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Artin-Mazur zeta functions of generalized beta-transformations

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Abstract. In this paper, we study the Artin-Mazur zeta function of a generalization of the well-known β-transformation introduced by Góra [7]. We show that the Artin-Mazur zeta function can be extended to a meromorphic function via an expansion of 1 defined by using the transformation. As an application, we relate its analytic properties to algebraic properties of β.

1. Introduction

Let $\beta > 1$. The so-called β-transformation $\tau_\beta : [0,1] \to [0,1]$ is defined by

$$\tau_\beta(x) = \beta x - [\beta x]$$

for $x \in [0,1]$, where $[y]$ denotes the integral part of $y \in \mathbb{R}$. It is known that the β-expansion of x is given by

$$x = \sum_{n=0}^{\infty} \frac{I_\beta(\tau_\beta^n(x))}{\beta^{n+1}},$$

where $I_\beta(x) = [\beta x]$ for $x \in [0,1]$ (e.g.[14],[16]). In this paper, we consider the generalized β-transformation $\tau_{\beta,E} : [0,1] \to [0,1]$ introduced by Góra in [7]. It is defined as follows: Let $\beta > 1$ be a non-integer. Let us denote by

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$E = (E(0), \ldots, E([\beta]))$ a $([\beta]+1)$-dimensional vector with $E(i) \in \{0,1\}$ for $0 \leqq i \leqq [\beta]$. For $x \in [0,1]$, we put

$$\tau_{\beta,E}(x) = E(I_\beta(x)) + (-1)^{E(I_\beta(x))}\tau_\beta(x).$$

We note that if $E(i) = 0$ for $0 \leqq i \leqq [\beta]$, the generalized β-transformation $\tau_{\beta,E}$ is the β-transformation and if $E(i) = 1$ for $0 \leqq i \leqq [\beta]$, $\tau_{\beta,E}$ is the negative β-transformation whose dynamical properties are studied in [12], [10] and the references therein.

Similar to the β-transformation, we can define an expansion of numbers in $[0,1]$ associated with the generalized β-transformation. We define the $([\beta]+1)$ dimensional vector $D = (D(0), \cdots, D([\beta]))$ by $D(i) = i + E(i)$ for $0 \leqq i \leqq [\beta]$. For $x \in [0,1]$ and every non-negative integer $n \geqq 0$ we define 'digits' $d_n(\beta, E, x)$ and 'signs' $e_n(\beta, E, x)$ by

$$\begin{aligned} d_n(\beta,E,x) &= E(I_\beta(\tau^n_{\beta,E}(x))) + I_\beta(\tau^n_{\beta,E}(x)), \\ e_n(\beta,E,x) &= (-1)^{E(I_\beta(\tau^n_{\beta,E}(x)))}, \end{aligned}$$

and 'cumulative signs' $s_n(\beta, E, x)$ by

$$s_n(\beta,E,x) = \begin{cases} 1, & n = 0, \\ s_{n-1}(\beta,E,x)e_{n-1}(\beta,E,x), & n \geqq 1. \end{cases}$$

Then the definition of $\tau_{\beta,E}$ we can represent $x \in [0,1]$ by

(1.1)

$$
\begin{aligned}
x &= \frac{s_0(\beta,E,x)d_0(\beta,E,x)}{\beta} + \frac{s_1(\beta,E,x)\tau_{\beta,E}(x)}{\beta} \\
&= \frac{s_0(\beta,E,x)d_0(\beta,E,x)}{\beta} + \frac{s_1(\beta,E,x)d_1(\beta,E,x)}{\beta^2} + \frac{s_2(\beta,E,x)\tau_{\beta,E}^2(x)}{\beta^2} \\
&= \cdots \\
&= \sum_{i=0}^{n-1} \frac{s_i(\beta,E,x)d_i(\beta,E,x)}{\beta^{i+1}} + \frac{s_n(\beta,E,x)\tau_{\beta,E}^n(x)}{\beta^n}
\end{aligned}
$$

for every positive integer $n \geqq 1$ (see Proposition 1 in [7]). Since $\tau_{\beta,E}^n(x) \in [0,1]$ and $s_n(\beta,E,x) \in \{-1,1\}$, we have

$$x = \sum_{i=0}^{\infty} \frac{s_i(\beta,E,x)d_i(\beta,E,x)}{\beta^{i+1}}.$$

We call this expansion the $\tau_{\beta,E}$-expansion of x. We note that if $\tau_{\beta,E}$ is the β-transformation, the expansion is equal to the β-expansion and if $\tau_{\beta,E}$ is the negative β-transformation, the expansion is equal to the negative β-expansion (see [12] and the references therein). If there exists a positive integer $n \geqq 1$ such that $\tau_{\beta,E}^n(x) \in \{1/\beta, \cdots, [\beta]/\beta\}$, by the equation (1.1) we have

$$x = \sum_{i=0}^{n-1} \frac{s_i(\beta,E,x)d_i(\beta,E,x)}{\beta^{i+1}} + \frac{s_n(\beta,E,x)n_0}{\beta^{n+1}},$$

where $n_0 = \beta\tau_{\beta,E}^n(x) \in \{1, \cdots, [\beta]\}$. In this case, we call the $\tau_{\beta,E}$-expansion of x finite. If the $\tau_{\beta,E}$-expansion of 1 is finite then we call β simple.

In [7] Góra showed that every map $\tau_{\beta,E}$ has a unique absolutely continuous invariant probability measure $\mu_{\beta,E}$ and its density function $h_{\beta,E}$ can be expressed by using the coefficients of the $\tau_{\beta,E}$-expansion of 1. Precisely, the density function $h_{\beta,E}$ is expressed as

$$h_{\beta,E}(x)=\frac{1}{F(\beta,E)}\sum_{n=0}^{\infty}\frac{s_n(\beta,E,1)\chi_{[0,\tau_{\beta,E}^n(1))}(x)}{\beta^{n+1}},\quad x\in[0,1],$$

where χ_A denotes the characteristic function of A and $F(\beta,E)$ denotes the normalizing factor:

$$F(\beta,E)=\sum_{n=0}^{\infty}\frac{s_n(\beta,E,1)\tau_{\beta,E}^n(1)}{\beta^{n+1}}.$$

We remark that Parry showed that the above result for β-transformations in [14].

The purpose of this paper is to give the analytic continuation of the Artin-Mazur zeta function of a generalized β-transformation $\zeta_{\tau_{\beta,E}}(z)$ via the generating function for the coefficients of the $\tau_{\beta,E}$-expansion of 1. The Artin-Mazur zeta function is given by

$$\zeta_{\tau_{\beta,E}}(z)=\exp\Big(\sum_{n=1}^{\infty}\frac{z^n}{n}\sharp\mathrm{Fix}\tau_{\beta,E}^n\Big),$$

where $\mathrm{Fix}\tau_{\beta,E}^n=\{x\in[0,1]\ ;\ \tau_{\beta,E}^n(x)=x\}$ for $n\geq 1$ and $\sharp A$ denotes the cardinality of A. We denote by $\rho_{\beta,E}(z)$ the generating function for the coefficients of the $\tau_{\beta,E}$-expansion of 1:

$$\rho_{\beta,E}(z)=\sum_{n=0}^{N-1}s_n(\beta,E,1)d_n(\beta,E,1)z^n$$
$$+k_0s_N(\beta,E,1)z^N$$

if β is simple, where N is the minimal positive integer with $\tau_{\beta,E}^N(1) \in \{1/\beta, \ldots, [\beta]/\beta\}$ and $k_0 = \beta\tau_{\beta,E}^N(1)$, and

$$\rho_{\beta,E}(z) = \sum_{n=0}^{\infty} s_n(\beta, E, 1)d_n(\beta, E, 1)z^n$$

if β is not simple. We set $\phi_{\beta,E}(z) = z\rho_{\beta,E}(z)$. The main theorem in this paper is:

Theorem 1.1. *The Artin-Mazur zeta function $\zeta_{\tau_{\beta,E}}(z)$ of the generalized β-transformation $\tau_{\beta,E}$ converges absolutely in $|z| < 1/\beta$ and $\phi_{\beta,E}(z)$ has a radius of convergence greater than or equal to 1. In addition, for $z \in \mathbb{C}$ with $|z| < 1/\beta$, we have*

$$\zeta_{\tau_{\beta,E}}(z) = \frac{p_{\beta,E}(z)}{1 - \phi_{\beta,E}(z)}, \tag{1.2}$$

where $p_{\beta,E}(z)$ is a 'cyclotomic factor', i.e., a rational function whose numerator and denominator are the products of cyclotomic polynomials. Therefore, $\zeta_{\tau_{\beta,E}}(z)$ can be extended to a meromorphic function in the unit open disc. In particular, if the sequence of integers $\{s_n(\beta, E, 1)d_n(\beta, E, 1)\}_{n=0}^{\infty}$ is eventually periodic, then $\zeta_{\tau_{\beta,E}}(z)$ can be extended to a rational function. Otherwise $\zeta_{\tau_{\beta,E}}(z)$ can not be extended to a meromorphic function beyond the unit circle. Furthermore, $\zeta_{\tau_{\beta,E}}(z)$ has a simple pole at $1/\beta$.

