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ON LOGARITHMIC $K3$ SURFACES

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Introduction. By surfaces we mean non-singular algebraic surfaces defined over the field of complex numbers $\boldsymbol{C}$. *A logarithmic $K3$ surface S* is by definition a surface S with $\bar{p}_g(S)=1$, $\bar{\kappa}(S)=\bar{q}(S)=0$, in which $\bar{p}_g(S)$ is the *logarithmic geometric genus*, $\bar{\kappa}(S)$ is the *logarithmic Kodaira dimension*, and $\bar{q}(S)$ is the *logarithmic irregularity*. These notions will be explained in §1.

Let $\bar{S}$ be a completion of S with *ordinary boundary D, i.e.*, $\bar{S}$ is a non-singular complete surface and D is a divisor with normal crossings on $\bar{S}$ such that $S=\bar{S}-D$. We write D as a sum of irreducible components: $D=C_1+\cdots+C_s$.

Logarithmic $K3$ surfaces are classified into the following three types:
Type I) $p_g(\bar{S})=1$; Then $\bar{S}$ is a $K3$ surface and D consists of non-singular rational curves C_i with negative-definite intersection matrix $[(C_i, C_j)]$.
Type II$_a$) $p_g(\bar{S})=0$ and a component C_1 of D is a non-singular elliptic curve; Then $\bar{S}$ is a rational surface and the graph of D has no cycles.
Type II$_b$) $p_g(\bar{S})=0$ and D consists of rational curves C_j; Then $\bar{S}$ is a rational surface and the graph of D has one cycle.

We define *A-boundary D_A* and *B-boundary D_B* of $(\bar{S}, D)$ as follows: 1) If S is of type I, then $D_A=\phi$ and $D_B=D$. 2) If S is of type II$_a$, then $D_A=C_1$ (a non-singular elliptic curve) and $D_B=C_2+\cdots+C_s$. 3) If S is of type II$_b$, then $D_A=C_1+\cdots+C_r$ that is a *circular boundary* (for definition, see §1 v)) and $D_B=C_{r+1}+\cdots+C_s$.

Theorem 1. *If $\bar{S}-D_A$ has no exceptional curves of the first kind, then* $K(\bar{S})+D_A\sim 0$.

Next, consider the case where $\bar{S}-D_A$ has exceptional curves. Let ρ: $\bar{S}\rightarrow\bar{S}_*$ be a contraction of exceptional curves of the first kind on $\bar{S}-D_A$, *i.e.*, $\bar{S}_*$ is a complete surface and ρ is biregular around D_A such that $\bar{S}_*-\rho(D_A)$ has no exceptional curves of the first kind. By Theorem 1, $K(\bar{S}_*)+\rho(D_A)\sim 0$.

Theorem 2. *$\rho(D_B)$ is a divisor with simple normal crossings. Let $\mathcal{Z}_1, \cdots, \mathcal{Z}_u$ be the connected components of $\rho(D_B)$. Then* 1) *if $\mathcal{Z}_i\cap\rho(D_A)\neq\phi$, $\mathcal{Z}_i$ is an exceptional curve of the first kind such that $(\mathcal{Z}_i, \rho(D_A))=1$.* 2) *If $\mathcal{Z}_i\cap\rho(D_A)=\phi$,* *then $\mathcal{Z}_i$ is a curve of Dynkin type ADE on $\bar{S}-\rho(D_A)$. In case S is of type* II, *$\mathcal{Z}_i$ is a curve of Dynkin type* A.

For definition of *curves of Dynkin type* ADE, see § 1. iv).

Theorem 3. *Suppose that $K(\bar{S})+D_A\sim 0$ and D_B is a curve of Dynkin type* ADE. *If S is of type* II_a, *then $(\bar{S}, D)$ is obtained from one of 4 classes in Table* II_a *by 1/2-point attachments. If S is of type* II_b, *then $(\bar{S}, D)$ is obtained from one of 15 classes in TABLE* II_b *by canonical blowing ups and attaching several 1/2-points.*

Theorem 4. *Let $(\bar{S}, D)$ be a ∂-surafce of which interior S satisfies that $\bar{\kappa}(S)=p_g(\bar{S})=0$ and $\bar{p}_g(S)=1$. Suppose that a component C_1 of D is not rational. Then $\bar{q}(S)=0$. Next, assume that D consists of rational curves. If $\bar{q}(S)=0$, then there exists an open subset S_1 of S such that $\bar{\kappa}(S_1)=0$ and $\bar{q}(S_1)=1$. Furthermore, if $\bar{q}(S)=1$, then there exists an open subset S_2 of S such that $\bar{\kappa}(S_2)=0$ and $\bar{q}(S_2)=2$.*

Theorem 5. *Let S be a surface with $\bar{\kappa}(S)=p_g(\bar{S})=0$ and $\bar{p}_g(S)=1$. Then there exists an algebraic pencil $\{C_u\}$ on S whose general member C_u is isomorphic to $\boldsymbol{C}^*$. Hence, S is not measure-hyperbolic. Moreover, the connected component of* Aut(S) *is* $\{1\}$ *or* $\boldsymbol{C}^*$ *or* $\boldsymbol{C}^{*2}$. *Further,*

$$\dim \mathrm{Aut}(S)^0 \leqq \bar{q}(S) .$$

Theorem 6. *Let $(\bar{S}, D)$ be a ∂-surface whose interior S satisfies that $\bar{\kappa}(S)=0$ and $\bar{p}_g(S)=1$. Then, there exists a proper birational morphism $\rho\colon \bar{S}\rightarrow\bar{S}_*$ such that* i) *$\bar{S}_*$ is relatively minimal,* ii) *$P_m(\bar{S}_*-\rho_*(D))=1$ for any $m\geqq 1$,* iii) *$\rho_*(D)=\Delta+Y$ has only normal crossings with $K(\bar{S}_*)+\Delta\sim 0$, Y being a curve of Dynkin type.*

$(\bar{S}_*, \rho_*(D))$ might be called a *supermodel* of S (or of $(\bar{S}, D)$). In the study of non-complete surfaces, minimal model (and even ∂-minimal model) is not helpful. Instead, supermodel will play the important role. For full discussion of the classification theory of surfaces of non-complete surfaces, see Kawamata's recent article [18].

EXAMPLE 1. Let $\bar{S}$ be a non-singular cubic surface in $\boldsymbol{P}^3$. Let E be a general hyperplane section on $\bar{S}$. Then $\bar{S}-E$ is a logarithmic $K3$ surface of type II_a and the fundamental group $\pi_1(\bar{S}-E)\cong\{1\}$. Contracting exceptional curves of the first kind, we obtain a proper birational morphism $\rho\colon \bar{S}\rightarrow\bar{S}_*$ in which $\bar{S}_*=\boldsymbol{P}^2$. $E_1=\rho(E)$ is a non-singular elliptic curve on $\boldsymbol{P}^2$. Then $\pi_1(\bar{S}_*-E_1)\cong\boldsymbol{Z}/(3)$ and $\bar{S}-E\supset\bar{S}_*-E_1$.

EXAMPLE 2. Let $\varphi(y)$ be a polynomial of degree $n+1$ such that $\varphi(0)\neq 0$. Let Γ be the graph $(\subset\boldsymbol{C}^2)$ of a rational function $\varphi(y)/y^{n-m}$ $(0<m<n)$. By

C we denote the closure of Γ in $\boldsymbol{P}^2$. Then $\boldsymbol{P}^2-\Gamma$ is a logarithmic $K3$ surface of type $\mathrm{II_b}$.

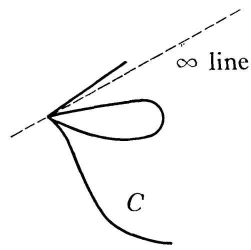


Figure 1.

EXAMPLE 3. Let Φ: $\boldsymbol{C}[x, y]\rightarrow\boldsymbol{C}[x, y]$ be a $\boldsymbol{C}$-automorphism. Put $X(x, y)=\Phi(x)$ and $Y(x, y)=\Phi(y)$. Let $F(x, y)=Y(x, y)^{n-m}X(x, y)-\varphi(Y(x, y))$, φ being as in Example 3. Then the closure C_Φ of $V(F)=\mathrm{Spec}\ k[x, y]/(F)$ in $\boldsymbol{P}^2$ is a complement of a logarithmic $K3$ surface of type $\mathrm{II_b}$ if C_Φ has an analytically reducible (i.e., non-cusp) singular point.

For instance, let $\varphi(y)=y^3+1$ and $\Phi(x)=x$, $\Phi(y)=y+x^2$. Then $F=(y+x^2)x-(y+x^2)^3-1$. Thus letting Γ be the closure of $V(F)$ in $\boldsymbol{P}^2$, $\boldsymbol{P}^2-\Gamma$ is a logarithmic $K3$ surface of type $\mathrm{II_b}$.

EXAMPLE 4. Let $C=V((y-x^2)^2-xy^2)$ in $\boldsymbol{C}^2$. Denote by Γ the closure of C in $\boldsymbol{P}^2$. Then $S=\boldsymbol{P}^2-C$ has the following numerical characters: $\bar{p}_g=0$, $\bar{P}_2=1$, $\bar{\kappa}=1$, and $\bar{q}=0$.

1. Basic notions, notations and conventions

i) *∂-manifold and* 1/2*-point attachment.* A pair $(\bar{V}, D)$ consisting of a complete non-singular algebraic variety $\bar{V}$ and a divisor D with normal crossings on $\bar{V}$ is called a *∂-manifold.* The dimension of $(\bar{V}, D)$ is understood as the dimension of $\bar{V}$. A 2-dimensional ∂-manifold is called a *∂-surface.* We have a theory of minimal models for ∂-manifolds (see [12]). Let $(\bar{S}, D)$ be a ∂-surface. Then D is not a *minimal boundary* if and only if there is an irreducible component E of D which is an exceptional curve of the first kind such that $(E, D')=1$ or 2, D' being defined by $D=D'+E$. We say that $(\bar{S}, D)$ is *relatively ∂-minimal* if $S-D$ has no exceptional curves of the first kind and if D is a minimal boundary.

Let $(\bar{V}_1, D_1)$ and $(\bar{V}_2, D_2)$ be ∂-manifolds. We say that a morphism f: $\bar{V}_1\rightarrow\bar{V}_2$ is a *∂-morphism* when $f^{-1}D_2\subset D_1$. Here $f^{-1}(D_2)$ is the reduced divisor of the pull back f^*D_2.

Let $(\bar{S}, D)$ be a ∂-surface and take a point $p\in D$. By λ: $\bar{S}^1=Q_p(\bar{S})\rightarrow\bar{S}$ denote the blowing up at p. Defining $D^1=\lambda^{-1}(D)$, we have a ∂-morphism λ: $(\bar{S}^1, D^1)\rightarrow(\bar{S}, D)$. If p is a double point of D, λ is called a *canonical blowing* *up*. Then we have the linear equivalence:

$$K(\bar{S}^1)+D^1 \sim \lambda^*(K(\bar{S})+D),$$

where $K(\bar{S}^1)$ and $K(\bar{S})$ denote canonical divisors on $\bar{S}^1$ and $\bar{S}$, respectively. If p is a simple point of D, define D^* by $D^1=\lambda^{-1}(p)+D^*$. $S^*=\bar{S}^1-D^*$ contains S as an open subset. S^* is called a *1/2-point attachment to S* at p. Conversely, S is called a *1/2-point detachment from* S^*. To make things clear, we may say that $(\bar{S}^*, D^*)$ is obtained from $(\bar{S}, D)$ by attaching a 1/2-point $\lambda^{-1}(p)-D^*$([10]). It is easy to check that

$$K(\bar{S}^1)+D^* \sim \lambda^*(K(\bar{S})+D).$$

Hence, *$K(\bar{S})+D$ modulo linear equivalence is invariant under canonical blowing ups and 1/2-point attachments.*

In general letting $(\bar{S}, D)$ be a ∂-surface, we consider an irreducible curve E on $\bar{S}$ satisfying that E is an exceptional curve of the first kind, $E \not\subset D$, and $(E, D)=1$. Such an E is called a *D-exceptional curve of the first kind.* Note that $E-D \cong A^1$, which is called a *1/2-point*. $S-D$ is a 1/2-point attachment to $\bar{S}-D-E$.

ii) *logarithmic genera.* Let V be an algebraic variety. Then there exists a non-singular algebraic variety V^* such that there exists a proper birational morphism $\mu: V^* \to V$. Let $(\bar{V}^*, D^*)$ be a ∂-manifold such that $V^*=\bar{V}^*-D^*$. Then we say that *$\bar{V}^*$ is a completion of V^* with ordinary boundary D^*.* According to Deligne [3], we have a sheaf $\Omega^1(\log D^*)$ of logarithmic 1-forms on $\bar{V}^*$. We have the spaces of logarithmic forms:

$$T_i(V^*) = H^0(\bar{V}^*, \Omega^i(\log D^*)), \qquad 1 \leqq i \leqq n;$$

and

$$H^0(\bar{V}^*, (\Omega^n \log D^*)^m) \qquad \text{for } m=1, 2, \cdots,$$

where $\Omega^i(\log D^*)=\wedge^i(\Omega^1 \log D^*)$ and $n=\dim V$. These spaces depend only on V. Hence, define

$$\bar{q}_i(V) = \dim T_i(V^*)$$

and

$$\bar{P}_m(V) = \dim H^0(\bar{V}^*, (\Omega^n \log D^*)^m).$$

We call $\bar{q}_i(V)$ the *logarthmic i-th irregularity* of V and call $\bar{P}_m(V)$ *the logarithmic m-genus* of V. Writing $\bar{q}(V)=\bar{q}_1(V)$ and $\bar{p}_g(V)=\bar{q}_n(V)=\bar{P}_1(V)$, we call them the *logarithmic irregularity* and the *logarithmetic geometric genus* of V, respectively (see [4], [5]).

iii) *D-dimension and logarithmic Kodaira dimension.* In general, let $\bar{V}$ be a normal complete algebraic variety and D a divisor on $\bar{V}$. By Φ_m we denote the rational map associated with $|mD|$ under the assumption that $|mD| \neq \phi$. We define

$$\kappa(D, \bar{V}) = \max\{\dim \Phi_m(\bar{V}); \text{ when } |mD| \neq \phi\},$$

which is said to be the *D-dimension* of $\bar{V}$. If $|mD|$ is empty for any $m \geqq 1$, we put $\kappa(D, \bar{V}) = -\infty$. The following two facts ([6]) are very useful in the study of varieties and divisors.

