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^k}(1/2) \\
&= \left(\frac{k+1}{2^k} - k - 2\right)\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} + \sum_{i=1}^{k}\left\{\frac{2^k-1}{2^k} - (-1)^i\binom{k+1}{i}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p} \\
&= \frac{1-2^{k+1}}{2^k}\frac{B_{p-k-1}}{k+1}\boldsymbol{p}.
\end{aligned}$$

Since the equality $2^{\boldsymbol{p}} = 2(1+q_{\boldsymbol{p}}(2)\boldsymbol{p})$ holds in $\mathcal{A}_2$ and $\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^k}(1/2)\boldsymbol{p} = 0$ by Proposition 16.2 (142), we obtain the equality (150). We can also calculate by Corollary 12.14 (80). $\square$

Remark 16.6. The cases $k=2$ of Theorem 16.5 have already been given by Z. W. Sun–L. L. Zhao [51], Meštrović [27], and Tauraso–J. Zhao [54]. Indeed, the equality (148) is [54, Proposition 7.1 (78)] and the equality (149) which is equivalent to Proposition 16.10 (195) below is [27, Theorem 1.1 (1)] or [54, Proposition 7.1(77)]. The equality (151) was conjectured by Z. W. Sun [50, Conjecture 1.1] and proved by Z. W. Sun–L. L. Zhao [51]. Meštrović gave another proof of Sun's conjecture in [27] and our proof of the equality (151) is similar to his proof.

Theorem 16.7. *Let k, k_1, k_2, and k_3 be positive integers and $\bullet \in \{\emptyset, \star\}$. If $k = k_1 + k_2$ and k is odd, then we have*

$$\pounds^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = (-1)^{k_1}\frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{155}$$

If $k = k_1 + k_2$ and k is odd, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}) = (-1)^{k_1}\frac{2^{k-1}-1}{2^{k-1}}\binom{k}{k_1}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{156}$$

If $k = k_1 + k_2$ and k is even, then we have

$$\pounds^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-1}) = 0. \tag{157}$$

If $k = k_1 + k_2$ and k is even and k_1, k_2 are greater than or equal to 2, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-1}) = -\frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{k-3}k_1k_2}B_{\boldsymbol{p}-k_1}B_{\boldsymbol{p}-k_2}. \tag{158}$$

If k is odd and greater than or equal to 3, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^{k+1}}(\{1\}^{k-1}, 2, 1/2) = \frac{1-2^{k-1}}{2^{k-2}k}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-k}. \tag{159}$$

If k is odd and greater than or equal to 3, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^{k+1}}(2, 1/2, \{1\}^{k-1}) = \frac{1-2^{k-1}}{2^{k-2}k}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-k}. \tag{160}$$

If $k = k_1 + k_2$, k_1 is odd and greater than or equal to 3, and k_2 is even, then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(2, \{1\}^{k_1-2}, 1/2, 2, \{1\}^{k_2-1}) = \frac{2^{k-1}-1}{2^{k-1}} \left\{ \binom{k}{k_1} - 1 \right\} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{161}$$

If k is odd and greater than or equal to 3, then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(2, 1/2, 2, \{1\}^{k-3}) = \frac{(1-2^{k-1})(k^2-k+2)}{2^k} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{162}$$

If $k = k_1 + k_2$ is odd and greater than or equal to 3 and k_2 is greater than or equal to 2, then we have

$$\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 2, 1/2, \{1\}^{k_2-2}, 2) = \frac{1-2^{k-1}}{2^{k-1}} \left\{ 1 + (-1)^{k_1-1} \binom{k}{k_1} \right\} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{163}$$

If $k = k_1 + k_2$, k_1 is even, and k_2 is odd and greater than or equal to 3. Then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1}, 1/2, 2, \{1\}^{k_2-2}, 1/2) = \frac{1-2^{k-1}}{2^k} \left\{ \binom{k}{k_1} - 1 \right\} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{164}$$

If k is odd and greater than or equal to 3, then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k-3}, 1/2, 2, 1/2) = \frac{(2^{k-1}-1)(k^2-k+2)}{2^{k+1}} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{165}$$

If $k = k_1 + k_2$ is odd and greater than or equal to 3 and k_1 is greater than or equal to 2, then we have

$$\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^k}(1/2, \{1\}^{k_1-2}, 2, 1/2, \{1\}^{k_2-1}) = \frac{2^{k-1}-1}{2^k} \left\{ 1 + (-1)^{k_1} \binom{k}{k_1} \right\} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{166}$$

If k is odd and greater than or equal to 3, then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(1, 2, \{1\}^{k-2}) = \frac{(2^{k-1}-1)(k-1)}{2^{k-1}} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{167}$$