The equation (1.2) enables us to relate the analytic properties of the Artin-Mazur zeta function $\zeta_{\tau_{\beta,E}}(z)$ to the algebraic properties of β.

In [6], Flatto et al. proved the equation (1.2) in the case of β-transformations. We note that Theorem 1.1 is extension of this result (see also [9]).

In [13], Milnor and Thurston introduced the kneading matrix for a piecewise monotone continuous map in the interval and proved that its Artin-Mazur zeta function can be expressed by using the reciprocal of its kneading determinant. After that Preston [15] extended the result to the case where the map has a finite number of discontinuities (see also [2]). Since generalized β-transformations are included in Preston's situation, it seems that the analytic continuation of the Artin-Mazur zeta function $\zeta_{\tau_{\beta,E}}(z)$ is calculated explicitly by the kneading determinant. However, the relation between the kneading determinant and the generating function $\rho_{\beta,E}(z)$ is not clear, for now.

This paper is organized as follows. In Section 2, we summarize the notions we need to prove Theorem 1.1 and we give the proof in Section 3. In Section 4, we study the analytic properties of $\zeta_{\tau_{\beta,E}}(z)$ and show that if $\zeta_{\tau_{\beta,E}}(z)$ has no pole in the unit open disk except $z = 1/\beta$ then β is a Pisot or Salem number (see Proposition 4.2). In Section 5, we consider the Artin-Mazur zeta function of a negative β-transformation and see that the Artin-Mazur zeta function has no pole in the disk $\{z \in \mathbb{C}; |z| \leqq 1/\beta\}$ except $z = 1/\beta$ (see Proposition 5.1). In addition, we show that the Artin-Mazur zeta function converges to a meromorphic function associated with the Thue-Morse sequence in the unit open disk as $\beta \searrow 1$ (see Theorem 5.3). In Section 6, we discuss Chebyshev maps considered as a generalization of Chebyshev polynomials. They are defined by

$$T_\beta(x) = \cos(\beta \arccos x)$$

for $x \in [-1, 1]$, where $\beta > 1$. As an application of Theorem 1.1, we prove that a dynamical zeta function of this map is analytic in the unit open disk and it can be extended to a meromorphic function defined in the open disk $\{z \in \mathbb{C}; |z| < \beta\}$ (see Theorem 6.4).

2. Preliminaries

We introduce basic notions which are used in the proof of Theorem 1.1. Let X be a non-empty set and $S : X \to X$ a map. We set $\mathrm{Fix}S = \{x \in X; Sx = x\}$ and assume that $\sharp\mathrm{Fix}S^n < +\infty$ for every positive integer $n \geqq 1$. The Artin-Mazur zeta function $\zeta_S(z)$ is formally defined by

$$\zeta_S(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} \sharp\,\mathrm{Fix}\, S^n\Big).$$

Next, we give basic notions about symbolic dynamics. It is well known that Artin-Mazur zeta functions of subshift of finite types can be calculated by Bowen-Lanford Formula. We summarize them in the following.

Let $Y = \{0, 1, \ldots, N-1\}$ be a finite set endowed with the discrete topology and $\Sigma^+ = Y^{\mathbb{Z}_{\geqq 0}}$ with the product topology. The left shift $\sigma : \Sigma^+ \to \Sigma^+$ is defined by

$$\sigma\{x_n\}_{n=0}^{\infty} = \{x_{n+1}\}_{n=0}^{\infty}$$

for $\{x_n\}_{n=0}^{\infty} \in \Sigma^+$. Then the left shift is a continuous transformation. The pair (Σ^+, σ) is called the full shift.

Let $M = (m_{ij})_{i,j=0}^{N-1}$ be an $N \times N$ matrix with $m_{ij} \in \{0, 1\}$ for $0 \leqq i, j \leqq N-1$. Put

$$\Sigma_M^+ = \{\{x_n\}_{n=0}^{\infty} \in \Sigma^+ |\ m_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}_{\geqq 0}\}.$$

Then Σ_M^+ is a closed and σ-invariant subset of Σ^+. We denote by σ_M the restriction of σ to Σ_M^+. The pair (Σ_M^+, σ_M) is called the subshift of finite type determined by a structure matrix M. It is clear that σ_M has a finite number of n-fixed points. The next proposition is known as Bowen-Lanford Formula.

Proposition 2.1 (Bowen-Lanford [3])**.** *Let M be an $N \times N$ matrix with $m_{ij} \in \{0,1\}$ for $0 \leqq i,j \leqq N-1$ and (Σ_M^+, σ_M) the subshift of finite type determined by M. Let $\lambda_1, \ldots, \lambda_N$ be eigenvalues of M. For $z \in \mathbb{C}$ with $|z| < 1/\max_{1 \leqq i \leqq N}\{\lambda_i\}$, we have*

$$\zeta_{\sigma_M}(z) = \frac{1}{\det(I - zM)}.$$

Now, we define the set of monotone pieces of the map $\tau_{\beta,E}^n$. For $0 \leqq i \leqq [\beta]$ we put

$$J_i = \begin{cases} [\frac{i}{\beta}, \frac{i+1}{\beta}), & 0 \leqq i \leqq [\beta]-1, \\ [\frac{[\beta]}{\beta}, 1], & i = [\beta]. \end{cases}$$

For every positive integer $n \geqq 1$, we define the set of open intervals $\mathcal{A}_n(\beta, E)$ by

$$\mathcal{A}_1(\beta, E) = \left\{ \mathring{J}_0, \mathring{J}_1, \cdots, \mathring{J}_{[\beta]} \right\}$$

and

$$\mathcal{A}_n(\beta, E) = \left\{ I \cap \tau_{\beta,E}^{-1} J \neq \emptyset \ ; \ I \in \mathcal{A}_1(\beta, E), J \in \mathcal{A}_{n-1}(\beta, E) \right\}$$

for $n \geqq 2$. We note that if $I \in \mathcal{A}_n(\beta, E)$ then there exists a unique word $p_0 \cdots p_{n-1} \in \{0, 1, \cdots, [\beta]\}^n$ such that $I = \bigcap_{i=0}^{n-1} \tau_{\beta,E}^{-i} J_{p_i}$ and $\tau_{\beta,E}^n$ is continuous on I.

Finally, we introduce a linear operator on the set of all functions from $[0, 1]$ to $\mathbb{R}$ associated with a generalized β-transformation.

Definition 2.2. Let $\mathcal{F}$ be the set of all functions from $[0, 1]$ to $\mathbb{R}$. We define the operator $\mathcal{L}_{\tau_{\beta,E}} : \mathcal{F} \to \mathcal{F}$ by

$$\mathcal{L}_{\tau_{\beta,E}} f(x) = \sum_{y : \tau_{\beta,E}(y) = x} f(y)$$

for $x \in [0, 1]$.

We remark that the operator $(1/\beta)\mathcal{L}_{\tau_{\beta,E}}$ is the Perron-Frobenius operator for a generalized β-transformation $\tau_{\beta,E}$. Perron-Frobenius operators associated with piecewise expanding maps on the interval are useful for studying its ergodic properties (e.g. [4], [11]).

3. The proof of Theorem 1.1

The next lemma shows a basic property of the operator $\mathcal{L}_{\tau_{\beta,E}}$.

