Doctor thesis

# Aspects of the gauge/gravity correspondence: holographic superconductor and geodesic Witten diagram

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**Abstract**

The gauge/gravity correspondence is a conjecture of the duality between field theory and gravity theory. One approach of the study about the gauge/gravity correspondence is calculation based on assumption of the correspondence. By using a dual description method, we can obtain a new physical perspective. Another approach is to compare corresponding objects in the two theories which are dual to each other. Such a test is important for confirming and extending the application range of the gauge/gravity correspondence.

In part 1, we study a three-scalar holographic superconductor model which can describe frustration. We analyze solutions of this model and compute the free energy of the solutions. We find that there are chiral ground states in this model. This holographic model will be useful for study of multicomponent superconductivity in strongly correlated systems from the viewpoint of the gauge/gravity correspondence.

In part 2, we study the correspondence between geodesic Witten diagrams and conformal partial waves with an external symmetric traceless tensor field. We construct an amplitude of the geodesic Witten diagrams and show that it is consistent with the properties and the formulas of the conformal partial waves. Construction of the geodesic Witten diagrams gives us a novel expression of the conformal partial waves.

We construct and analyze the holographic model by the bottom-up process in part 1 and we verify the correspondence between the objects in conformal field theory and AdS spacetime in part 2. These results lead to understand the holographic description of various objects.

# Contents

# 1 Introduction

The gauge/gravity correspondence [1] is a duality between field theory and AdS gravity theory. This correspondence is one of well-studied research topics in theoretical physics. The most famous example of the gauge/gravity correspondence is the duality between $\mathcal{N} = 4$ super Yang-Mills theory and type IIB super string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. One evidence of this duality is that these theories have the same symmetry. In particular, superconformal symmetry and R-symmetry of the $\mathcal{N} = 4$ super Yang-Mills theory are the same as isometries of $\mathrm{AdS}_5$ and $\mathrm{S}^5$. Especially, the gauge/gravity correspondence in the large $N$ and $\lambda$ limit implies that the strong coupling gauge theory can be calculated by the classical gravity theory. $N$ is the number of colors and $\lambda$ is the 't Hooft coupling. This surprising statement is a reason that many researchers are interested in the gauge/gravity correspondence.

Nowadays it is expected that the gauge/gravity correspondence occurs in various field theories and gravity theories. Based on this conjecture, many studies have been conducted. One research method in such studies is to compute physical quantities with the assumption of the gauge/gravity correspondence. When we cannot calculate one side of the field theories and the gravity theories, we can pull out the information from the other side by assuming the gauge/gravity correspondence. A test of the gauge/gravity correspondence is another research method. Construction of a concrete example of which we can calculate the both sides is important for justification of the gauge/gravity correspondence.

One example of studies towards application of the gauge/gravity correspondence is holographic QCD [2, 3, 4]. The holographic QCD is a holographic model which is constructed by D-branes. Effective theory of such D-brane systems has the QCD-like behavior. By using the holographic QCD models, we can interpret the properties of QCD such as the confinement and the chiral phase transition as geometrical properties of the D-branes. Another example is holographic entanglement entropy [5]. Entanglement entropy is a quantity which measures entanglement between quantum states and its geometrical interpretation was proposed in [5]. Moreover, by applying this proposal, the study of quantum gravity from the entanglement has been challenged [6, 7, 8].

One example of tests of the gauge/gravity correspondence is comparing degrees of freedom in $\mathrm{AdS}_4/\mathrm{CFT}_3$ by using a localization technic [9, 10]. The localization is a non-perturbative computation method by super symmetry [11]. The free energy of ABJM theory [12] on the three-dimensional sphere was computed by this localization technic and it is consistent with the prediction of the gravity side. Another example is a numerical computation of finite N gauge theories [13]. From the viewpoint of the gauge/gravity correspondence, finite N gauge theories are candidates of definitions for quantum gravity theories. The authors of [13] analyzed one-dimensional U($N$) super Yang-Mills theory numerically and their computation reproduced the quantum gravity effect by the $1/N^2$ correction.

Motivations to study the gauge/gravity correspondence for the author are as follows. First motivation is interest in classical equations which appear in researches of the gauge/gravity correspondence. If the gauge/gravity correspondence is true, there are the holographic models which correspond to various physical phenomena. It is expected that such holographic models have interesting solutions. Conversely, we may predict properties of condensed matter systems from the holographic models. Second motivation is to understand how to construct conformal field theory (CFT) objects which can be determined by conformal symmetry only in terms of AdS geometry. Generally, it is difficult to show the gauge/gravity correspondence exactly because the spectra of the theory are complicated. However, we can verify the correspondence about CFT objects which do not depend on the detail of the theory such as conformal partial wave by using conformal symmetry and isometry of AdS spacetime. Developing a systematic way to construct such CFT objects by using AdS geometry is an important research theme.

In part 1, we analyze a three-scalar holographic superconductor model [14]. Holographic superconductor model [15, 16] is a model that describes phase transition in the AdS black hole. By lowering the Hawking temperature, the phase transition occurs in the holographic superconductor model and its phase transition is similar to superconductor phase transition. The holographic superconductor models are expected to be related to strongly correlated systems . We examine the equations of motion for the three-scalar holographic superconductor model and evaluate their solutions. Because of the interactions between the scalar fields, this holographic superconductor model can express the frustration.

In part 2, we explore the correspondence between geodesic Witten diagrams (GWD) and conformal partial waves (CPW) with an external symmetric traceless tensor field [17]. CPW is a fundamental object in CFT and it can be determined by the conformal symmetry only. Recently, GWD has been proposed as the gravity dual of CPW [18]. We construct an amplitude of GWD with an external arbitrary symmetric traceless tensor field and verify that our construction agrees with the known result of CPW. Our approach is useful to find an unknown expression of CPW with spinning fields. This work is collaboration with Kotaro Tamaoka.

This thesis is organized as follows. In section 2, we review the GKP-W relation and its application for the holographic superconductor. We analyze the s-wave holographic super conductor model in section 3 (review) and analyze the three-scalar holographic superconductor model in section 4. In section 5, we review the conformal partial wave (CPW) and the geodesic Witten diagram (GWD). We verify the correspondence between the scalar exchange CPW and GWD with an external spinning field and three external scalar fields in section 6. Section 7 is conclusion and discussion of this doctor thesis.

# Part I
# Three-scalar holographic superconductor model for frustration

## 2 Review of GKP-W relation

In this section, we review GKP-W relation [19, 20] and how to compute physical quantities in the holographic superconductor model by using the GKP-W relation. This review is based on [21, 22] and my master thesis [23].

### 2.1 GKP-W relation

GKP-W relation [19, 20] is the most fundamental and important concept in gauge /gravity correspondence. This relation is written as

$$\left\langle \exp\left(\int d^d x \phi^{(0)} \mathcal{O}\right)\right\rangle = \exp\left(-S_{\text{onshell}}[\phi]\right), \tag{1}$$

$$\phi_{r\to\infty} = \frac{\phi^{(0)}}{r^{d-\Delta}} + \cdots . \tag{2}$$

Our notation is as below:

- gauge theory side (left hand side)
  - $\mathcal{O}$: an operator in the gauge theory
  - $\phi^{(0)}$: a source term for $\mathcal{O}$
  - $\langle\cdots\rangle = \int \mathcal{D}\mathcal{O}\cdots\exp(-S_{\text{gauge}})$
  - $S_{\text{gauge}}$: the action of the gauge theory
- gravity theory side (right hand side)
  - $S_{\text{onshell}}$: the classical action of the gravity theory
  - asymptotic AdS metric $ds^2|_{r\to\infty} = r^2(dt^2 + d\mathbf{x}^2) + \frac{1}{r^2}dr^2$
  - $\phi$: a field which obeys the EOM
  - $\Delta$ is a constant which is determined from the EOM. This constant is related to conformal dimension of $\mathcal{O}$ as we will see later.

(1) means equality between a generating functional of the gauge theory and a partition function of the gravity theory. (2) means equality between sources for operators of the gauge theory and coefficients of the classical solution at the AdS boundary in the gravity theory. Usually, the GKP-W relation is formulated by Euclidean formalism, therefore we use a Euclidean action.

Study of the gauge/gravity correspondence in this thesis is roughly divided into two types. First type is application of the GKP-W relation. In this type of study such as holographic superconductor, only the one side of gauge and gravity theories is computed and the other side is estimated by assumption of the GKP-W relation. In part 1, we will analyze a holographic superconductor model by assuming the GKP-W relation. Second type is rigorous checking of the gauge/gravity correspondence between objects in the both sides. In this type of study, we compute the both sides of gauge and gravity theories and show equality of the objects. In part 2, we will show the equality between conformal partial waves in the CFT side and geodesic Witten diagrams in the AdS side.

## 2.2 Computation of one-point function by the GKP-W relation

Consider a perturbative source $\delta\phi^{(0)}$ as $\phi^{(0)}$ in (1). In this case, deviation of the expectation value $\delta\langle\mathcal{O}\rangle$ by $\delta\phi^{(0)}$,

$$\delta\langle\mathcal{O}\rangle \equiv \big\langle \mathcal{O} \exp\big( \int d^dx\ \delta\phi^{(0)}\mathcal{O}\big)\big\rangle - \langle\mathcal{O}\rangle \tag{3}$$

can be computed from the gravity side by assuming the GKP-W relation,

$$\left\langle \exp\left( \int d^dx \delta\phi^{(0)}\mathcal{O}\right)\right\rangle = \exp\left(-S_{\text{onshell}}[\phi]\right). \tag{4}$$

In order to see it, let us compute some examples.

### 2.2.1 Massless scalar field

Starting from a four-dimensional massless scalar field action,

$$S = \frac{1}{2}\int d^4x\sqrt{g}(\nabla_\mu\phi)^2, \tag{5}$$

we derive $\delta\langle\mathcal{O}\rangle$ by using (3) and (4). For simplicity, we assume

$$\phi = \phi(r). \tag{6}$$

We consider a metric like AdS black hole,

$$\sqrt{g} = r^2, \qquad g^{rr} = \begin{cases} r^2 & r \to \infty \\ 0 & r \to r_0 \end{cases}, \tag{7}$$

where $r_0$ is a horizon radius. Since we use Euclidean metric, the integration range of $r$ in (5) is from $r_0$ to $\infty$. From (5), (6) and (7), we get

$$\begin{aligned} S &= \int d^3x \int_{r_0}^{\infty} dr \frac{r^2 g^{rr}}{2} \phi'^2 \\ &= \int d^3x \frac{r^4}{2} \phi\phi' \Big|_{r=\infty} - \int d^3x \int_{r_0}^{\infty} dr \Big( \frac{r^2 g^{rr}}{2} \phi' \Big)' \phi, \end{aligned} \tag{8}$$

where $'$ represents a differential with respect to $r$ and we use partial integration.

The classical equation of motion for (5) is

$$\Big( r^2 g^{rr} \phi' \Big)' = 0. \tag{9}$$

Since the second term in (8) is zero under (9), we obtain a classical action $S_{\text{onshell}}$,

$$S_{\text{onshell}} = \int d^3x \frac{r^4}{2} \phi\phi' \Big|_{r=\infty}. \tag{10}$$

(5) is a four-dimensional integral, however, (10) is a three-dimensional integral at $r \to \infty$ (AdS boundary) by the equation of motion. The GKP-W relation (4) relates this classical action to a generating functional of the three-dimensional field theory.

Next, we consider an explicit solution of (9) at $r \to \infty$. At $r \to \infty$, (9) becomes

$$\Big( r^4 \phi' \Big)' = 0, \tag{11}$$

therefore, an asymptotic form of the solution at $r \to \infty$ is

$$\phi = \delta\phi^{(0)} (1 + \frac{\phi^{(1)}}{r^3}) \qquad\qquad (r \to \infty). \tag{12}$$

Substituting (12) to (10), we get

$$\begin{aligned} S_{\text{onshell}} &= \int d^3x \big(\delta\phi^{(0)}\big)^2 \frac{r^4}{2} \Big( 1 + \frac{\phi^{(1)}}{r^3} \Big) \Big( -3 \frac{\phi^{(1)}}{r^4} \Big) \Big|_{r=\infty} \\ &= - \int d^3x \frac{3}{2} \big(\delta\phi^{(0)}\big)^2 \phi^{(1)}. \end{aligned} \tag{13}$$

From (3), (4) and (13), we obtain

$$\begin{aligned}
\delta\langle\mathcal{O}\rangle &= \Big\langle \mathcal{O}\exp\big(\int d^3x\ \delta\phi^{(0)}\mathcal{O}\big)\Big\rangle - \langle\mathcal{O}\rangle \\
&= \frac{\delta}{\delta(\delta\phi^{(0)})}\Big\langle \exp\big(\int d^3x\ \delta\phi^{(0)}\mathcal{O}\big)\Big\rangle - \frac{\delta}{\delta(\delta\phi^{(0)})}\Big\langle \exp\big(\int d^3x\ \delta\phi^{(0)}\mathcal{O}\big)\Big\rangle\Big|_{\delta\phi^{(0)}=0} \\
&= 3\delta\phi^{(0)}\phi^{(1)}\exp\big(\int d^3x\frac{3}{2}\big(\delta\phi^{(0)}\big)^2\phi^{(1)}\big) - 3\delta\phi^{(0)}\phi^{(1)}\exp\big(\int d^3x\frac{3}{2}\big(\delta\phi^{(0)}\big)^2\phi^{(1)}\big)\Big|_{\delta\phi^{(0)}=0} \\
&\approx 3\delta\phi^{(0)}\phi^{(1)}, \qquad (14)
\end{aligned}$$

where we ignore higher order terms of $\delta\phi^{(0)}$ and (12) becomes

$$\phi = \delta\phi^{(0)} + \frac{\delta\langle\mathcal{O}\rangle}{3r^3} \qquad (r\to\infty). \qquad (15)$$

Therefore, with assumption of the GKP-W relation, coefficients of the classical solution at the AdS boundary in gravity theory side correspond to source and its response in field theory side.

In order to derive (12), we assume that the second term also vanishes if the first term vanishes. We can also consider a solution as

$$\phi = \delta\phi^{(0)} + \frac{\phi^{(1)}}{r^3} \qquad (r\to\infty). \qquad (16)$$

With $\delta\phi^{(0)} = 0$, we obtain

$$\begin{aligned}
S_{\text{onshell}} &= -\int d^3x\frac{3}{2}\delta\phi^{(0)}\phi^{(1)}, \qquad (17)\\
\langle\mathcal{O}\rangle &= \frac{3}{2}\phi^{(1)}\exp\big(\int d^3x\frac{3}{2}\delta\phi^{(0)}\phi^{(1)}\big)\Big|_{\delta\phi^{(0)}=0} \\
&= \frac{3}{2}\phi^{(1)} \qquad (18)
\end{aligned}$$

and thus, (16) becomes

$$\phi = \frac{2\langle\mathcal{O}\rangle}{3r^3} \qquad (r\to\infty). \qquad (19)$$

### 2.2.2 Massive scalar field

Consider an action of a massive scalar field in four-dimensional spacetime,

$$S = \frac{1}{2}\int d^4x\sqrt{g}[(\nabla_\mu\phi)^2 + m^2\phi^2] \qquad (20)$$

and we assume (6) and (7) again. Classical action $S_{\text{onshell}}$ of (20) is the same as (10), but the equation of motion is

$$m^2 r^2 \phi - \Big(r^2 g^{rr} \phi'\Big)' = 0. \tag{21}$$

Asymptotic form of the solution at $r \to \infty$ is

$$\phi = \delta\phi^{(0)}\Big(\frac{1}{r^{\Delta_-}} + \frac{\phi^{(1)}}{r^{\Delta_+}}\Big) \qquad (r \to \infty), \tag{22}$$

$$\Delta_\pm = \frac{3 \pm \sqrt{9 + 4m^2}}{2}. \tag{23}$$

In scalar field theories on flat spacetime, a field with $m^2 < 0$ corresponds to tachyon and its existence is related to instability of the theory. However, on AdS spacetime, there is possibility that the theory is stable with $m^2 < 0$ since the equation of motion includes the AdS metric. The bound of $m^2$ such that the theory is stable is called as Breitenlohner-Freedman bound [24]. In particular, the theory is stable if $m^2 > -\frac{d^2}{4}$ on $d+1$-dimensional AdS spacetime. This condition corresponds to the fact that $\Delta_\pm$ has no imaginary part. On four-dimensional AdS spacetime, the theory is stable if $m^2 > -\frac{9}{4}$.