1) *If* $\kappa(D_1, \bar{V}) \geqq 0, \cdots, \kappa(D_l, \bar{V}) \geqq 0$, *then for any* $\alpha_1 > 0, \cdots, \alpha_l > 0$, *we have*

$$\kappa(\sum D_j, \bar{V}) = \kappa(\sum \alpha_j D_j, \bar{V}).$$

2) *Let* $f: \bar{V} \to W$ *be a surjective morphism of* $\bar{V}$ *onto a normal complete variety* W. *For a divisor* D *on* W *and an effective divisor* E *which is f-exceptional* (*i.e.,* $\operatorname{codim} f(E) \geqq 2$), *we have*

$$\kappa(f^{-1}D + E, \bar{V}) = \kappa(D, W).$$

When $\bar{V}$ is non-singular, we denote by $K(\bar{V})$ a canonical divisor on $\bar{V}$. The *Kodaira dimension* $\kappa(\bar{V})$ of $\bar{V}$ is defined to be $\kappa(K(\bar{V}), \bar{V})$.

Let $(\bar{V}, D)$ be a ∂-manifold of dimension n. $V = \bar{V} - D$ is called the *interior* of $(\bar{V}, D)$. We see that

$$\bar{P}_m(V) = \dim H^0(\bar{V}, \mathcal{O}(m(K(\bar{V}) + D))).$$

The *logarithmic Kodaira dimension* of V is defined to be

$$\bar{\kappa}(V) = \kappa(K(\bar{V}) + D, \bar{V}),$$

which does not depend on the choice of the smooth completion $\bar{V}$ of V with ordinary boundary D.

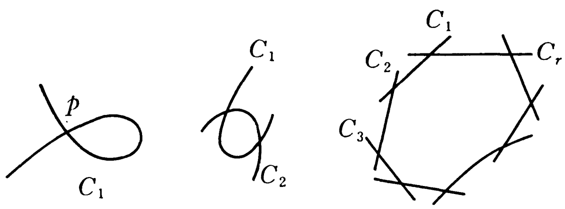
iv) *W^2PB-equivalence.* If there exists a proper birational morphism $f: V_1 \to V_2$, then $\bar{P}_m(V_1) = \bar{P}_m(V_2)$ and $\bar{q}_i(V_1) = \bar{q}_i(V_2)$. A *proper birational map* is by definition a composition of a proper birational morphism and an inverse of a proper birational morphism. If there is a proper birational map $f: V_1 \to V_2$, then we say that V_1 is *proper birationally equivalent* to V_2. In this case, $\bar{P}_m(V_1) = \bar{P}_m(V_2)$ and $\bar{q}_i(V_1) = \bar{q}_i(V_2)$.

Moreover, when V is non-singular and F a closed subset of V of codim $\geqq 2$, $\bar{P}_m(V - F) = \bar{P}_m(V)$ and $\bar{q}_i(V - F) = \bar{q}_i(V)$. In such a case, we say that i: $V - F \hookrightarrow V$ is a *strict open immersion.*

A *WPB-map* $f: V_1 \to V_2$ is by definition a birational map which is a composition of proper birational maps, strict open immersions, and inverses of strict open immersions. If there exists a *WPB*-map $f: V_1 \to V_2$, we say that V_1 is *WPB-equivalent to* V_2.

Now define $\mathcal{W}=\{f\colon V_1\to V_2$ birational morphism; there exist a morphism $g\colon V_2\to V_3$ such that $g\cdot f$ is a *WPB*-map or a morphism $h\colon U\to V_1$ such that $f\cdot h$ is a *WPB*-map$\}$. A birational map which is a composition $f_1 f_2^{-1} f_3 \cdots f_l^{\pm 1}$, $f_j \in \mathcal{W}$, is called a *W^2PB-map*. If there is a W^2PB-map $f\colon V_1\to V_2$, then we say that V_1 is *W^2PB-equivalent to* V_2 and $\bar{P}_m(V_1)=\bar{P}_m(V_2)$, $\bar{q}_i(V_1)=\bar{q}_i(V_2)$. Recall that *a surface S is W^2PB-equivalent to a quasi-abelian surface if and only if* $\bar{\kappa}(S)=0$ *and* $\bar{q}(S)=2$ ([10]).

v) *circular boundary*. Let $(\bar{S}, D)$ be a ∂-surface. We say that D is a *circular boundary* if D is a rational curve with only one ordinary double point p such that $D-\{p\}$ is non-singular or if D is a sum of non-singular rational curves $C_1, C_2, \cdots, C_r$ such that when $r=2$, we have $(C_1, C_2)=2$ and when $r\geqq 3$, we have $(C_i, C_j)=1$ for $i-j\equiv\pm 1 \bmod r$, and $(C_i, C_j)=0$ for $i-j\not\equiv 0, \not\equiv\pm 1 \bmod r$.



Figures 3.

vi) *curve of Dynkin type*. Let $(\bar{S}, Y)$ be a ∂-surface. We say that Y is a *curve of Dynkin type* ADE if Y is a sum of non-singular rational curves Y_j such that $Y_j^2=-2$ and the intersection matrix $[(Y_i, Y_j)]$ corresponds to a direct sum of Dynkin diagrams A_n, D_m, E_l. Similarly, we can define a *curve of extended Dynkin type* $\tilde{A}\tilde{D}\tilde{E}$ (, which are not necessarily reduced divisors).

2. Logarithmic $K3$ surfaces of type I

Let S be a logarithmic $K3$ surface, *i.e.*, $\bar{p}_g(S)=1$. $\bar{q}(S)=\bar{\kappa}(S)=0$. Let $(\bar{S}, D)$ be a ∂-surface of which interior is S. Then $\kappa(\bar{S})\leqq\bar{\kappa}(S)=0$, $p_g(\bar{S})\leqq\bar{p}_g(S)=1$. Hence, $p_g(\bar{S})=1$ or 0.

First, assume that $p_g(\bar{S})=1$. Combining this with $\kappa(\bar{S})\leqq\bar{\kappa}(S)=0$, $\bar{q}(\bar{S})\leqq q(\bar{S})=0$, we see that $\bar{S}$ is a $K3$ surface which may not be minimal. By contracting exceptional curves of the first kind on $\bar{S}$ successively, we obtain a minimal $K3$ surface $\bar{S}_*$ and a birational morphism $\mu\colon \bar{S}\to\bar{S}_*$. If $\mu(D)$ is a finite set of points, then, putting $\bar{S}_0=\bar{S}-\mu^{-1}(\mu(D))$ and $S_*=\bar{S}_*-\mu(D)$, we have a proper birational morphism $\mu_0=\mu\,|\,S_0\colon S_0\to S_*$. We obtain the following commutative diagram:

$$\begin{array}{ccccc} S_0 & \subset & S & \subset & \bar{S} \\ & {}_{\mu_0}\searrow & & \swarrow^{\mu} & \\ & S_* & \subset & \bar{S}_* & \end{array}$$

Figure 4.

Hence, by definition (see § 1 iv)) $S_0 \subset S$ and $S \subset \bar{S}$ are both W^2PB-morphisms. Hence S is W^2PB-equivalent to $\bar{S}_*$.

If $\mu(D)$ contains a curve, we let D_* be a purely 1-dimensional part of $\mu(D)$. Then by the previous argument, we see that S is W^2PB-equivalent to $\bar{S} - \mu^{-1}(D_*) \cap D$. Thus we may assume $D_* = \mu(D)$.

Lemma 1. *Let $\bar{V}$ be a complete non-singular algebraic variety and D a reduced divisor on $\bar{V}$. Let μ: $\bar{V}^* \to \bar{V}$ be a birational morphism such that $(\bar{V}^*, \mu^{-1}(D))$ is a ∂-manifold. Denote by D^* the proper transform of D by μ^{-1}. Suppose that $\kappa(\bar{V}) \geqq 0$. Then*

$$\begin{aligned} \bar{\kappa}(\bar{V}^* - D^*) &= \bar{\kappa}(\bar{V}^* - \mu^{-1}(D)) = \bar{\kappa}(\bar{V} - D) \\ &= \kappa(K(\bar{V}) + D, \bar{V}) . \end{aligned}$$

For a proof, see [6]. A generalization of this is the following Lemma 6, whose proof will be given there. By the above lemma, we get

$$\begin{aligned} 0 = \bar{\kappa}(S) &= \bar{\kappa}(\bar{S} - D) = \bar{\kappa}(\bar{S}_* - D_*) \\ &= \kappa(K(\bar{S}_*) + D_*, \bar{S}_*) = \kappa(D_*, S_*) . \end{aligned}$$

Proposition 1. *Let $\bar{S}$ be a minimal K3 surface and Y a reduced divisor on $\bar{S}$ such that $\kappa(Y, \bar{S}) = 0$. Then Y turns out to be a curve of Dynkin type ADE. Moreover, $\bar{P}_m(\bar{S} - Y) = 1$ for any $m \geqq 1$ and $\bar{q}(S - Y) = 0$. Hence $\bar{S} - Y$ is a logarithmic K3 surface.*

Proof. Let $\sum Y_j$ be the irreducible decomposition of Y. Then for any $m_j \geqq 0$, we have $\kappa(\sum m_j Y_j, \bar{S}) = 0$ by the fact 1) in § 1 iii). By making use of Riemann Roch Theorem on $\bar{S}$ we have

$$0 = \dim |(\sum m_j Y_j)| \geqq (\sum m_j Y_j)^2/2 + 1$$

except for $m_1 = \cdots = m_s = 0$. Hence

$$(\sum m_j Y_j)^2 \leqq -2 .$$

In particular, $Y_j^2 \leqq -2$. In view of the adjunction formula, we have

$$-2 \leqq 2\pi(Y_j) - 2 = Y_j^2 .$$

Here $\pi(Y)$ denotes the *virtual genus* of Y. Thus $Y_j^2=-2$ and $\pi(Y_j)=0$. More generally, letting $\mathcal{Y}$ be a connected reduced curve in Y, we have the exact sequences

$$0 \to \mathcal{O}(-\mathcal{Y}) \to \mathcal{O} \to \mathcal{O}_{\mathcal{Y}} \to 0$$

and

$$\begin{aligned} 0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_{\mathcal{Y}}) \to H^1(\mathcal{O}(-\mathcal{Y})) \to H^1(\mathcal{O}) \\ \to H^1(\mathcal{O}_{\mathcal{Y}}) \to H^2(\mathcal{O}(-\mathcal{Y})) \to H^2(\mathcal{O}) \to 0 \,. \end{aligned}$$

From this, it follows that $H^1(\mathcal{O}(-\mathcal{Y}))=0$ and

$$\begin{aligned} \dim H^0(\mathcal{O}(\mathcal{Y})) &= \dim H^2(\mathcal{O}(-\mathcal{Y})) = \dim H^1(\mathcal{O}_{\mathcal{Y}})+1 \\ &= \pi(\mathcal{Y})+1 = \mathcal{Y}^2/2+2 \,. \end{aligned}$$

Hence $\mathcal{Y}^2=-2$. In particular, if $Y_i \neq Y_j$, we have $(Y_i, Y_j)=0$ or 1. It is easy to see that the intersection-matrix $[(Y_i, Y_j)]$ $(Y_i \leqq \mathcal{Y})$ corresponds to the Dynkin diagram of type A_n, D_m, E_l. Eence, Y is a curve of Dynkin type ADE. Therefore,

$$\bar{\kappa}(\bar{S}-Y) = \kappa(K(\bar{S})+Y, \bar{S}) = 0$$

and $\bar{p}_g(\bar{S}-Y) \geqq p_g(\bar{S})=1$. These imply that $\bar{P}_m(\bar{S}-Y)=1$ for any $m \geqq 1$.