If k is odd and greater than or equal to 3, then we have

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k-2}, 1/2, 1) = -\frac{(2^{k-1}-1)(k-1)}{2^k} \frac{B_{\boldsymbol{p}-k}}{k}. \tag{168}$$

If k is odd and greater than or equal to 3, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^k}(\{1\}^{k-2},2,1) = \frac{(2^{k-1}-1)(k-1)}{2^{k-1}}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{169}$$

If k is odd and greater than or equal to 3, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^k}(1,1/2,\{1\}^{k-2}) = -\frac{(2^{k-1}-1)(k-1)}{2^{k}}\frac{B_{\boldsymbol{p}-k}}{k}. \tag{170}$$

If $k = k_1+k_2+k_3$, k_1 is even, and k_2+k_3 is odd, then we have

$$\begin{aligned}&\pounds^{*}_{\mathcal{A},(\{1\}^{k_1+k_2-1},2,\{1\}^{k_3-1})}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2+k_3-2})\\ &= \frac{1}{2}\left\{(-1)^{k_3}\binom{k}{k_3} - \frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned} \tag{171}$$

If $k = k_1+k_2+k_3$, k_1 is even, and k_2+k_3 is odd, then we have

$$\begin{aligned}&\pounds^{*,\star}_{\mathcal{A},(\{1\}^{k_1+k_2-1},2,\{1\}^{k_3-1})}(\{1\}^{k_1-1},2,1/2,\{1\}^{k_2+k_3-2})\\ &= \frac{1}{2}\left\{(-1)^{k_3}\binom{k}{k_3} - \frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned} \tag{172}$$

If $k = k_1+k_2+k_3$, k_1+k_2 is odd, and k_3 is even, then we have

$$\begin{aligned}&\pounds^{*}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2+k_3-1})}(\{1\}^{k_1+k_2-2},1/2,2,\{1\}^{k_3-1})\\ &= -\frac{1}{2}\left\{(-1)^{k_1}\binom{k}{k_1} - \frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned} \tag{173}$$

If $k = k_1+k_2+k_3$, k_1+k_2 is odd, and k_3 is even, then we have

$$\begin{aligned}&\pounds^{*,\star}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2+k_3-1})}(\{1\}^{k_1+k_2-2},2,1/2,\{1\}^{k_3-1})\\ &= -\frac{1}{2}\left\{(-1)^{k_1}\binom{k}{k_1} - \frac{1-2^{k-1}}{2^{k-1}}\binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned} \tag{174}$$

If $k = k_1+k_2+k_3$, k_1 is even, k_2 is odd and greater than 1, and k_3 is even, then we have

$$\begin{aligned}&\pounds^{*}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-2},1/2,2,\{1\}^{k_3-1})\\ &= \frac{1-2^{k-1}}{2^{k}}\left\{\binom{k}{k_1} - \binom{k}{k_3}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned} \tag{175}$$

If $k=k_1+k_2+k_3$, k_1 is even, k_2 is odd and greater than 1, and k_3 is even, then we have

$$\begin{aligned}&\pounds^{*,\star}_{\mathcal{A},\{1\}^k}(\{1\}^{k_1-1},2,1/2,\{1\}^{k_2-2},2,1/2,\{1\}^{k_3-1})\\&=\frac{1-2^{k-1}}{2^k}\left\{\binom{k}{k_3}-\binom{k}{k_1}\right\}\frac{B_{\boldsymbol{p}-k}}{k}.\end{aligned}\tag{176}$$

If $k=k_1+k_2$ and k_1,k_2 are even, then we have

$$\pounds^{*}_{\mathcal{A},\{1\}^{k+1}}(\{1\}^{k_1-1},1/2,1,2,\{1\}^{k_2-1})=\frac{1-2^k}{2^{k+1}}\left\{\binom{k+1}{k_1}-\binom{k+1}{k_2}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}.\tag{177}$$

If $k=k_1+k_2$ and k_1,k_2 are even, then we have

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^{k+1}}(\{1\}^{k_1-1},2,1,1/2,\{1\}^{k_2-1})=\frac{1-2^k}{2^{k+1}}\left\{\binom{k+1}{k_2}-\binom{k+1}{k_1}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}.\tag{178}$$

Proof. First, we prove the star-cases. By Corollary 12.13 (75), we have

$$\begin{aligned}&\pounds^{\text{ш},\star}_{\mathcal{A},(k_1,k_2)}(s,t)\\&=\pounds^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1-s,\{1\}^{k_2-1},1-t)-\pounds^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1-s,\{1\}^{k_2}),\end{aligned}\tag{179}$$

where s and t are indeterminates. If we substitute -1 and 1 into s and t of the functional equation (179) respectively, then we see that

$$(\text{L. H. S. of (179)})=\pounds^{\text{ш},\star}_{\mathcal{A},(k_1,k_2)}(-1,1)=\pounds^{*,\star}_{\mathcal{A},(k_1,k_2)}(-1,-1)$$

and

$$\begin{aligned}(\text{R. H. S. of (179)})&=-\pounds^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,\{1\}^{k_2})\\&=-\pounds^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,1/2,\{1\}^{k_2-1}).\end{aligned}$$