Lemma 3.1. *Let $n \geqq 0$ be a non-negative integer. Then we have*

$$\begin{aligned}
&\mathcal{L}_{\tau_{\beta,E}} \chi_{(0, \tau_{\beta,E}^n(1))}(x) \\
&= e_n(\beta, E, 1) \chi_{(0, \tau_{\beta,E}^{n+1}(1))}(x) + d_n(\beta, E, 1) \chi_{(0,1)}(x)
\end{aligned}$$

for $x \in (0, 1) \setminus \{\tau_{\beta,E}^{n+1}(1)\}$.

Proof. By the definition of $\mathcal{L}_{\tau_{\beta,E}}$, if $e_n(\beta,E,1)=+1$ we have

$$\begin{aligned}\mathcal{L}_{\tau_{\beta,E}}\chi_{(0,\tau_{\beta,E}^n(1))}(x) &= \sharp\{y\in(0,\tau_{\beta,E}^n(1))\ ;\ x=\tau_{\beta,E}(y)\}\\ &= I_\beta(\tau_{\beta,E}^n(1))+1\end{aligned}$$

for $x\in(0,\tau_{\beta,E}^{n+1}(1))$ and

$$\begin{aligned}\mathcal{L}_{\tau_{\beta,E}}\chi_{(0,\tau_{\beta,E}^n(1))}(x) &= \sharp\{y\in(0,\tau_{\beta,E}^n(1))\ ;\ x=\tau_{\beta,E}(y)\}\\ &= I_\beta(\tau_{\beta,E}^n(1))\end{aligned}$$

for $x\in(\tau_{\beta,E}^{n+1}(1),1)$ since the map $\tau_{\beta,E}$ is bijective in J_i for $0\leqq i\leqq[\beta]-1$.

Similar to the above calculation, if $e_n(\beta,E,1)=-1$ then we have

$$\mathcal{L}_{\tau_{\beta,E}}\chi_{(0,\tau_{\beta,E}^n(1))}(x)=I_\beta(\tau_{\beta,E}^n(1))$$

for $x\in(0,\tau_{\beta,E}^{n+1}(1))$ and

$$\mathcal{L}_{\tau_{\beta,E}}\chi_{(0,\tau_{\beta,E}^n(1))}(x)=I_\beta(\tau_{\beta,E}^n(1))+1$$

for $x\in(\tau_{\beta,E}^{n+1}(1),1)$.

By definition, we have

$$d_n(\beta,E,1)=I_\beta(\tau_{\beta,E}^n(1))$$

if $e_n(\beta,E,1)=+1$ and

$$d_n(\beta,E,1)=I_\beta(\tau_{\beta,E}^n(1))+1$$

if $e_n(\beta,E,1)=-1$. Hence we get the assertion. □

The following lemma states that non-simple β-numbers are approximated by simple β-numbers.

Lemma 3.2 (Góra [7]). *Let $n \geqq 1$ be a positive integer and $E = (E(0), \dots, E(n))$ an $(n+1)$-dimensional vector with $E(i) \in \{0, 1\}$ for $0 \leqq i \leqq n$. Then the set of all simple numbers is dense in $(n, n+1)$.*

The next lemma shows that if a real number $\beta_0 > 1$ is not simple, we can approximate the coefficients of the $\tau_{\beta_0,E}$-expansion of 1 by those of the $\tau_{\beta,E}$-expansion of 1 where β is sufficiently close to β_0.

Lemma 3.3. *Let $n \geqq 1$ be a positive integer and $E = (E(0), \dots, E(n))$ an $(n+1)$-dimensional vector with $E(i) \in \{0, 1\}$ for $0 \leqq i \leqq n$. Assume that $\beta_0 \in (n, n+1)$ is not simple. Then for every positive integer $m \geqq 1$ there exists $\delta > 0$ such that*

$$\{s_i(\beta, E, 1)d_i(\beta, E, 1)\}_{i=0}^m = \{s_i(\beta_0, E, 1)d_i(\beta_0, E, 1)\}_{i=0}^m$$

whenever $|\beta - \beta_0| < \delta$.

Proof. We note that since $\beta_0 \in (n, n+1)$ is not simple, we have

$$\frac{I_{\beta_0}(\tau^i_{\beta_0,E}(1))}{\beta_0} < \tau^i_{\beta_0,E}(1) < \frac{I_{\beta_0}(\tau^i_{\beta_0,E}(1)) + 1}{\beta_0}$$

for every $i \geqq 1$.

For $\beta \in (n, n+1)$, we put

$$\begin{aligned} P_1(\beta) &= \beta \cdot (E(n) + (-1)^{E(n)}\tau_\beta(1)) \\ &= \beta\tau_{\beta,E}(1). \end{aligned}$$

By the definition of the polynomial P_1, we have $P_1(\beta_0) = \beta_0\tau_{\beta_0,E}(1)$ and $n < P_1(\beta_0) < n+1$. Since $P_1(\beta)$ is continuous at β_0, there exists $\delta > 0$ such that $n < P_1(\beta) < n+1$ whenever $|\beta - \beta_0| < \delta$. So we have $n/\beta < \tau_{\beta,E}(1) <$

$(n+1)/\beta$ whenever $|\beta-\beta_0|<\delta$. Hence, we obtain $e_0(\beta,E,1)=e_0(\beta_0,E,1)$ and $I_\beta(\tau_{\beta,E}(1))=I_\beta(\tau_{\beta_0,E}(1))$, namely,

$$s_1(\beta,E,1)d_1(\beta,E,1)=s_1(\beta_0,E,1)d_1(\beta_0,E,1)$$

whenever $|\beta-\beta_0|<\delta$.

For $\beta\in(\beta_0-\delta,\beta_0+\delta)$, we put

$$P_2(\beta)=\beta\tau^2_{\beta,E}(1).$$

Similar to the above argument, there exists a positive number $\eta>0$ such that

$$s_2(\beta,E,1)d_2(\beta,E,1)=s_2(\beta_0,E,1)d_2(\beta_0,E,1)$$

whenever $|\beta-\beta_0|<\eta$. The assertion is obtained by repeating the above argument inductively. $\square$

The next lemma states that if β_0 is not simple, we can approximate the number of all n-fixed points of $\tau_{\beta_0,E}$ by that of $\tau_{\beta,E}$ where β is sufficiently close to β_0.

Lemma 3.4. *Let $n\geqq 1$ be a positive integer and $E=(E(0),\dots,E(n))$ an $(n+1)$-dimensional vector with $E(i)\in\{0,1\}$ for $0\leqq i\leqq n$. Assume that $\beta_0\in(n,n+1)$ is not simple. Then for a positive integer $m\geqq 1$ there exists $\delta>0$ such that*

$$\sharp\,\mathrm{Fix}\,\tau^i_{\beta,E}=\sharp\,\mathrm{Fix}\,\tau^i_{\beta_0,E}$$

for $1\leqq i\leqq m$ whenever $|\beta-\beta_0|<\delta$.

Proof. Recall that since $\beta_0\in(n,n+1)$ is not simple, we have

$$\text{(3.1)} \qquad \frac{I_{\beta_0}(\tau^i_{\beta_0,E}(1))}{\beta_0} < \tau^i_{\beta_0,E}(1) < \frac{I_{\beta_0}(\tau^i_{\beta_0,E}(1))+1}{\beta_0}$$

for every $i \geqq 1$. By the proof of Lemma 3.3, for a positive integer $m \geqq 1$ there exists $\delta > 0$ such that

$$\text{(3.2)} \qquad I_{\beta}(\tau^i_{\beta,E}(1)) = I_{\beta_0}(\tau^i_{\beta_0,E}(1))$$

and

$$\text{(3.3)} \qquad \frac{I_{\beta}(\tau^i_{\beta,E}(1))}{\beta} < \tau^i_{\beta,E}(1) < \frac{I_{\beta}(\tau^i_{\beta,E}(1))+1}{\beta}$$

for $1 \leqq i \leqq m$ whenever $|\beta - \beta_0| < \delta$. We put the polynomials $f_1, \ldots, f_m$ with integral coefficients as

$$f_i(\beta) = \tau^i_{\beta,E}(1)$$

for $1 \leqq i \leqq m$ and $|\beta - \beta_0| < \delta$. By the equality (3.2) and the inequality (3.3) we have

$$f_i(\beta_0) = \tau^i_{\beta_0,E}(1)$$

for $1 \leqq i \leqq m$.