Then, we continue the computation with $m^2 = -2$. With $m^2 = -2$, (22) is

$$\phi = \delta\phi^{(0)}\Big(\frac{1}{r} + \frac{\phi^{(1)}}{r^2}\Big) \qquad (r \to \infty). \tag{24}$$

Substituting it to (10), we get

$$\begin{aligned} S_{\text{onshell}} &= \int d^3x \big(\delta\phi^{(0)}\big)^2 \frac{r^4}{2}\Big(\frac{1}{r} + \frac{\phi^{(1)}}{r^2}\Big)\Big(-\frac{1}{r^2} - 2\frac{\phi^{(1)}}{r^3}\Big)\Big|_{r=\infty} \\ &= -\int d^3x \frac{1}{2}\big(\delta\phi^{(0)}\big)^2 r\Big|_{r=\infty} \to -\infty, \end{aligned} \tag{25}$$

and we encountered the divergence. In the case of massless scalar field, the first term of $\phi$ becomes zero because of the derivative with respect to $r$ and $S_{\text{onshell}}$ is finite. In the case of massive scalar field, however, the first term of $\phi$ does not vanish and $S_{\text{onshell}}$ diverges.

Holographic renormalization is a way to solve this problem about the divergence and it is a method to remove the divergence by adding counter terms to the action (for example, see [25]). In particular, we add boundary terms which have symmetry of the gravity theory (e.g. general coordinate transformation symmetry) to the action for cancellation of the divergence. In order not to change the EOM under the holographic renormalization, we use the boundary terms.

In the example of (20), we use a counter term as

$$S_{\rm CT} = \frac{1}{2}\int d^3x\sqrt{\gamma}\phi^2\Big|_{r=\infty}, \tag{26}$$

where $\gamma_{\mu\nu}$ is a three-dimensional induced metric of $g_{\mu\nu}$ at $r \to \infty$. In asymptotic AdS spacetime as (7), we get

$$\sqrt{\gamma}\big|_{r=\infty} = r^3. \tag{27}$$

Adding $S_{\rm CT}$ to the action, we obtain

$$\begin{aligned} S_{\rm onshell} + S_{\rm CT} &= \frac{1}{2}\int d^3x\big(\delta\phi^{(0)}\big)^2 r^3\Big(\frac{1}{r} + \frac{\phi^{(1)}}{r^2}\Big)\Big(-\frac{1}{r} - 2\frac{\phi^{(1)}}{r^2} + \frac{1}{r} + \frac{\phi^{(1)}}{r^2}\Big)\Big|_{r=\infty} \\ &= -\int d^3x\frac{1}{2}\big(\delta\phi^{(0)}\big)^2\phi^{(1)}, \end{aligned} \tag{28}$$

and thus, we can remove the divergence. Similar computation as the massless scalar field gives

$$\delta\langle\mathcal{O}\rangle \approx \delta\phi^{(0)}\phi^{(1)}, \tag{29}$$

$$\phi = \frac{\delta\phi^{(0)}}{r} + \frac{\delta\langle\mathcal{O}\rangle}{r^2} \qquad (r \to \infty). \tag{30}$$

#### 2.2.3 U(1) gauge field

Consider an action of a U(1) gauge field in four-dimensional spacetime,

$$S = \frac{1}{4}\int d^4x\sqrt{g}F_{\mu\nu}^2. \tag{31}$$

We assume that $A_t$ is only nonzero as

$$A_t = A_t(r) \tag{32}$$

and we also assume (7) and

$$A_t(r)|_{r=\infty} = 0. \tag{33}$$

(33) corresponds to a boundary condition in the next section. Under these assumptions, the EOM and $S_{\rm onshell}$ are

$$(r^2A_t')' = 0 \qquad (r \to \infty), \tag{34}$$

$$S_{\rm onshell} = \frac{1}{2}\int d^3x\, r^2A_tA_t'\Big|_{r=\infty}. \tag{35}$$

Assuming a solution of (34) as

$$A_t = \delta A_t^{(0)}\Big(1 + \frac{A_t^{(1)}}{r}\Big) \qquad (r \to \infty), \tag{36}$$

we get

$$S_{\text{onshell}} = -\frac{1}{2}\int d^3x \big(\delta A_t^{(0)}\big)^2 A_t^{(1)}. \tag{37}$$

Since we consider a massless gauge field, there is no divergence. Define an operator $J^t$ which corresponds to $\delta A_t^{(0)}$ or consider a generating functional

$$\langle \exp\Big(\int d^3x\ \delta A_t^{(0)} J^t\Big)\rangle, \tag{38}$$

in the gauge theory side. Similar computation as (14) gives

$$\delta\langle J^t\rangle \approx \delta A_t^{(0)} A_t^{(1)}, \tag{39}$$

$$A_t = \delta A_t^{(0)} + \frac{\delta\langle J^t\rangle}{r} \qquad (r \to \infty). \tag{40}$$

For later convenience, we consider a relation between Euclidean and Lorentzian formalism. They are related as

$$S_E = -iS_L, \quad t_E = it_L, \quad A_{t_E} = -iA_{t_L}, \quad \langle J^{t_E}\rangle = i\langle J^{t_L}\rangle, \quad g_E^{tt} = -g_L^{tt}. \tag{41}$$

Thus, in Lorentzian formalism, (40) becomes

$$A_{t_L} = \delta A_{t_L}^{(0)} - \frac{\delta\langle J^{t_L}\rangle}{r} \qquad (r \to \infty). \tag{42}$$

Instead of (32), if we consider

$$A_x = A_x(r), \tag{43}$$

similar computation gives

$$\delta\langle J^x\rangle \approx \delta A_x^{(0)} A_x^{(1)}, \tag{44}$$

$$A_x = \delta A_x^{(0)} + \frac{\delta\langle J^x\rangle}{r} \qquad (r \to \infty). \tag{45}$$

## 2.3 Relation between mass and dimension of operators

From the above computation, we saw that a classical solution of the gravity theory corresponds to an expectation value of an operator and its source in the field theory. In this subsection, we consider which coefficient of the classical solution corresponds to the expectation value of the operator based on [26].

As a condition for the expectation value of the operator, we impose finiteness of the action. In particular, we need finiteness at the AdS boundary. In addition, we need a boundary condition for finiteness at the horizon.

For example, consider a massive scalar field in $d+1$-dimensional asymptotic AdS spacetime

$$ds^2 = \frac{1}{z^2}\Big(dz^2 + \sum_{i=1}^{d}(dx^i)^2\Big) \qquad (z \to 0), \tag{46}$$

where we used a change of variables

$$r = \frac{1}{z}. \tag{47}$$

The AdS boundary is $z = 0$. The action is the same as (20),

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}[(\nabla_\mu\phi)^2 + m^2\phi^2]. \tag{48}$$

Now, $\phi$ depends on $z$ and also $x^i$. Near $z = 0$, (48) becomes

$$\frac{1}{2}\int d^dx\; dz\; z^{-d+1}\Big[(\partial_z\phi)^2 + (\partial_i\phi)^2 + \frac{m^2}{z^2}\phi^2\Big]. \tag{49}$$

A classical solution of this action can be written as

$$\phi = z^\Delta A(\vec{x}) \qquad (z \to 0). \tag{50}$$

$\Delta$ is a solution of

$$\Delta(\Delta - d) = m^2, \tag{51}$$

and its explicit form is

$$\Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2}. \tag{52}$$

At $z \to 0$, the solution of $\Delta_+$ is smaller, and therefore, we generally interpret it as the expectation value of the operator. In fact, substituting (50) to (49), we get

$$\frac{1}{2}\int d^dx\; dz\; z^{2\Delta-d-1}\Big[(\Delta A)^2 + (z\partial_i A)^2 + (mA)^2\Big]. \tag{53}$$

Because of $2\Delta_+ - d - 1 > -1$, there is no divergence at $z \to 0$ in the case of the solution of $\Delta_+$.

In the case of the solution of $\Delta_-$, there is the divergence at $z \to 0$ because of $2\Delta_- - d - 1 < -1$. We can not interpret it as the expectation value of the operator in this situation. In order to loosen the condition for the expectation value of the operator, we use holographic renormalization. In particular, we perform a partial integration of $z^{-d+1}(\partial_z \phi)^2$ and remove the boundary term by the counter term. Then, (49) becomes

$$\begin{aligned}
&\frac{1}{2}\int d^dx\, dz\, z^{-d+1}\Big[-\frac{\phi}{z^{-d+1}}\partial_z(z^{-d+1}\partial_z\phi) + (\partial_i\phi)^2 + \frac{m^2}{z^2}\phi^2\Big] \\
=&\frac{1}{2}\int d^dx\, dz\, z^{-d+1}\Big[-\frac{\Delta(\Delta-d)}{z^2}\phi^2 + (\partial_i\phi)^2 + \frac{m^2}{z^2}\phi^2\Big] \\
=&\frac{1}{2}\int d^dx\, dz\, z^{2\Delta-d+1}\Big[(\partial_i A)^2\Big], \qquad (54)
\end{aligned}$$

and there is no divergence if $2\Delta_- - d + 1 > -1$. Thus, we can interpret the solution of $\Delta_-$ as the expectation value of the operator with this holographic renormalization.

Summarizing the above, if

$$-\frac{d^2}{4} \leq m^2 < -\frac{d^2}{4} + 1, \qquad (55)$$

we can interpret the two solution of $\Delta_\pm$ as the expectation value of the operator. If

$$-\frac{d^2}{4} + 1 \leq m^2, \qquad (56)$$

we can interpret the solution of $\Delta_+$ only as the expectation value of the operator. In addition,

$$\Delta = \frac{d-2}{2} \qquad (57)$$

is mass dimension of a scalar field in $d$-dimensional spacetime and it is known as unitarity bound in CFT (for example, see [27, 28, 29]). Therefore, modification by holographic renormalization is considered to be correct.

# 3 Review of s-wave holographic superconductor model

In this section, we review [15, 16] by S. A. Hartnoll, C. P. Herzog and G. T. Horowitz. Their papers are the origin of the study of holographic superconductor and various models are computed after these papers. This review is based on my master thesis [23].

## 3.1 S-wave holographic superconductor model

In the study of holographic superconductor, we analyze classical solutions of holographic models whose phase transition is similar to superconductor. From now, we consider a 3+1 dimensional holographic model which corresponds to 2+1 dimensional s-wave superconductor.

### 3.1.1 Correspondence between superconductor and black hole

Rigorous correspondence between superconductor and gravity theory is not understood well. Then, we consider a gravity theory which has characteristic physical quantity of superconductivity as a bottom up approach.

Since superconductivity is a phase transition phenomenon when temperature is lowered, we need the temperature in the holographic model. Moreover, we need an electromagnetic field for an electric current and an order parameter to distinguish between a normally conducting phase and a superconducting phase. In order to satisfy these property, we introduce a holographic model on AdS black hole spacetime with a scalar flied and a U(1) gauge field. We consider AdS black hole because there is Hawking temperature in black hole. The scalar field represents the order parameter of s-wave superconductivity with angular momentum $l = 0$. As we will see later, this scalar field is zero at high temperature, however it becomes nonzero at low temperature. Therefore, it is suitable for the order parameter of superconductivity.

### 3.1.2 Action of the model

We consider an action of the s-wave holographic superconductor model as

$$S = \int d^4x\sqrt{-g}\Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2|\Psi|^2 - |\partial\Psi - iA\Psi|^2\Big], \tag{58}$$

where $\Psi$ is a scalar field, $A_\mu$ is a U(1) gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We consider AdS black hole spacetime as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \tag{59}$$

$$f(r) = r^2 - \frac{M}{r}, \tag{60}$$

where $M$ is a black hole mass. At $r \to \infty$, (59) becomes AdS metric

$$ds^2 = -r^2dt^2 + \frac{dr^2}{r^2} + r^2(dx^2 + dy^2). \tag{61}$$

The horizon radius of the black hole $r_0$ which is defined by $f(r_0) = 0$ is

$$r_0 = M^{1/3}. \tag{62}$$

Hawking temperature of the black hole $T$ is

$$T = \frac{3M^{1/3}}{4\pi}. \tag{63}$$

We set mass of the scalar field $m^2$ as

$$m^2 = -2. \tag{64}$$

This value is larger than Breitenlohner-Freedman bound -9/4 in four-dimension. In the study of holographic superconductor model, (64) is commonly used as mass of a scalar field on four-dimensional spacetime. The reasons are as follows: $\Delta_\pm$ is integer, mass of the four-dimensional scalar field which is derived by compactification of supergravity is (64), etc. However, (58) is not derived from supergravity directly, (64) is not always necessary.

In order to justify (58), we use approximation which is called as the probe limit. The probe limit means that we ignore the effect of the scalar field and the U(1) gauge field for the metric. The action before using the probe limit is (58) with Einstein-Hilbert action

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G}(R-2\Lambda) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2|\Psi|^2 - |\partial\Psi - iqA\Psi|^2\Big], \tag{65}$$

where $q$ is charge of the scalar field and $G$ is the Newton constant. With redefinition of the scalar field and the U(1) gauge field such as $\Psi \to \Psi/q$ and $A_\mu \to A_\mu/q$, we get

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R - 2\Lambda + \frac{16\pi G}{q^2}\Big\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2|\Psi|^2 - |\partial\Psi - iA\Psi|^2\Big\}\Big]. \tag{66}$$

At the limit of $G/q^2 \to 0$, the latter part of (66) is smaller than the former part and (66) comes down to Einstein-Hilbert action. Therefore, in the probe limit, we can derive (58) from (66) by choosing (59) as a solution of Einstein-Hilbert action and considering the latter part of (66) as a perturbative action of the scalar field and the U(1) gauge field.

In [15, 16], the authors used the action in Lorentzian formalism as (58). In order to analyze the EOM and its solutions, we can use Lorentzian formalism because the EOM is the same in Lorentzian and Euclidean formalism. However, we need to use Euclidean formalism to compute the free energy of the solutions.

### 3.1.3 Derivation of Hawking temperature

We can derive Hawking temperature by using periodicity of Euclidean metric. In this subsection, we derive (63) by this method.

With $t_E = it$, (59) becomes

$$ds_E^2 = \frac{dr^2}{f(r)} + f(r)dt_E^2, \tag{67}$$

where we ignore $x$ and $y$ components. Since Hawking temperature is related to radiation from the black hole, we approximate $f(r)$ near the horizon $r_0$. With a coordinate transformation, we obtain

$$\begin{aligned} ds_E^2 &\simeq \frac{dr^2}{f'(r_0)(r-r_0)} + f'(r_0)(r-r_0)dt_E^2 \\ &= d\rho^2 + \rho^2 d\Big(\frac{f'(r_0)}{2}t_E\Big)^2, \qquad (68) \\ \rho &\equiv 2\sqrt{(r-r_0)/f'(r_0)}. \qquad (69) \end{aligned}$$

Next, we impose that (68) represents polar coordinates. In other words, we impose that a period of $f'(r_0)t_E/2$ is $2\pi$ as an angle of two-dimensional flat space. This condition is the same that a period of imaginary time $t_E$ is

$$\beta = \frac{4\pi}{f'(r_0)}. \tag{70}$$

The inverse number of $\beta$ corresponds to Hawking temperate $T$,

$$T = \frac{1}{\beta} = \frac{f'(r_0)}{4\pi} = \frac{3M^{\frac{1}{3}}}{4\pi}. \tag{71}$$

Imposing the periodicity in the direction of imaginary time is a technique of finite temperature field theory. From this derivation, we can imagine that Hawking temperature of black hole corresponds to temperature of finite temperature field theory. If we impose that $\rho$ is real number as a radius of two-dimensional plane, we get $r \geq r_0$ from (69). Therefore, a range of $r$ is from $r_0$ to $\infty$ in Euclidean formalism.