Therefore, we obtain the equality (156), (158), and (159) by Lemma 15.6 (109), (110), and (111), respectively. The equality (160) is obtained by Corollary 12.13 (74). Next, if we substitute -1 into s and t of the functional equation (179), then we see that

$$(\text{L. H. S. of (179)})=\pounds^{\text{ш},\star}_{\mathcal{A},(k_1,k_2)}(-1,-1)=\pounds^{\star}_{\mathcal{A},(k_1,k_2)}(-1)$$

and

$$(\text{R. H. S. of (179)}) = \mathcal{L}^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,\{1\}^{k_2-1},2) - \mathcal{L}^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,\{1\}^{k_2}).$$

Therefore, by combining Lemma 15.2 (101) and the equality (156) which has obtained just before, we have the explicit value of $\mathcal{L}^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,\{1\}^{k_2-1},2)$ if k_1+k_2 is odd. By translating $\mathcal{A}$-FSSMP into $\mathcal{A}$-FHSMP, we have the equalities (163) and (169). The equalities (166) and (170) are obtained by Corollary 12.13 (74).

By Corollary 12.13 (75), we have the following equality:

$$\mathcal{L}^{\text{ш},\star}_{\mathcal{A},(k_1,k_2,k_3)}(-1,1,1) = -\mathcal{L}^{\text{ш},\star}_{\mathcal{A},(\{1\}^{k_1+k_2-1},2,\{1\}^{k_3-1})}(\{1\}^{k_1-1},2,\{1\}^{k_2+k_3-1}), \tag{180}$$

since $(k_1,(k_2,k_3))^* = (\{1\}^{k_1+k_2-1},2,\{1\}^{k_3-1})$. After translating $\mathcal{A}$-FSSMPs into $\mathcal{A}$-FHSMPs, we have the equality (172) by Lemma 15.6 (113) when k_1 is even and k_2+k_3 is odd. The equality (174) is obtained by Corollary 12.13 (74).

By Corollary 12.13 (75), we have the following equality:

$$\mathcal{L}^{\text{ш},\star}_{\mathcal{A},(k_1,k_2,k_3)}(-1,-1,1) = -\mathcal{L}^{\text{ш},\star}_{\mathcal{A},\{1\}^{k_1+k_2+k_3}}(\{1\}^{k_1-1},2,\{1\}^{k_2-1},2,\{1\}^{k_3}). \tag{181}$$

After translating $\mathcal{A}$-FSSMPs into $\mathcal{A}$-FHSMPs, we have the equalities (176) and (178) by Lemma 15.6 (117) when k_1 is even, k_2 is odd, and k_3 is even.

Next, we prove non-star cases. By Corollary 12.13 (76), we have

$$\begin{aligned}&(-1)^{k_1+k_2-1}\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-1})\\ &= \mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-1}) + (-1)^{k_1}\widetilde{\mathcal{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2)\widetilde{\mathcal{L}}^{\star}_{\mathcal{A},\{1\}^{k_2}}(2).\end{aligned} \tag{182}$$

Suppose $k_1,k_2 \geq 2$. By Proposition 16.2 (141) and (142), we have

$$\widetilde{\mathcal{L}}_{\mathcal{A},\{1\}^{k_1}}(1/2)\widetilde{\mathcal{L}}^{\star}_{\mathcal{A},\{1\}^{k_2}}(2) = -\frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{k_1+k_2-3}k_1k_2}B_{\boldsymbol{p}-k_1}B_{\boldsymbol{p}-k_2}.$$

If k_1+k_2 is odd, then

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-1}) = \mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-1})$$

holds and if k_1+k_2 is even, then

$$\begin{aligned}
&\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-1})\\
&=\frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{k_1+k_2-3}k_1k_2}B_{\boldsymbol{p}-k_1}B_{\boldsymbol{p}-k_2}+(-1)^{k_1}\frac{(2^{k_1-1}-1)(2^{k_2-1}-1)}{2^{k_1+k_2-3}k_1k_2}B_{\boldsymbol{p}-k_1}B_{\boldsymbol{p}-k_2}=0
\end{aligned}$$

holds by the equality (158). The case $k_1=1$ and $k_2=1$ are similar. Therefore we obtain the equalities (155) and (157).