Since $[(Y_i, Y_j)]$ is negative-definite, $Y_1, \cdots, Y_s$ are linearly independent in Pic $(\bar{S})$. We make use of the following

Lemma 2. *Let $\bar{V}$ be a non-singular complete algebraic variety with $q(\bar{V})=0$ and Y a reduced divisor on $\bar{V}$. Let $\sum Y_j$ be the irreducible decomposition of Y. Then, putting $V=\bar{V}-Y$, we get*

$$\bar{q}(V) = \dim \mathrm{Ker}(\bigoplus_j \boldsymbol{Q} Y_j \to \mathrm{Pic}(\bar{V}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}) \,.$$

Proof. We have the exact sequence:

$$0 = H^1(\bar{V}, \boldsymbol{Q}) \to H^1(V, \boldsymbol{Q}) \to \oplus \boldsymbol{Q} Y_j \xrightarrow{\delta} H^2(\bar{V}, \boldsymbol{Q}) \,.$$

Since $q(\bar{V})=0$, it follows that $\mathrm{Im}\,\delta \subset \mathrm{Pic}(\bar{V}) \otimes \boldsymbol{Q} \subset H^2(\bar{V}, \boldsymbol{Q})$. Thus we obtain

$$\bar{q}(V) = \dim \mathrm{Ker}(\oplus \boldsymbol{Q} Y_j \xrightarrow{\delta} \mathrm{Pic}(\bar{V}) \otimes \boldsymbol{Q}) \,. \qquad \text{Q.E.D.}$$

We proceed with the proof of Proposition 1. By the lemma above we conclude that $\bar{q}(\bar{S}-Y)=0$. Q.E.D.

Thus we obtain the following

Theorem I. *Let $(\bar{S}, D)$ be a ∂-surface whose interior is a logarithmic $K3$ surface S of type I. Then there exists a birational morphism μ: $\bar{S} \to \bar{S}_*$ such that*

$\bar{S}_*$ is a minimal K3 surface and such that $\mu(D)$ is a union of a curve Y of Dynkin type and a finite set F, and hence

$$S_0 = \bar{S} - \mu^{-1}(Y) - \mu^{-1}(F) \subset S \subset \bar{S}.$$

In other words, S is W^2PB-equivalent to $\bar{S}_* - Y$.

Note that D and Y may be empty.

Table I. $\bar{S}_*$ being a minimal compact K3 surface

class	D	$\bar{S}_* - D$
i)	ϕ	compact
i)*	curve of Dynkin type ADE	non-compact

3. Logarithmic *K*3 surfaces of type II. We begin by recalling the elementary result, called *$\bar{p}_g$-formula.*

Lemma 3. *Let $(\bar{S}, D)$ be a ∂-surface with $q(\bar{S})=0$. Let $\sum^s C_j$ be the irreducible decomposition of D. Then*

$$\bar{p}_g(\bar{S}-D) = p_g(\bar{S}) + \sum g(C_j) + h(\Gamma(D)),$$

where $\Gamma(D)$ is the (dual) *graph of the intersection* of $D = \sum C_j$, *$h(\Gamma)$ is the cyclotomic number of the graph Γ, and the $g(C_i)$ denote the genera of the C_i.*

For a proof see ([7], the Appendix).

With the notation being in Lemma 3, we further assume that S is a logarithmic $K3$ surface of type II. Hence $p_g(\bar{S})=0$ and $\bar{p}_g(S)=1$. By the formula in Lemma 3, we have

$$1 = \bar{p}_g(S) = \sum g(C_j) + h(\Gamma(D)).$$

Hence, there are the following two types;

$$\text{Type II}_a;\ g(C_1) = 1,\ g(C_2) = \cdots = g(C_s) = 0 \text{ and } h(\Gamma(D)) = 0.$$
$$\text{Type II}_b;\ g(C_1)) = g(C_2) = \cdots = g(C_s) = 0 \text{ and } h(\Gamma(D)) = 1.$$

Proposition 2. *If S is a logarithmic K3 surface of type* II, *then S is a rational surface.*

First, assume $\kappa(\bar{S})$ to be 0. Recalling $p_g(\bar{S}) = q(\bar{S}) = 0$, we see that $\bar{S}$ is an Enriques surface. Hence, there exists an étale covering $\pi\colon \tilde{S} \to \bar{S}$ where $\tilde{S}$ is a $K3$ surface. Let $\tilde{D} = \pi^{-1}(D)$. Since $\tilde{S} - \tilde{D} \to \bar{S} - D$ is étale, we have $\bar{\kappa}(\tilde{S} - \tilde{D}) = \bar{\kappa}(\bar{S} - D) = 0$ by Theorem 3 [5]. Hence, $\tilde{S} - \tilde{D}$ is a logarithmic

$K3$ surface of type I. By Theorem I, $\tilde{D}$ consists of rational non-singular curves whose intersection matrix is negative-definite. Hence D has the same property as $\tilde{D}$. Thus $h(\Gamma(D))=0$. This contradicts the fact that S is of type II. Therefore, it follows that $\kappa(\bar{S})=-\infty$. Recalling Castelnuovo's criterion, $\bar{S}$ is a rational surface, because $q(\bar{S})=0$. Q.E.D.

4. Logarithmic $K3$ surfaces of type $\mathrm{II_a}$. Employing the notation in § 3, we assume S to be a logarithmic $K3$ surface of type $\mathrm{II_a}$. Putting $D_A=C_1$ and $D_B=C_2+\cdots+C_s$, we have $D=D_A+D_B$ and $g(D_A)=1$. Hence, $\bar{p}_g(S-D_A)=1$, $\bar{\kappa}(\bar{S}-D_A)\leqq\bar{\kappa}(\bar{S}-D)=0$, and $\bar{q}(\bar{S}-D_A)\leqq\bar{q}(\bar{S}-D_A)=0$. These show that $\bar{S}-D_A$ is a logarithmic $K3$ surface of type $\mathrm{II_a}$. Contracting exceptional curves of the first kind in $\bar{S}-D_A$, successively, we have a birational morphism $\mu\colon \bar{S}\to\bar{S}_*$ such that μ is isomorphic around $D_A\cong\mu(D_A)$ and $\bar{S}_*-\mu(D_A)$ has no exceptional curves of the first kind, i.e., $(\bar{S}_*, \mu(D_A))$ is a relatively ∂-minimal model of $(\bar{S}, D_A)$.

Proposition 3. *Let $(\bar{S}, C)$ be a relatively ∂-minimal ∂-surface such that C is a non-singular elliptic curve with $\bar{\kappa}(\bar{S}-C)=\bar{q}(\bar{S}-C)=0$. Then $K(\bar{S})+C\sim 0$.*

Proof. By Proposition 2, $\bar{S}$ is a rational surface.

If $K(\bar{S})+C$ were linearly equivalent to an effective divisor $\Delta=\sum_{i=1}^{s}r_iE_i$ $(r_i>0)$, we would derive a contradiction. Since $\kappa(\Delta, \bar{S})=\bar{\kappa}(\bar{S}-C)=0$, we know that the intersection matrix $[(E_i, E_j)]$ is negative semi-definite. In particular $E_j^2\leqq 0$ for any $1\leqq j\leqq s$. If $E_j=C$, then $K(=K(\bar{S}))\sim\Delta-E_j=\Delta-C_1\geqq 0$. This is a contradiction. Therefore $E_j\neq C$, which implies $(\Delta, C)\geqq 0$. Since $\Delta^2\leqq 0$, we may assume that $(\Delta, E_1)\leqq 0$. Hence, $(K, E_1)\leqq -(C, E_j)\leqq 0$. By the adjunction formula,

$$-2\leqq 2\pi(E_1)-2 = E_1^2+(K, E_1)\leqq 0\,.$$

Hence, $\pi(E_1)=0$ or 1. We shall examine various cases, separately.

1) If $\pi(E_1)=1$, we have $E_1^2=(K, E_1)=0$.

Hence $(C, E_1)=0$. Thus $C\cap E_1=\phi$ and $(\Delta, E_1)=0$.

2) If $\pi(E_1)=0$ and $(C, E_1)\geqq 1$, it follows that $(K, E_1)\leqq -1$ and $-2=E_1^2+(K, E_1)\leqq -1$. Hence, α) $E_1^2=(K, E_1)=-1$ or β) $E_1^2=0$ and $(K, E_1)=-2$. In the case of α), we have $1\leqq(C, E_1)=(\Delta, E_1)-(K, E_1)\leqq 1$. Hence $(\Delta, E_1)=0$, $-(C, E_1)=(K, E_1)=-1$. This implies that E_1 is a C-exceptional curve. Hence, we can contract E_1. Note that $K+C$ is invariant under 1/2-point detachments (see § 1 i)). Thus we may assume that this case does not occur.

In the case of β), we use the following

Lemma 4. *Let $\bar{S}$ be a complete surface with $p_g(\bar{S})=q(\bar{S})=0$ and E a curve* *on $\bar{S}$ such that $\pi(E)=0$. Then*

$$\dim|E|\geqq 1+E^2 .$$

Proof. By Riemann Roch Theorem,

$$\dim|E|\geqq (E, E-K)/2,\ K \text{ being } K(\bar{S}) .$$

On the other hand, $(E, E+K)=2\pi(E)-2=-2$. Hence, follows the assertion. Q.E.D.

Therefore letting $S=\bar{S}-C$,

$$0=\bar{p}_g(S)-1=\dim|\Delta|\geqq\dim|E_1|\geqq 1 .$$

Thus we have arrived at a contradiction.

3) If $\pi(E_1)=(C, E_1)=0$, then $E_1^2\leqq -1$ and $(K, E_1)=-1$ or 0. Suppose $(K, E_1)=-1$. We have $E_1^2=-1$ and $E_1\cap C=\phi$. This yields that E_1 is an exceptional curve of the first kind on $\bar{S}-C$. This contradicts the hypothesis. Suppose that $(K, E_1)=0$. We have $E_1^2=-2$. Thus $E_1\cap C=\phi$ and $(\Delta, E_1)=0$.

Consequently, after a finite succession of 1/2-point detachments, we have $(\Delta, E_j)=0$, and i) $E_j^2=0$, $\pi(E_j)=1$ or ii) $E_j^2=-2$, $\pi(E_j)=0$. Hence $(K, E_j)=0$ for any irreducible components E_j of Δ. Thus letting $\mathcal{D}_1, \cdots, \mathcal{D}_c$ be the connected components of Δ, we have $\Delta=\sum\mathcal{D}_j$ and $\Delta^2=\sum\mathcal{D}_j^2=0$. Since $\Delta^2=0$ and $\mathcal{D}_j^2\leqq 0$ for any j, it follows that $\mathcal{D}_1^2=\cdots=\mathcal{D}_c^2=0$. Recalling that $(K, E_i)=0$, for any i we have $(K, \mathcal{D}_j)=0$. Therefore, the $\mathcal{D}_j$ are curves of extended Dynkin type $\tilde{A}\tilde{D}\tilde{E}$.

Lemma 5. *Let $\bar{S}$ be a complete surface with $p_g(\bar{S})=q(\bar{S})=0$. For an effective divisor F $(\neq 0)$ on S, we have*

$$\dim|F+K|=\dim H^1(\mathcal{O}_F)-1\geqq(F, F+K)/2 .$$

Moreover, if $\dim H^0(\mathcal{O}_F)=1$, *then*

$$H^1(\mathcal{O}(F+K))=0, \textit{ and so } \pi(F)=\dim H^1(\mathcal{O}_F) .$$

Hence,

$$\dim|F+K|=(F, F+K)/2 .$$

Proof. From the exact sequence:

$$\begin{aligned}0\to \boldsymbol{C}=H^0(\mathcal{O})&\to H^0(\mathcal{O}_F)\to H^1(\mathcal{O}(-F))\\ \to 0=H^1(\mathcal{O})&\to H^1(\mathcal{O}_F)\to H^1(\mathcal{O}(-F))\to 0=H^2(\mathcal{O}) ,\end{aligned}$$

follows the assertion. Q.E.D.

By this, we have

$$\dim|\mathcal{D}_i+K|\geqq(\mathcal{D}_i,\ \mathcal{D}_i+K)/2=0\,.$$

But since $\bar{P}_2(S)-1\geqq\dim|\Delta+K|\geqq\dim|\mathcal{D}_i+K|$, it follows that $\dim|\mathcal{D}_i+K|=0$. Putting $K(\mathcal{D}_i)=(\mathcal{D}_i+K)|\mathcal{D}_i$, we get the following exact sequence:

$$0=H^0(\mathcal{O}(K))\to H^0(\mathcal{O}(K+\mathcal{D}_i))\to H^0(\mathcal{O}(K(\mathcal{D}_i)))$$
$$\to H^1(\mathcal{O}(K))=H^1(\mathcal{O})=0\,.$$

Hence,

$$\begin{aligned}\dim|K(\mathcal{D}_i)|&=\dim H^0(\mathcal{O}(K(\mathcal{D}_i)))-1\\&=\dim H^0(\mathcal{O}(K+\mathcal{D}_i))-1\\&=\dim|K+\mathcal{D}_i|=0\,.\end{aligned}$$

Similarly, we have

$$\dim|K(C)|=0,\text{ where }K(C)=(K+C)|C\,,$$

since $\bar{p}_g(\bar{S}-C)-1=\dim|K+C|=0$. Furthermore,

$$\begin{aligned}0=\bar{p}_g(\bar{S}-C)-1&\leqq\dim|K+C+\mathcal{D}_i|\\&\leqq\dim|2\Delta|=\bar{P}_2(\bar{S}-C)-1=0\,.\end{aligned}$$

Hence, $\dim|K+C+\mathcal{D}_i|=0$. Thus,

*) $$\dim|K(C+\mathcal{D}_i)|=\dim|K+C+\mathcal{D}_i|=0\,.$$

By the way, since $C\cap\mathcal{D}_i=\phi$, it follows that

$$\begin{aligned}K(C+\mathcal{D}_i)&=(K+C+\mathcal{D}_i)|(C+\mathcal{D}_i)\\&=(K+C)|C\oplus(K+\mathcal{D}_i)|\mathcal{D}_i\\&=K(C)\oplus K(\mathcal{D}_i)\,.\end{aligned}$$

Thus, $\dim|K(C+\mathcal{D}_i)|=\dim|K(C)|+\dim|K(\mathcal{D}_i)|+1=1$. This contradicts *).
Q.E.D.