## 3.2 Phase transition of the scalar field

In this subsection, we analyze the phase transition of the scalar field which corresponds to the order parameter of superconductivity.

### 3.2.1 Equations of motion

The EOM of $\Psi$ from (58) is

$$D^{\mu}D_{\mu}\Psi - m^2\Psi = 0, \tag{72}$$

and the EOM of $A_\nu$ is

$$\nabla_\mu F^{\mu\nu} + i\Psi(D^\nu\Psi)^* - i\Psi^*(D^\nu\Psi) = 0, \tag{73}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ and $D_\mu \equiv \nabla_\mu - iA_\mu$. Next, we consider a solution of these equations with the ansatz as

$$\Psi = \Psi(r), \ \ A_t = \Phi(r), \ \ A_r = A_x = A_y = 0, \tag{74}$$

where $\Psi$ and $\Phi$ are real functions. $r$ dependence of $\Psi$ and $\Phi$ means uniformity of the field theory side. With this ansatz, let us see explicit forms of the EOM. Christoffel symbols for the computation are

$$\Gamma^r_{rr} = \frac{-f'}{2f}, \quad \Gamma^x_{rx} = \Gamma^y_{ry} = \frac{1}{r}, \quad \Gamma^r_{xx} = \Gamma^r_{yy} = -fr, \quad \Gamma^t_{rt} = \frac{f'}{2f}, \quad \Gamma^r_{tt} = \frac{ff'}{2}. \tag{75}$$

Substituting (74) to (72), we obtain

$$\begin{aligned} &g^{\mu\nu}(\partial_\mu D_\nu\Psi - \Gamma^\lambda_{\mu\nu}D_\lambda\Psi - iA_\mu D_\nu\Psi) - m^2\Psi \\ =&f\Psi'' + \left(f' + \frac{2f}{r}\right)\Psi' + \frac{\Phi^2}{f}\Psi - m^2\Psi = 0. \end{aligned} \tag{76}$$

Dividing (76) by $f$, we get

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{f}\Psi = 0, \tag{77}$$

where $m^2 = -2$. Similarly, (73) becomes

$$\partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\lambda}F^{\lambda\nu} + i\Psi(D^\nu\Psi)^* - i\Psi^*(D^\nu\Psi) = 0. \tag{78}$$

In the case of $\nu = x, y$, the left hand side is zero. In the case of $\nu = r$, (78) is

$$i\Psi\left(\Psi'\right)^* - i\Psi^*\Psi' = 0, \tag{79}$$

and the left hand side is zero with $\Psi = \Psi^*$. In the case of $\nu = t$, (78) is

$$\Phi'' + \frac{2}{r}\Phi' - \frac{2\Psi^2}{f}\Phi = 0. \tag{80}$$

### 3.2.2 Asymptotic solution

We consider an asymptotic solution of (77) and (80) at $r \to \infty$. Assume that $\Phi$ is finite at $r \to \infty$. With this assumption, (77) is

$$\Psi'' + \frac{4}{r}\Psi' + \frac{2}{r^2}\Psi = 0 \qquad (r \to \infty). \tag{81}$$

Therefore, the asymptotic solution is

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} \qquad (r \to \infty). \tag{82}$$

Similarly, (80) at $r \to \infty$ is

$$\Phi'' + \frac{2}{r}\Phi' = 0 \qquad (r \to \infty), \tag{83}$$

and the asymptotic solution is

$$\Phi = \mu - \frac{\rho}{r}, \tag{84}$$

where we put a minus sign because of (42).

As discussed in section 2, we interpret the coefficients in (82) and (84) as the source and its response in the field theory side. Since the mass of the scalar field (64) satisfies the condition (55), we can interpret both $\Psi^{(1)}$ and $\Psi^{(2)}$ as the expectation value of the operator. In addition, we interpret $\mu$ as a chemical potential (source) and $\rho$ as a charge density (response).

### 3.2.3 Boundary conditon

In order to solve the EOM, we need to decide boundary conditions. Now, we impose three boundary conditions.

First boundary condition is

$$\Phi = 0 \qquad (r \to r_0). \tag{85}$$

Since $g^{tt}$ in $-\sqrt{-g}g^{tt}\Phi^2\Psi^2$ diverges at $r \to r_0$, this condition prevents the divergence of $-\sqrt{-g}g^{tt}\Phi^2\Psi^2$ in (58).

Second condition is

$$3r_0\Psi' + 2\Psi = 0 \qquad (r \to r_0). \tag{86}$$

This condition is related to (77) and finiteness of $\Psi$ at $r \to r_0$.

Third condition is

$$\Psi^{(1)} = 0 \quad \text{or} \quad \Psi^{(2)} = 0. \tag{87}$$

This condition means that the source of the scalar field in the field theory side is zero. For example, in the case of $\Psi^{(1)} = 0$, $\Psi^{(1)}$ is the source and $\Psi^{(2)}$ is its response. If the temperature is enough low, $\Psi^{(2)}$ can take a nonzero value because of $\mu$.

Since there are the two differential equations with second order, their solution has four parameters. With the three boundary conditions, the freedom of the solution decreases and we can describe the solution by one parameter.

### 3.2.4 Phase transition

Before the numerical computations, consider how the phase transition occurs qualitatively.

For any value of Hawking temperature $T$, there is a solution of (77) and (80) as

$$\Psi = 0, \qquad \Phi = \mu(1 - \frac{r_0}{r}). \tag{88}$$

In fact, we can check that this solution satisfies (80) and the boundary condition (85). Since $\Psi$ of this solution is zero, this solution represents normal conductivity.

If Hawking temperature $T$ is low enough, the scalar field can have a nonzero value and there is a solution as

$$\Psi = \frac{\Psi^{(1)}}{r} \text{ or } \frac{\Psi^{(2)}}{r^2}, \qquad \Phi = \mu - \frac{\rho}{r} \qquad (r \to \infty). \tag{89}$$

This solution represents super conductivity because of the nonzero scalar field. The reason for such a solution is the coefficient of $|\Psi|^2$ in (58). We denote this coefficient by $m_{\text{eff}}^2$. Substituting (12) to $m_{\text{eff}}^2$, we get

$$\begin{aligned} m_{\text{eff}}^2 = m^2 + g^{tt}\Phi^2 &= m^2 - \frac{\mu^2(1 - \frac{r_0}{r})^2}{r^2\left(1 - \frac{r_0^3}{r^3}\right)} \\ &= m^2 - \left(\frac{3\mu}{4\pi T}\right)^2 \frac{1 - \frac{r_0}{r}}{1 + \frac{r}{r_0} + \frac{r^2}{r_0^2}}. \end{aligned} \tag{90}$$

Thus, if $T$ is low, $m_{\text{eff}}^2$ becomes little and the phase transition can occur such as a phase transition by Higgs potential. For this phase transition, the shape of the metric and the existence of the scalar field and the U(1) field are important.

In the low temperature region, the Euclidean classical action $S_{\rm onshell}$ can determine which solution is favored. In the holographic superconductor model, the right hand side of (1) corresponds to the generating functional of the field theory. Therefore, we consider $S_{\rm onshell}$ as a free energy and the solution with smaller $S_{\rm onshell}$ is favored. Generally, the free energy of the super conductivity solution is smaller than one of the normal conductivity solution and the phase transition occurs. We will compute the free energy in the next section.

#### 3.2.5 Result of numerical computation

We solve the simultaneous equations (77) and (80) with the boundary conditions (85), (86) and (87). We perform numerical computations by Mathematica. We use a numerical code on the homepage of C. P. Herzog [30].

Denote the operators in the field theory by $\mathcal{O}_i$. We consider these operators as order parameters. We normalize these operators as

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)}, \qquad i = 1, 2. \tag{91}$$

Figure 1 and figure 2 show numerical result of $\mathcal{O}_i$. The horizontal axes are Hawking temperature $T$, and the vertical axes are $\langle \mathcal{O}_1 \rangle$ and $\sqrt{\langle \mathcal{O}_2 \rangle}$. We normalize the horizontal and vertical axes by the critical temperature $T_c$ at which the phase transition occurs. As it can be seen from the figures, $\langle \mathcal{O}_i \rangle$ has nonzero value below the critical temperature $T_c$ and the phase transition occurs.

For the numerical computations, we need to determine two parameters: Hawking temperature $T$ and the coefficient of (89) (for example, $\rho$). However, we can consider only one parameter by using the symmetry of the metric (59). In order to see it, consider a coordinate transformation as

$$r = ar', \qquad t = \frac{t'}{a}, \qquad x = \frac{x'}{a}, \qquad y = \frac{y'}{a}. \tag{92}$$

Under this transformation, (59) becomes

$$ds^2 = -f(r')dt'^2 + \frac{dr'^2}{f(r')} + r'^2(dx'^2 + dy'^2), \tag{93}$$

$$f(r') = r'^2 - \frac{M/a^3}{r'}. \tag{94}$$

Hawking temperature of this coordinate system is

$$T' = \frac{3M^{1/3}}{4\pi a} = \frac{T}{a}. \tag{95}$$

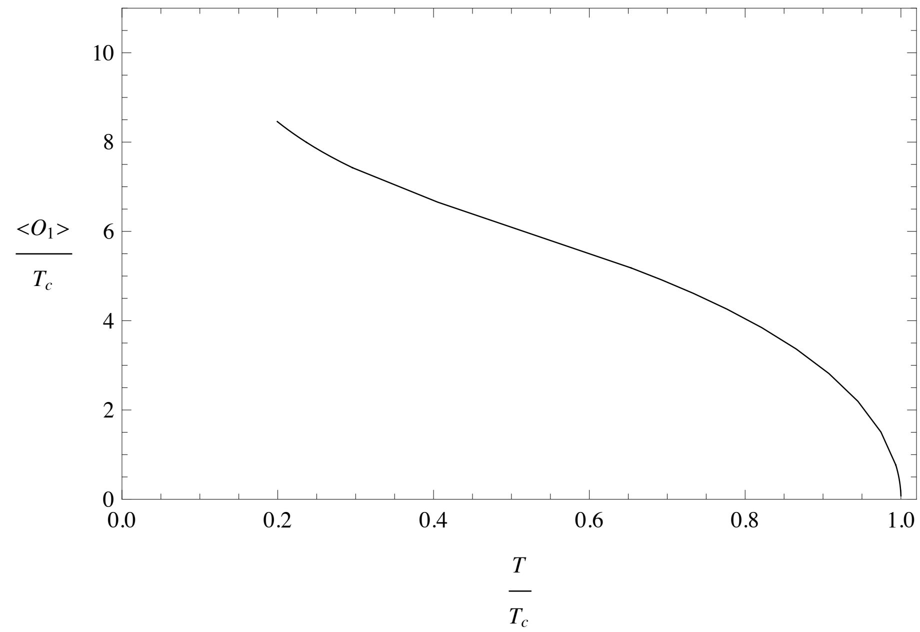


Figure 1: $T$ dependence of $\mathcal{O}_1$

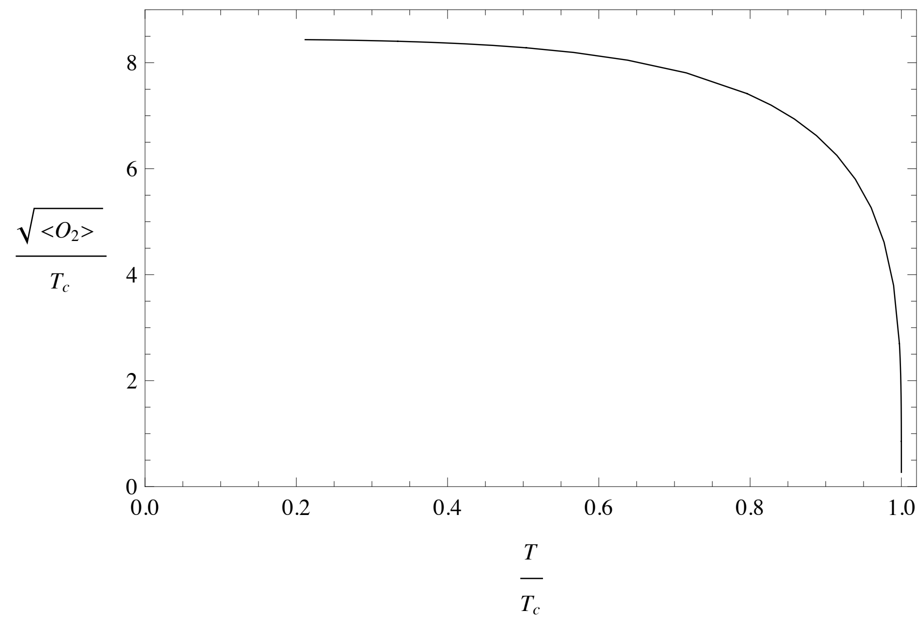


Figure 2: $T$ dependence of $\mathcal{O}_2$

Moreover, (89) changes as

$$\Psi'(r') = \frac{\Psi'^{(1)}}{r'} + \frac{\Psi'^{(2)}}{r'^2} = \Psi(r) = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} = \frac{\Psi^{(1)}/a}{r'} + \frac{\Psi^{(2)}/a^2}{r'^2}, \tag{96}$$

$$\Phi'(r') = \mu' - \frac{\rho'}{r'} = \frac{\Phi(r)}{a} = \mu/a - \frac{\rho/a}{r} = \mu/a - \frac{\rho/a^2}{r'}, \tag{97}$$

where $'$ means the physical quantity in the $r'$ coordinate system. Thus, some relations hold such as

$$\frac{\rho}{T^2} = \frac{\rho'}{T'^2}. \tag{98}$$

After all, the solutions which satisfy (98) are equivalent under the coordinate transformation (92). In the numerical computations, we first computed with $T = 3/4\pi$, and we changed the coordinate system to $\rho = 1$ by using (98). In fact, we used

$$\frac{\rho}{(3/4\pi)^2} = \frac{1}{T^2}. \tag{99}$$

$\rho$ in the left hand side is the value in the numerical computation and $T$ in the right hand side is the value with $\rho = 1$ after the coordinate transformation.

### 3.3 Electric conductivity

In the holographic superconductor model, we can also calculate electric conductivity. In order to compute electrical conductivity, we need a source of a electric field. Therefore, we introduce a perturbation of $A_x$. In addition, we use an ansatz as

$$A_x = A_x(r)\exp(-i\omega t), \tag{100}$$

where $\omega$ is frequency of the electric field, $\omega = 0$ corresponds to a direct current and $\omega \neq 0$ corresponds to an alternating current. In first order perturbation of $A_x$, the EOM of $A_x$ is

$$A_x'' + \frac{f'}{f}A_x' + \Big(\frac{\omega^2}{f^2} - \frac{2\Psi^2}{f}\Big)A_x = 0. \tag{101}$$

We use the classical solution which we computed in the previous section as $\Psi$.

From (45), the asymptotic solution is

$$A_x = A_x^{(0)} + \frac{A_x^{(1)}}{r}. \tag{102}$$

Moreover, we impose a boundary condition as

$$A_x(r_0) \propto f^{-i\omega/3r_0} \tag{103}$$

This condition means that there is no flow from inside of the black hole. In fact, (103) satisfies (101) as

$$-i\omega(-i\omega - 3r_0)f^{-i\omega/3r_0-2} - 3r_0 i\omega f^{-i\omega/3r_0-2} + \omega^2 f^{-i\omega/3r_0-2} = 0. \tag{104}$$

Near $r = r_0$, the metric (59) becomes

$$ds^2 \approx -fdt^2 + fdr_*^2 + r^2(dx^2 + dy^2), \tag{105}$$

$$r_* = \frac{\log f(r)}{3r_0}, \tag{106}$$

under the approximation of $f' = 3r_0$. In this coordinate system, (100) becomes

$$A_x \propto \exp[-i\omega(r_* + t)], \tag{107}$$

and this solution is flow into the black hole under the time evolution.