Suppose that k_1+k_2 is odd. By Corollary 12.13 (76), we have

$$\begin{aligned}
\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(2,\{1\}^{k_1-2},1/2,2,\{1\}^{k_2-1})&=\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-2},2)\\
&\quad+\sum_{j=1}^{k_1-1}(-1)^j\mathcal{L}_{\mathcal{A},\{1\}^j}(2)\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2-j}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-j-1})\\
&\quad+(-1)^{k_1}\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1}}(2,\{1\}^{k_1-2},1/2)\widetilde{\mathcal{L}}^{\star}_{\mathcal{A},\{1\}^{k_2}}(2).
\end{aligned}\tag{183}$$

Suppose that k_2 is even. $\widetilde{\mathcal{L}}^{\star}_{\mathcal{A},\{1\}^{k_2}}(2)=0$ holds by Proposition 16.2 (141). Furthermore, if j is even, then $\mathcal{L}_{\mathcal{A},\{1\}^j}(2)=0$ holds by Proposition 16.2 (140) and if j is odd, then $\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2-j}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-j-1})=0$ holds by the equality (158). Hence,

$$\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(2,\{1\}^{k_1-2},1/2,2,\{1\}^{k_2-1})=\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-2},2)$$

holds and we have the equality (161) by the equality (163). If $k\geq 5$ is odd, we have

$$\begin{aligned}
&\mathcal{L}^{*}_{\mathcal{A},\{1\}^{k}}(2,1/2,2,\{1\}^{k-3})-\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k}}(\{1\}^{k-3},2,1/2,2)\\
&=-\mathcal{L}_{\mathcal{A},1}(2)\mathcal{L}^{*,\star}_{\mathcal{A},\{1\}^{k-1}}(\{1\}^{k-3},2,1/2)+\mathcal{L}^{*}_{\mathcal{A},\{1\}^{2}}(2,1/2)\widetilde{\mathcal{L}}^{\star}_{\mathcal{A},\{1\}^{k-2}}(2)\\
&=-(-2q_{\boldsymbol{p}}(2))\frac{1-2^{k-3}}{2^{k-4}(k-2)}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-k+2}+(-2q_{\boldsymbol{p}}(2)^2)\frac{1-2^{k-3}}{2^{k-4}}\frac{B_{\boldsymbol{p}-k+2}}{k-2}\\
&=0
\end{aligned}$$

by the equality (183), Lemma 16.1 (134), the equality (159), Proposition 16.14 (222) below, and Proposition 16.2 (141). The case $k=3$ is similar. Therefore, we have the equality (162). The equalities (164) and (165) are obtained by Corollary 12.13 (74). The equality (167) (resp. (168)) is obtained by Corollary 12.12 (73) and the equality (169) (resp. (170)).

Since the equality (171) is obtained by Corollary 12.13 (74), we prove the equality (173). Suppose that k_1+k_2 is odd and k_3 even. By Corollary 12.13 (76), we have

$$\begin{aligned}
&\pounds^{*}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2+k_3-1})}(\{1\}^{k_1+k_2-2},1/2,2,\{1\}^{k_3-1})\\
&=-\pounds^{*,\star}_{\mathcal{A},(\{1\}^{k_2+k_3-1},2,\{1\}^{k_1-1})}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_1+k_2-2})\\
&\quad-(-1)^{k_1+k_2-1}\widetilde{\pounds}_{\mathcal{A},(\{1\}^{k_1-1},2,\{1\}^{k_2-1})}(1/2)\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^{k_3}}(2).
\end{aligned}$$

Since $\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^{k_3}}(2)=0$ by Proposition 16.2 (141), we have the equality (173) by the equality (172).

Suppose that k_1 is even, k_2 is odd and greater than 1, and k_3 is even. By Corollary 12.13 (76), we have

$$\begin{aligned}
&\pounds^{*}_{\mathcal{A},\{1\}^{k_1+k_2+k_3}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-2},1/2,2,\{1\}^{k_3-1})\\
&=\pounds^{*,\star}_{\mathcal{A},\{1\}^{k_1+k_2+k_3}}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-2},2,1/2,\{1\}^{k_1-1})\\
&\quad+(-1)^{k_1}\widetilde{\pounds}_{\mathcal{A},\{1\}^{k_1}}(1/2)\pounds^{*,\star}_{\mathcal{A},\{1\}^{k_2+k_3}}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-2},2)\\
&\quad+\sum_{j=0}^{k_2-2}(-1)^{k_1+j+1}\pounds^{*}_{\mathcal{A},\{1\}^{k_1+j+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^{j})\pounds^{*,\star}_{\mathcal{A},\{1\}^{k_2+k_3-j-1}}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-j-2})\\
&\quad+(-1)^{k_1+k_2}\pounds^{*}_{\mathcal{A},\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-2},1/2)\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^{k_3}}(2).
\end{aligned}$$