The following lemma is a generalization of Lemma 1.

Lemma 6. *Let $(\bar{V},D)$ be a ∂-manifold and put $V=\bar{V}-D$. Assume that $\bar{\kappa}(V)\geqq0$. Let Y be a reduced divisor on V and denote by $\bar{Y}$ the closure of Y in in $\bar{V}$. Take a proper birational morphism ρ: $\bar{V}^*\to\bar{V}$ such that $(V^*,\rho^{-1}(\bar{Y}+D))$ is a ∂-manifold. $\mu=\rho|V^*$: $V^*=\bar{V}^*-\rho^{-1}(D)\to V$ is a proper birational morphism. Then letting Y^* be the proper transform of Y by μ^{-1}, we obtain*

$$\bar{\kappa}(V^*-Y^*)=\bar{\kappa}(V-Y)=\kappa(K(\bar{V})+D+\bar{Y},\ \bar{V})\,.$$

Proof. Denoting by $Z^\sharp$ the closure of Z in $\bar{V}^*$, we have $(\mu^{-1}(Y))^\sharp=Y^\sharp+\mathcal{E}$, $\mathcal{E}$ being an effective divisor which is ρ-exceptional. Similarly,

$$(\mu^*(Y))^\# = Y^\# + \mathcal{F},\ \mathcal{F} \text{ being effective and } \mathcal{F}_{\text{red}} = \mathcal{E}.$$

Recall the logarithmic ramification formula ([5]):

$$K(\bar{V}^*) + \rho^{-1}(D) = \rho^*(K(\bar{V}) + D) + \bar{R}_\mu,$$

where $\bar{R}_\mu$ is the logarithmic ramification divisor for μ. By definition, we have

$$\begin{aligned}\bar{\kappa}(V-Y) &= \bar{\kappa}(V^* - \mu^{-1}(Y)) \geqq \bar{\kappa}(V^* - Y^*)\\ &= \kappa(K(\bar{V}^*) + \rho^{-1}(D) + Y^\#, \bar{V}^*)\\ &= \kappa(\rho^*(K(\bar{V}) + D) + \bar{R}_\mu + Y^\#, \bar{V}^*)\\ &= \kappa(\rho^*(K(\bar{V}) + D) + N\bar{R}_\mu + Y^\#, \bar{V}^*),\ N \gg 0.\end{aligned}$$

This follows from $\bar{\kappa}(V) \geqq 0$ by using 2) of § 1. iii). On the other hand, $\bar{R}_\mu | V^* = R_\mu$ and $\mu^{-1}(Y) \leqq Y^* + N_1 R_\mu$ for some $N_1 > 0$. Hence, we have $(\mu^* Y)^\# \leqq Y^\# + N_2 (R_\mu)^\#$ for some $N_2 > 0$. Choosing $N \gg 0$, we obtain

$$\begin{aligned}&\kappa(\rho^*(K(\bar{V}) + D) + N\bar{R}_\mu + Y^\#, \bar{V}^*)\\ \geqq\ &\kappa(\rho^*(K(\bar{V}) + D) + (\mu^* Y)^\#, \bar{V}^*).\end{aligned}$$

We note that

$$\rho^*(D) + (\mu^* Y)^\# = \rho^*(D + \bar{Y}).$$

Hence,

$$\begin{aligned}\kappa(\rho^*(K(\bar{V}) + D) + (\mu^* Y)^\#, \bar{V}^*) &= \kappa(\rho^*(K(\bar{V}) + D + \bar{Y}), \bar{V}^*)\\ &= \kappa(K(\bar{V}) + D + \bar{Y}, \bar{V}^*).\end{aligned}$$

It is easily seen that

$$\kappa(K(\bar{V}) + D + \bar{Y}, \bar{V}^*) \geqq \bar{\kappa}(V - Y) \geqq \bar{\kappa}(V^* - Y^*).$$

Thus we obtain the desired equality. Q.E.D.

We come back to the study of a logarithmic $K3$ surface S of type II_a. Writing $D_A = \mu(D_A)$ and $Y = \mu_*(D_B)$, we have by Lemma 6

$$\bar{\kappa}(\bar{S}_* - D_A - Y) = \bar{\kappa}(\bar{S} - D) = 0.$$

Since $K(\bar{S}_*) + D_A \sim 0$, we make use of the following proposition.

Proposition 4. *With the notation being as in Proposition 3, let Y be a reduced divisor on $\bar{S}$ which does not contain C. Suppose that $\bar{\kappa}(\bar{S} - C - Y) = 0$. Then $\kappa(Y, \bar{S}) = 0$. Moreover, letting $\mathcal{Y}_1, \cdots, \mathcal{Y}_u$ be the connected components of Y, we have the following assertions, separately.*

1) *If $\mathcal{Y}_j \cap C \neq \phi$, then $(\mathcal{Y}_j, C) = 1$ and $\mathcal{Y}_j$ is an exceptional curve of the first kind in $\bar{S}$.*

2) *If $\mathcal{Y}_j \cap C = \phi$, then $\mathcal{Y}_j$ is a curve of Dynkin type ADE.*

Proof. Letting $Y_0 = Y \cap S$, $S = \bar{S} - C$, we have $\bar{Y}_0$ (the closure of Y_0 in $\bar{S}$) $= Y$. Take a proper birational morphism $\rho\colon \bar{S}^* \to S$ such that $(\bar{S}^*, \rho^{-1}(C+Y))$ is a ∂-surface. By Lemma 6, we have

$$\kappa(K(\bar{S})+C+Y, \bar{S}) = \bar{\kappa}(\bar{S}-C-Y) = 0\,.$$

Recalling Proposition 3, we get $\kappa(Y, \bar{S}) = 0$. Let $\sum Y_j$ be the irreducible decomposition of Y and let $\mathcal{Y}_1, \cdots, \mathcal{Y}_u$ be the connected components of Y. By Lemma 5, letting $\mathcal{Y}$ be a connected reduced divisor in Y, we have

$$\begin{aligned} 0 &= \dim|\mathcal{Y}| = \dim|K+C+\mathcal{Y}| \\ &= \dim H^1(\mathcal{O}_{C+\mathcal{Y}})-1 \geqq (C+\mathcal{Y}, K+C+\mathcal{Y})/2\,. \end{aligned}$$

Hence, $(C+\mathcal{Y}, \mathcal{Y}) \leqq 0$. If $C+\mathcal{Y}$ is connected,

$$\begin{aligned} 0 &= \dim|K+C+\mathcal{Y}| = (C+\mathcal{Y}, K+C+\mathcal{Y})/2 \\ &= \pi(C+\mathcal{Y})-1 = \pi(C)+\pi(\mathcal{Y})+(C, \mathcal{Y})-2 \\ &= \pi(\mathcal{Y})+(C, \mathcal{Y})-1 \geqq \pi(\mathcal{Y})\,. \end{aligned}$$

From this, it follows that $\pi(\mathcal{Y}) = 0$ and $(C, \mathcal{Y}) = 1$. If $C+\mathcal{Y}$ is not connected, then

$$\begin{aligned} 0 &= \dim|K+C+\mathcal{Y}| = \dim H^1(\mathcal{O}_{C+\mathcal{Y}})-1 \\ &= \dim H^1(\mathcal{O}_C)+\dim H^1(\mathcal{O}_{\mathcal{Y}})-1 \\ &= \dim H^1(\mathcal{O}_{\mathcal{Y}}) = \pi(\mathcal{Y}) = (\mathcal{Y}, K+\mathcal{Y})/2+1\,. \end{aligned}$$

On the other hand, $(C, \mathcal{Y}) = 0$ yields $(K, \mathcal{Y}) = 0$, since $K+C \sim 0$. Hence, $\mathcal{Y}^2 = -2$. In particular, if $Y_j \cap C \neq \phi$, then Y_j is a C-exceptional curve, and if $Y_j \cap C = \phi$, then $Y_j^2 = -2$ and $(K, Y_j) = 0$.

For any $m_j \geqq 0$, define $Z = \sum m_j Y_j \neq 0$. We write $Z = \mathcal{Z}_1 + \cdots + \mathcal{Z}_v$ where $\operatorname{Supp}(\mathcal{Z}_1), \cdots, \operatorname{Supp}(\mathcal{Z}_v)$ are the connected components of $\operatorname{Supp} Z$. By Lemma 5,

$$\begin{aligned} 0 &= \dim|Z| = \dim|Z+C+K| = \dim H^1(\mathcal{O}_{C+Z})-1 \\ &\geqq (C+Z, C+K+Z)/2 = ((C, Z)+Z^2)/2\,. \end{aligned}$$

If $(C, Z) > 0$, then $Z^2 \leqq -1$. Next, assume $(C, Z) = 0$. Then $(C, \mathcal{Z}_1) = \cdots = (C, \mathcal{Z}_v) = 0$. This implies $(K, \mathcal{Z}_1) = \cdots = (K, \mathcal{Z}_v) = 0$. Hence,

$$1 = \dim H^1(\mathcal{O}_{C+Z}) = \dim H^1(\mathcal{O}_C)+\sum \dim H^1(\mathcal{O}_{\mathcal{Z}_i})\,.$$

Thus $\dim H^1(\mathcal{O}_{\mathcal{Z}_i}) = 0$. Recalling Riemann Roch Theorem on $\bar{S}$, we have

$$(\mathcal{Z}_i, \mathcal{Z}_i+K)/2 = \dim H^1(\mathcal{O}_{\mathcal{Z}_i}) - \dim H^0(\mathcal{O}_{\mathcal{Z}_i}) \leqq -1\,.$$

Since $(\mathcal{Z}_i, K) = 0$, we have $\mathcal{Z}_i^2 \leqq -2$. Hence $\operatorname{Supp} \mathcal{Z}_i$ is a curve of Dynkin type and so the intersection matrix $[(Y_i, Y_j)]$ is negative-definite. Thus we complete the proof of Proposition 4.

Proposition 5. *Let $\bar{S}$ be a complete surface and C a non-singular elliptic curve on $\bar{S}$. Suppose that $q(\bar{S})=0$ and $K(\bar{S})+C\sim 0$. Then $\bar{q}(\bar{S}-C)=0$, and $(\bar{S}, C)$ is obtained from one of the following three ∂-surfaces by attaching 1/2-points:*
a-i) *$(\boldsymbol{P}^2, E)$ where E is a non-singular curve of degree 3,*
a-i) *$(\boldsymbol{P}^1\times\boldsymbol{P}^1, E)$ where E is a non-singular curve of degree (2, 2),*
a-iii) *$(\sum_2, E)$ where $\sum_2$ is a Hirzebruch surface of degree 2 and E a non-singular elliptic curve such that $K(\sum_2)+E\sim 0$.*

Proof. $\bar{q}(\bar{S}-C)=0$ follows from Lemma 2. First assume that $\bar{S}=\boldsymbol{P}^2$ or $\sum_0=\boldsymbol{P}^1\times\boldsymbol{P}^1$ or $\sum_b$ $(b\geqq 2)$, that is the Hirzebruch surface of degree b.

Lemma 7. *A Hirzebruch surface $\sum_b$ $(b\geqq 1)$ is a non-trivial $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ on which there exists one and only one irreducible curve Δ_∞ with negative self-intersection number $-b$. Δ_∞ is a section of $\sum_b\to\boldsymbol{P}^1$, whose fiber is denoted by F. Any section $C\neq\Delta_\infty$ is linearly equivalent to $\Delta_\infty+\alpha F$ $(\alpha\geqq b)$. Then $C^2=2\alpha-b$ and $(C, \Delta_\infty)=\alpha-b$. The smallest C^2 is b. Since $\dim|\Delta_\infty+bF|=1+b$, we have sections Δ_λ (λ being a point of $\boldsymbol{C}^{1+b}$), which satisfy $\Delta_\lambda\cap\Delta_\infty=\phi$ and $\Delta_\lambda^2=b$. Moreover, $-K(\sum_b)\sim\Delta_\infty+\Delta_\lambda+2F$.*

Proof. The verification is easy and omitted.

We continue the proof of Proposition 5. If $\bar{S}=\sum_b$, and $E\sim -K(\sum_b)\sim \Delta_\lambda+\Delta_\infty+2F$, then $(E, \Delta_\infty)=-b+2$. By the way, $E\neq\Delta_\infty$. Hence, $(E, \Delta_\infty)\geqq 0$, which implies $b\leqq 2$. We have to show that there exists a non-singular member in $|-K(\sum_2)|$.