Let $i \in \{1, \cdots, m\}$ and $\mathcal{A}_i(\beta, E)$ be the set of open intervals defined in Section 2. Remark that for $J \in \mathcal{A}_i(\beta, E)$ the two endpoints of the image $\tau^i_{\beta,E}(J)$ are in the set $\{0, \tau_{\beta,E}(1), \cdots, \tau^i_{\beta,E}(1), 1\}$. Therefore, the equality (3.2) and the inequality (3.1) and (3.3) yield that

$$\sharp\mathcal{A}_i(\beta, E) = \sharp\mathcal{A}_i(\beta_0, E)$$

for $1 \leqq i \leqq m$ whenever $|\beta - \beta_0| < \delta$.

Let $I \in \mathcal{A}_i(\beta_0, E)$ and put $I = \big(a(\beta_0), b(\beta_0)\big)$. Then by the definition of the set of open intervals $\mathcal{A}_i(\beta_0, E)$, we have

$$\lim_{x \searrow a(\beta_0)} \tau^i_{\beta_0,E}(x), \lim_{x \nearrow b(\beta_0)} \tau^i_{\beta_0,E}(x) \in \{0, \tau_{\beta_0,E}(1), \dots, \tau^i_{\beta_0,E}(1), 1\}.$$

If there exists an i-fixed point $y \in \bar{I}$ then we get $y \in \mathring{I}$ because β_0 is not simple. In addition, if $\tau^i_{\beta_0,E}$ is increasing on I, we have

$$\lim_{x \searrow a(\beta_0)} \tau^i_{\beta_0,E}(x) < a(\beta_0), \lim_{x \nearrow b(\beta_0)} \tau^i_{\beta_0,E}(x) > b(\beta_0).$$

If $\tau^i_{\beta_0,E}$ is decreasing on I then we have

$$\lim_{x \searrow a(\beta_0)} \tau^i_{\beta_0,E}(x) > a(\beta_0), \lim_{x \nearrow b(\beta_0)} \tau^i_{\beta_0,E}(x) < b(\beta_0).$$

Assume that $\lim_{x \searrow a(\beta_0)} \tau^i_{\beta_0,E}(x) < a(\beta_0)$ and $\lim_{x \nearrow b(\beta_0)} \tau^i_{\beta_0,E}(x) > b(\beta_0)$. Since $f_i(\beta) = \tau^i_{\beta,E}(1)$ is continuous at β_0 we obtain that there exists a positive number η such that

$$\lim_{x \searrow a(\beta)} \tau^i_{\beta,E}(x) < a(\beta), \lim_{x \nearrow b(\beta)} \tau^i_{\beta,E}(x) > b(\beta)$$

whenever $|\beta - \beta_0| < \eta$. The other case where $\lim_{x \searrow a(\beta_0)} \tau^i_{\beta_0,E}(x) > a(\beta_0)$ and $\lim_{x \nearrow b(\beta_0)} \tau^i_{\beta_0,E}(x) < b(\beta_0)$ is similar. Since $\sharp \mathcal{A}_i(\beta_0, E) < +\infty$ for $1 \leqq i \leqq m$, we can get the assertion. $\square$

Proof of Theorem 1.1. Let $n \geqq 1$ be a positive integer and $\mathcal{A}_n(\beta, E)$ the set of open intervals defined in Section 2. Since $\tau^n_{\beta,E}$ is strictly monotone on $I \in \mathcal{A}_n(\beta, E)$, we have

$$\sharp \mathrm{Fix} \tau^n_{\beta_0,E} \leqq \sharp \mathcal{A}_n(\beta, E).$$

We put $\tilde{\mathcal{L}}_{\tau_{\beta,E}} = (1/\beta)\mathcal{L}_{\tau_{\beta,E}}$. Then the operator $\tilde{\mathcal{L}}_{\tau_{\beta,E}}$ is known to be the Perron-Frobenius operator for $\tau_{\beta,E}$. As a consequence of the Lasota-Yorke inequality for $\tau_{\beta,E}$ applied to the constant function 1, we have that there exists a constant $C' > 0$ such that

$$\bigvee_0^1 \tilde{\mathcal{L}}_{\tau_{\beta,E}^n} 1 \leqq C'$$

for $n \geqq 1$, where $\bigvee_0^1 f$ denotes the total variation of a function f on $[0,1]$ (see the proof of Theorem 1 in [11]). Since

$$\sup_{x\in[0,1]} |f(x)| \leqq \bigvee_0^1 f + \int_0^1 f \ dm$$

for a function of bounded variation f on $[0,1]$, where m denotes the Lebesgue measure on $[0,1]$ (see Lemma 2.3.1 in [4]), and

$$\int_0^1 \tilde{\mathcal{L}}_{\tau_{\beta,E}^n} 1 \ dm = \int_0^1 1 \ dm = 1$$

by the definition of the Perron-Frobenius operator, there exists a constant $C > 0$ such that

$$\sup_{x\in[0,1]} |\tilde{\mathcal{L}}_{\tau_{\beta,E}^n} 1(x)| \leqq C$$

for $n \geqq 1$. Then we have

$$\begin{aligned}
\sharp\mathcal{A}_n(\beta, E) &\leqq \sum_{i=0}^{n}(\mathcal{L}_{\tau_{\beta,E}^n}1)(\tau_{\beta,E}^i(1)) + (\mathcal{L}_{\tau_{\beta,E}^n}1)(0) \\
&= \beta^n \cdot \Big(\sum_{i=0}^{n}(\tilde{\mathcal{L}}_{\tau_{\beta,E}^n}1)(\tau_{\beta,E}^i(1)) + (\tilde{\mathcal{L}}_{\tau_{\beta,E}^n}1)(0)\Big) \\
&\leqq \beta^n \cdot (n+2) \cdot \sup_{x\in[0,1]} |\tilde{\mathcal{L}}_{\tau_{\beta,E}}1(x)| \\
&\leqq C(n+2)\beta^n.
\end{aligned}$$

Therefore, we get

$$\limsup_{n\to\infty} \sqrt[n]{\frac{\sharp\mathrm{Fix}\tau_{\beta,E}^n}{n}} \leqq \beta.$$

Hence we obtain $\zeta_{\tau_{\beta,E}}(z)$ converges absolutely in $\{z \in \mathbb{C} \mid |z| < 1/\beta\}$.

Since the sequence of integers $\{s_n(\beta, E, 1)d_n(\beta, E, 1)\}_{n=0}^{\infty}$ is bounded, it is clear that $\phi_{\beta,E}(z)$ has the radius of convergence greater than or equal to 1. Furthermore, we obtain that $\phi_{\beta,E}(z)$ is a rational function or has the unit circle as the natural boundary by the theorem of Pólya and Carleson, which states that a formal power series $\sum_{n=0}^{\infty} a_n z^n$ with integral coefficients which converges in the unit disk is a rational function or has the unit circle as the natural boundary (see Theorem 5.3 in [17]). Furthermore, if $\{a_n\}_{n=0}^{\infty}$ is bounded, we can show that $\sum_{n=0}^{\infty} a_n z^n$ is a rational function if and only if $\{a_n\}_{n=0}^{\infty}$ is eventually periodic. Since $\{s_n(\beta, E, 1)d_n(\beta, E, 1)\}_{n=0}^{\infty}$ is bounded, we can apply this argument to $\phi_{\beta,E}(z)$ directly.

We are in the position to prove the equation (1.2). Assume that β is simple and take the minimal positive integer N such that $\tau_{\beta,E}^N(1) \in$

$\{1/\beta, \cdots, [\beta]/\beta\}$ and set $k_0 = \beta \cdot \tau_{\beta,E}^N(1)$. We put

$$\left\{0, \frac{1}{\beta}, \ldots, \frac{[\beta]}{\beta}, 1\right\} \cup \{\tau_{\beta,E}(1), \ldots, \tau_{\beta,E}^{N-1}(1)\}$$

$$= \{a_0, \ldots, a_{N+[\beta]} \ ; \ a_i < a_{i+1} \text{ for } 0 \leqq i \leqq N+[\beta]-1\}$$

and $I_i = (a_i, a_{i+1})$ for $0 \leqq i \leqq N+[\beta]-1$. Then $\{I_i\}_{i=0}^{N+[\beta]-1}$ is a Markov partition for $\tau_{\beta,E}$.