For $j=0,\dots,k_2-2$, if j is odd, then $\pounds^{*}_{\mathcal{A},\{1\}^{k_1+j+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^{j})=0$ holds by the equality (157) and if j is even, then

$$\pounds^{*,\star}_{\mathcal{A},\{1\}^{k_2+k_3-j-1}}(\{1\}^{k_3-1},2,1/2,\{1\}^{k_2-j-2})=(\text{a certain element of }\mathcal{A})\times B_{\boldsymbol{p}-k_3}=0$$

holds by the equality (158). Furthermore, $\widetilde{\pounds}_{\mathcal{A},\{1\}^{k_1}}(1/2)=\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^{k_3}}(2)=0$ holds by Proposition 16.2 (141) and (142). Therefore, we obtain the equality (175) by the equality (176). Similarly, we see that the equality (177) holds. $\square$

Remark 16.8. The case $k=3$ of the equalities (167) and (168)

$$\pounds^{*}_{\mathcal{A},\{1\}^3}(1,2,1)=\frac{1}{2}B_{\boldsymbol{p}-3}, \tag{184}$$

$$\pounds^{*}_{\mathcal{A},\{1\}^3}(1,1/2,1)=-\frac{1}{4}B_{\boldsymbol{p}-3} \tag{185}$$

also have been obtained by Tauraso–J. Zhao [54, Proposition 7.1 (85)].

Theorem 16.9. *Let k_1 and k_2 be positive even integers. Put $k:=k_1+k_2$. Then we have*

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^k}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-1})=-\frac{1}{2}\left\{1+\frac{2^k-1}{2^k}\binom{k+1}{k_2}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}, \tag{186}$$

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^k}(\{1\}^{k_1-1},2,1/2,\{1\}^{k_2-1})=\frac{1}{2}\left\{1+\frac{2^k-1}{2^k}\binom{k+1}{k_1}\right\}\frac{B_{\boldsymbol{p}-k-1}}{k+1}\boldsymbol{p}. \tag{187}$$

Proof. By Corollary 12.15 (82), we have

$$\begin{aligned}&\pounds^{*,\star}_{\mathcal{A}_2,(k_1,k_2)}(-1,-1)+\Big(\pounds^{*,\star}_{\mathcal{A}_2,(1,k_1,k_2)}(1,-1,-1)-\pounds^{*,\star}_{\mathcal{A}_2,(k_1+1,k_2)}(-1,-1)\Big)\boldsymbol{p}\\&=-\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},2,1/2,\{1\}^{k_2-1}).\end{aligned} \tag{188}$$

Therefore, the equality (187) is obtained by Lemma 15.6 (109), (115), and Lemma 15.7 (121). By Corollary 12.15 (83), we have

$$\begin{aligned}\pounds^{*}_{\mathcal{A}_2,\{1\}^{k_1+k_2}}(\{1\}^{k_1-1},1/2,2,\{1\}^{k_2-1})&=-\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^{k_1+k_2}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-1})\\&\quad-\sum_{j=1}^{k_1-1}(-1)^j\zeta_{\mathcal{A}_2}(\{1\}^j)\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^{k_1+k_2-j}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-j-1})\\&\quad-(-1)^{k_1}\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^{k_1}}(1/2)\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^{k_2}}(2)\\&\quad-\sum_{i=0}^{k_2-2}(-1)^{k_1+i+1}\pounds^{*}_{\mathcal{A}_2,\{1\}^{k_1+i+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^i)\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k_2-i-1}).\end{aligned}$$

For $j=1,\dots,k_1-1$, if j is odd, then $\zeta_{\mathcal{A}_2}(\{1\}^j)=0$ by Proposition 7.1 (18) and if j is even, then

$$\zeta_{\mathcal{A}_2}(\{1\}^j)\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^{k_1+k_2-j}}(\{1\}^{k_2-1},2,1/2,\{1\}^{k_1-j-1})=(\text{a certain element of }\mathcal{A}_2)\times B_{\boldsymbol{p}-k_2}=0$$

by Theorem 16.7 (158). By Theorem 16.5, we have

$$\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^{k_1}}(1/2)\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^{k_2}}(2)=0.$$

For $i=0,\dots,k_2-2$, if i is even, then $\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k_2-i-1})=0$ holds by Proposition 7.1 (19) and if i is odd, then

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^{k_1+i+1}}(\{1\}^{k_1-1},1/2,2,\{1\}^{i})\zeta^{\star}_{\mathcal{A}_2}(\{1\}^{k_2-i-1})=0$$

by Theorem 16.7 (157). Therefore, we have the equality (186) by the equality (187). $\square$