Lemma 8. *Let $V=\boldsymbol{P}^1\times\boldsymbol{P}^2$. Then $\sum_b$ $(b\geqq 1)$ is isomorphic to a non-singular hypersurface of degree $(b, 1)$ of V.*

Proof. Letting h be a line on $\boldsymbol{P}^2$, we put $L=p\times\boldsymbol{P}^2$ and $M=\boldsymbol{P}^1\times h$. Then, by the adjunction formula,

$$-K(V)\sim 2L+3M.$$

Since $bL+M$ is very ample $(b\geqq 1)$, a general member W of $|bL+M|$ is non-singular and

$$-K(W)\sim(2L+3M-M-bL)|W.$$

Hence $K(W)^2=8$. Moreover, the projection $\pi: V\to\boldsymbol{P}^1$ induces the fibered surface $\pi'=\pi|W: W\to\boldsymbol{P}^1$, whose fiber is linearly equivalent to $L|W$. Clearly, $(L|W)^2=0$ and $L|W\simeq\boldsymbol{P}^1$. Hence, $\pi|W: W\to\boldsymbol{P}^1$ is a $\boldsymbol{P}^1$-bundle. $M|W$ is a section which satisfies $(M|W)^2=b$. Hence $W\simeq\sum_b$. Employing the notation in Lemma 7, we see that $\Delta_\infty\sim(M-bL)|W$ and $\Delta_\lambda\sim M|W$. Q.E.D.

When $b=2$, $-K(\sum_2)$ is linearly equivalent to $2M|W$. $(2M|W)^2=8$ and $2M|W$ has no base points. Therefore a general member of $|-K(\sum_2)|$ is a non-singular elliptic curve. A curve E on $\boldsymbol{P}^2$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$ which satisfies the condition of Proposition 5 is a non-singular curve of degree 3 or degree (2, 2), respectively.

Recalling that a relatively minimal rational surface $\bar{S}$ is isomorphic to $\boldsymbol{P}^2$, $\boldsymbol{P}^1\times\boldsymbol{P}^1$ or $\sum_b$, we have only to consider the case where there is an exceptional curve L of the first kind on $\bar{S}$. Since $L\neq C$ and $L^2=(K(\bar{S}), L)=-1$, we have $(C, L)=-(K(\bar{S}), L)=1$. Hence, L is a C-exceptional curve. Contracting such L successively, we complete the proof.

With the notation being as in Proposition 5, let Y be a curve of Dynkin type in $S=\bar{S}-C$. Corresponding to the 1/2-point attachments, we have a proper birational morphism $\mu\colon \bar{S}\to\bar{S}_*$, $\bar{S}_*=\boldsymbol{P}^2$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$ or $\sum_2$. By Lemma 6, writing $Z=\mu_*(Y)$, we have $\bar{\kappa}(\bar{S}_*-\mu(C)-Z)=\bar{\kappa}(\bar{S}-C-Y)=\kappa(Y, \bar{S})=0$. Hence, Z is a sum of exceptional curves and a curve of Dynkin type. Since $\bar{S}_*$ is relatively minimal, Z is a curve of Dynkin type such that $Z\cap\mu(C)=\phi$. Thus, $Z=\Delta_\infty$ in $\sum_2$. Accordingly, $\mu(Y)$ is a union of a finite set of points in $\mu(C)$ and $\Delta_\infty\subset\bar{S}_*=\sum_2$.

Therefore, Y is a curve of Dynkin type A. Summarizing the argument above, we obtain the following proposition.

Proposition 6. *Let $(\bar{S}, D)$ be a relatively ∂-minimal surface such that $S=\bar{S}-D$ is a logarithmic K3 surface. Suppose that $(\bar{S}, D_A)$ is relatively ∂-minimal and that there are no D-exceptional curves of the first kind on $\bar{S}$. Then such ∂-surfaces $(\bar{S}, D)$ are classified into the following table. There, $D=\sum C_i$ is the irreducible decomposition and C_1 is a non-singular elliptic irreducible curve.*

Table II_a.

class	$\bar{S}$	D with the self-intersection numbers	$\pi_1(S)$	S
a-i)	$\boldsymbol{P}^2$	C_1 9	$\boldsymbol{Z}/(3)$	affine
a-ii)	$\boldsymbol{P}^1\times\boldsymbol{P}^1$	C_1 8	$\boldsymbol{Z}/(2)$	
a-iii)	$\sum_2$	C_1 8	$\boldsymbol{Z}/(2)$	non-affine
a-iii)		C_1 8 C_2 -2	?	

We have the following

Theorem II_a. *Let $(\bar{S}, D)$ be a ∂-surface whose interior S is a logarithmic K3 surface of type II_a. Then there exists a birational morphism $\mu\colon \bar{S}\to\bar{S}_*$ such that*

1) $\bar{S}_*=\boldsymbol{P}^2$ *or* $\boldsymbol{P}^1\times\boldsymbol{P}^1$ *or* $\sum_2$ 2) $C=\mu(D_A)$ *is a non-singular curve,* 3) $\mu(D_B)$ *is a finite set or a union of a finite set and* $Z=\Delta_\infty$ *on* $\sum_2$. *The latter case occurs only when* $\bar{S}_*=\sum_2$.

Structure of logarithmic $K3$ surfaces of type $\mathrm{II_a}$ is studied precisely by examining each class of a-i) through a-iii)* separately. We use the following notion: Let S be a surface and let μ be a proper birational morphism: $S^*\to S$ such that there exists a dominant morphism f: $S^*\to J$, J being a curve, whose general fiber $f^{-1}(u)$ is $\boldsymbol{C}^*$. Then we say that *S is a $\boldsymbol{C}^*$-fibered surface or S has the structure of $\boldsymbol{C}^*$-fibered surface.*

Proposition 7. *Every surface of the class* a-ii) *or* a-iii) *has a structure of $\boldsymbol{C}^*$-fibered surface.*

Proof is easy.

Proposition 8. *Let S be a surface of the class* a-i) *or* a-iii)*. *Then S does not admit the structure of $\boldsymbol{C}^*$-fibered surface.*

Proof. First we let S be a surface of the class a-iii)*. Suppose that there exist a proper birational morphism μ: $S^*\to S$ and a dominant morphism f: $S^*\to J$, J being a complete curve, whose general fiber is $\boldsymbol{C}^*$. Choosing a suitable completion S^* of S^* with smooth boundary D^*, we assume that μ defines a morphism $\bar{\mu}$: $\bar{S}^*\to\sum_2$ and $D^*=\bar{\mu}^{-1}(C_1+C_2)$ and that f defines a morphism $\bar{f}$: $\bar{S}^*\to J$. By C_1^* we denote the proper transform of C_1 by μ^{-1}, which is a non-singular elliptic curve. Since a general fiber of $\bar{f}$ is $\boldsymbol{P}^1$, C_1^* is not contained in a fiber of $\bar{f}$. Hence $\bar{f}(C_1^*)=J$. Since $\bar{S}^*$ is rational, J is $\boldsymbol{P}^1$. This implies that $\bar{f}\,|\,C_1^*$: $C_1^*\to\boldsymbol{P}^1$ is a two-sheeted covering. Hence, $\bar{f}(C_2^*)$ is a point, because $\bar{f}^{-1}(u)\cap D^*=\{p_1, p_2\}$ for a general point $u\in J$. Therefore, $g=\bar{f}\cdot\bar{\mu}^{-1}$: $\sum_2\to J$ turns out to be a morphism. Moreover, $g(C_2)$ is a point a. Hence, C_2 is a part of the singular fiber $g^{-1}(a)$. Since $C_2^2=-2$, there is another component C_3 in $g^{-1}(a)$ such that $C_3^2=-1$. This contradicts the fact that $\sum_2$ is a relatively minimal surface. It is easier to prove the same result for surfaces of the class a-i). Q.E.D.

Proposition 9. *There exists an algebraic pencil $\{C_u\}$ on each surface of the classes* a-i) *and* a-iii)* *whose general member C_u is $\boldsymbol{C}^*$.*

Here, an *algebraic pencil* $\{C_u\}$ on S is understood as follows: there exist an algebraic surface S^* and a proper birational morphism ρ: $S^*\to S$ in which ψ: $S^*\to J$ is a fibered surface whose general fiber C_u^*. $\{C_u=\rho(C_u^*)\}$ is the algebraic pencil on S.

We omit the proof of Proposition 9.

If there is a proper birational map f: $S_1 \to S_2$ then the existence of the algebraic pencil $\{C_u\}$, $C_u \simeq \boldsymbol{C}^*$, on S_1, induces the existence of the same thing on S_2. Moreover, when S_1 is an open set of S_2 with $\bar{\kappa}(S_2) \geqq 0$, the existence of an algebraic pencil of $C_u \simeq \boldsymbol{C}^*$ on S_1 implies the existence of the same thing on S_2. In fact, there are a proper birational morphism ρ: $S_1^* \to S_1$ and a morphism ψ: $S_1^* \to J$ with $C_u = \rho(\psi^{-1}(u)) \simeq \boldsymbol{C}^*$ for a general $u \in J$. Let Γ_u be the closure of C_u in S_2. Then $\bar{\kappa}(\Gamma_u) \leqq 0$. If $\bar{\kappa}(\Gamma_u) = -\infty$, it would imply that $\bar{\kappa}(S) = -\infty$, a contradiction.

Accordingly we get

Proposition 10. *There is an algebraic pencil* $\{C_u\}$ *with the general member* $C_u \simeq \boldsymbol{C}^*$ *on any logarithmic* $K3$ *surface of type* $\mathrm{II_a}$.

Corollary. *A logarithmic* $K3$ *surface of type* $\mathrm{II_a}$ *is not measure-hyperbolic.*

Proof follows from the fact that $\boldsymbol{C}^*$ is not measure-hyperbolic.

Proposition 11. *Let* S *be a surface in the TABLE* IIa. *Then, Aut* (S) *is a finite group.*

Proof. We give a proof for a surface of the class a-iii)*. Let $\varphi \in \mathrm{Aut}(S)$. Then φ extends to an isomorphism of $\bar{S} = \sum_2$, since $g(C_1) = 1$ and $C_2^2 = -2 \leqq -2$ ([12]). Thus $\mathrm{Aut}(S) \subset \mathrm{Aut}_D(\sum_2) = \{\varphi \in \mathrm{Aut} \sum_2;\ \varphi(D) = D\}$. Let π: $\sum_2 \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-bundle structure of $\sum_2$. We have the group extension:

$$1 \to G_1 \to \mathrm{Aut}\,(\textstyle\sum_2) \to PGL\,(1, k) = \mathrm{Aut}\,(\boldsymbol{P}^1) \to 1\,.$$

It is well known that G_1 is an algebraic group of dimension 4. Moreover, G_1 is an affine group. Hence Aut $(\sum_2)$ is an affine algebraic group. And so is $\mathrm{Aut}_D(\sum_2)$. Furthermore, we have the group homomorphism γ: $\mathrm{Aut}_D(\sum_2) \to \mathrm{Aut}(C_1)$ which is the restriction, i.e., $\gamma(\varphi) = \varphi\,|\,C_1$. Therefore, $\mathrm{Im}\,\gamma$ is finite, since Aut (C_1) is a finite union of elliptic curves. Put $G_2 = \mathrm{Ker}\,\gamma$, which turns out to be a finite group. Thus $\mathrm{Aut}_D\,(\sum_2)$ is finite and so is $\mathrm{Aut}\,(S)$. Q.E.D.

Proposition 12. *Let* $\bar{S}$ *be a rational surface and* C *a non-singular elliptic curve on* $\bar{S}$. *Let* Y *be a reduced divisor on* S *such that* $\bar{\kappa}(\bar{S} - (C \cup Y)) = 0$. *Then* $\bar{q}(\bar{S} - (C \cup Y)) = 0$, i.e., $\bar{S} - (C \cup Y)$ is a logarithmic $K3$ surface of type $\mathrm{II_a}$.

A proof follows from the arguments in the proofs of Propositions 3 and 4. Actually, the intersection matrix of Y is negative-definite and hence we can use Lemma 2.

Propostion 13. *Let* $(\bar{S}, D)$ *be a* ∂*-surface whose interior* S *is a logarithmic* $K3$ *surface of type* $\mathrm{II_a}$. *Suppose that* 1) $(\bar{S}, D)$ *is relatively* ∂*-minimal,* 2) S *has no* $1/2$*-points, and* 3) D *is connected. Then* $(\bar{S}, D)$ *is one of* a-i)$\sim$a-iii) *in Proposition* 5.

Proof. At the beginning of §4 we have had the decomposition: $D=D_A+D_B$. Suppose that there exists an irreducible exceptional curve E of the first kind on $\bar{S}-D_A$. In view of Preposition 4, by contracting E we have a proper birational ∂-morphism λ: $(\bar{S}, D)\rightarrow(\bar{S}_1, D_1)$. We have the following cases: 1) If $E\subset D_B$ or $E\cap D_B=\phi$, this contradicts the hypothesis. 2) If $E\cap D_B\neq\phi$, then λ: $(\bar{S}, D+E)\rightarrow(S_1, D_1)$ is a non-canonical blowing up. In fact if λ were canonical, D would be disconnected. Thus $E-D_B\subset S$ is a 1/2-point. This is also a contradiction. Accordingly, we conclude that $\bar{S}-D_A$ is relatively minimal. By Proposition 4, D_B is a union of exceptional curves of the first kind. Hence $D_B=\phi$. Since, there are no D-exceptional curves, it follows that $\bar{S}$ is a relatively minimal surface. Q.E.D.