As section 3.2, we consider the coefficients of (102) as the physical quantity in the field theory side. $A_x^{(0)} = A_x$ is a source and $A_x^{(1)} = \langle J^x \rangle$ is a current density. With this assumption, the electric conductivity $\sigma(\omega)$ is

$$\sigma(\omega) = \frac{\langle J^x \rangle}{E_x} = -\frac{\langle J^x \rangle}{\dot{A}_x} = -\frac{i\langle J^x \rangle}{\omega A_x} = -\frac{iA_x^{(1)}}{\omega A_x^{(0)}}. \tag{108}$$

In [15], the authors found that the DC electric conductivity (108) at $\omega = 0$ of the holographic model diverges. This divergence is the origin of the name "holographic superconductor".

Moreover, the DC electric conductivity with a lattice effect is studied in [31]. The authors of [31] considered a neutral scalar field source

$$\phi_1(x) = A_0 \cos(k_0 x), \tag{109}$$

on a charged black hole. This source violates transformation invariance. They found that the electric conductivity has a power-law behavior

$$|\sigma(\omega)| = \frac{B}{\omega^{2/3}} + C, \tag{110}$$

at intermediate frequency. This property is similar to the property of the cuprate high-temperature superconductor (normal phase) [32]. Because of this coincidence, it is considered that the holographic models can describe the high-temperature superconductor.

# 4 Frustrated holographic superconductor

In this section, we consider a three-scalar holographic superconductor model. We analyze its equations of motion and solutions. By choosing specific values of parameters, we find that there are two chiral solutions whose free energy is minimum. Therefore, this holographic model can describe frustration. This part is based on our paper [14].

## 4.1 Motivation

Frustration is a situation where there are several competing constraints (see, for example, [33, 34]). This competition can trigger degeneracy of ground states. Because of this degeneracy, frustrated systems can have a rich phase structure. Antiferromagnets on a triangular lattice is an example of the frustrated systems.

For example, consider a potential

$$f(\theta) = \cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1), \tag{111}$$

and compute its minimum. We can minimize two of the potential terms in (111), however, we cannot minimize all terms together. Thus, the potential (111) is one example which describes the frustration. Configurations of $\theta$ which minimize the potential (111) are

$$\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{2\pi}{3}, \tag{112}$$

and

$$\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{4\pi}{3}. \tag{113}$$

We can define chirality of $\theta$ to distinguish these two minima as we will see later. As this example shows, the frustrated systems tend to have some ground states.

In condensed matter physics, a three-band superconductor model was proposed as a frustrated systems [35]. This model is considered to study multiband superconductivity such as Fe-based superconductor [36]. In order to cause frustration, the authors of [35] introduced three scalar order parameters and Josephson coupling between the scalar fields as (111). In this model, $\theta_i$ are complex phases of the scalar fields. They found that their model has chiral ground states which correspond to (112) and (113). These states have nonzero chirality and their free energy is the same.

Based on [35], we expected that a three-scalar holographic superconductor model also has the similar solutions which correspond to the chiral ground states. Moreover, the Fe-based superconductor is one of high temperature superconductors by strongly correlated effects. Therefore, we may reveal the physics of the Fe-based superconductor from holographic superconductor models with multi order parameters This is a motivation to study the three-scalar holographic superconductor model.

## 4.2 Three-scalar holographic superconductor model

We consider a three-scalar holographic superconductor model as

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |D_\mu\varphi_1|^2 - |D_\mu\varphi_2|^2 - |D_\mu\varphi_3|^2 \\ & - m_1^2|\varphi_1|^2 - m_2^2|\varphi_2|^2 - m_3^2|\varphi_3|^2 \\ & - \epsilon_{12}(\varphi_1^*\varphi_2 + \varphi_1\varphi_2^*) - \epsilon_{23}(\varphi_2^*\varphi_3 + \varphi_2\varphi_3^*) - \epsilon_{31}(\varphi_3^*\varphi_1 + \varphi_3\varphi_1^*) \\ & - \eta(|\varphi_1|^4 + |\varphi_2|^4 + |\varphi_3|^4)\Big], \end{aligned} \tag{114}$$

$$D_\mu = \nabla_\mu - iA_\mu, \tag{115}$$

where we introduce a U(1) field $A_\mu$, three complex scalar fields $\varphi_i$, three nonzero Josephson coupling $\epsilon_{ij}$, and a nonnegative constant $\eta$. The potential term $\epsilon_{12}(\varphi_1^*\varphi_2+\varphi_1\varphi_2^*)+\epsilon_{23}(\varphi_2^*\varphi_3+\varphi_2\varphi_3^*)+\epsilon_{31}(\varphi_3^*\varphi_1+\varphi_3\varphi_1^*)$ with $\epsilon_{12} = \epsilon_{23} = \epsilon_{31} > 0$ corresponds to (111). In this section, we use the AdS black hole metric as

$$ds^2 = \frac{1}{z^2}(-f(z)dt^2 + dx^2 + dy^2 + \frac{dz^2}{f(z)}), \tag{116}$$

$$f(z) = 1 - \Big(\frac{z}{z_h}\Big)^3, \tag{117}$$

where $z = 1/r$ and $z_h$ is the horizon radius of the black hole. We consider the probe limit and fix this metric. The quartic potential in (114) is one choice where the chiral ground states exist.

## 4.3 Solutions with $\eta = 0$

First, we analyze the three-scalar holographic superconductor model with $\eta = 0$. Its equations of motion of $\varphi_i$ are

$$D^\mu D_\mu\varphi_1 - m_1^2\varphi_1 - \epsilon_{12}\varphi_2 - \epsilon_{31}\varphi_3 = 0, \tag{118}$$

$$D^\mu D_\mu\varphi_2 - m_2^2\varphi_2 - \epsilon_{23}\varphi_3 - \epsilon_{12}\varphi_1 = 0, \tag{119}$$

$$D^\mu D_\mu\varphi_3 - m_3^2\varphi_3 - \epsilon_{31}\varphi_1 - \epsilon_{23}\varphi_2 = 0. \tag{120}$$

There are three types of the solutions in these equations:

Sol.1 $\varphi_1 = \varphi_2 = \varphi_3 = 0$.

Sol.2 One scalar field is zero and the others are nonzero such as $\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0$.

Sol.3 $\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 \neq 0$.

In order to study frustration between the three scalar fields, we examine Sol.3. For simplicity, we rewrite $\varphi_i$ as

$$\varphi_i = \psi_i e^{i\theta_i}, \tag{121}$$

where $\psi_i > 0$ and we use an ansatz

$$A_t = A_t(z), \qquad \psi_i = \psi_i(z), \qquad \theta_i = \text{const.}, \tag{122}$$

and the other components of $A_\mu$ are zero. With this ansats, the equations of motion are

$$\nabla_\mu F^{\mu\nu} - 2\psi_1^2 A^\nu - 2\psi_2^2 A^\nu - 2\psi_3^2 A^\nu = 0, \tag{123}$$
$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - m_1^2 \psi_1 - \epsilon'_{12}\psi_2 - \epsilon'_{31}\psi_3 = 0, \tag{124}$$
$$\nabla_\mu \nabla^\mu \psi_2 - A_\mu A^\mu \psi_2 - m_2^2 \psi_2 - \epsilon'_{23}\psi_3 - \epsilon'_{12}\psi_1 = 0, \tag{125}$$
$$\nabla_\mu \nabla^\mu \psi_3 - A_\mu A^\mu \psi_3 - m_3^2 \psi_3 - \epsilon'_{31}\psi_1 - \epsilon'_{23}\psi_2 = 0, \tag{126}$$
$$\epsilon_{12}\psi_1\psi_2 \sin(\theta_1 - \theta_2) + \epsilon_{31}\psi_1\psi_3 \sin(\theta_1 - \theta_3) = 0, \tag{127}$$
$$\epsilon_{23}\psi_2\psi_3 \sin(\theta_2 - \theta_3) + \epsilon_{12}\psi_2\psi_1 \sin(\theta_2 - \theta_1) = 0, \tag{128}$$
$$\epsilon_{31}\psi_3\psi_1 \sin(\theta_3 - \theta_1) + \epsilon_{23}\psi_3\psi_2 \sin(\theta_3 - \theta_2) = 0, \tag{129}$$
$$\epsilon'_{12} \equiv \epsilon_{12}\cos(\theta_1 - \theta_2), \quad \epsilon'_{23} \equiv \epsilon_{23}\cos(\theta_2 - \theta_3), \quad \epsilon'_{31} \equiv \epsilon_{31}\cos(\theta_3 - \theta_1). \tag{130}$$

Let us focus on the equations of $\theta_i$. There are two types of the solutions:

Sol.3a $\sin(\theta_1 - \theta_2) \neq 0, \sin(\theta_2 - \theta_3) \neq 0, \sin(\theta_3 - \theta_1) \neq 0$.

Sol.3b $\sin(\theta_1 - \theta_2) = \sin(\theta_2 - \theta_3) = \sin(\theta_3 - \theta_1) = 0$.

As we will see later, Sol.3a has nonzero chirality.

Consider a condition for the existence of Sol.3a. From (128) and (129), we get

$$\psi_2 = -\frac{\epsilon_{31}\sin(\theta_3 - \theta_1)}{\epsilon_{23}\sin(\theta_3 - \theta_2)}\psi_1, \quad \psi_3 = -\frac{\epsilon_{12}\sin(\theta_2 - \theta_1)}{\epsilon_{23}\sin(\theta_2 - \theta_3)}\psi_1. \tag{131}$$

Because of $\psi_i > 0$, we need

$$\frac{\epsilon_{31}\sin(\theta_3-\theta_1)}{\epsilon_{23}\sin(\theta_3-\theta_2)} < 0, \quad \frac{\epsilon_{12}\sin(\theta_2-\theta_1)}{\epsilon_{23}\sin(\theta_2-\theta_3)} < 0. \tag{132}$$

Substituting (131) to (124), (125) and (126), we get

$$\nabla_\mu\nabla^\mu\psi_1 - A_\mu A^\mu\psi_1 - \left(m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right)\psi_1 = 0, \tag{133}$$

$$\nabla_\mu\nabla^\mu\psi_1 - A_\mu A^\mu\psi_1 - \left(m_2^2 - \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right)\psi_1 = 0, \tag{134}$$

$$\nabla_\mu\nabla^\mu\psi_1 - A_\mu A^\mu\psi_1 - \left(m_3^2 - \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}}\right)\psi_1 = 0. \tag{135}$$

Therefore, Sol.3a exists only if

$$m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} = m_2^2 - \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} = m_3^2 - \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}}. \tag{136}$$

Let us consider the other solutions with (136).

Sol.1 $\underline{\varphi_1 = \varphi_2 = \varphi_3 = 0.}$

Generally, nonzero scalar fields decreases the free energy. Therefore, the free energy of this solution is not minimum if Hawking temperature is low enough.

Sol.2 $\underline{\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0.}$

From (120), we get

$$\psi_2 = -\frac{\epsilon_{31}}{\epsilon_{23}}\psi_1 e^{i(\theta_1-\theta_2)}, \tag{137}$$

and

$$\theta_1 - \theta_2 = 0 \text{ or } \pi, \tag{138}$$

are the solutions of the equations of $\theta_i$. Substituting (137) to (125) and (126), we get

$$\nabla_\mu\nabla^\mu\psi_1 - A_\mu A^\mu\psi_1 - \left(m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right)\psi_1 = 0, \tag{139}$$

$$\nabla_\mu\nabla^\mu\psi_1 - A_\mu A^\mu\psi_1 - \left(m_2^2 - \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right)\psi_1 = 0, \tag{140}$$

and the solution with $\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0$ can exist if (136) holds. Similarly, we can derive other solutions with $\varphi_1 = 0, \varphi_2 \neq 0, \varphi_3 \neq 0$ and $\varphi_1 \neq 0, \varphi_2 = 0, \varphi_3 \neq 0$. Since the mass squared of $\psi_1$ in (133) and (139) is the same, the free energy of these solutions is the same as that of solution 3a.

Sol.3b $\underline{\sin(\theta_1 - \theta_2) = \sin(\theta_2 - \theta_3) = \sin(\theta_3 - \theta_1) = 0.}$

In this case, we can use the diagonalization. We diagonalize a matrix as

$$\begin{pmatrix} m_1^2 & \epsilon'_{12} & \epsilon'_{31} \\ \epsilon'_{12} & m_2^2 & \epsilon'_{23} \\ \epsilon'_{31} & \epsilon'_{23} & m_3^2 \end{pmatrix}. \tag{141}$$

If (136) holds, we can transform this matrix to a diagonal matrix as

$$\begin{pmatrix} m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} & 0 & 0 \\ 0 & m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} & 0 \\ 0 & 0 & m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \end{pmatrix}. \tag{142}$$

In order to analyze the free energy, we assume that the scalar fields with $m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}$ and $m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}$ can not coexist in this model. We will explain this assumption later. If $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ holds, the free energy of this solution is the same as that of the solution 3a since $m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} < m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}$. However, if $\epsilon_{12}\epsilon_{23}\epsilon_{31} < 0$ holds, the free energy of the solution corresponds to $m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}$ is smaller than that of the solution 3a.

In short summary, we found that there are some solutions whose free energy is same if $\eta = 0$, $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ and (136) hold.

## 4.4 Solutions with $\eta > 0$ and their free energy

Second, we study the three-scalar model with $\eta > 0$, $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ and (136).[1] In particular, we compute the free energy with $\eta > 0$ by substituting the solutions with $\eta > 0$ to the action.

We can derive the equations of motion of $\psi_i$ as

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - m_1^2 \psi_1 - \epsilon'_{12}\psi_2 - \epsilon'_{31}\psi_3 - 2\eta\psi_1^3 = 0, \tag{143}$$
$$\nabla_\mu \nabla^\mu \psi_2 - A_\mu A^\mu \psi_2 - m_2^2 \psi_2 - \epsilon'_{23}\psi_3 - \epsilon'_{12}\psi_1 - 2\eta\psi_2^3 = 0, \tag{144}$$
$$\nabla_\mu \nabla^\mu \psi_3 - A_\mu A^\mu \psi_3 - m_3^2 \psi_3 - \epsilon'_{31}\psi_1 - \epsilon'_{23}\psi_2 - 2\eta\psi_3^3 = 0, \tag{145}$$

[1] We would like to thank K. Hashimoto for useful suggestion about this subsection.

and the equations of motion of $A_\mu$ and $\theta_i$ are the same as (123), (127), (128) and (129). From now, we set

$$m_1^2 = m_2^2 = m_3^2, \quad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} > 0. \tag{146}$$

In this choice, the chiral ground states is symmetric with respect to $\theta_i$ such as (112) and (113).

Sol.2 $\underline{\varphi_1 \neq 0, \varphi_2 \neq 0, \varphi_3 = 0.}$

Substituting (137) to (143) and (144), we get

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi_1 - 2\eta \psi_1^3 = 0, \tag{147}$$

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_2^2 - \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \right) \psi_1 - 2\eta \left( \frac{\epsilon_{31}}{\epsilon_{23}} \right)^2 \psi_1^3 = 0, \tag{148}$$

and these equations are the same with (146). This equation is the same equation in [37, 38]. Substituting (137) to (114), the on-shell action becomes

$$\begin{aligned} S_{\text{on-shell}} &= \int d^4x \sqrt{-g} \Big[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - 2|D_\mu \psi_1|^2 - 2 \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi_1^2 - 2\eta \psi_1^4 \Big] \\ &= \int d^4x \sqrt{-g} \Big[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu \psi'|^2 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi'^2 - \frac{\eta}{2} \psi'^4 \Big], \end{aligned} \tag{149}$$

where a new scalar field $\psi'$ is redefined as

$$\psi'^2 \equiv 2\psi_1^2, \tag{150}$$

for comparison of the free energy. The coefficient of the quartic potential in (149) is $\eta/2$. In the case of $\varphi_1 = 0, \varphi_2 \neq 0, \varphi_3 \neq 0$ and $\varphi_1 \neq 0, \varphi_2 = 0, \varphi_3 \neq 0$, we can obtain the similar on-shell action.