16.3 Calculations in low weights

Proposition 16.10. *Let $\bullet\in\{\emptyset,\star\}$. Then the following equalities hold:*

$$\pounds_{\mathcal{A}_2,\{1\}^2}(-1)=q_{\boldsymbol{p}}(2)^2-\left(q_{\boldsymbol{p}}(2)^3+\frac{13}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{189}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^2}(-1)=-q_{\boldsymbol{p}}(2)^2+\left(q_{\boldsymbol{p}}(2)^3+\frac{13}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{190}$$

$$\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^2}(-1)=-q_{\boldsymbol{p}}(2)^2+\left(q_{\boldsymbol{p}}(2)^3+\frac{1}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{191}$$

$$\pounds^{\star}_{\mathcal{A}_2,\{1\}^2}(-1)=q_{\boldsymbol{p}}(2)^2-\left(q_{\boldsymbol{p}}(2)^3+\frac{1}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{192}$$

$$\pounds_{\mathcal{A}_2,\{1\}^2}(1/2)=\frac{1}{2}q_{\boldsymbol{p}}(2)^2-\frac{1}{2}q_{\boldsymbol{p}}(2)^3\boldsymbol{p}, \tag{193}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^2}(1/2)=-\frac{1}{2}q_{\boldsymbol{p}}(2)^2+\frac{1}{2}q_{\boldsymbol{p}}(2)^3\boldsymbol{p}, \tag{194}$$

$$\widetilde{\pounds}_{\mathcal{A}_2,\{1\}^2}(2)=q_{\boldsymbol{p}}(2)^2-\left(\frac{2}{3}q_{\boldsymbol{p}}(2)^3+\frac{1}{12}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{195}$$

$$\pounds^{\star}_{\mathcal{A}_2,\{1\}^2}(2)=-q_{\boldsymbol{p}}(2)^2+\left(\frac{2}{3}q_{\boldsymbol{p}}(2)^3+\frac{1}{12}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{196}$$

$$\pounds_{\mathcal{A},(1,2)}(1/2) = -\frac{1}{6}q_{\boldsymbol{p}}(2)^3 - \frac{25}{48}B_{\boldsymbol{p}-3}, \tag{197}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(2,1)}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{25}{48}B_{\boldsymbol{p}-3}, \tag{198}$$

$$\widetilde{\pounds}_{\mathcal{A},(2,1)}(2) = \frac{1}{3}q_{\boldsymbol{p}}(2)^3 + \frac{25}{24}B_{\boldsymbol{p}-3}, \tag{199}$$

$$\pounds^{\star}_{\mathcal{A},(1,2)}(2) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{25}{24}B_{\boldsymbol{p}-3}, \tag{200}$$

$$\pounds_{\mathcal{A},(2,1)}(1/2) = -\frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{23}{48}B_{\boldsymbol{p}-3}, \tag{201}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(1,2)}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 - \frac{23}{48}B_{\boldsymbol{p}-3}, \tag{202}$$

$$\widetilde{\pounds}_{\mathcal{A},(1,2)}(2) = \frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{23}{24}B_{\boldsymbol{p}-3}, \tag{203}$$

$$\pounds^{\star}_{\mathcal{A},(2,1)}(2) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 + \frac{23}{24}B_{\boldsymbol{p}-3}, \tag{204}$$

$$\pounds^{\bullet}_{\mathcal{A},\{1\}^3}(-1) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{24}B_{\boldsymbol{p}-3}, \tag{205}$$

$$\widetilde{\pounds}^{\bullet}_{\mathcal{A},\{1\}^3}(-1) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{24}B_{\boldsymbol{p}-3}, \tag{206}$$

$$\pounds_{\mathcal{A},\{1\}^3}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{7}{48}B_{\boldsymbol{p}-3}, \tag{207}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^3}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{7}{48}B_{\boldsymbol{p}-3}, \tag{208}$$

$$\widetilde{\pounds}_{\mathcal{A},\{1\}^3}(2) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{24}B_{\boldsymbol{p}-3}, \tag{209}$$

$$\pounds^{\star}_{\mathcal{A},\{1\}^3}(2) = -\frac{1}{3}q_{\boldsymbol{p}}(2)^3 - \frac{7}{24}B_{\boldsymbol{p}-3}. \tag{210}$$

Proof. We can calculate $\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^2}(-1)$, $\pounds^{\star}_{\mathcal{A}_2,\{1\}^2}(2)$, $\pounds^{\star}_{\mathcal{A},(1,2)}(2)$, $\pounds^{\star}_{\mathcal{A},(2,1)}(2)$, $\widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^3}(-1)$, and