5. Logarithmic $K3$ surfaces of type $\mathrm{II_b}$.

In §5, let S be a logarithmic $K3$ surface and let $(\bar{S}, D)$ be a ∂-surface such that $S=\bar{S}-D$. By $C_1, \cdots, C_s$ we denote the irreducible components of D. Since $h(\Gamma(D))=1$, there is a circular boundary $D_A=C_1+\cdots+C_r\leqq D$. $\bar{p}_A(\bar{S}-D_A)=1$ induces that $\bar{S}-D_A$ is also a logarithmic $K3$ surface of type $\mathrm{II_b}$. Contracting exceptional curves of the first kind in $\bar{S}-D_A$ successively, we have a non-singular complete surface $\bar{S}_*$ and a birational morphism μ: $\bar{S}\rightarrow\bar{S}_*$ such that μ is isomorphic around $D_A\xrightarrow{\sim}\mu(D_A)$ and such that $\bar{S}-\mu(D_A)$ has no exceptional curves of the first kind. After choosing D to be a minimal boundary, we have a minimal boundary $D_A=\mu(D_A)$. Then $(\bar{S}_*, D_A)$ is a relatively ∂-minimal ∂-surface.

We write $D=D_A+D_B$ and $Y=\mu_*(D_B)$. By Lemma 6 we have

$$0=\bar{\kappa}(\bar{S}-D)=\bar{\kappa}(\bar{S}_*-D_A-Y)\,.$$

From the condition $h(\Gamma(D_A))=1$, we infer readily that $\bar{p}_g(\bar{S}_*-D_A)=1$. Hence, $\bar{P}_i(\bar{S}_*-D_A)=1$ for any $i\geqq 1$. However, $\bar{q}(\bar{S}_*-D_A)\geqq 0$.

Proposition 14. *Let $(\bar{S}, D)$ be a circular ∂-surface* (i.e., D is circular) *which is relatively ∂-minimal. Suppose that $\bar{\kappa}(\bar{S}-D)=0$. Then $K(S)+D\sim 0$.*

Proof. It is easy to check that $\bar{S}$ is a rational surface. Assuming that $|K(\bar{S})+D|$ has a non-trivial member $\Delta=\sum r_iE_i$ $(r_i>0)$ we shall derive a contradiction.

Now, $0=\kappa(\bar{S}-D)=\kappa(K(\bar{S})+D, \bar{S})=\kappa(\Delta, \bar{S})=\kappa(\sum E_i, \bar{S})$ implies that the intersection matrix $[(E_i\,E_j)]$ is negative semi-definite. We assume $(\Delta, E_1)\leqq 0$ and $E_1\not\subset D$ Then by the same reasoning as in the proof of Proposition 4, we have the following cases:

Case 1: $\pi(E_1)=1$. Then $E_1\cap D=\phi$ and $E_1^2=(K, E_1)=0$.

Case 2: $\pi(E_1)=0$ and $(D, E_1)\geqq 1$. Then $E_1^2=(K, E_1)=-1$ and $(E_1, D)=1$. Hence E_1 is D-exceptional. By detaching 1/2-points, we may assume that this case does not occur.

Case 3: $\pi(E_1)=0$ and $(D, E_1)=0$. Then $E_1\cap D=\phi$ and $E_1^2=-2$, $(K, E_1)=0$.
In all cases we have $(\Delta_1, E_1)=0$. If $E_1\subset D$ and $r\geqq 2$, we have $D'+E_1=D$, $E_1=\boldsymbol{P}^1$ and $(D', E_1)=2$. Hence

$$\dim|K+D'|=\bar{p}_g(\bar{S}-D')-1=h(\Gamma(D'))-1\,.$$

On the other hand, $|K+D'|\ni(r_1-1)E_1+r_2E_2+\cdots$. This is a contradiction.

Thus, $\Delta^2=\sum r_i(\Delta, E_i)\geqq 0$. Since $\kappa(\Delta, \bar{S})=0$, we have $\Delta^2=0$. By the similar argument to the proof of Proposition 4, we derive a contradiction. Q.E.D.

Proposition 15. *With the notation being as in Proposition* 14, *let Y be a reduced divisor on $\bar{S}$ which does not contain any components of D. Suppose that $\bar{\kappa}(\bar{S}-D-Y)=0$. Then $\kappa(Y, \bar{S})=0$. By $\mathcal{Y}_1, \cdots, \mathcal{Y}_u$, we denote the connected components of Y. If $\mathcal{Y}_j\cap D\neq\phi$, then $(\mathcal{Y}_j, D)=1$ and $\mathcal{Y}_j$ is an exceptional curve of the first kind. If $\mathcal{Y}_j\cap D=\phi$, then $\mathcal{Y}_j$ is a curve of Dynkin type* A.

The proof of Proposition 4 can be used again here.

Proposition 16. *Let $(\bar{S}, D)$ be a circular ∂-surface such that $K(\bar{S})+D\sim 0$. Then $(\bar{S}, D)$ is obtained from one of the following ∂-surfaces by attaching several 1/2-points and canonical blowing ups.*

b-i) $\bar{S}=\boldsymbol{P}^2$, $D=H_1+H_2+H_3$ where *each H_i is a line on $\boldsymbol{P}^2$,*

b-ii) $\bar{S}=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $D=H_1+H_2+G_1+G_2$, where *each H_i is a line of degree* (1, 0) *and each G_j is a line of degree* (0, 1),

b-iii) $\bar{S}=\sum_\beta$, $D=\Delta_\lambda+\Delta_\infty+F_1+F_2$, where *each F is a fiber,*

b-iv) $\bar{S}=\boldsymbol{P}^2$, $D=H+C$, where H is *a line and C is a conic,*

b-v) $\bar{S}=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $D=C_1+C_2$ where *each C_i is a curve of degree* (1, 1),

b-vi) $\bar{S}=\sum_2$, $D=\Delta_0+\Delta_\lambda$ $(\lambda\neq 0)$, where *the Δ_λ is a section which is different from Δ_∞,*

b-vii)$_\beta$ $\bar{S}=\sum_\beta$, $D=F+\Delta_\infty+C_3$ where *C_3 is a non-singular rational curve which is linearly equivalent to Δ_0+F,*

b-viii) $\bar{S}=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $D=H_1+G_1+C$, where *H is a line of degree* (1, 0), *G_1 is a line of degree* (0, 1), *and C is a curve of degree* (1, 1),

b-ix) $\bar{S}=\boldsymbol{P}^2$, $D=C$, where *C is a cubic curve with one ordinary double point,*

b-x) $\bar{S}=\boldsymbol{P}^2$ $D=C$, where *C is a curve of degree* (2, 2) *which has one ordinary double point,*

b-xi) $\bar{S}=\sum_2$, $D=C$, where *C is a rational curve with only one ordinary double point which is linearly equivalent to $2\Delta_\lambda$,*

b-xii) $\bar{S}=\boldsymbol{P}^1\times\boldsymbol{P}^1$, $D=G+C$, where *G is a line of degree* (0, 1) *and C is a curve of degree* (2, 1),

b-xiii)$_\beta$ $\bar{S}=\sum_\beta$, $D=\Delta_\infty+C$, where *C is a curve which is linearly equivalent to Δ_0+2F.*

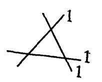
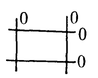
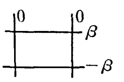
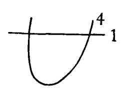
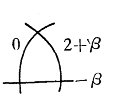
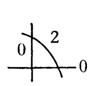
Proof is easy and left to the reader.

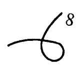
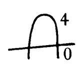
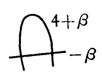
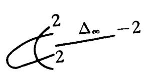
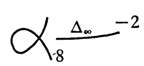
In the following Table II_b, we exhibit $\bar{q}$ and configurations of components of D of b-i)$\sim$b-xiii).

Proposition 17. *Let* $(\bar{S}, D)$ *be a circular* ∂*-surface whose interior* S *is a logarithmic K3 surface or a surface satisfying the following conditions:* 1) $\bar{S}$ *is rational,* 2) $\bar{\kappa}(S)=0$, 3) $\bar{p}_g(S)=1$, *and* 4) $\bar{q}(S)=1$ *or* 2. *Suppose that* i) $(\bar{S}, D)$ *is relatively* ∂*-minimal,* ii) D *is connected, and* iii) S *has no* 1/2*-points. Then* $(\bar{S}, D)$ *is one of* b-i)$\sim$b-xiii)$_\beta$ *in TABLE* II_b.

Proof is similar to that of Proposition 13.

Table II_b of $(\bar{S}, D)$, $S=\bar{S}-D$

$\bar{q}$	class	$\bar{S}$	configuration of D	$\pi_1(S)$	S
	b-i)	$\boldsymbol{P}^2$		$\boldsymbol{Z}^2$	
2	b-ii)	$\boldsymbol{P}^1\times\boldsymbol{P}^1$		$\boldsymbol{Z}^2$	$\boldsymbol{C}^*$
	b-iii)$_\beta$ ($\beta\geqq 2$)	Σ_β		$\boldsymbol{Z}^2$	
	b-iv)	$\boldsymbol{P}^2$		$\boldsymbol{Z}$	
	b-v)	$\boldsymbol{P}^1\times\boldsymbol{P}^1$		$\boldsymbol{Z}$	
1	b-vi)	Σ_2		$\boldsymbol{Z}$	
	b-vii)$_\beta$ ($\beta\geqq 2$)	Σ_β		$\boldsymbol{Z}$	
	b-viii)	$\boldsymbol{P}^1\times\boldsymbol{P}^1$		$\boldsymbol{Z}$	

$\bar{q}$	class	$\bar{S}$	configuration of D	$\pi_1(S)$	S
	b-ix)	$\boldsymbol{P}^2$		$\boldsymbol{Z}/(3)$	
	b-x)	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$		$\boldsymbol{Z}/(2)$	
0	b-xi)	Σ_2		$\boldsymbol{Z}/(2)$	
	b-xii)	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$		$\boldsymbol{Z}/(2)$	
	b-xiii)$_\beta$ ($\beta \geqq 2$)	Σ_β		?	
1	b-vi)*	Σ_2		?	
0	b-xi)*			?	

Next we treat the ∂-surface (S, D) whose boundary is not connected. As in § 4, we have to look for a curve Z of Dynkin type on $\bar{S}-D$ where $(\bar{S}, D)$ is one of b-i) through b-xiii)$_\beta$. Such Z exists only in the cases b-vi) and b-xi). Then Z turns out to be Δ_∞ of Σ_2. We write b-vi)* or b-xi)* in the case of disconnected boundaries. Therefore we obtain the following

Theorem II$_\text{b}$. *Let $(\bar{S}, D)$ be a ∂-surface whose interior S is a logarithmic $K3$ surface of type* II$_\text{b}$. *Then, there exists a birational morphism μ: $\bar{S} \to \bar{S}_*$ such that $(\bar{S}_*, \mu(D_A))$ is one of* b-i) *through* b-xiii)$_\beta$ *in TABLE* II$_\text{b}$. *Moreover, $\mu(D_B)$ is a finite set or a union of a finite set and $Z=\Delta_\infty$ on Σ_2. The latter case occurs only when $(\bar{S}, \mu_*(D)-Z)$ is the class* b-vi) *or* b-xi).

REMARK. In the above theorem the hypothesis that S is a logarithmic $K3$ surface of type II$_\text{b}$ is replaced by the following condition that 1) $\bar{p}_g(S)=1$ and $\bar{\kappa}(S)=0$, 2) $\bar{S}$ is rational, 3) D consists of rational curves.

In order to prove the generalized Theorem II$_\text{b}$, we have only to note that

Propositions 14, 15 and 16 were proved without the logarithmic irregularity condition to the effect $\bar{q}=0$.

6. Surfaces with $\bar{\kappa}=0$ and $\bar{p}_g=1$. In general, let $(\bar{S}, D)$ be a ∂-surface such that the interior S satisfies $\bar{p}_g(S)=1$ and $\bar{\kappa}(S)=0$. Then $p_g(\bar{S})\leqq 1$ and $\kappa(\bar{S})\leqq 0$.

Proposition 18. *If $p_g(\bar{S})=0$, then $\kappa(\bar{S})=-\infty$. Hence, $\bar{S}$ is a ruled surface.*

Proof. In view of Proposition 2, it suffices to derive a contradiction from the hypothesis that $\kappa(\bar{S})=0$, $p_g(\bar{S})=0$, and $q(\bar{S})\geqq 1$. Such a surface $\bar{S}$ is birationally equivalent to a hyperelliptic surface, whose universal covering surface is an abelian surface. Namely, contracting exceptional curves of the first kind on $\bar{S}$ succssively, we get a hyperelliptic surface $\bar{S}_*$ and a birational morphism μ: $\bar{S}\to\bar{S}_*$. Then by Lemma 1,

$$\begin{aligned} 0 &= \bar{\kappa}(S) = \kappa(K(\bar{S})+D, \bar{S}) = \kappa(K(\bar{S}_*)+\mu_*(D), \bar{S}_*) \\ &= \kappa(\mu_* D, \bar{S}) . \end{aligned}$$

This implies that $\mu_* D=0$. Thus

$$\begin{aligned} H^0(\mathcal{O}(K(\bar{S})+D)) &= H^0(\mathcal{O}(\mu^*(K(\bar{S}_*))+R_\mu+D)) \\ &\tilde{\to} H^0(\mathcal{O}(K(\bar{S}_*))) = 0 . \end{aligned}$$

This contradicts $\bar{p}_g(S)=\dim H^0(\mathcal{O}(K(\bar{S})+D))=1$. Q.E.D.

Consequently, we have the following cases to examine separately.