Sol.3a $\underline{\sin(\theta_1 - \theta_2) \neq 0, \sin(\theta_2 - \theta_3) \neq 0, \sin(\theta_3 - \theta_1) \neq 0.}$

Substituting (131) to (143), (144) and (145), we get

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi_1 - 2\eta \psi_1^3 = 0, \tag{151}$$

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_2^2 - \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \right) \psi_1 - 2\eta \left( \frac{\epsilon_{31} \sin(\theta_3 - \theta_1)}{\epsilon_{23} \sin(\theta_3 - \theta_2)} \right)^2 \psi_1^3 = 0, \tag{152}$$

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_3^2 - \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} \right) \psi_1 - 2\eta \left( \frac{\epsilon_{12} \sin(\theta_2 - \theta_1)}{\epsilon_{23} \sin(\theta_2 - \theta_3)} \right)^2 \psi_1^3 = 0. \tag{153}$$

If (146) holds, the solution 3a is possible only if

$$1 = \left( \frac{\sin(\theta_3 - \theta_1)}{\sin(\theta_3 - \theta_2)} \right)^2 = \left( \frac{\sin(\theta_2 - \theta_1)}{\sin(\theta_2 - \theta_3)} \right)^2 . \tag{154}$$

There are two solutions of $\theta_i$ which satisfy this condition with (132):

$$\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{2\pi}{3}, \tag{155}$$

and

$$\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = \frac{4\pi}{3}. \tag{156}$$

These configurations are the same as (112) and (113). We define chirality as the sign of $i(\varphi_1^* \varphi_2 - \varphi_1 \varphi_2^*) = 2\psi_1 \psi_2 \sin(\theta_1 - \theta_2)$. With this definition, the solutions (155) and (156) are chiral as figure 3. Substituting (131) and (154) to (114), the on-shell action can be obtained as

$$\begin{aligned} S_{\text{on-shell}} &= \int d^4x \sqrt{-g} \Big[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - 3|D_\mu \psi_1|^2 - 3 \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi_1^2 - 3\eta \psi_1^4 \Big] \\ &= \int d^4x \sqrt{-g} \Big[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - |D_\mu \psi'|^2 - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi'^2 - \frac{\eta}{3} \psi'^4 \Big], \end{aligned} \tag{157}$$

where a new scalar field $\psi'$ is redefined as

$$\psi'^2 \equiv 3\psi_1^2, \tag{158}$$

and the coefficient of the quartic potential in (157) is $\eta/3$.

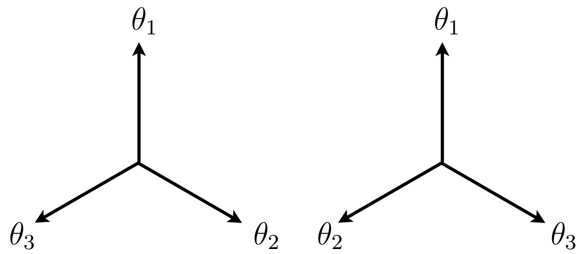


Figure 3: Chiral solutions (155) and (156). Three allows describe the phase angles of the three scalar fields. By using a mirror operation, we can interchange the two solutions.

Sol.3b $\underline{\sin(\theta_1-\theta_2)=\sin(\theta_2-\theta_3)=\sin(\theta_3-\theta_1)=0.}$

Consider the solution with $\cos(\theta_2-\theta_3)=1, \cos(\theta_1-\theta_2)=\cos(\theta_3-\theta_1)=-1$ as an example. Towards the diagonalization, we redefine $\psi_i'$ as

$$\psi_1' \equiv \frac{2}{\sqrt{6}}\psi_1+\frac{1}{\sqrt{6}}\psi_2+\frac{1}{\sqrt{6}}\psi_3, \tag{159}$$

$$\psi_2' \equiv -\frac{1}{\sqrt{2}}\psi_2+\frac{1}{\sqrt{2}}\psi_3, \tag{160}$$

$$\psi_3' \equiv -\frac{1}{\sqrt{3}}\psi_1+\frac{1}{\sqrt{3}}\psi_2+\frac{1}{\sqrt{3}}\psi_3. \tag{161}$$

With this definition and (146), (114) becomes

$$S=\int d^4x\sqrt{-g}\Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-|D_\mu\psi_1'|^2-|D_\mu\psi_2'|^2-|D_\mu\psi_3'|^2 \\ -\left(m_1^2-\frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right)\psi_1'^2-\left(m_1^2-\frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}}\right)\psi_2'^2-\left(m_1^2+\frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}}+\frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right)\psi_3'^2 \\ -\eta(\psi_1^4+\psi_2^4+\psi_3^4)\Big]. \tag{162}$$

Since rigorous proof of the existence of solutions is difficult, we assume that there are only four types of solutions with $\eta=0$:

- $\psi_2'\neq 0, \psi_1'=\psi_3'=0$ or $\psi_1'\neq 0, \psi_2'=\psi_3'=0$.
- $\psi_1'=A\psi_2'\neq 0, \psi_3'=0$ ($A$ is constant).
- $\psi_1'=\psi_2'=0, \psi_3'\neq 0$.
- $\psi_1'=\psi_2'=\psi_3'=0$.

This assumption is reasonable because the scalar fields with different mass squared do not become nonzero together in a two-scalar model

[39] as we explained before. With this assumption, we can derive the solution with $\eta = 0$ as

$$\psi_1' = \psi, \psi_2' = A\psi, \psi_3' = 0, \tag{163}$$

$$\psi_1 = \frac{2}{\sqrt{6}}\psi, \psi_2 = \frac{1-\sqrt{3}A}{\sqrt{6}}\psi, \psi_3 = \frac{1+\sqrt{3}A}{\sqrt{6}}\psi, \tag{164}$$

$$\psi > 0, -\frac{1}{\sqrt{3}} < A < \frac{1}{\sqrt{3}}. \tag{165}$$

Then, we consider whether (164) is the solution with $\eta > 0$. Substituting (146) and (164) to (143), (144) and (145), we get

$$\nabla_\mu \nabla^\mu \psi - A_\mu A^\mu \psi - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi - \frac{4}{3}\eta\psi^3 = 0, \tag{166}$$

$$\nabla_\mu \nabla^\mu \psi - A_\mu A^\mu \psi - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi - \frac{(1-\sqrt{3}A)^2}{3}\eta\psi^3 = 0, \tag{167}$$

$$\nabla_\mu \nabla^\mu \psi - A_\mu A^\mu \psi - \left( m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} \right) \psi - \frac{(1+\sqrt{3}A)^2}{3}\eta\psi^3 = 0. \tag{168}$$

Since these equations are not the same with $\eta > 0$, (164) is not the solution with $\eta > 0$. In the case of $\cos(\theta_1 - \theta_2) = 1, \cos(\theta_2 - \theta_3) = \cos(\theta_3 - \theta_1) = -1$ and $\cos(\theta_3 - \theta_1) = 1, \cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_3) = -1$, we can obtain the same conclusion.

In the case of $\cos(\theta_1 - \theta_2) = \cos(\theta_2 - \theta_3) = \cos(\theta_3 - \theta_1) = 1$, we can find the solution as

$$\psi_1 = \psi_2 = \psi_3. \tag{169}$$

In fact, by substituting (146) and (169) to (143), (144) and (145), we get

$$\nabla_\mu \nabla^\mu \psi_1 - A_\mu A^\mu \psi_1 - \left( m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \right) \psi_1 - 2\eta\psi_1^3 = 0, \tag{170}$$

$$\nabla_\mu \nabla^\mu \psi_2 - A_\mu A^\mu \psi_2 - \left( m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \right) \psi_2 - 2\eta\psi_2^3 = 0, \tag{171}$$

$$\nabla_\mu \nabla^\mu \psi_3 - A_\mu A^\mu \psi_3 - \left( m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} \right) \psi_3 - 2\eta\psi_3^3 = 0, \tag{172}$$

and we obtain the on-shell action as

$$S_{\text{on-shell}} = \int d^4x\sqrt{-g}\Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - 3|D_\mu\psi_1|^2 - 3\left(m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right)\psi_1^2 - 3\eta\psi_1^4\Big]$$
$$= \int d^4x\sqrt{-g}\Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |D_\mu\psi'|^2 - \left(m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}}\right)\psi'^2 - \frac{\eta}{3}\psi'^4\Big], \tag{173}$$

where a new scalar field $\psi'$ is redefined as

$$\psi'^2 \equiv 3\psi_1^2, \tag{174}$$

and the coefficient of the quartic potential in (173) is $\eta/3$.

Third, we compute the free energy of the three solutions by a numerical computation. For the numerical calculation, we set the parameters as $m_1^2 + \frac{\epsilon_{31}\epsilon_{23}}{\epsilon_{12}} + \frac{\epsilon_{23}\epsilon_{12}}{\epsilon_{31}} = 0, m_1^2 - \frac{\epsilon_{12}\epsilon_{31}}{\epsilon_{23}} = -2$ and $\eta = 1/2$. We impose boundary conditions as

$$\psi'(z) = \langle\mathcal{O}_2\rangle z^2 \ \text{(Sol.2 and Sol.3a)}, \ \ \psi'(z) = \langle\mathcal{O}_3\rangle z^3 \ \text{(Sol.3b)}, \ \ A_t(z) = \mu - \rho z, \tag{175}$$

and we fix $\mu = 1$ as section 3.2. We can derive the equations of motion of $\psi'$ from (149), (157) and (173).

Figure 4 shows a plot of the three solutions. Sol.2, Sol.3a and Sol.3b correspond to the blue, red and green curves, respectively. We define $T_c$ as a critical temperature of Sol.3a. Figure 5 shows a plot of the free energy density $S_E/\int dtdxdy$. In order to calculate $S_E$, we use the Euclidian action of (149), (157) and (173). From this figure, we can conclude that the free energy of Sol.3a (the red curve) is minimum . Therefore, the solutions (155) and (156) are the chiral ground states if $\eta > 0$ and (146) hold.

## 4.5 Short summary

Summarizing the above, we have analyzed the three holographic superconductor model and its equations of motion. This holographic model can describes the frustration between the scalar fields because of the Josephson coupling such as $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$. We have studied the classical solutions of our holographic model with $\eta > 0$ and (146). Especially, we have found that there are the chiral ground states (155) and (156). The degeneracy and nonzero chirality of the ground states are strongly related to the frustration. Because there is a holographic model [31] which has the property

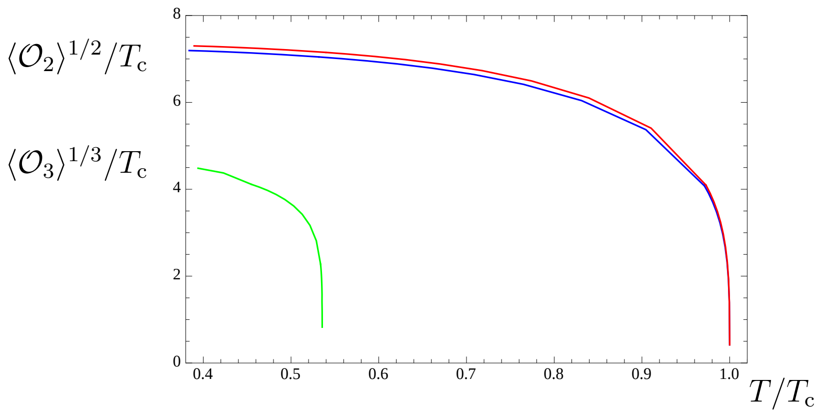


Figure 4: Plot of the coefficients of the scalar field in each solution. correspond to The blue, red and green curves represent Sol.2, Sol.3a and Sol.3b, respectively. $T_{\rm c}$ is a critical temperature of Sol.3a.

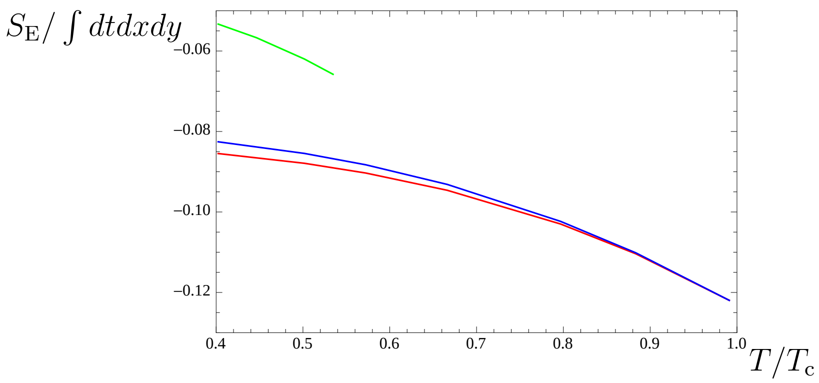


Figure 5: Plot of the free energy density. The free energy of Sol.3a which corresponds to the red curve is minimum.

of the cuprate high-temperature superconductor, our holographic model also may predict the property of frustrated superconductors in strongly correlated systems. Analysis with other parameters values or other ansatz is interesting as a future work in order to find a rich phase structure.

# Part II

# Correspondence between conformal partial wave and geodesic Witten diagram

## 5 Review of conformal partial wave and geodesic Witten diagram

In this section, we review conformal partial wave (CPW) and geodesic Witten diagram (GWD). In particular, we review the correspondence between CPW and GWD with external scalar fields in [18]. This review is also based on [22, 40, 41, 42, 43] and our paper [17].

### 5.1 Conformal partial wave

Conformal partial wave (CPW) is a basis of correlation functions of CFT. It can be determined by conformal symmetry only and does not depend on the detail of the theory.

#### 5.1.1 Definition

Consider a $d$-dimensional Euclidean CFT. It is well-known that four point functions of primary fields $\mathcal{O}_i$ in CFT are expanded by CPW $W_{\Delta,\ell}(x_i)$,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}} C_{12\mathcal{O}}{C^{\mathcal{O}}}_{34} W_{\Delta,\ell}(x_i), \tag{176}$$

where $\mathcal{O}$ is a primary field with conformal dimension $\Delta$ and spin $\ell$, $C_{12\mathcal{O}}$ and ${C^{\mathcal{O}}}_{34}$ are the operator product expansion (OPE) coefficients. The OPE coefficients are related to the coefficients of three point functions and they depend on the detail of the theory. On the other hand, CPW $W_{\Delta,\ell}(x_i)$ is defined as a basis of correlation functions such as (176) and it does not depend on the detail of the theory. This universality of CPW comes from constraints of correlation functions in CFT.

#### 5.1.2 Property

CPW has the following three properties [40]. We can also define CPW as a function which has these properties. However, we can not determine the normalization of CPW by using these property only. For simplicity, we consider scalar primary fields $\mathcal{O}_i$ with conformal dimension $\Delta_i$.

1. Transformation law under conformal transformation

   By definition, the scalar primary fields $\mathcal{O}_i$ with conformal dimension $\Delta_i$ are transformed under conformal transformation as

$$\mathcal{O}_i'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_i}{d}} \mathcal{O}_i(x), \tag{177}$$

   where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the conformal transformation. From this transformation law of the scalar primary operators, CPW of $\mathcal{O}_i$ is transformed under the conformal transformation as

$$W_{\Delta,\ell}(x_i') = \left(\Pi_{i=1}^4 \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_i}{d}}\right) W_{\Delta,\ell}(x_i). \tag{178}$$

2. Solution of the conformal Casimir equation

   Since the conformal group in $d$-dimension is equivalent to the Lorentz group in $d+2$-dimension as $SO(d+1,1)$, we use the Lorentz generators $L_{AB}$ in $d+2$-dimension instead of the generators of conformal symmetry. $L_{AB}$ acts on a local field $\mathcal{O}(x)$ as

$$[L_{AB}, \mathcal{O}(x)] = (L_x)_{AB}\mathcal{O}(x), \tag{179}$$

   where $(L_x)_{AB}$ is the differential operator of $x$ and its explicit form depends on $\mathcal{O}(x)$.