We also put $A = \bigcup_{n=0}^{\infty} \tau_{\beta,E}^{-n}\{a_0, \ldots, a_{N+[\beta]}\}$ and $\Sigma^+ = \{0, 1, \ldots, [\beta]+N-1\}^{\mathbb{Z}_{\geqq 0}}$. We define the coding map $T : [0,1] \setminus A \to \Sigma^+$ by $T(x) = (y_i)_{i=0}^{\infty}$, where $y_i = k$ if $\tau_{\beta,E}^i(x) \in I_k$ for $i \geqq 0$. Then the closure of $T([0,1] \setminus A)$ in Σ^+ is a shift-invariant set and a subshift of finite type whose structure matrix $M = (m_{ij})_{0 \leqq i,j \leqq N+[\beta]-1}$ satisfies

$$\mathcal{L}_{\tau_{\beta,E}} \chi_{I_i} = \sum_{j=0}^{N+[\beta]-1} m_{ij} \chi_{I_j} \tag{3.4}$$

for $0 \leqq i \leqq N+[\beta]-1$.

Let $\mathbb{A}$ be the subspace of $\mathcal{F}$ spanned by $\alpha = \{\chi_{I_j}\}_{j=0}^{N+[\beta]-1}$. It is clear that $\mathcal{L}_{\tau_{\beta,E}} \mathbb{A} \subset \mathbb{A}$. Since $I_i \cap I_j = \emptyset$ if $i \neq j$, so α is linearly independent in $\mathcal{F}$ and a basis for $\mathbb{A}$. By the equation (3.4), M is the matrix representation of $\mathcal{L}_{\tau_{\beta,E}} : \mathbb{A} \to \mathbb{A}$ relative to the basis α. We define the basis $\tilde{\alpha}$ for $\mathbb{A}$ by

$$\chi_{(0,1)}, \chi_{(0,\tau_{\beta,E}(1))}, \ldots, \chi_{(0,\tau_{\beta,E}^{N-1}(1))}, \chi_{(0,\tau_{\beta,E}^{N}(1))}$$

and

$$\chi_{(\frac{k}{\beta}, \frac{k+1}{\beta})}$$

for $k \in \{0, \ldots, [\beta]-1\} \setminus \{k_0\}$. By Lemma 3.1 and

$$\mathcal{L}_{\tau_{\beta,E}} \chi_{(\frac{k}{\beta}, \frac{k+1}{\beta})} = \chi_{(0,1)},$$

we have that the $(N+[\beta]) \times (N+[\beta])$ matrix L, where

$$L = \begin{pmatrix} d_0(\beta,E,1) & e_0(\beta,E,1) & & & \text{\huge 0} \\ \vdots & & \ddots & & \\ \vdots & & & & \\ d_{N-1}(\beta,E,1) & & & e_{N-1}(\beta,E,1) & \\ k_0 & & & & \\ 1 & & & & \\ \vdots & & \text{\huge 0} & & \\ \vdots & & & & \\ 1 & & & & \end{pmatrix}$$

is the matrix representation of $\mathcal{L}_{\tau_{\beta,E}} : \mathbb{A} \to \mathbb{A}$ for the basis $\tilde{\alpha}$. By Proposition 2.1, we get

$$\begin{aligned} \zeta_{\sigma_M}(z) &= \det(I - zM)^{-1} \\ &= \det(I - zL)^{-1}. \end{aligned}$$

By repeating the Laplace expansion along the first row, we get

$$\begin{aligned} \det(I - zL) &= 1 - \sum_{n=0}^{N-1} s_n(\beta,E,1) d_n(\beta,E,1) z^{n+1} \\ &\quad + k_0 s_N(\beta,E,1) z^{N+1} \\ &= 1 - \phi_{\beta,E}(z). \end{aligned}$$

Therefore, we obtain

$$\zeta_{\sigma_M}(z) = \frac{1}{1 - \phi_{\beta,E}(z)}.$$

We remark that $\underline{x} \in \mathrm{Fix}\sigma_M^n$ if and only if $T^{-1}(\underline{x}) \in \mathrm{Fix}\tau_{\beta,E}^n$ and $T^{-1}(\underline{x})$ is an interior point of $I \in \mathcal{A}_n(\beta, E)$ or there exists an end point x of $I \in \mathcal{A}_n(\beta, E)$ such that $\underline{x} = \lim_{y \to x} T \circ \tau_{\beta,E}^n(y)$.

If $\tau_{\beta,E}$ satisfies $\lim_{x \nearrow 1} \tau_{\beta,E}^{N+1}(x) = \tau_{\beta,E}^{N+1}(1)$, we have

$$\sharp \mathrm{Fix}\tau_{\beta,E}^i = \sharp \mathrm{Fix}\sigma_M^i$$

for $i \geqq 1$. Therefore we have $p_{\beta,E}(z) \equiv 1$.

In the case where $\lim_{x \nearrow 1} \tau_{\beta,E}^{N+1}(x) \neq \tau_{\beta,E}^{N+1}(1)$, there are only four cases to consider at the end since $\tau_{\beta,E}^{N+1} \in \{0, 1\}$ and $\lim_{x \nearrow 1} \tau_{\beta,E}^{N+1}(x) \in \{0, 1\}$.

Case 1. We consider the case where $\lim_{x \nearrow 1} \tau_{\beta,E}^{N+1}(x) = 1, \tau_{\beta,E}^{N+1}(1) = 0$ and $E(0) = 0$. Since 1 is not a fixed point of $\tau_{\beta,E}^{N+1}$ and $\lim_{y \to 1} T \circ \tau_{\beta,E}^n(y)$ is a fixed point of σ_M^{N+1}, we get

$$\sharp \mathrm{Fix}\tau_{\beta,E}^{(N+1)i} + N + 1 = \sharp \mathrm{Fix}\sigma_M^{(N+1)i}$$

for $i \geqq 1$. Therefore we have

$$\begin{aligned}
\zeta_{\tau_{\beta,E}}(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} \sharp \,\mathrm{Fix}\, \tau_{\beta,E}^{(N+1)i}\Big) \\
&= \zeta_{\sigma_M}(z) \cdot \exp\Big(-\sum_{i=1}^{\infty} \frac{z^{(N+1)i}}{i}\Big) \\
&= \frac{1 - z^{N+1}}{1 - \phi_{\beta,E}(z)}.
\end{aligned}$$

Hence we obtain

$$p_{\beta,E}(z) = 1 - z^{N+1}.$$

Case 2. We consider the case where $\lim_{x\nearrow 1}\tau_{\beta,E}^{N+1}(x)=1, \tau_{\beta,E}^{N+1}(1)=0$ and $E(0)=1$. Since 1 is not a fixed point of $\tau_{\beta,E}^{N+1}$ but is a fixed point of $\tau_{\beta,E}^{N+2}$, we get

$$\sharp\mathrm{Fix}\tau_{\beta,E}^{(N+1)i}+N+1=\sharp\mathrm{Fix}\sigma_M^{(N+1)i}$$

and

$$\sharp\mathrm{Fix}\tau_{\beta,E}^{(N+2)i}=\sharp\mathrm{Fix}\sigma_M^{(N+2)i}+N+2$$

for $i\geqq 1$. Hence we obtain

$$p_{\beta,E}(z)=\frac{1-z^{N+1}}{1-z^{N+2}}.$$

Case 3. We consider the case where $\lim_{x\nearrow 1}\tau_{\beta,E}^{N+1}(x)=0, \tau_{\beta,E}^{N+1}(1)=1$ and $E(0)=0$. Then 1 is a fixed point of $\tau_{\beta,E}^{N+1}$ and $\lim_{y\to 1}T\circ\tau_{\beta,E}^{n}(y)$ is not a fixed point of σ_M^{N+1}. Therefore we get