1) If $p_g(\bar{S})=0$ and $q(\bar{S})=0$, then $\bar{S}$ is a rational surface. Hence, letting $\sum_{j=1}^{s} C_j$ be the irreducible decomposition of D,

α) if $g(C_1)=1$, then put $D_A=C_1$,

β) if $g(C_1)=\cdots=g(C_s)=0$, then there is a circular boundary $D_A=C_1+\cdots+C_r\subset D$.

2) If $p_g(\bar{S})=0$ and $q(\bar{S})\geqq 1$, then $\bar{S}$ is a ruled surface of genus 1. Let f: $\bar{S}\to J$ be the Albanese map of $\bar{S}$, J being an elliptic curve, since $p_g(\bar{S})=0$. For a general point $y\in J$, $f^{-1}(y)$ turns out to be a non-singular rational curve. Define $C_y=f^{-1}(y)-D\cap f^{-1}(y)$. Then by Kawamata's Theorem ([14]), we obtain

$$0 = \bar{\kappa}(S)\geqq\bar{\kappa}(C_y)+\bar{\kappa}(J) = \bar{\kappa}(C_y) .$$

Hence, $\bar{\kappa}(C_y)=0$ follows. This implies that $C_y\simeq \boldsymbol{C}^*$ and $(D, f^{-1}(y))=2$. Hence, the horizontal component D_A defined to be $\{\sum C_j ; f(C_j)=J\}$ satisfies that $(D_A, f^{-1}(y))=2$. Referring to the following lemma, we have

$$\dim|K(\bar{S})+D_A| = 0 , \qquad \text{i.e.,}\ \bar{p}_g(\bar{S}-D_A) = 1 .$$

Lemma 9. *Let $\bar{V}$ be a complete normal variety and let A, B be divisors on $\bar{V}$ such that $\kappa(A, \bar{V})\geqq 0$, $|A+B|\neq\phi$, B is effective, and $\kappa(A+B, \bar{V})=0$. Then $|A|\neq\phi$.*

Proof. Choose $i>0$ such that $|iA|\neq\phi$ and take $X\in|A+B|$ and $Z\in|iA|$. Then $Z+iB\sim iX$. By $\kappa(X, \bar{V})=0$, we have $Z+iB=iX$. Hence, $Z=i(X-B)$ is effective. This implies that $X-B$ is effective. Q.E.D.

3) If $p_g(\bar{S})=1$, then put $D_A=0$.

In all cases above, we define D_B by $D=D_A+D_B$

Theorem III. *With the notation being as above, we suppose that $\bar{S}-D_A$ has no exceptional curves of the first kind. Then $K(\bar{S})+D_A\sim 0$.*

Proof. Recalling Propositions 3 and 14, it suffices to prove under the assumption that $\bar{S}$ is a ruled surface with $q(\bar{S})=1$. Take $\Delta\in|K+D_A|$ and we shall derive a contradiction from the hypothesis $\Delta\neq 0$. Let $\sum r_jE_j$ be the irreducible decomposition of Δ. $[(E_i, E_j)]$ is negative semi-definite. In particular, $E_j^2\leqq 0$. First assume that $(\Delta, E_1)\leqq 0$, since $\Delta^2\leqq 0$. If $E_1\subset D_A$, then, putting $D_A=E_1+D'$, we would have $(f^{-1}(y), D')\leqq 1$. This would imply $\bar{\kappa}(\bar{S}-D')=-\infty$ while $\bar{\kappa}(S-D')=\kappa(K(\bar{S})+D', \bar{S})=\kappa(K(\bar{S})+D_A-E_1, \bar{S})=\kappa(\Delta-E_1, \bar{S})=\kappa((r_1-1)E_1+, \cdots, \bar{S})=0$. Therefore, $E_1\not\subset D_A$. Hence $(D_A, E_1)\geqq 0$. Since $(\Delta, E_1)=(K, E_1)+(D_A, E_1)\leqq 0$, we have $E_1^2\leqq 0$ and $(K, E_1)\leqq 0$. As in the proof of Proposition 3 we have the following cases to examine separately.

1) If $E_1^2=-2$, $(K, E_1)=0$, then $\pi(E_1)=0$ and $(D_A, E_1)=0$.

2) If $E_1^2=-1$, $(K, E_1)=-1$, then $(D_A, E_1)=0$ or 1. In this case, $(D_A, E_1)=0$ contradicts the hypothesis that $\bar{S}-D_A$ has no exceptional curves of the first kind. In the case when $(D_A, E_1)=1$, contracting E_1 corresponds to a 1/2-point detachment.

3) If $E_1^2=0$, $(K, E_1)=-2$, then $(D_A, E_1)=2$. Since $\pi(E_1)=0$, $f(E_1)=p\in J$. Hence, $E_1=f^{-1}(p)$. Therefore, by Kawamata's Theorem ([14]), $\bar{\kappa}(S-E_1)\geqq\bar{\kappa}(C_y)+\bar{\kappa}(J-\{p\})=1$. On the other hand, $\kappa(K(\bar{S})+D_A+E_1, \bar{S})\geqq\bar{\kappa}(S-E_1)\geqq 1$. Since $E_1\leqq\Delta\in|K(\bar{S})+D_A|$, we have

$$\kappa(K(\bar{S})+D_A+E_1, \bar{S})=0\,.$$

This is a contradiction. Hence, we conclude that the case 3) does not occur.

4) If $E_1^2=0$ and $(K, E_1)=0$, then $\pi(E_1)=1$ and $(D_A, E_1)=0$. In all cases, we have $(D_A, E_1)=0$ and $(\Delta, E_1)=0$. Therefore, $(\Delta, E_j)=0$ for all j, hence $\Delta^2=\sum r_j(\Delta, E_j)=0$. Letting $\mathscr{D}_1, \cdots, \mathscr{D}_u$ be the connected components of Δ, we can easily see that these are curves of extended Dynkin type $\tilde{A}\tilde{D}\tilde{E}$. In particular, $\mathscr{D}_1^2=\cdots=\mathscr{D}_u^2=0$.

α) If $\mathscr{D}_1$ consists of one irreducible component, then $\mathscr{D}_1$ is an elliptic curve. Hence $f(\mathscr{D}_1)=J$, and so $(\mathscr{D}_1+D_A, f^{-1}(y))\geqq 3$. This implies $\bar{\kappa}(\bar{S}-\mathscr{D}_1)\geqq 1$ by

Kawamata's Theorem. By the way,

$$\kappa(K(\bar{S})+D_A+\mathcal{D}_1, \bar{S}) \geqq \bar{\kappa}(\bar{S}-\mathcal{D}_1) \geqq 1$$

and

$$\kappa(K(\bar{S})+D_A+\mathcal{D}_1, \bar{S}) = \kappa(\Delta+\mathcal{D}_1, \bar{S}) = 0 .$$

This is a contradiction.

β) If $\mathcal{D}_1$ has more than 1 irreducible components, $f(\mathcal{D}_1)$ is a point. Hence $\mathcal{D}_1$ is a reducible member of $|f^*(y)|$. This implies $h(\Gamma(\mathcal{D}_1))=0$, a contradiction. Q.E.D.

Next, we shall consider the counterparts of Propositions 4 and 15 in the case of $q(\bar{S})=1$.

Proposition 19. *Let $\bar{S}$ be a ruled surface of $q(\bar{S})=1$ with the Albanese fibered surface $f: \bar{S}\to J$. Let D_A be a divisor with normal crossings consisting of horizontal components such that $K(\bar{S})+D_A\sim 0$. Suppose that a reduced divisor Y on $\bar{S}$, each component of which is not contained in D_A, satisfies the condition that $\bar{\kappa}(\bar{S}-D_A-Y)=0$. Then $\kappa(Y, \bar{S})=0$. Moreover, letting $\mathcal{Y}_1, \cdots, \mathcal{Y}_u$ be the connected components, we see that if $\mathcal{Y}_j\cap D_A\neq\phi$, $\mathcal{Y}_j$ is an exceptional curve of the first kind such that $(\mathcal{Y}_j, D_A)=1$ and that if $\mathcal{Y}_j\cap D_A=\phi$, then $\mathcal{Y}_j$ is a curve of Dynkin type* A.

Proof. Let $\sum Y_j$ be the irreducible decomposition of Y. If Y_j is horizontal with respect to f, then $(Y_j+D_A, f^{-1}(u))\geqq 3$ for a general $u\in J$. By Kawamata's Theorem, we get

$$\bar{\kappa}(S-Y_j)\geqq\bar{\kappa}(f^{-1}(u)-Y_j-D_A)+\bar{\kappa}(J) = 1 ,$$

where $S=\bar{S}-D_A$.

This contradicts $\bar{\kappa}(S-Y)=0$. Hence, $f(Y)$ is a finite set of points. For a connected reduced curve $\mathcal{Y}\subset Y$, we have a point $p=f(\mathcal{Y})$, and so $\mathcal{Y}\subset f^{-1}(p)$. In view of $\bar{\kappa}(S-\mathcal{Y})\neq 1$, we see that $\mathcal{Y}\neq f^{-1}(p)$. Therefore, $\mathcal{Y}$ consists of nonsingular rational curves Y_j with negative-definite intersection matrix $[(Y_i, Y_j)]$, $Y_i\subset\mathcal{Y}$. If $Y_j\cap D_A=\phi$, then $(D_A, Y_j)=0$ and so $(K, Y_j)=-(D_A, Y_j)=0$. Combining this with $Y_j^2\leqq -1$, we have $Y_j^2=-2$ and $\pi(Y_j)=0$. If $Y_j\cap D_A\neq\phi$, then $(Y_j, D_A)=-(Y_j, K)>0$. Hence Y_j is an exceptional curve of the first kind and $(Y_j, D_A)=1$. Q.E.D.

Proposition 20. *With the same notation as in Proposition* 19, *we further assume that $\bar{S}$ is relatively minimal. Then*

c-i) $\bar{S}=\boldsymbol{P}^1\times J$, $D_A=p_1\times J+p_2\times J$,

or

c-ii) *$\bar{S}\to J$ is a $\boldsymbol{C}^*$-bundle of degree* 0 *which is not $\boldsymbol{P}^1\times J$, and $D_A=\Gamma_0+\Gamma_\infty$,*

Γ_0 and Γ_∞ being sections with $\Gamma_0^2=\Gamma_\infty^2=(\Gamma_0, \Gamma_\infty)=0$. Note that Γ_0 is cohomologously equivalent to Γ_∞.
Further,

c-iii) *$\bar{S}\to J$ is a C^*-bundle of degree $m>0$ and $D_A=\Gamma_0+\Gamma_\infty$, Γ_0 and Γ_∞ being sections with $\Gamma_0^2=m$ and $\Gamma_\infty^2=-m$.*

In order to prove this, we need the following lemma.

Lemma 10. *Let f: $\bar{S}\to J$ be a P^1-bundle over an elliptic curve J. Then we have the following table.*

Table III

class	$\bar{S}\to J$	$\dim\lvert -K(\bar{S})\rvert$	a member of $\lvert -K(\bar{S})\rvert$	$\bar{q}(\bar{S}-D_A)$
i)	$P^1\times J$	2	$D_A=p_1\times J+p_2\times J$	2
ii)	C^*-bundle of degree 0	0	$D_A=\Gamma_0+\Gamma_\infty$ $(\Gamma_0^2=\Gamma_\infty^2=(\Gamma_0, \Gamma_\infty)=0)$	2
iii)	C^*-bundle of degree m, $m\geqq 1$	m	$D_A=\Gamma_0+\Gamma_\infty$ $(\Gamma_0^2=m, \Gamma_\infty^2=-m, (\Gamma_0, \Gamma_\infty)=0)$	1
iv)	affine bundle A_0	0	$2\Gamma_\infty$ $(\Gamma_\infty^2=0)$	D_A does not exist.
v)	affine bundle A_{-1}	$-\infty$	ϕ	

For the notation used above, we refer the reader to [2] and [18]. Explicit constructions of $\bar{S}$ in [18] are used to compute $\dim\lvert -K(\bar{S})\rvert$ and to find a normal crossing divisor in $\lvert -K(\bar{S})\rvert$. We omit the details.

Proposition 20 follows from the lemma above. In the case of the class c-i) or c-ii), $\bar{S}-D_A$ is a quasi-abelian surface. Attaching several 1/2-points to $\bar{S}-D_A$ at points of D_A, we have surfaces with $\bar{\kappa}=0$ and $q=\bar{q}=1$.

Proposition 21. *Let $(\bar{S}, D)$ be a ∂-surface with the interior S. Suppose that $\bar{p}_g(S)=1$, $\bar{\kappa}(S)=0$, and $q(\bar{S})=1$. Then $\bar{S}$ is a ruled surface of genus 1. Moreover, D is disconnected. D_A consists of two sections of the Albanese fibered surface f: $\bar{S}\to J$ of $\bar{S}$. In particular, S cannot be affine.*

Proof. If $\kappa(S)=0$, it would follow that $p_g(\bar{S})=0$ from the classification theory of projective surfaces. Combined with Proposition 18, this would imply $\kappa(\bar{S})=-\infty$, a contradiction. Thus, $\bar{S}$ turns out to be a ruled surface of genus 1. In view of Lemma 6, by contracting exceptional curves of the first kind on $\bar{S}-D_A$, we may assume that $K(\bar{S})+D_A\sim 0$. Then we contract successively connected exceptional curves $\mathcal{Y}$ of the first kind $\leqq D_B$ such that $(\mathcal{Y}, D_A)=1$. Thus we arrive at the situation that $D_B \cap D_A=\phi$. Detaching several half-points in $\bar{S}-D_A$, we have a relatively minimal surface $\bar{S}_*$ and a proper birational map $\mu: \bar{S}\to\bar{S}_*$. By Lemma 6, $\bar{\kappa}(\bar{S}-\mu(D_A)-\mu_*(D_B), S)=0$. Hence $\mu_*(D_B)\subset\mu(D_A)$. Thus we can apply Proposition 21. Especially D and D_A are disconnected. Q.E.D.