   For simplicity, consider a scalar exchange CPW $W_{\Delta,0}(x_i)$. By inserting the complete set into (176), we can express $W_{\Delta,0}(x_i)$ as

$$W_{\Delta,0}(x_i) = \frac{1}{C_{12\mathcal{O}}C^{\mathcal{O}}{}_{34}} \sum_\alpha \langle 0| \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) |\alpha\rangle \langle\alpha| \mathcal{O}_3(x_3)\mathcal{O}_4(x_4) |0\rangle\,, \tag{180}$$

   where $|\alpha\rangle$s are states for the conformal family of $O(x)$ and $\mathcal{O}(x)$ is a scalar primary field with conformal dimension $\Delta$. Consider the quadratic Casimir of the Lorentz group $\frac{1}{2}L_{AB}L^{AB}$. $|\alpha\rangle$s are the eigenstates of $\frac{1}{2}L_{AB}L^{AB}$ and their eigenvalues are [44]

$$\frac{1}{2}L_{AB}L^{AB} |\alpha\rangle = C_2(\Delta,0) |\alpha\rangle = -\Delta(\Delta-d) |\alpha\rangle\,. \tag{181}$$

Let us derive the conformal Casimir equation for CPW. By using (179) and the conformal invariance of the vacuum, we obtain

$$\begin{aligned}
&\frac{1}{2}(L^{(0)}_{x_1}+L^{(0)}_{x_2})_{AB}(L^{(0)}_{x_1}+L^{(0)}_{x_2})^{AB}\left\langle 0\right| \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\left|\alpha\right\rangle \\
=&\frac{1}{2}\left\langle 0\right| [L^{AB},[L_{AB},\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)]]\left|\alpha\right\rangle \\
=&\left\langle 0\right| \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\frac{1}{2}L_{AB}L^{AB}\left|\alpha\right\rangle ,
\end{aligned} \tag{182}$$

where $L^{(0)}_x$ is the differential operator for a scalar primary field. From (180), (181) and (182), we obtain a second order differential equation

$$\frac{1}{2}(L^{(0)}_{x_1}+L^{(0)}_{x_2})_{AB}(L^{(0)}_{x_1}+L^{(0)}_{x_2})^{AB}W_{\Delta,0}(x_i)=C_2(\Delta,0)W_{\Delta,0}(x_i). \tag{183}$$

This equation is the conformal Casimir equation for CPW.

3. Boundary condition

   Since the conformal Casimir equation is a second order differential equation, we need a boundary condition to obtain the solution. This boundary condition can be determined by OPE. Consider OPE of the scalar primary fields $\mathcal{O}_i(x_i)$ as

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\sim C_{12\mathcal{O}}|x_{12}|^{\Delta-\Delta_1-\Delta_2}\mathcal{O}(x_2)+\cdots , \tag{184}$$

   where $x_{12}\equiv x_1-x_2$. The power of $x_{12}$ in the OPE is determined by the conformal dimensions of the scalar fields. From this OPE, we obtain the boundary condition of CPW as

$$\lim_{x_{12}\to 0}W_{\Delta,0}(x_i)\propto\frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}}. \tag{185}$$

#### 5.1.3 How to construct CPW

In this subsection, we introduce some methods to construct CPW explicitly.

1. Solving the conformal Casimir equation directly

   First method is solving the conformal Casimir equation and finding its solution which satisfies the properties of CPW in the last subsection. For example, in [45], the authors solved the conformal Casimir equation and found an expression of CPW in terms of hypergeometric functions.

2. Shadow formalism

   Second method is so-called shadow formalism [47, 48, 49, 50, 51]. Define the shadow operator $\widetilde{\mathcal{O}}(x)$ of a scalar primary operator $\mathcal{O}(x)$ with conformal dimension $\Delta$ as

   $$\widetilde{\mathcal{O}}(x) \equiv \int d^d x' \frac{\mathcal{O}(x')}{|x'-x|^{2(d-\Delta)}}. \tag{186}$$

   This shadow operator $\widetilde{\mathcal{O}}(x)$ is a scalar operator with conformal dimension $d-\Delta$. Let us construct an integral

   $$\int d^d x \mathcal{O}(x)|0\rangle\langle 0|\widetilde{\mathcal{O}}(x). \tag{187}$$

   and insert (187) into (176). Then, we obtain

   $$\int d^d x \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \propto W_{\Delta,0}(x_i)+K_{\mathcal{O}}W_{d-\Delta,0}(x_i), \tag{188}$$

   where $K_{\mathcal{O}}$ is a constant. Since the integral in (188) satisfies the conformal Casimir equation and conformal dimension of (187) is zero, we can obtain CPW by computing the integral in (188). $W_{d-\Delta,0}(x_i)$ is the shadow CPW and it is a solution of the conformal Casimir equation with a boundary condition different from $W_{\Delta,0}(x_i)$. In particular, the authors of [50, 51] derived a double integral representation for the scalar CPW by using the shadow formalism.

3. Construction of the amplitude of geodesic Witten digram

   Third method is constructing the amplitude of GWD which we will define in the next subsection. In [18], the authors proposed GWD as the gravity dual of CPW. In particular, they constructed and computed the amplitude of GWD with external scalar fields and showed the correspondence between the amplitude of GWD and CPW up to normalization. It is expected that this correspondence holds for any fields such as symmetric tensor fields, antisymmetric tensor fields, and fermionic fields. For justification of the construction, it is better to confirm that the amplitude of GWD which we constructed satisfies the properties of CPW.

## 5.2 Geodesic Witten diagram

In this subsection, we explain what is GWD.

GWD is a diagram which describes the reaction process on AdS spacetime such as Witten diagram. However, the integration range in GWD is different from the integration range in the Witten diagram. For the Witten diagram, we integrate interaction points over all AdS spacetime. On the other hand, for GWD, we integrate interaction points over the geodesics between external fields at the AdS boundary. Witten diagram was proposed as the gravity dual of correlation functions of CFT in [20], and GWD was proposed as the gravity dual of CPW in [18].

For example, consider a scalar exchange GWD with four external scalar fields as Figure 6. We define the amplitude $\mathcal{W}_{\Delta,0}(x_i;\Delta_i)$ of this GWD as

$$\begin{aligned}\mathcal{W}_{\Delta,0}(x_i;\Delta_i) \equiv \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta)\\ \times G_{b\partial}(y(\lambda'),x_3;\Delta_3)G_{b\partial}(y(\lambda'),x_4;\Delta_4), \qquad (189)\end{aligned}$$

where $\lambda$ and $\lambda'$ are proper time coordinates of the geodesics $\gamma_{12}$ and $\gamma_{34}$. $\gamma_{ij}$ are the geodesics between boundary points $x_i$ and $x_j$. $y(\lambda)$ and $y(\lambda')$ are coordinates of $\gamma_{12}$ and $\gamma_{34}$. $G_{b\partial}$ and $G_{bb}$ are the bulk-boundary propagator and the bulk-bulk propagator on AdS spacetime, respectively, (see, for example, [22, 52])

$$G_{b\partial}(y,x_i;\Delta_i) \equiv \left(\frac{u}{u^2+|x-x_i|^2}\right)^{\Delta_i}, \qquad (190)$$

$$G_{bb}(y,y';\Delta) \equiv \xi^{\Delta}{}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2},\Delta+1-\frac{d}{2};\xi^2\right), \qquad (191)$$

$$\xi \equiv \frac{2uu'}{u^2+u'^2+|x-x'|^2}. \qquad (192)$$

We use $d+1$-dimensional AdS metric as

$$ds^2 = \frac{du^2+dx^a dx^a}{u^2}, \qquad y^\mu = \{u, x^a\}. \qquad (193)$$

The explicit form of $y(\lambda)$ is

$$u(\lambda) = \frac{|x_1-x_2|}{2\cosh\lambda}, \qquad (194)$$

$$x^a(\lambda) = \frac{x_1^a+x_2^a}{2} - \frac{x_1^a-x_2^a}{2}\tanh\lambda. \qquad (195)$$

We note that (189) coincides with a double integral expression of CPW in [50] ($d=4$).

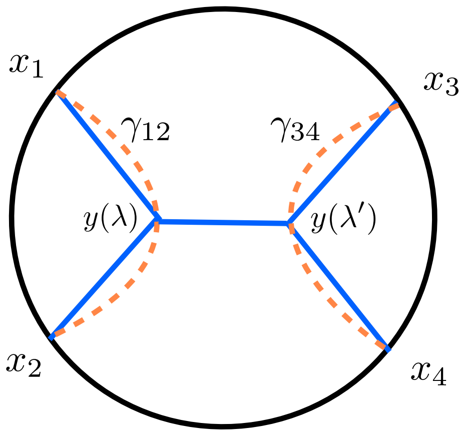


Figure 6: Scalar exchange geodesic Witten diagram with four external scalar fields. The orange dot curves describe the geodesics $\gamma_{ij}$ between the boundary points $x_i$ and $x_j$. The blue straight lines represent the scalar propagator. The interaction points are integrated over $y$ on the geodesics $\gamma_{ij}$.

## 5.3 Embedding formalism

In order to check the correspondence between CPW and GWD, embedding formalism is useful (see, for example, [53, 54, 55, 56, 57, 58, 59, 60, 61, 62]) . In this subsection, we review this formalism.

Since conformal symmetry in $d$-dimension and isometry of $\mathrm{AdS}_{d+1}$ are equivalent to Lorentz symmetry in $d$+2-dimension, we can use $d$+2-dimensional embedding Minkowski spacetime to express a $d$-dimensional CFT and a theory on $\mathrm{AdS}_{d+1}$. This method is called the embedding formalism.

Let us embed $\mathrm{AdS}_{d+1}$ and $d$-dimensional flat space $\mathbb{R}^d$ into $d$+2-dimensional Minkowski spacetime $\mathbb{R}^{1,d+1}$ explicitly. $\mathrm{AdS}_{d+1}$ coordinates $y^\mu = \{u, x^a\}$ are embedded into $Y^A$ such that

$$Y^A \equiv (Y^+, Y^-, Y^a) \tag{196}$$

$$= \frac{1}{u}(1, u^2 + x^2, x^a) \tag{197}$$

where $Y^A$ are coordinates of $\mathbb{R}^{1,d+1}$. On the other hand, $d$-dimensional flat space $\mathbb{R}^d$ at the AdS boundary is embedded into $X^A$ such that

$$\begin{aligned} X^A &\equiv (X^+, X^-, X^a) \\ &= (1, x^2, x^a), \end{aligned} \tag{198}$$

where $X^A$ are also coordinates of $\mathbb{R}^{1,d+1}$.

In the embedding formalism, we introduce fields on $d+2$-dimensional Minkowski spacetime $\mathbb{R}^{1,d+1}$ and impose constraints of the fields for the $d$-dimensional CFT and the theory on AdS$_{d+1}$. First, we impose the transverse condition to traceless symmetric tensor fields as

$$X_{A_1} T_{\partial}^{A_1 A_2 \cdots A_l}(X) = 0, \quad Y_{A_1} T_{b}^{A_1 A_2 \cdots A_l}(Y) = 0, \tag{199}$$

where $T_\partial$ is a tensor field in the boundary CFT and $T_b$ is a tensor field in AdS$_{d+1}$. Second, we impose the condition for primary traceless symmetric tensor fields in the boundary CFT as

$$T_{\partial}^{A_1 A_2 \cdots A_l}(\lambda X) = \lambda^{-\Delta} T_{\partial}^{A_1 A_2 \cdots A_l}(X). \tag{200}$$

Moreover, we introduce the index-free notation [53, 54, 61] to express the tensor fields simply. In this notation, we use auxiliary fields $Z$ and $W$ to contract vector indices,

$$T_\partial(X; Z) \equiv Z_{A_1} \cdots Z_{A_l} T_{\partial}^{A_1 A_2 \cdots A_l}(X), \quad T_b(Y; W) \equiv W_{A_1} \cdots W_{A_l} T_{b}^{A_1 A_2 \cdots A_l}(Y). \tag{201}$$

We can restrict $Z$ to $Z^2 = Z \cdot X = 0$ and $W$ to $W^2 = W \cdot Y = 0$ [54].

The propagator in the embedding space [54] is expressed as

$$G_{b\partial}(X, Y; \Delta) \equiv \frac{1}{(-2X \cdot Y)^{\Delta}}, \tag{202}$$

$$G_{bb}(Y_1, Y_2; \Delta) \equiv \xi^{\Delta}{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta + 1 - \frac{d}{2}; \xi^2\right), \tag{203}$$

$$\xi^{-1} \equiv -Y_1 \cdot Y_2. \tag{204}$$

By using these propagators, we can express the amplitude of the scalar GWD (189) as

$$\begin{aligned}\mathcal{W}_{\Delta,0}(X_i; \Delta_i) = \int_{-\infty}^{\infty} d\lambda' \left[\int_{-\infty}^{\infty} d\lambda\, G_{b\partial}(Y_1(\lambda), X_1, \Delta_1)\, G_{b\partial}(Y_1(\lambda), X_2; \Delta_2)\, G_{bb}(Y_1(\lambda), Y_2(\lambda'); \Delta)\right] \\ \times G_{b\partial}(Y_2(\lambda'), X_3; \Delta_3)\, G_{b\partial}(Y_2(\lambda'), X_4; \Delta_4),\end{aligned} \tag{205}$$

where $Y_{1A}(\lambda)$ and $Y_2(\lambda')$ are

$$Y_{1A}(\lambda) = \frac{e^{-\lambda} X_{1A} + e^{\lambda} X_{2A}}{\sqrt{-2X_1 \cdot X_2}}, \quad Y_{2A}(\lambda') = \frac{e^{-\lambda'} X_{3A} + e^{\lambda'} X_{4A}}{\sqrt{-2X_3 \cdot X_4}}. \tag{206}$$

## 5.4 Correspondence between scalar conformal partial wave and scalar geodesic Witten diagram

In this subsection, we show that the amplitude of the scalar GWD (189), (205) satisfies the properties of the scalar CPW based on [18]. This check means the correspondence between the scalar CPW and the scalar GWD up to normalization.

1. Transformation law under conformal transformation

   For simplicity, consider a scale transformation

   $$x' = \alpha x. \tag{207}$$

   Under this transformation, the scalar CPW is transformed as

   $$W_{\Delta,0}(x_i') = \left(\Pi_{i=1}^4 \alpha^{-\Delta_i}\right) W_{\Delta,0}(x_i). \tag{208}$$

   The isometric transformation of AdS which corresponds to the scale transformation is

   $$u' = \alpha u, \qquad x' = \alpha x. \tag{209}$$

   By definition, the bulk-boundary propagator $G_{b\partial}$ (190) is transformed under this isometric transformation as

   $$G_{b\partial}(y', x_i'; \Delta_i) = \alpha^{-\Delta_i} G_{b\partial}(y, x_i; \Delta_i). \tag{210}$$

   The bulk-bulk propagator $G_{bb}$ (191) is invariant under the isometric transformation since $\xi$ is invariant under the isometric transformation. In addition, the geodesics between the boundary points are transformed to the geodesics between the new boundary points under the isometric transformation. This proposition can be proved from the facts that the geodesic is the shortest path between two points and the length of a path is invariant under the isometric transformation. Therefore, the amplitude of the scalar GWD (189) is transformed as

   $$\begin{aligned}\mathcal{W}_{\Delta,0}(x_i'; \Delta_i) = \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\partial}(y'(\lambda), x_1'; \Delta_1) G_{b\partial}(y'(\lambda), x_2'; \Delta_2) G_{bb}(y'(\lambda), y'(\lambda'); \Delta) \\ \times G_{b\partial}(y'(\lambda'), x_3'; \Delta_3) G_{b\partial}(y'(\lambda'), x_4'; \Delta_4) \end{aligned} \tag{211}$$

   $$= \left(\Pi_{i=1}^4 \alpha^{-\Delta_i}\right) \mathcal{W}_{\Delta,0}(x_i; \Delta_i). \tag{212}$$

   This transformation law is consistent with the transformation law of the scalar CPW (208). We can also show the correspondence of the transformation law under the other conformal transformation.