$$\sharp\mathrm{Fix}\tau_{\beta,E}^{(N+1)i}=\sharp\mathrm{Fix}\sigma_M^{(N+1)i}+N+1$$

for $i\geqq 1$. Hence we obtain

$$p_{\beta,E}(z)=\frac{1}{1-z^{N+1}}.$$

Case 4. We consider the case where $\lim_{x\nearrow 1}\tau_{\beta,E}^{N+1}(x)=0, \tau_{\beta,E}^{N+1}(1)=1$ and $E(0)=1$. Then $\lim_{y\to 1}T\circ\tau_{\beta,E}^{n}(y)$ is not a fixed point of σ_M^{N+1} but is a fixed point of σ_M^{N+2}. Therefore we get

$$\sharp\mathrm{Fix}\tau_{\beta,E}^{(N+1)i}=\sharp\mathrm{Fix}\sigma_M^{(N+1)i}+N+1$$

and

$$\sharp\mathrm{Fix}\tau_{\beta,E}^{(N+2)i} + N + 2 = \sharp\mathrm{Fix}\sigma_M^{(N+2)i}$$

for $i \geqq 1$. Hence we obtain

$$p_{\beta,E}(z) = \frac{1 - z^{N+2}}{1 - z^{N+1}}.$$

Therefore, for $z \in \mathbb{C}$ with $|z| < 1/\beta$ we obtain

$$\zeta_{\tau_{\beta,E}}(z) = \frac{p_{\beta,E}(z)}{1 - \phi_{\beta,E}(z)}$$

If β is not simple, by Lemma 3.2, we can take a sequence $\{\beta_n\}_{n=1}^{\infty}$ such that each β_n is simple and $\beta_n \to \beta$ as $n \to \infty$.

By Lemma 3.3 and Lemma 3.4, for $z \in \mathbb{C}$ with $|z| < 1/\beta$ we obtain

$$\phi_{\beta_n,E}(z) \to \phi_{\beta,E}(z)$$

and

$$\zeta_{\tau_{\beta_n,E}}(z) \to \zeta_{\tau_{\beta,E}}(z)$$

as $n \to +\infty$. Furthermore, for $z \in \mathbb{C}$ with $|z| < 1/\beta$ we have

$$p_{\beta_n,E}(z) \to 1$$

as $n \to +\infty$. Hence we get

$$\zeta_{\tau_{\beta,E}}(z) = \frac{1}{1 - \phi_{\beta,E}(z)}$$

for $z \in \mathbb{C}$ with $|z| < 1/\beta$.

The fact that $\zeta_{\beta,E}(z)$ has a simple pole at $z = 1/\beta$ is proved in the next section. $\square$

4. Analytic properties of $\zeta_{\tau_{\beta,E}}(z)$

Now we can study the analytic properties of $\zeta_{\tau_{\beta,E}}(z)$. We put

$$\psi_{\beta,E}(z) = \sum_{n=0}^{N} s_n(\beta, E, 1)\tau^n_{\beta,E}(1)z^n$$

if β is simple, where N is the minimal number with $\tau^N_{\beta,E}(1) \in \{1/\beta, \ldots, [\beta]/\beta\}$ and otherwise we take $N = \infty$.

Proposition 4.1. (1) *$\zeta_{\tau_{\beta,E}}(z)$ can be expressed as*

$$\zeta_{\tau_{\beta,E}}(z) = \frac{p_{\beta,E}(z)}{(1-\beta z)\psi_{\beta,E}(z)}$$

for $z \in \mathbb{C}$ with $|z| < 1/\beta$.

(2) *$\zeta_{\tau_{\beta,E}}(z)$ has a simple pole at $1/\beta$ and its residue can be expressed as*

$$-\frac{p_{\beta,E}(1/\beta)}{\beta\psi_{\beta,E}(1/\beta)}.$$

(3) *For $z \in \mathbb{C}$ with $0 \leqq |z| < 1/2$, we have $\psi_{\beta,E}(z) \neq 0$.*

Proof. (1). We shall prove the equation:

$$1 - \phi_{\beta,E}(z) = (1-\beta z)\psi_{\beta,E}(z).$$

By the definition of $e_n(\beta, E, 1)$ and $d_n(\beta, E, 1)$, we have

$$\tau^{n+1}_{\beta,E}(1) = e_n(\beta, E, 1)(\beta\tau^n_{\beta,E}(1) - d_n(\beta, E, 1))$$

for $n \in \mathbb{Z}_{\geqq 0}$. Therefore we get

$$d_n(\beta, E, 1) = \beta \cdot \tau^n_{\beta,E}(1) - e_n(\beta, E, 1)\tau^{n+1}_{\beta,E}(1).$$

If β is simple, it holds that

$$\begin{aligned}\phi_{\beta,E}(z) &= \sum_{n=0}^{N-1} s_n(\beta,E,1)d_n(\beta,E,1)z^{n+1} + k_0 s_N(\beta,E,1)z^{N+1} \\ &= \sum_{n=0}^{N-1} s_n(\beta,E,1)\big(\beta\tau_{\beta,E}^n(1) - e_n(\beta,E,1)\tau_{\beta,E}^{n+1}(1)\big) \\ &= (\beta z - 1)\sum_{n=0}^{N} s_n(\beta,E,1)\tau_{\beta,E}^n(1)z^{n+1} + 1.\end{aligned}$$

Hence we have

$$1 - \phi_{\beta,E}(z) = (1-\beta z)\psi_{\beta,E}(z).$$

If β is not simple, it holds that

$$\begin{aligned}\phi_{\beta,E}(z) &= \sum_{n=0}^{\infty} s_n(\beta,E,1)d_n(\beta,E,1)z^{n+1} \\ &= (\beta z - 1)\sum_{n=0}^{\infty} s_n(\beta,E,1)\tau_{\beta,E}^n(1)z^{n+1} + 1.\end{aligned}$$

Hence we have

$$1 - \phi_{\beta,E}(z) = (1-\beta z)\psi_{\beta,E}(z).$$

(2). From (1), we get

$$\Big(z - \frac{1}{\beta}\Big)\frac{p_{\beta,E}(z)}{(1-\beta z)\psi_{\beta,E}(z)} = -\frac{1}{\beta}\frac{p_{\beta,E}(z)}{\psi_{\beta,E}(z)}.$$

Note that if β is not simple, we know that $p_{\beta,E}(z) = 1$ from the proof of Theorem 1.1 and $\psi_{\beta,E}(1/\beta) = \beta F(\beta,E) > 0$, where $F(\beta,E)$ denotes the normalizing factor for the invariant density $h_{\beta,E}(x)$.

We consider the case where β is simple. Since $p_{\beta,E}(1/\beta) \neq 0$ from the proof of Theorem 1.1, it is enough to show that $\psi_{\beta,E}(1/\beta) \neq 0$. Let N be the minimal positive integer with $\tau_{\beta,E}^{N}(1) \in \{1/\beta, \ldots, [\beta]/\beta\}$. Put

$$h^*_{\beta,E}(x) = \sum_{n=0}^{N} \frac{s_n(1)\chi_{[0,\tau_{\beta,E}(1))}(x)}{\beta^{n+1}}$$

for $x \in [0,1]$. Then from Lemma 3.1, we obtain

$$\begin{aligned}
\frac{1}{\beta}\mathcal{L}_{\tau_{\beta,E}} h^*_{\beta,E}(x) &= \sum_{n=0}^{N} \frac{s_n(1)\mathcal{L}_{\tau_{\beta,E}}\chi_{[0,\tau_{\beta,E}(1))}(x)}{\beta^{n+1}} \\
&= \sum_{n=1}^{N} \frac{s_n(1)\chi_{[0,\tau_{\beta,E}(1))}(x)}{\beta^{n+1}} + \frac{1}{\beta} \\
&= h^*_{\beta,E}(x).
\end{aligned}$$

This shows that $h^*_{\beta,E}(x)$ is a fixed point of the Perron-Frobenius operator for $\tau_{\beta,E}$, that is, this function is the $\tau_{\beta,E}$-invariant density. Similar to Lemma 10 in [7], we get $h^*_{\beta,E}$ is non-negative. Since $\psi_{\beta,E}(1/\beta) = \beta \cdot \int_0^1 h^*_{\beta,E}dm > 0$, we get the conclusion.