Proposition 22. *Let $(\bar{S}, D)$ be a ∂-surface whose interior S satisfies that $\bar{p}_g(S)=1$, $\bar{\kappa}(S)=0$, $p_g(\bar{S})=0$, and $q(\bar{S})=1$. Suppose that $\bar{q}(S)=2$. Then there are a relatively minimal ruled surface $\bar{S}_*$ and a birational morphism $\mu: \bar{S}\to\bar{S}_*$ such that $\mu(D_B)$ is a finite set and $(\bar{S}_*, \mu(D_A))$ is c-i) or c-ii) in Proposition 20. Moreover, if $\mu(D_B)\subset\mu(D_A)$, S is proper birationally equivalent to a quasi-abelian surface.*

By these theorem and propositions, we have another proof of Theorem I in [10].

Theorem IV. *Let S be a logarithmic abelian surface, i.e., $\bar{\kappa}(S)=0$, $\bar{q}(S)=2$. Then S is W^2PB-equivalent to a quasi-abelian surface.*

Proof. Let $\alpha: S\to\mathcal{A}_S$ be a quasi-Albanese map. Let J be the closure of $\alpha(S)$ in $\mathcal{A}_S$. Then by Kawamata's Theorem, J turns out to be a surface $\mathcal{A}_S$. Hence, $\bar{p}_g(S)\geqq\bar{p}_g(\mathcal{A}_S)=1$. Therefore, we can apply Theorem III and Propositions 20, 22. We omit the details.

Corollary 1. *Let S be an affine normal surface with $\bar{\kappa}(S)=0$ and $\bar{q}(S)=2$. Then S is isomorphic to $\boldsymbol{C}^{*2}$.*

Corollary 2. *Let S be a surface with $\bar{\kappa}(S)=q(S)=0$ and $\bar{q}(S)=2$. Then S is W^2PB-equivalent to $\boldsymbol{C}^{*2}$.*

The above two corollaries are found in [10].

Proposition 23. *Let $(\bar{S}, D)$ be any ∂-surface in TABLE II_b. If $\bar{q}(S)=0$, then there is a reduced divisor R on S such that $\bar{\kappa}(S-R)=0$ and $\bar{q}(S-R)=1$. Similarly, if $\bar{q}(S)=1$, then there is R' on S such that $\bar{\kappa}(S-R')=0$ and $\bar{q}(S-R')=2$. Hence $S-R'\cong\boldsymbol{C}^{*2}$.*

Proof. We use the notation in Proposition 16 and we shall look for R in each case, separately.
i) If S is the class b-iv), take a line $\bar{R}$ on $\boldsymbol{P}^2$ such that $\bar{R}\cap C=\{p\}$ and $H\cap C=\{p\}$. Then $\bar{S}-D-\bar{R}\cong\boldsymbol{C}^{*2}$.
ii) If S is the class b-v), take two curves C_3 and C_4 of degree (1, 0) such that, denoting by $\{p_1, p_2\}$ the intersection $C_1\cap C_2$, $C_3\ni p_1$ and $C_4\ni p_2$. Defining $\bar{R}=C_3+C_4$, we have $S-\bar{R}\cong\boldsymbol{C}^{*2}$.

iii) If S is the class b-vi), write $C_1 \cap C_2 = \{p_1, p_2\}$. Take two fibres C_3 and C_4 of $\sum_2 \to \boldsymbol{P}^1$ of such that $C_3 \ni p_1$ and $C_4 \ni p_2$. Then defining $\bar{R} = C_3 + C_4$, we have $S - \bar{R} = \boldsymbol{C}^{*2}$.
iv) If S is the class b-vii)$_\beta$, write $C_3 \cap \Delta_\infty = \{p\}$. Take a fiber $\bar{R}$ passing through p. Then $\bar{S} - \bar{R} \cong \boldsymbol{C}^{*2}$.
v) If S is the class b-viii), write $H_1 \cap C = \{p\}$. Take a curve $\bar{R} = G_2$ of degree (0, 1) passing through p. Then $S - \bar{R} = \boldsymbol{C}^{*2}$.
vi) If S is the class b-ix), by p we denote the singular point of C. Take two lines C_1, C_2 which are tangential to C at p. Putting $\bar{R} = C_1 + C_2$, we have $S - \bar{R} = \boldsymbol{C}^{*2}$. Moreover, $S - C_1$ is a surface of the class b-vii)$_2$.
vii) If S is the class b-x), by p we denote the singular point of C. Take two curves C_2 and C_3 of degree (1, 0) and (0, 1), respectively, such that $C_2 \ni p$ and $C_3 \ni p$. Then, putting $\bar{R} = C_2 + C_3$, we see $S - \bar{R}$ is a surface of the class b-iv).
viii) If S is the class b-xi), by p we denote the singular point of C. Take a fiber C_2 passing through p. Defining $\bar{R} = C_2 + \Delta_\infty$, we see $S - \bar{R}$ is a surface of the class b-iv).
ix) If S is the class b-xii), take a curve $\bar{R}$ of degree (1, 0) passing through a point $\in G \cap C$. Then $S - \bar{R}$ is a surface of class b-iv).
x) If S is the class b-xiii)$_\beta$, take a fiber $\bar{R}$ passing through a point $\in \Delta_\infty \cap C$. Then $S - \bar{R}$ is a surface of the class b-vii)$_{\beta+1}$.
xi) If S is the class b-vi)*, take a fiber C_4. Then $S - C_4 = \boldsymbol{C}^{*2}$.
xii) If S is the class b-xi)*, take a fiber C_4 which passes through the singular point of C. Then $S - C_4$ is a surface of the class b-iv). Q.E.D.

Therefore, we establish the following

Proposition 24. *Let S be a surface with $\bar{\kappa}(S) = 0$, $\bar{p}_g(S) = 1$ and $p_g(\bar{S}) = q(\bar{S}) = 0$. Suppose that S is not a logarithmic K3 surface of type II_a. If $\bar{q}(S) = 0$, then there is an open subset S_1 of S such that $\bar{\kappa}(S_1) = \bar{\kappa}(S) = 0$ and $\bar{q}(S_1) = 1$. Moreover if $\bar{q}(S) = 1$, then there is an open subset S_2 of S such that $\bar{\kappa}(S_2) = 0$ and $\bar{q}(S_2) = 2$.*

Corollary. *Let S be a surface in Proposition 24. Then there is a surjective morphism $\psi\colon S \to J$ whose general fiber $\psi^{-1}(u) \cong \boldsymbol{C}^*$. Here $J \cong \boldsymbol{P}^1$ or $\boldsymbol{A}^1$, if $\bar{q}(S) = 0$. And $J \cong \boldsymbol{C}^*$, if $\bar{q}(S) = 1$ or 2.*

A proof follows from the fact that S_2 with $\bar{\kappa}(S_2) = q(S_2) = 0$ and $\bar{q}(S_2) = 2$ is W^2PB-equivalent to $\boldsymbol{C}^{*2}$.

EXAMPLE. Let C be an irreducible curve with a non-cuspidal singular point. Then $\boldsymbol{P}^2 - C$ is a logarithmic $K3$ surface, i.e., $\bar{\kappa}(\boldsymbol{P}^2 - C) = 0$ if and only if there exist two irreducible curves C_1 and C_2 such that $\boldsymbol{P}^2 - C - C_1 - C_2 \cong \boldsymbol{C}^{*2}$.

Proposition 25. *Let $C = V(\varphi)$, φ being an irreducible polynomial, be a* *curve on A^2 and let $S=A^2-C$. Suppose $\bar{\kappa}(S)=0$. Then, choosing an appropriate system of coordinates (x, y) of A^2, φ is written as follows:*

$$\varphi = x^l y+a_0+a_1x+\cdots+a_sx^s .$$

Proof. Since $\bar{q}(S)=1$ and $\bar{\kappa}(S)=0$, it follows that $\bar{p}_g(S)=1$. Actually, assume that $\bar{p}_g(S)=0$. Then $\bar{C}$ (the closure of C in P^2) is a rational curve whose singularities are cuspidal. If C were singular, then a general member C_λ of the fiber space φ: $S\to C^*$ would be of hyperbolic type, i.e., $\bar{\kappa}(C_\lambda)=1$. Kawamata's Theorem would assert that $\bar{\kappa}(S)\geqq\bar{\kappa}(C_\lambda)+\bar{\kappa}(G_m)=1$, a contradiction. Thus C is non-singular and hence $C\overset{\sim}{\to}A^1$. By the imbedding theorem of A^1 due to Abhyankar and Moh [1], we know that $S\cong A^1\times G_m$, which implies that $\bar{\kappa}(S)=-\infty$.

Accordingly, we conclude that $\bar{p}_g(S)=1$ and $\bar{\kappa}(S)=0$. Applying Proposition 24, we have an irreducible curve C_3 such that $P^2-C_1\cup C_2\cup C_3\cong C^{*2}$, where $C_1=P^2-A^2$ and $C_2=\bar{C}$. Since $\bar{p}_g(S-C_3)=1$, C_2 or C_3 has only cuspidal singularities. We may assume that C_3 has only cuspidal singularities. Hence, applying Kawamata's Theorem and Abhyankar and Moh Theorem, we can assume that $A^2\cap C_3$ is $V(x)$, i.e., the y-axis of the affine plane. Therefore

$$\operatorname{Spec} k[x, y, x^{-1}, \varphi^{-1}] \cong C^{*2} .$$

From this it follows that $y\in k[x, y, x^{-1}, \varphi^{-1}]=k[x, \varphi, x^{-1}, \varphi^{-1}]$. Hence

$$y = f(x, \varphi)/x^m\varphi^n$$

where, $m, n>0$ and $f(x, Y)$ is a polynomial. Then consider the y-derivative $\partial_y=\partial/\partial_y$. Thus,

$$x^m\varphi^n+nx^m\varphi^{n-1}\partial_y\varphi = \partial_Y f(x, \varphi)\partial_y\varphi .$$

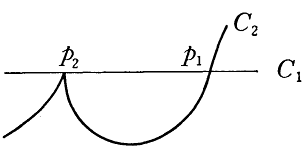
Hence,

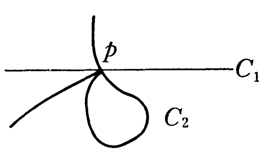
$$x^m\varphi^n = \partial_y\varphi\{\partial_Y f(x, \varphi)-nx^m\varphi^{n-1}\} .$$

Since φ is irreducible, $\partial_y\varphi=\alpha x^l$ for some $\alpha\neq 0$, $l\geqq 0$. This yields that $\varphi=\psi(x)+\alpha x^l y$, ψ being a polynomial. We may assume $\alpha=1$ and hence

$$\varphi = x^l y+a_0+a_1x+\cdots+a_sx^s . \qquad \text{Q.E.D.}$$

In the above, we may assume that $a_0=1$ and $a_s\neq 0$. We have the following cases: 1) If $l+1\geqq s$, then writing $C_1\cap C_2=\{p_1, p_2\}$, C_2 has the cusp singularity





at p_1 and C_1+C_2 has normal crossings at p_2. 2) If $2+l\leqq s$, then C_2 has two (analytically irreducible) branches at p, the singular point of C_2. Hence $\boldsymbol{P}^2-C_2$ is a logarithmic $K3$ surface of type $\mathrm{II_b}$.

Proposition 26. *If S satisfies that $\bar{\kappa}(S)=0$, $\bar{p}_g(S)=1$ and $p_g(\bar{S})=0$. Then there exists an algebraic pencil $\{C_u\}$* whose general member C_u is $\boldsymbol{C}^*$. Hence S is not measure-hyperbolic.

This follows from Corollary to Proposition 24 and Propositions 9, 21.

Proposition 27. *Let $(\bar{S}, D)$ be a ∂-surface in the* TABLE $\mathrm{II_b}$. *Define* $\mathrm{Aut}(\bar{S}, D)=\{\varphi\in\mathrm{Aut}(\bar{S}); \varphi D=D\}$. *Then* $\mathrm{Aut}(\bar{S}, D)$ *is a finite group if $\bar{q}(S)=0$.*

Proof. First assume that $(\bar{S}, D)$ is the class b-ix). A point p of inflexion of D(a nodal cubic curve), is characterized by the existence of a line L on $\boldsymbol{P}^2$ such that $L\cap D=\{p\}$. There are three such points. Hence $\varphi\in\mathrm{Aut}(\bar{S}, D)$ preserves the set of points of inflexion. Therefore the image of the homomorphism $\mathrm{Aut}(S, D)\rightarrow\mathrm{Aut}(D)$ is a finite group. Using the similar argument to the proof of Proposition 11, we complete the proof. We can check the finiteness of $\mathrm{Aut}(\bar{S}, D)$ for the other classes. Q.E.D.

From the above, we infer the following Proposition, whose proof is not given here.

Proposition 28. *Let S be a logarithmic $K3$ surface. Then, Aut(S) has at most countably many elements.*

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