2. Solution of the conformal Casimir equation

In the embedding formalism, the Lorentz generators are expressed as

$$(L_X^{(\ell)})_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} + S_{AB}^{(\ell)}, \tag{213}$$

where $S_{AB}^{(\ell)}$ with $\ell = 0, 1$ are

$$S_{AB}^{(0)} = 0, \quad (S_{AB}^{(1)})_{CD} = \eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}. \tag{214}$$

For the index-free notation, we define $S_{AB}^{(1)}$ as

$$S_{AB}^{(1)} \equiv Z_C (S_{AB}^{(1)})^{CD} \frac{\partial}{\partial Z^D} = Z_A \frac{\partial}{\partial Z^B} - Z_B \frac{\partial}{\partial Z^A}. \tag{215}$$

We will use the fact that the Laplacian on AdS and the quadratic Casimir are related [63] as

$$\frac{1}{2}(L_Y^{(0)})_{AB}(L_Y^{(0)})^{AB}\, f(Y) = -\nabla_Y^2 f(Y), \tag{216}$$

where $f(Y)$ is an arbitary scalar function on AdS and the covariant derivative in the embedding formalism is [54]

$$\nabla_A \equiv \frac{\partial}{\partial Y^A} + Y_A \left( Y \cdot \frac{\partial}{\partial Y} \right) + W_A \left( Y \cdot \frac{\partial}{\partial W} \right). \tag{217}$$

Then, let us show that the scalar GWD $\mathcal{W}_{\Delta,0}(x_i; \Delta_i)$ satisfies the conformal Casimir equation. Define $F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2)$ as

$$F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2) \equiv \int_{-\infty}^{\infty} d\lambda\, G_{b\partial}(Y_1, X_1; \Delta_1)\, G_{b\partial}(Y_1, X_2; \Delta_1)\, G_{bb}(Y_1, Y_2; \Delta), \tag{218}$$

$$\mathcal{W}_{\Delta,0}(x_i; \Delta_i) = \int_{-\infty}^{\infty} d\lambda'\, F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2(\lambda')) G_{b\partial}(Y_2(\lambda'), X_3; \Delta_3)\, G_{b\partial}(Y_2(\lambda'), X_4; \Delta_4). \tag{219}$$

Since $F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2)$ does not have a vector index, it is invariant under the $SO(d + 1, 1)$ rotation. Therefore, we obtain

$$(L_{X_1}^{(0)} + L_{X_2}^{(0)} + L_{Y_2}^{(0)})_{AB}\, F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2) = 0. \tag{220}$$

From (216) and (220), we get

$$\frac{1}{2}(L_{X_1}^{(0)}+L_{X_2}^{(0)})_{AB}(L_{X_1}^{(0)}+L_{X_2}^{(0)})^{AB}\, F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2) = -\nabla_{Y_2}^2\, F_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2). \tag{221}$$

Because $G_{bb}(Y_1,Y_2;\Delta)$ is a eigenfunction of $\nabla^2_{Y_2}$ and its eigenvalue is $\Delta(\Delta-d)$ [64], we get

$$\frac{1}{2}(L^{(0)}_{X_1}+L^{(0)}_{X_2})_{AB}(L^{(0)}_{X_1}+L^{(0)}_{X_2})^{AB}\,\mathcal{W}_{\Delta,0}(x_i;\Delta_i)=-\Delta(\Delta-d)\mathcal{W}_{\Delta,0}(x_i;\Delta_i). \tag{222}$$

For this derivation, we assume that the two geodesics do not intersect each other. This assumption is related to the convergence of the OPE and $\mathcal{W}_{\Delta,0}(x_i;\Delta_i)$. (222) is just the conformal Casimir equation (183).

3. Boundary condition

   A the limit of $x_{12}\to 0$, the propagators behave as

$$\lim_{x_{12}\to 0}G_{b\partial}(y,x_1;\Delta_1)\propto|x_{12}|^{-\Delta_1},\quad \lim_{x_{12}\to 0}G_{b\partial}(y,x_2;\Delta_2)\propto|x_{12}|^{-\Delta_2}, \tag{223}$$

$$\lim_{x_{12}\to 0}\xi\propto|x_{12}|,\quad \lim_{x_{12}\to 0}G_{bb}(y,y';\Delta)\propto|x_{12}|^{\Delta}. \tag{224}$$

   Therefore, we get

$$\begin{aligned}\lim_{x_{12}\to 0}\mathcal{W}_{\Delta,0}(x_i;\Delta_i)&\propto\lim_{x_{12}\to 0}G_{b\partial}(y,x_1;\Delta_1)G_{b\partial}(y,x_2;\Delta_2)G_{bb}(y,y';\Delta)\\&\propto\frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}}.\end{aligned} \tag{225}$$

   Thus, the boundary condition of $\mathcal{W}_{\Delta,0}(x_i;\Delta_i)$ is the same as the boundary condition of the scalar CPW (185).

## 5.5 Merit of geodesic Witten diagram for conformal partial wave

In this subsection, we list merits of GWD for CPW.

1. The properties of CPW is manifest in terms of the AdS propagator.

   As we saw in the previous subsection, we can show that the expression of CPW which constructed from the amplitude of GWD satisfies the properties of CPW in terms of the AdS propagator.

2. A systematic way to find an expression of CPW by the AdS propagator

   If we know the explicit form of the AdS propagator and three point interactions in AdS spacetime, we can define the amplitude of GWD systematically. Therefore, the construction of the amplitude of GWD is a systematic way to find an expression of CPW.

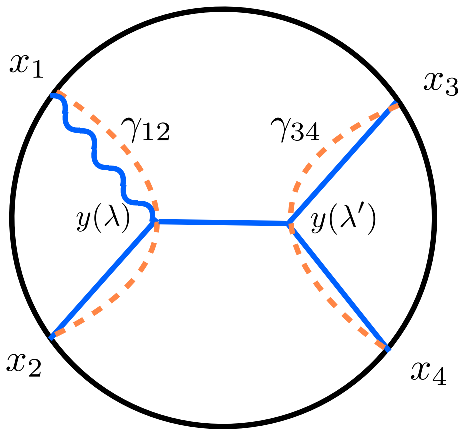


Figure 7: Scalar exchange geodesic Witten diagram with an external spin-$n$ field and three external scalar fields. The blue wave line represents the propagator of the spin-$n$ field. The meaning of the other lines is explained in the caption of figure 6.

3. Calcualtion of the amplitude is familiar for theoretical particle physicist.

   Any theoretical particle physicist learns how to calculate the amplitude of Feynman diagram. Therefore, calculation of the amplitude by using the propagator is friendly for theoretical particle physicist. Moreover, we may apply a numerical technic to calculate Feynman diagram for calculation of GWD.

# 6 Geodesic Witten diagram with an external spinning field

In this section, we construct the amplitude of scalar exchange GWD with an external spinning field as figure 7 and show that its amplitude corresponds to conformal partial wave (CPW) up to normalization based on our paper [17].

## 6.1 Motivation

Since CPW is an important and fundamental object in CFT, deriving an expression of CPW is an interesting research project of CFT (see, for example, [44, 45, 46, 47, 49, 48, 50, 51]). In particular, the CPW which includes external operators with spin is important because the stress tensor plays a central role in CFT. However, in [18], the authors considered GWD with external scalar fields only as the gravity dual of CPW. Therefore, towards the generalization to external fields in any representation, we study the correspondence between CPW and GWD with an external spinning field as the simplest case.

## 6.2 Direct proof of the correspondence with an external spin-1 field

In this subsection, we derive an expression of CPW with an external spin-1 field explicitly by using (189). We check that this expression corresponds to the amplitude of GWD.

In order to derive the expression of CPW, we use the shadow formalism. In preparation for the shadow formalism, consider the relational expression between three point functions. In a CFT, the forms of the three point functions are determined by conformal symmetry,

$$
\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{226}
$$

$$
\langle \mathcal{J}^a(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}} \times \left(\frac{x_{12}^a}{|x_{12}|^2} - \frac{x_{13}^a}{|x_{13}|^2}\right), \tag{227}
$$

where $\mathcal{O}_i(x_i)$ are scalar primary fields with conformal dimension $\Delta_i$ and $\mathcal{J}^a(x_1)$ is a spin-1 primary field with conformal dimension $\Delta_1+1$. We ignore the OPE coefficients since we would like to show the correspondence up to normalization. We can derive the relation between (226) and (227)

$$
\left(\frac{\partial}{\partial x_1^a} + \frac{2\Delta_1 (x_{12})_a}{|x_{12}|^2}\right)\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = (\Delta_3+\Delta_1-\Delta_2)\langle \mathcal{J}_a(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle. \tag{228}
$$

From now, we denote CPW which includes four external primary fields with conformal dimension $\Delta_i$ and spin $\ell_i$ as $W_{\Delta,\ell}^{(\ell_1,\ell_2,\ell_3,\ell_4)}(x_i;\Delta_i)$. $\Delta$ and $\ell$ represent conformal dimension and spin of an exchanging primary operator. Similarly, we denote the amplitude of GWD as $\mathcal{W}_{\Delta,\ell}^{(\ell_1,\ell_2,\ell_3,\ell_4)}(x_i;\Delta_i)$. From

(188) and (228), we obtain (up to normalization)

$$\begin{aligned}
\left(W^{(1,0,0,0)}_{\Delta,0}(x_i;\widetilde{\Delta}_i)\right)_a &= \int d^dx \langle \mathcal{J}_a(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle|_{\mathrm{BC}} \\
&= \left(\frac{\partial}{\partial x_1^a}+\frac{2\Delta_1(x_{12})_a}{|x_{12}|^2}\right)\int d^dx \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}(x)\rangle\langle\widetilde{\mathcal{O}}(x)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle|_{\mathrm{BC}} \\
&= \left(\frac{\partial}{\partial x_1^a}+\frac{2\Delta_1(x_{12})_a}{|x_{12}|^2}\right) W^{(0,0,0,0)}_{\Delta,0}(x_i;\Delta_i) \\
&= \left(\frac{\partial}{\partial x_1^a}+\frac{2\Delta_1(x_{12})_a}{|x_{12}|^2}\right) \mathcal{W}^{(0,0,0,0)}_{\Delta,0}(x_i;\Delta_i), \qquad (229)
\end{aligned}$$

where $\widetilde{\Delta}_i = \Delta_i + \delta_{i1}$. $|_{\mathrm{BC}}$ means imposing the boundary condition for the CPW to ignore the shadow CPW and the explicit boundary conditions for CPW are

$$\lim_{x_{12}\to 0} W^{(0,0,0,0)}_{\Delta,0}(x_i;\Delta_i) \propto \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}}, \qquad (230)$$

$$\lim_{x_{12}\to 0} \left(W^{(1,0,0,0)}_{\Delta,0}(x_i;\widetilde{\Delta}_i)\right)_a \propto \frac{1}{|x_{12}|^{\Delta_1+\Delta_2-\Delta}} \times \frac{(x_{12})_a}{|x_{12}|^2}. \qquad (231)$$

Thus, we have obtained the expression of CPW with an external spin-1 field (229) by using (189).

However, the relation between the expression (229) of CPW and the spin-1 AdS propagator is not manifest. In order to examine it, we deform (229) by using the spin-1 AdS propagator. In particular, we will show

$$\begin{aligned}
&\left(\frac{\partial}{\partial x_1^a}+2\Delta_1\frac{(x_{12})_a}{|x_{12}|^2}\right)\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta) \\
&\qquad\qquad \times G_{b\partial}(y(\lambda'),x_3;\Delta_3)G_{b\partial}(y(\lambda'),x_4;\Delta_4) \\
&= \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \left(G^1_{b\partial}(y(\lambda),x_1;\Delta_1+1)\right)^\mu_a G_{b\partial}(y(\lambda),x_2;\Delta_2) \\
&\qquad\qquad \times u(\lambda)^2\frac{\partial}{\partial y^\mu(\lambda)}\left(G_{bb}(y(\lambda),y(\lambda');\Delta)\right) \\
&\qquad\qquad \times G_{b\partial}(y(\lambda'),x_3;\Delta_3)G_{b\partial}(y(\lambda'),x_4;\Delta_4), \qquad (232)
\end{aligned}$$

where $G^1_{b\partial}(y,x_1;\Delta_1+1)$ is the spin-1 bulk-boundary propagator (see, for example, [54, 65]),

$$\left(G^1_{b\partial}(y,x_1;\Delta_1+1)\right)^\mu_a \equiv \left(\frac{u}{u^2+|x-x_1|^2}\right)^{\Delta_1}\left(\frac{\delta^\mu_a}{u^2+|x-x_1|^2}-2\frac{(y-x_1)_a(y-x_1)^\mu}{(u^2+|x-x_1|^2)^2}\right). \qquad (233)$$

L.h.s of (232) is the explicit form of the last line in (229). On the other hand, r.h.s of (232) is a definition of the amplitude of GWD $\mathcal{W}^{(1,0,0,0)}_{\Delta,0}(x_i;\widetilde{\Delta}_i)$. For this definition, we introduce a three point interaction coefficient $u^2\frac{\partial}{\partial y^\mu}$ which is the usual coupling of the scalar QED such as $A_\mu g^{\mu\nu}\phi\partial_\nu\phi^\dagger$. Therefore, the relation (232) shows the correspondence between CPW and GWD with an external spin-1 field.

In order to prove (232), we transform

$$\frac{\partial}{\partial x_1^a}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta). \tag{234}$$

From (194), (195) and the definitions of the propagators, we get

$$\begin{aligned}
&\frac{\partial}{\partial x_1^a}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta)\\
=&-(\Delta_1+\Delta_2)\frac{(x_{12})_a}{|x_{12}|^2}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta)\\
&+\int_{-\infty}^{\infty}d\lambda\left(G^1_{b\partial}(y(\lambda),x_1;\Delta_1+1)\right)^{\mu}_{a}G_{b\partial}(y(\lambda),x_2;\Delta_2)u(\lambda)^2\frac{\partial}{\partial y^\mu(\lambda)}\left(G_{bb}(y(\lambda),y(\lambda');\Delta)\right)\\
&-\frac{(x_{12})_a}{|x_{12}|^2}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)\frac{\partial y^\mu(\lambda)}{\partial\lambda}\frac{\partial}{\partial y^\mu(\lambda)}G_{bb}(y(\lambda),y(\lambda');\Delta).
\end{aligned}\tag{235}$$

By using integration by parts, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_1^a}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta)\\
=&-2\Delta_1\frac{(x_{12})_a}{|x_{12}|^2}\int_{-\infty}^{\infty}d\lambda G_{b\partial}(y(\lambda),x_1;\Delta_1)G_{b\partial}(y(\lambda),x_2;\Delta_2)G_{bb}(y(\lambda),y(\lambda');\Delta)\\
&+\int_{-\infty}^{\infty}d\lambda\left(G^1_{b\partial}(y(\lambda),x_1;\Delta_1+1)\right)^{\mu}_{a}G_{b\partial}(y(\lambda),x_2;\Delta_2)u(\lambda)^2\frac{\partial}{\partial y^\mu(\lambda)}\left(G_{bb}(y(\lambda),y(\lambda');\Delta)\right),
\end{aligned}\tag{236}$$

where we assume $|\Delta_1-\Delta_2|<\Delta$ to use the integration by parts. Then, we obtain the final expression (232) after integrating (236) by $\lambda'$ with multiplying $G_{b\partial}(y(\lambda'),x_3;\Delta_3)G_{b\partial}(y(\lambda'),x_4;\Delta_4)$.