(3). If β is simple then we have

$$\begin{aligned}
|1 - \psi_{\beta,E}(z)| &= \left|\sum_{n=1}^{N} s_n(1)\tau_{\beta,E}^{n}(1)z^{n+1}\right| \\
&\leqq \sum_{n=1}^{N} |z|^n \\
&< \frac{|z|}{1-|z|} \\
&< 1
\end{aligned}$$

if $|z| \leqq 1/2$.

If β is not simple then we have

$$\begin{aligned}|1-\psi_{\beta,E}(z)| &= \Big|\sum_{n=1}^{\infty} s_n(1)\tau_{\beta,E}^n(1)z^{n+1}\Big| \\ &< \sum_{n=1}^{\infty} |z|^n \\ &= \frac{|z|}{1-|z|} \\ &< 1\end{aligned}$$

if $|z| \leqq 1/2$. Therefore we get the conclusion. □

Finally, we relate the analytic properties of $\zeta_{\tau_{\beta,E}}(z)$ to the algebraic properties of β. Let $n \geqq 1$ be a positive integer and $E = (E(0), \ldots, E(n))$ an $(n+1)$-dimensional vector with $E(i) \in \{0, 1\}$ for $0 \leqq i \leqq n$. For $\beta \in (n, n+1)$, we denote by $M(\beta, E)$ the minimum modulus of any poles of $\zeta_{\tau_{\beta,E}}(z)$ in the unit open disk $\{z \in \mathbb{C}; |z| < 1\}$ except $z = 1/\beta$. If no other pole exists in the unit open disk, we put $M(\beta, E) = 1$.

Recall that a Perron number is a real algebraic integer $\beta > 1$ whose Galois conjugates have an absolute value less than β. A Pisot number is a real algebraic integer greater than 1 whose Galois conjugates have an absolute value less than 1 and a Salem number is a real algebraic integer greater than 1 whose Galois conjugates have an absolute value not greater than 1, and one of which has an absolute value 1.

Proposition 4.2. (1) *Let $n \geqq 2$ be a positive integer and $\beta \in (n, n+1)$. If 1 is an eventually periodic points of $\tau_{\beta,E}$, then β is a Perron number.*

(2) *If* $M(\beta,E)=1$ *and* 1 *is an eventually periodic points for* $\tau_{\beta,E}$*, then* β *is a Pisot or Salem number.*

Proof. (1). Assume that β is not simple. Then there exists positive integers N and k such that

$$\begin{aligned}1-\phi_{\beta,E}(z)=&1-\sum_{n=0}^{N+k-1}s_n(\beta,E,1)d_n(\beta,E,1)z^{n+1}\\&-\sum_{n=N+k}^{2N+k-1}s_n(\beta,E,1)d_n(\beta,E,1)z^{n+1}\cdot\frac{1}{1-z^N}.\end{aligned}$$

Hence we have $z^{N+k-1}(z^N-1)(1-\phi_{\beta,E}(1/z))$ is a monic polynomial whose coefficients are integers having a zero at β. We remark that this polynomial has no zero in the set $\{z\in\mathbb{C};|z|>\beta\}$ by the equation (1.2). If β is simple, by the definition of $\phi_{\beta,E}(z)$, we know that $z^{N+1}(1-\phi_{\beta,E}(1/z))$ is a monic polynomial whose coefficients are integers with a zero at β. In addition, this polynomial has no zero in $\{z\in\mathbb{C};|z|>\beta\}$ by the equation (1.2). From Proposition 4.1 (1) and Proposition 4.1 (3), $1-\phi_{\beta,E}(1/z)$ has no zero in the circle $\{z\in\mathbb{C};|z|=\beta\}$ except $z=\beta$ since $\beta>2$ by the assumption $n\geqq 2$. This shows that β is a Perron number.

(2). By the proof of (1), there exist non-negative integers m, n such that $z^m(z^n-1)(1-\phi_{\beta,E}(1/z))$ is a monic polynomial whose coefficients are integers having a zero at β. Since $1-\phi_{\beta,E}(z)$ has no zero in $\{z\in\mathbb{C};1/\beta\leqq|z|<1\}$ except $z=1/\beta$, we get $1-\phi_{\beta,E}(1/z)$ has no zero in $\{z\in\mathbb{C};1<|z|\leqq\beta\}$ except $z=\beta$. This shows that β is a Pisot or Salem number. $\square$

5. Negative β-transformations

In this section, we study the analytic properties of the Artin-Mazur zeta function of a negative β-transformation.

Let $\beta > 1$ be a non-integer and $E_1 = (E_1(0), \ldots, E_1([\beta]))$ a $([\beta]+1)$-dimensional vector with $E_1(i) = 1$ for $0 \leqq i \leqq [\beta]$. We note that τ_{β,E_1} is a negative β-transformation.

Proposition 5.1. *The Artin-Mazur zeta function $\zeta_{\tau_{\beta,E_1}}(z)$ has no pole in the disk $\{z \in \mathbb{C}; |z| \leqq 1/\beta\}$ except $z = 1/\beta$.*

Proof. Let μ_{β,E_1} be the τ_{β,E_1}-invariant measure defined in Section 1. Note that the associated operator $U_{\tau_{\beta,E_1}} : L^1(\mu_{\beta,E_1}) \to L^1(\mu_{\beta,E_1})$ defined by

$$U_{\tau_{\beta,E_1}} f = f \circ \tau_{\beta,E} \quad a.e. \quad \mu_{\beta,E_1}$$

has a simple eigenvalue at 1 and no eigenvalue in the unit circle if and only if $(\tau_{\beta,E_1}, \mu_{\beta,E_1})$ is exact.

In addition, since μ_{β,E_1} is the unique ergodic absolutely continuous τ_{β,E_1}-invariant probability measure (see [7]), by the theorem of Baladi and Keller (Theorem 2 in [1]) and the theorem of Keller (Theorem 1 and 2 in [8]), we know that $\zeta_{\tau_{\beta,E_1}}(z)$ has no pole in the circle $\{z \in \mathbb{C}; |z| = 1/\beta\}$ except $z = 1/\beta$ if and only if $(\tau_{\beta,E_1}, \mu_{\beta,E_1})$ is exact. Therefore the statement follows from the fact that $(\tau_{\beta,E_1}, \mu_{\beta,E_1})$ is exact (see Corollary 2.3 in [12]) and Theorem 1.1. $\square$

From Proposition 5.1 and Theorem 1.1, we get the conclusion that every Yrrap number is a Perron number (It was first proved in [12]). Note that we call a real number $\beta > 1$ a Yrrap number if 1 is an eventually periodic point of τ_{β,E_1}.

Proposition 5.2. *Every Yrrap number is a Perron number.*

Proof. In the proof of Proposition 4.2 (2), we assume that $n \geqq 2$ only to reach the result that $1 - \phi_{\beta,E}(1/z)$ has no zero in the circle $\{z \in \mathbb{C}; |z| = \beta\}$ except $z = \beta$. Therefore by Proposition 5.1, we get the statement similar to the proof of Theorem 4.2 (2). $\square$

Now, we consider an asymptotic behavior of the Artin-Mazur zeta function $\zeta_{\beta,E_1}(z)$ as $\beta \to 1$. In [12], Liao and Steiner showed that

$$\lim_{\beta \to 1} d(\beta, E_1, 1) = \{m_{n+1} + 1\}_{n=0}^{\infty},$$

where $\{m_n\}_{n=0}^{\infty}$ denotes the Thue-Morse sequence $0100111\cdots$. This limit means that for all $N \geqq 1$, there exists a positive number $\delta > 0$ such that

$$\{d_n(\beta, E_1, 1)\}_{n=0}^{N} = \{m_{n+1} + 1\}_{n=0}^{N}$$

whenever $1 < \beta < 1 + \delta$. By the above fact, for $z \in \mathbb{C}$ with $|z| < 1$ we have

$$\lim_{\beta \to 1} (1 - \phi_{\beta,E_1}(z)) = \sum_{n=0}^{\infty} (m_n + 1)(-z)^n$$

since the coefficients of the power series $1 - \phi_{\beta,E_1}(z)$ are bounded.