$\pounds^{\star}_{\mathcal{A},\{1\}^3}(2)$ by the relations

$$\widetilde{\pounds}^{\star}_{\mathcal{A}_2,\{1\}^2}(-1)=\zeta^{\star}_{\mathcal{A}_2}(\{1\}^2)+\pounds_{\mathcal{A}_2,2}(2)+\big(\widetilde{\pounds}^{\star}_{\mathcal{A}_2,(1,2)}(2)-\pounds_{\mathcal{A}_2,3}(2)\big)\boldsymbol{p},$$

$$\pounds^{\star}_{\mathcal{A}_2,\{1\}^2}(2)=\pounds_{\mathcal{A}_2,\{1\}^2}(2)+\pounds_{\mathcal{A}_2,2}(2),\ \pounds^{\star}_{\mathcal{A},(1,2)}(2)=\pounds_{\mathcal{A},(1,2)}(2)+\pounds_{\mathcal{A},3}(2),$$

$$\pounds^{\star}_{\mathcal{A},(2,1)}(2)=\pounds_{\mathcal{A},(2,1)}(2)+\pounds_{\mathcal{A},3}(2),\ \widetilde{\pounds}^{\star}_{\mathcal{A},\{1\}^3}(-1)=\pounds_{\mathcal{A},3}(2),\ \pounds^{\star}_{\mathcal{A},\{1\}^3}(2)=\pounds^{\star}_{\mathcal{A},\{1\}^3}(-1),$$

respectively and by Proposition 7.1 (19), Lemma 16.1, Proposition 16.2, and Proposition 16.3. Here, we have used Corollary 12.14 (78), Lemma 13.1 (85), and Corollary 13.4 (93). All other values obtained by Corollary 12.12 and 12.14. □

Remark 16.11. Note that all of the values that appear in the above proposition essentially have been given by Meštrović [27, Theorem 1.1] and Tauraso–J. Zhao [54, Proposition 7.1]. We have determined all values of the form $\overline{\pounds}^{\bullet}_{\mathcal{A}_n,\mathbf{k}}(m)$ for $-\in\{\emptyset,\sim\}$, $\bullet\in\{\emptyset,\star\}$, and $m\in\{-1,2^{\pm1}\}$ when $n+\mathrm{wt}(\mathbf{k})\leq 4$.

Furthermore, we have the following some special values of $\mathcal{A}$-FMPs of weight 4.

Proposition 16.12. *Let* $\bullet\in\{\emptyset,\star\}$. *Then, the following equalities hold*:

$$\pounds^{\bullet}_{\mathcal{A},(1,3)}(-1)=\frac{1}{2}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3},\tag{211}$$

$$\widetilde{\pounds}^{\bullet}_{\mathcal{A},(3,1)}(-1)=-\frac{1}{2}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3},\tag{212}$$

$$\pounds_{\mathcal{A},(2,1,1)}(2)=-\frac{1}{2}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3},\tag{213}$$

$$\widetilde{\pounds}^{\star}_{\mathcal{A},(1,1,2)}(2)=-\frac{1}{2}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3},\tag{214}$$

$$\widetilde{\pounds}_{\mathcal{A},(1,1,2)}(1/2)=-\frac{1}{4}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3},\tag{215}$$

$$\pounds^{\star}_{\mathcal{A},(2,1,1)}(1/2)=-\frac{1}{4}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3}.\tag{216}$$

Proof. By [54, Proposition 6.1 (55)], we have $\widetilde{\pounds}_{\mathcal{A},(3,1)}(-1)=-\frac{1}{2}q_{\boldsymbol{p}}(2)B_{\boldsymbol{p}-3}$. All of the other values are obtained by Corollary 12.12. □

Proposition 16.13. *Let $\bullet \in \{\emptyset, \star\}$. Then we have*

$$\pounds^{*;\bullet}_{\mathcal{A},(1,2)}(2,1/2) = q_{\boldsymbol{p}}(2)^3 - \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{217}$$

$$\pounds^{*;\bullet}_{\mathcal{A},(2,1)}(2,1/2) = -q_{\boldsymbol{p}}(2)^3 + \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{218}$$

$$\pounds^{*;\bullet}_{\mathcal{A},(1,2)}(1/2,2) = -\frac{7}{8}B_{\boldsymbol{p}-3}, \tag{219}$$

$$\pounds^{*;\bullet}_{\mathcal{A},(2,1)}(1/2,2) = \frac{7}{8}B_{\boldsymbol{p}-3}, \tag{220}$$

$$\pounds^{*;\bullet}_{\mathcal{A},\{1\}^3}(2,1,1/2) = 0. \tag{221}$$

Proof. By applying the relation (180), we have the explicit value of $\pounds^{*,\star}_{\mathcal{A},(1,2)}(2,1/2)$ by Lemma 15.8 (125). $\pounds^{*,\star}_{\mathcal{A},(2,1)}(2,1/2)$ is obtained by Corollary 12.13 (74). The star cases of the equalities (219) and (220) are obtained by the following relations:

$$\pounds_{\mathcal{A},2}(2)\pounds_{\mathcal{A},1}(1/2) = \pounds^{*,\star}_{\mathcal{A},(2,1)}(2,1/2) + \pounds^{*,\star}_{\mathcal{A},(1,2)}(1/2,2) - \zeta_{\mathcal{A}}(3)$$

and

$$\pounds_{\mathcal{A},1}(2)\pounds_{\mathcal{A},2}(1/2) = \pounds^{*,\star}_{\mathcal{A},(1,2)}(2,1/2) + \pounds^{*,\star}_{\mathcal{A},(2,1)}(1/2,2) - \zeta_{\mathcal{A}}(3).$$