In the last definition of the amplitude, we used the minimal coupling $A_\mu g^{\mu\nu}\phi\partial_\nu\phi^\dagger$ since it is invariant under the isometric transformation. We can also use other couplings which are invariant under the isometric transformation such as $A_\mu g^{\mu\nu}\phi\partial_\nu\nabla^2\phi^\dagger$. The amplitude of GWD with this coupling is

$$
\begin{aligned}
&\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \left(G^{1}_{b\partial}(y(\lambda), x_1; \Delta_1+1)\right)^{\mu}_{a} G_{b\partial}(y(\lambda), x_2; \Delta_2) \\
&\qquad\qquad \times u(\lambda)^2 \frac{\partial}{\partial y^{\mu}(\lambda)} \nabla^2 \left(G_{bb}(y(\lambda), y(\lambda'); \Delta)\right) \\
&\qquad\qquad \times G_{b\partial}(y(\lambda'), x_3; \Delta_3) G_{b\partial}(y(\lambda'), x_4; \Delta_4) \\
= \Delta(\Delta-d) &\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\lambda' \left(G^{1}_{b\partial}(y(\lambda), x_1; \Delta_1+1)\right)^{\mu}_{a} G_{b\partial}(y(\lambda), x_2; \Delta_2) \\
&\qquad\qquad \times u(\lambda)^2 \frac{\partial}{\partial y^{\mu}(\lambda)} \left(G_{bb}(y(\lambda), y(\lambda'); \Delta)\right) \\
&\qquad\qquad \times G_{b\partial}(y(\lambda'), x_3; \Delta_3) G_{b\partial}(y(\lambda'), x_4; \Delta_4),
\end{aligned} \tag{237}
$$

where we used [22]

$$
(\nabla^2 - \Delta(\Delta-d)) G_{bb}(y(\lambda), y(\lambda'); \Delta) = -\frac{2^{\Delta+1}\pi^{d/2}\Gamma(\Delta - \frac{d-2}{2})}{\Gamma(\Delta)} \frac{1}{\sqrt{g}} \delta^{(d+1)}(y(\lambda) - y(\lambda')). \tag{238}
$$

(237) is the same as the amplitude of GWD with the minimal coupling up to normalization. As this example shows, we should use the coupling which is invariant under the isometric transformation to define the amplitude of GWD because of the transformation law of CPW under the conformal transformation.

In conclusion, we have proven the relation (232) between CPW $W^{(1,0,0,0)}_{\Delta,0}(x_i; \widetilde{\Delta}_i)$ and the amplitude of the scalar exchange GWD $\mathcal{W}^{(1,0,0,0)}_{\Delta,0}(x_i; \widetilde{\Delta}_i)$ which includes an external spin-1 field and three scalar fields.

## 6.3 Proof of the correspondence by the conformal Casimir equation

Let us check that the amplitude of GWD $\mathcal{W}^{(1,0,0,0)}_{\Delta,0}(x_i; \widetilde{\Delta}_i)$ satisfies the conformal Casimir equation by using the embedding formalism.

In the embedding formalism, the bulk-boundary propagator with spin $J$ [54] is expressed as

$$
G^{J}_{b\partial}(X, Y; Z, W; \Delta) \equiv \frac{\left((-2X\cdot Y)(Z\cdot W) + 2(Z\cdot Y)(X\cdot W)\right)^J}{(-2X\cdot Y)^{\Delta+J}}. \tag{239}
$$

With these expressions of the propagators, the amplitude of GWD $\mathcal{W}^{(1,0,0,0)}_{\Delta,0}(X_i; Z_1; \Delta_i)$

in the embedding formalism is

$$\begin{aligned}&\mathcal{W}^{(1,0,0,0)}_{\Delta,0}(X_i;Z_1;\Delta_i)\\&=\int_{-\infty}^{\infty}d\lambda'\left[\int_{-\infty}^{\infty}d\lambda\, G^0_{b\partial}(Y_1(\lambda),X_2;\Delta_2)\left\{G^1_{b\partial}(Y_1(\lambda),X_1;Z_1,\nabla_{Y_1};\Delta_1)\,G_{bb}(Y_1(\lambda),Y_2(\lambda');\Delta)\right\}\right]\\&\times G^0_{b\partial}(Y_2(\lambda'),X_3;\Delta_3)\,G^0_{b\partial}(Y_2(\lambda'),X_4;\Delta_4),\end{aligned} \tag{240}$$

where we denote the scalar bulk-boundary propagator as $G^0_{b\partial}$. By a similar discussion as section 5.4, we can show that (240) satisfies the conformal Casimir equation,

$$\frac{1}{2}(L^{(1)}_{X_1}+L^{(0)}_{X_2})_{AB}(L^{(1)}_{X_1}+L^{(0)}_{X_2})^{AB}\,\mathcal{W}^{(1,0,0,0)}_{\Delta,0}=-\Delta(\Delta-d)\mathcal{W}^{(1,0,0,0)}_{\Delta,0}, \tag{241}$$

where $L^{(1)}_X$ is

$$(L^{(1)}_X)_{AB}\equiv X_A\frac{\partial}{\partial X^B}-X_B\frac{\partial}{\partial X^A}+Z_A\frac{\partial}{\partial Z^B}-Z_B\frac{\partial}{\partial Z^A}. \tag{242}$$

## 6.4 Correspondence with an external spin-$n$ field

In this subsection, we define the amplitude of GWD with an external spin-$n$ field. We see that this definition of GWD is consistent with the formula of CPW in [53]. Moreover, we determine the three point interaction in AdS spacetime for this amplitude.

As an extension of (240), we define the amplitude of GWD with an external spin-$n$ field $\mathcal{W}^{(n,0,0,0)}_{\Delta,0}$ as

$$\begin{aligned}&\mathcal{W}^{(n,0,0,0)}_{\Delta,0}(X_i;Z_1;\Delta_i)\\&\equiv\int_{-\infty}^{\infty}d\lambda'\left[\int_{-\infty}^{\infty}d\lambda\, G^0_{b\partial}(Y_1(\lambda),X_2;\Delta_2)\left\{G^n_{b\partial}(Y_1(\lambda),X_1;Z_1,\nabla_{Y_1};\Delta_1)\,G_{bb}(Y_1(\lambda),Y_2(\lambda');\Delta)\right\}\right]\\&\qquad\times G^0_{b\partial}(Y_2(\lambda'),X_3;\Delta_3)\,G^0_{b\partial}(Y_2(\lambda'),X_4;\Delta_4).\end{aligned} \tag{243}$$

For convenience, we also define

$$\begin{aligned}F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}(X_1,X_2,Y_2;Z_1)&\equiv\int_{-\infty}^{\infty}d\lambda\, G^0_{b\partial}(Y_1,X_2;\Delta_2)\left\{G^n_{b\partial}(Y_1,X_1;Z_1,\nabla_{Y_1};\Delta_1)\,G_{bb}(Y_1,Y_2;\Delta)\right\},\\F^{(0,n;0)}_{\Delta_1,\Delta_2,\Delta}(X_1,X_2,Y_2;Z_2)&\equiv\int_{-\infty}^{\infty}d\lambda\, G^0_{b\partial}(Y_1,X_1;\Delta_1)\left\{G^n_{b\partial}(Y_1,X_2;Z_2,\nabla_{Y_1};\Delta_2)\,G_{bb}(Y_1,Y_2;\Delta)\right\}.\end{aligned} \tag{244}$$

From now, we check whether our definition of the amplitude of GWD (243) is reasonable for the gravity dual of CPW.

In [53], the authors derived a formula of CPW with symmetric traceless tensors by using the scalar CPW and the differential operators. They defined the following differential operators:

$$\begin{aligned}
D_{11} \equiv& (X_1 \cdot X_2)\left(Z_1 \cdot \frac{\partial}{\partial X_2}\right) - (Z_1 \cdot X_2)\left(X_1 \cdot \frac{\partial}{\partial X_2}\right) \\
&- (Z_1 \cdot Z_2)\left(X_1 \cdot \frac{\partial}{\partial Z_2}\right) + (X_1 \cdot Z_2)\left(Z_1 \cdot \frac{\partial}{\partial Z_2}\right), \\
D_{22} \equiv& (X_2 \cdot X_1)\left(Z_2 \cdot \frac{\partial}{\partial X_1}\right) - (Z_2 \cdot X_1)\left(X_2 \cdot \frac{\partial}{\partial X_1}\right) \\
&- (Z_2 \cdot Z_1)\left(X_2 \cdot \frac{\partial}{\partial Z_1}\right) + (X_2 \cdot Z_1)\left(Z_2 \cdot \frac{\partial}{\partial Z_1}\right), \\
D_{12} \equiv& (X_1 \cdot X_2)\left(Z_1 \cdot \frac{\partial}{\partial X_1}\right) - (Z_1 \cdot X_2)\left(X_1 \cdot \frac{\partial}{\partial X_1}\right) + (Z_1 \cdot X_2)\left(Z_1 \cdot \frac{\partial}{\partial Z_1}\right), \\
D_{21} \equiv& (X_2 \cdot X_1)\left(Z_2 \cdot \frac{\partial}{\partial X_2}\right) - (Z_2 \cdot X_1)\left(X_2 \cdot \frac{\partial}{\partial X_2}\right) + (Z_2 \cdot X_1)\left(Z_2 \cdot \frac{\partial}{\partial Z_2}\right).
\end{aligned} \tag{245}$$

By using these operators, we obtain the relation of $F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2; Z_1)$ and $F^{(0,n;0)}_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2; Z_2)$,

$$(D_{11})^n F^{(0,0;0)}_{\Delta_1+n,\Delta_2,\Delta} = \left(-\frac{1}{2}\right)^n F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}, \tag{246}$$

$$(D_{22})^n F^{(0,0;0)}_{\Delta_1,\Delta_2+n,\Delta} = \left(-\frac{1}{2}\right)^n F^{(0,n;0)}_{\Delta_1,\Delta_2,\Delta}, \tag{247}$$

$$(D_{12})^n F^{(0,0;0)}_{\Delta_1,\Delta_2+n,\Delta} = \left(-\frac{1}{2}\right)^n F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}, \tag{248}$$

$$(D_{21})^n F^{(0,0;0)}_{\Delta_1+n,\Delta_2,\Delta} = \left(-\frac{1}{2}\right)^n F^{(0,n;0)}_{\Delta_1,\Delta_2,\Delta}. \tag{249}$$

Since $\mathcal{W}^{(n,0,0,0)}_{\Delta,0}$ includes $F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2; Z_1)$, we can derive a formula between $\mathcal{W}^{(n,0,0,0)}_{\Delta,0}$ and $\mathcal{W}^{(0,0,0,0)}_{\Delta,0}$ from the relation of $F^{(n,0;0)}_{\Delta_1,\Delta_2,\Delta}(X_1, X_2, Y_2; Z_1)$ and this formula agrees with (3.40) of [53]. Therefore, we conclude that the amplitude of GWD with an external spin-$n$ field (243) corresponds to CPW.

Let us consider the three point interaction to construct the amplitude

(243). We can deform (243) as

$$
\begin{aligned}
&\mathcal{W}^{(n,0,0,0)}_{\Delta,0}(X_i; Z_1; \Delta_i) \\
&= \int_{-\infty}^{\infty} d\lambda' \left[ \int_{-\infty}^{\infty} d\lambda \, G^0_{b\partial}(Y_1(\lambda), X_2; \Delta_2) \right. \\
&\qquad\qquad \left. \times \, G^n_{b\partial}(Y_1(\lambda), X_1; Z_1; \Delta_1)^{A_1 \cdots A_n} \frac{\partial}{\partial Y_1^{A_1}} \cdots \frac{\partial}{\partial Y_1^{A_n}} G_{bb}(Y_1(\lambda), Y_2(\lambda'); \Delta) \right] \\
&\qquad\qquad \times \, G^0_{b\partial}(Y_2(\lambda'), X_3; \Delta_3) \, G^0_{b\partial}(Y_2(\lambda'), X_4; \Delta_4), \qquad (250) \\
&G^n_{b\partial}(Y, X; Z; \Delta)_{A_1 \cdots A_n} \equiv \frac{1}{n!} \frac{\partial}{\partial W^{A_1}} \cdots \frac{\partial}{\partial W^{A_n}} G^n_{b\partial}(Y, X; Z, W; \Delta). \qquad (251)
\end{aligned}
$$

From this expression, the three point interaction for the amplitude is identified as

$$
\begin{aligned}
S_{int} &= \int_{\text{AdS}} dY \, T_{\Delta_1}^{A_1 \cdots A_n} \phi_{\Delta_2} \left( \frac{\partial}{\partial Y^{A_1}} \cdots \frac{\partial}{\partial Y^{A_n}} \phi_{\Delta} \right) \qquad (252) \\
&= \int_{\text{AdS}} dY \, T_{\Delta_1}^{A_1 \cdots A_n} \phi_{\Delta_2} \, (\nabla_{A_1} \cdots \nabla_{A_n} \phi_{\Delta}), \qquad (253)
\end{aligned}
$$

where we used the transverse condition (199) and the traceless condition of $T^{A_1 \cdots A_n}$ for the replacement of $\frac{\partial}{\partial Y^A}$ with $\nabla_A$.

## 6.5 Short summary

Summarizing the above, we have defined the amplitude of the scalar exchange geodesic Witten diagram (GWD) with an external spinning field as (243). We have verified that our construction of the amplitude of GWD satisfies the formulas and the properties of the conformal partial wave (CPW) such as (232), (241) and (246). We also have studied the appropriate three point couplings in the AdS spacetime for the correspondence between GWD and CPW such as (237) and (253). Thus, we have been able to show the correspondence between the scalar exchange CPW and GWD with an external spinning field. Construction of the amplitude of GWD is a systematic way to find an expression of CPW as the solution of the conformal Casimir equation. Therefore, our study will lead to make a discovery of a novel expression of CPW with general external fields such as the stress tensors.

# 7 Conclusion and discussion

In part 1, we have studied the three-scalar holographic superconductor model (114) based on the three-band superconductor model which describes the frustration in condensed matter physics. We have found that there are several solutions whose free energy is the same in this holographic model if $\eta = 0$, $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ and (136) hold. The condition $\epsilon_{12}\epsilon_{23}\epsilon_{31} > 0$ is important for the frustration between the scalar fields. In addition, we have analyzed the three-scalar holographic superconductor model with $\eta > 0$ and (146). By computing the free energy, we have found that the solutions (155) and (156) which correspond to the chiral ground states exist. These solutions have nonzero chirality and their free energy is minimum. Our holographic model will be a hint to study frustrated superconductors in condensed matter physics.

Future direction of the study in part 1 is research of domain wall solutions of the holographic model. The existence of the domain wall is discussed in the three-band superconductor model in condensed matter physics [35] and a holographic two-band superconductor model [66]. It is a interesting problem whether there is the domain wall solutions which connect the chiral ground states (155) and (156) because topological solitons such as the domain wall are related to phase transitions in particle physics and condensed matter physics. Application to the Fe-based superconductor is another future work. For example, by deriving various classical solutions of the holographic model and comparing their free energies, we may predict the phase structure of the Fe-based superconductor. Also, finding their solution is interesting from the viewpoint of classical gravity.

In part 2, we have explored the correspondence between the conformal partial wave (CPW) and the geodesic Witten diagram (GWD) with an external spinning field. We have constructed the amplitude of GWD and found the three point interactions in the AdS spacetime for the correspondence. We have shown that the amplitude of GWD which we constructed satisfies the properties of CPW. Our result is a first step for the generalization to GWD with external fields in any representation.

Future direction of the study in part 2 is the generalization to combination of external fields such as symmetric tensor fields, antisymmetric tensor fields and fermionic fields. Since the degrees of freedom of three point functions in CFT is not generally one for the fields in any representation, the tensor structure of CPW for general fields cannot be determined uniquely. In order to clarify the correspondence for such CPW, it is important to examine the relation of the bases between the three point functions in CFT and the three point interactions in the AdS spacetime [61]. In particular, CPW of the stress tensors can be used to study the gravity theories [67] because the stress tensor in CFT corresponds to the graviton in the gravity side. Expressions of CPW in terms of GWD is useful for such study since the three point interactions in the AdS spacetime is used to construct the amplitude manifestly. Moreover, it is also interesting to construct super CPW [68] from GWD. For this construction, it is expected that an introduction of the AdS superspace [69, 70] is necessary.

For a future work of the gauge/gravity correspondence, the author is interested in the role of symmetry and consistency for the origin of the gauge/gravity correspondence. Discovery of the gauge/gravity correspondence is based on the superstring theory which is one candidate of consistent quantum gravity. Moreover, recent progress of conformal bootstrap [71] implies that the CFT spectrum is strongly constrained by conformal symmetry and consistency. Therefore, it is reasonable to imagine that symmetry and consistency have an important role in the gauge/gravity correspondence. In particular, we would like to know how much examples of the gauge/gravity correspondence which can be determined by symmetry and consistency only exist. Our study of the correspondence between GWD and CPW will be a clue to solve this question.

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