In addition, in [5] Dubickas gave the analytic continuation of the generating function for the sequence $\{m_n + 1\}_{n=0}^{\infty}$ explicitly, that is,

$$\sum_{n=0}^{\infty} (m_n + 1) z^n = \prod_{i=1}^{\infty} (1 + z^{(2^i + (-1)^{i-1})/3})$$

for $z \in \mathbb{C}$ with $|z| < 1$.

Therefore, as an application of Theorem 1.1, we have the following result:

Theorem 5.3. *For $z \in \mathbb{C}$ with $|z| < 1$, we have*

$$\lim_{\beta \to 1} \zeta_{\tau_{\beta,E_1}}(z) = \prod_{i=1}^{\infty} (1 - z^{(2^i + (-1)^{i-1})/3})^{-1}.$$

6. Chebyshev maps

In this section, we consider the Chebyshev maps $T_\beta : [-1, 1] \to [-1, 1]$ defined by

$$T_\beta(x) = \cos(\beta \arccos x)$$

for $x \in [-1, 1]$, where $\beta > 1$. If $\beta > 1$ is an integer, then the map is the well-known Chebyshev polynomial of nth order. For every $\beta > 1$, Góra [7] showed that the T_β-invariant density is expressed as a function associated with the orbit $\{T_\beta^n(-1)\}_{n=0}^{\infty}$. As an application of Theorem 1.1, we can show that the Artin-Mazur-Ruelle zeta functions of a Chebyshev map, which is defined by

$$Z_\beta(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x = T_\beta^n x} \frac{1}{|(T_\beta^n)'(x)|} \Big)$$

is analytic in the unit open disk $\{z \in \mathbb{C}; |z| < 1\}$ and can be extended to a meromorphic function in the disk $\{z \in \mathbb{C}; |z| < \beta\}$ associated with the orbit $\{T_\beta^n(-1)\}_{n=0}^{\infty}$. The results are based on the next proposition.

Proposition 6.1 (Proposition 17 in [7]). *Let* $\beta > 1$ *and put* $\Phi(x) = \cos(\pi x) : [0,1] \to [-1,1]$. *Then*

$$\tau_{\beta,E} = \Phi^{-1} \circ T_\beta \circ \Phi,$$

where $E = (E(0), \dots, E([\beta]))$ *is the vector with* $E(k) = 0$ *for even* k *and* $E(k) = 1$ *for odd* k.

The above proposition yields the next lemma.

Lemma 6.2. *Let* $\beta > 1$ *be a non-integer and* T_β *a Chebyshev map. Let* $\tau_{\beta,E}$ *be the map defined in Proposition 6.1. Then for* $x \in \operatorname{Fix} \tau_{\beta,E}^n \setminus \{1\}$, *we have*

$$(T_\beta^n)'(\Phi(x)) = \beta^n.$$

Proof. By Proposition 6.1, we have

$$\tau_{\beta,E}^n(x) = \Phi^{-1} \circ T_\beta^n \circ \Phi(x)$$

for $x \in [0,1]$. Since $T_\beta^n(1) = 1$ for every positive integer $n \geqq 1$ and $\Phi(0) = 1$, for $x \in \operatorname{Fix} \tau_{\beta,E}^n \setminus \{1\}$, we get

$$\begin{aligned}
(T_\beta^n)'(\Phi(x)) &= (\Phi \circ \tau_{\beta,E}^n \circ \Phi^{-1})'(\Phi(x)) \\
&= \Phi'(\tau_{\beta,E}^n(x)) \cdot (\tau_{\beta,E}^n)'(\Phi^{-1}(\Phi(x)) \cdot (\Phi^{-1})'(\Phi(x)) \\
&= \Phi'(x) \cdot (\tau_{\beta,E}^n)'(x) \cdot (\Phi^{-1})'(\Phi(x)) \\
&= (\tau_{\beta,E}^n)'(x) \\
&= \beta^n.
\end{aligned}$$

□

By simple calculation, we get the following lemma.

Lemma 6.3. *Let $\beta > 1$ and $T_\beta : [-1,1] \to [-1,1]$ be a Chebyshev map. Then the map T_β is differentiable in $(-1,1)$ and*

$$\lim_{x\to 1}(T_\beta^n)'(x) = \beta^{2n}.$$

We define a sequence $\{s_n^*(\beta,-1)\}_{n=0}^\infty$ as follows. Let $E^* = (E^*(0),\ldots,E^*([\beta]))$ be a $([\beta]+1)$-dimensional vector with $E^*(k) = 0$ if $[\beta]-k$ is even and $E^*(k) = 1$ if $[\beta]-k$ is odd, namely, $E^*(k) = 0$ if the kth branch of the map T_β is increasing and $E^*(k) = 1$ if the k-th branch of the map T_β is decreasing. We define

$$s^*(\beta,-1) = \begin{cases} 1 & (n=0) \\ s_{n-1}^*(\beta,-1)\cdot(-1)^{E^*(j(T_\beta^n(-1)))} & (n \geqq 1), \end{cases}$$

where $j(T_\beta^n(-1)) \in \{0,1,\ldots,[\beta]\}$ denotes the number of the branch to which $T_\beta^n(-1)$ belongs.

From Proposition 6.1, we know that

$$s_n(\beta,E,1) = s_n^*(\beta,-1)$$

for $n \in \mathbb{Z}_{\geqq 1}$, where E is the vector defined in Proposition 6.1.

Theorem 6.4. *Let $\beta > 1$ satisfy $T_\beta^n(-1) \notin \{-1,1\}$ for every positive integer $n \geqq 1$. Then $Z_\beta(z)$ is analytic in the unit open disk and for $z \in \mathbb{C}$ with $|z| < 1$ we have*

$$Z_\beta(z) = \frac{1 - z/\beta}{(1-z)(1-z/\beta^2)\left(\sum_{n=0}^\infty s_n^*(\beta,-1)\cdot(1/\pi)\arccos(T_\beta^n(-1))\cdot z^n/\beta^n\right)}.$$

Furthermore, the convergence radius of the formal power series $\sum_{n=0}^{\infty} s_n^*(\beta,-1)\cdot \arccos(T_\beta^n(-1))\cdot(z^n/\beta^n)$ *is* β*, so* $Z_\beta(z)$ *can be extended to the meromorphic function in* $\{z\in\mathbb{C}; |z|<\beta\}$.

Proof. By Lemma 6.2 and Lemma 6.3, we get

$$\begin{aligned}
Z_\beta(z) &= \exp\Big(\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{x=T_\beta^n x}\frac{1}{|(T_\beta^n)'(x)|}\Big)\\
&= \exp\Big(\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{y=\tau_{\beta,E}^n(y)}\frac{1}{|(\tau_{\beta,E}^n)'(y)|}-\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{1}{\beta^n}+\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{1}{\beta^{2n}}\Big)\\
&= \exp\Big(\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{\sharp\,\mathrm{Fix}\,\tau_{\beta,E}^n}{\beta^n}-\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{1}{\beta^n}+\sum_{n=1}^{\infty}\frac{z^n}{n}\frac{1}{\beta^{2n}}\Big).
\end{aligned}$$

Therefore, for $z\in\mathbb{C}$ with $|z|<1$ we have

$$Z_\beta(z)=\zeta_{\tau_{\beta,E}}\Big(\frac{z}{\beta}\Big)\cdot\frac{1-\frac{z}{\beta}}{1-\frac{z}{\beta^2}}. \tag{6.1}$$

From Proposition 4.1 (1), we have

$$\zeta_{\tau_{\beta,E}}\Big(\frac{z}{\beta}\Big)=\frac{1}{(1-z)\sum_{n=0}^{\infty}s_n(\beta,E,1)\tau_{\beta,E}^n(1)\cdot z^n/\beta^n}.$$

Hence the result follows from the relation

$$s_n(\beta,E,1)=s_n^*(\beta,-1)$$

and

$$\begin{aligned}
\tau_{\beta,E}^n(1) &= \Phi^{-1}\circ T_\beta^n\circ\Phi(1)\\
&= \frac{1}{\pi}\arccos(T_\beta^n(-1)).
\end{aligned}$$

□

Note that the analytic properties of $Z_\beta(z)$ are related to those of $\zeta_{\tau_{\beta,E}}(z)$ from the equation (6.1).

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