By applying the relation (181), we have the explicit value of $\pounds^{*,\star}_{\mathcal{A},\{1\}^3}(2,1,1/2)$ by Lemma 15.8 (126). The non-star cases are easily obtained from the star cases. □

Proposition 16.14.

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^2}(2,1/2) = -2q_{\boldsymbol{p}}(2)^2 + \left(q_{\boldsymbol{p}}(2)^3 - \frac{7}{8}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{222}$$

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(2,1/2) = -2q_{\boldsymbol{p}}(2)^2 + \left(q_{\boldsymbol{p}}(2)^3 - \frac{5}{24}B_{\boldsymbol{p}-3}\right)\boldsymbol{p}, \tag{223}$$

$$\pounds^{*}_{\mathcal{A}_2,\{1\}^2}(1/2,2) = \frac{5}{24}B_{\boldsymbol{p}-3}\boldsymbol{p}, \tag{224}$$

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(1/2,2) = \frac{7}{8}B_{\boldsymbol{p}-3}\boldsymbol{p}. \tag{225}$$

Proof. We can obtain the equality (223) by the relation (188), Lemma 15.6 (109), Lemma 15.8 (123), and Lemma 15.9 (130). The equalities (222), (224), and (225) are obtained by the following relations:

$$\pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(2,1/2) = \pounds^{*}_{\mathcal{A}_2,\{1\}^2}(2,1/2) + \zeta_{\mathcal{A}_2}(2),$$

$$\pounds_{\mathcal{A}_2,1}(2)\pounds_{\mathcal{A}_2,1}(1/2) = \pounds^{*}_{\mathcal{A}_2,\{1\}^2}(2,1/2) + \pounds^{*}_{\mathcal{A}_2,\{1\}^2}(1/2,2) + \zeta_{\mathcal{A}_2}(2),$$

$$\pounds_{\mathcal{A}_2,1}(2)\pounds_{\mathcal{A}_2,1}(1/2) = \pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(2,1/2) + \pounds^{*,\star}_{\mathcal{A}_2,\{1\}^2}(1/2,2) - \zeta_{\mathcal{A}_2}(2).$$

□

16.4 Some values of Ono–Yamamoto's finite multiple polylogarithms

In general, it is difficult to calculate each term in the right hand side of the relation (94) for $1 < i < r$. Therefore, it seems to hard to calculate special values of Ono–Yamamoto's FMPs. However, we can evaluate the following values:

Proposition 16.15.

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^2}(-1) = 2q_{\boldsymbol{p}}(2)^2, \tag{226}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^2}(2) = 2q_{\boldsymbol{p}}(2)^2, \tag{227}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^2}(1/2) = \frac{1}{2}q_{\boldsymbol{p}}(2)^2, \tag{228}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,2)}(-1) = 0, \tag{229}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(2,1)}(-1) = 0, \tag{230}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,2)}(2) = \frac{2}{3}q_{\boldsymbol{p}}(2)^3 - \frac{2}{3}B_{\boldsymbol{p}-3}, \tag{231}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(2,1)}(2) = \frac{2}{3}q_{\boldsymbol{p}}(2)^3 + \frac{4}{3}B_{\boldsymbol{p}-3}, \tag{232}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(1,2)}(1/2) = -\frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{1}{6}B_{\boldsymbol{p}-3}, \tag{233}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},(2,1)}(1/2) = -\frac{1}{6}q_{\boldsymbol{p}}(2)^3 - \frac{1}{3}B_{\boldsymbol{p}-3}, \tag{234}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^3}(-1) = -\frac{4}{3}q_{\boldsymbol{p}}(2)^3 - \frac{2}{3}B_{\boldsymbol{p}-3}, \tag{235}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^3}(2) = -\frac{4}{3}q_{\boldsymbol{p}}(2)^3 - \frac{2}{3}B_{\boldsymbol{p}-3}, \tag{236}$$

$$\pounds^{\mathrm{OY}}_{\mathcal{A},\{1\}^3}(1/2) = \frac{1}{6}q_{\boldsymbol{p}}(2)^3 + \frac{1}{12}B_{\boldsymbol{p}-3}. \tag{237}$$

Proof. All these values can be calculated by Corollary 14.4. All necessary special values of our $\mathcal{A}$-FMPs have already calculated. □

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