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ON LOGARITHMIC K3 SURFACES

SHIGERU IITAKA

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Introduction. By surfaces we mean non-singular algebraic surfaces defined over the field of complex numbers $\mathbb{C}$. A logarithmic K3 surface $S$ is by definition a surface $S$ with $p_g(S)=1$, $\kappa(S)=\bar{q}(S)=0$, in which $p_g(S)$ is the logarithmic geometric genus, $\kappa(S)$ is the logarithmic Kodaira dimension, and $\bar{q}(S)$ is the logarithmic irregularity. These notions will be explained in §1.

Let $\bar{S}$ be a completion of $S$ with ordinary boundary $D$, i.e., $\bar{S}$ is a non-singular complete surface and $D$ is a divisor with normal crossings on $\bar{S}$ such that $S=\bar{S}-D$. We write $D$ as a sum of irreducible components: $D=C_1+\cdots+C_r$.

Logarithmic K3 surfaces are classified into the following three types:

Type I) $p_g(S)=1$; Then $S$ is a K3 surface and $D$ consists of non-singular rational curves $C_i$ with negative-definite intersection matrix $\langle C_i, C_j \rangle$.

Type IIa) $p_g(S)=0$ and a component $C_i$ of $D$ is a non-singular elliptic curve; Then $S$ is a rational surface and the graph of $D$ has no cycles.

Type IIb) $p_g(S)=0$ and $D$ consists of rational curves $C_i$; Then $\bar{S}$ is a rational surface and the graph of $D$ has one cycle.

We define $A$-boundary $D_A$ and $B$-boundary $D_B$ of $(\bar{S}, D)$ as follows: 1) If $S$ is of type I, then $D_A=\phi$ and $D_B=D$. 2) If $S$ is of type IIa, then $D_A=C_i$ (a non-singular elliptic curve) and $D_B=C_2+\cdots+C_r$. 3) If $S$ is of type IIb, then $D_A=C_1+\cdots+C_r$ that is a circular boundary (for definition, see §1 v)) and $D_B=C_{r+1}+\cdots+C_s$.

Theorem 1. If $\bar{S}-D_A$ has no exceptional curves of the first kind, then $K(\bar{S})+D_A\sim 0$.

Next, consider the case where $\bar{S}-D_A$ has exceptional curves. Let $\rho: \bar{S}\to \bar{S}_*$ be a contraction of exceptional curves of the first kind on $\bar{S}-D_A$, i.e., $\bar{S}_*$ is a complete surface and $\rho$ is biregular around $D_A$ such that $\bar{S}_* \to \rho(D_A)$ has no exceptional curves of the first kind. By Theorem 1, $K(\bar{S}_*)+\rho(D_A)\sim 0$.

Theorem 2. $\rho(D_B)$ is a divisor with simple normal crossings. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_a$ be the connected components of $\rho(D_B)$. Then 1) if $\mathcal{Z}_i \cap \rho(D_A) = \phi$, $\mathcal{Z}_i$ is an exceptional curve of the first kind such that $(\mathcal{Z}_i, \rho(D_A))=1$. 2) If $\mathcal{Z}_i \cap \rho(D_A) = \phi$,
then \( \mathcal{Z}_i \) is a curve of Dynkin type ADE on \( \mathcal{S} - \rho(D_A) \). In case \( S \) is of type II, \( \mathcal{Z}_i \) is a curve of Dynkin type A.

For definition of curves of Dynkin type ADE, see \( \S \) 1. iv).

**Theorem 3.** Suppose that \( K(\mathcal{S}) + D_A \sim 0 \) and \( D_B \) is a curve of Dynkin type ADE. If \( S \) is of type \( \Pi_a \), then \( (\mathcal{S}, D) \) is obtained from one of 4 classes in Table \( \Pi_a \) by 1/2-point attachments. If \( S \) is of type \( \Pi_b \), then \( (\mathcal{S}, D) \) is obtained from one of 15 classes in TABLE \( \Pi_b \) by canonical blowing ups and attaching several 1/2-points.

**Theorem 4.** Let \( (\mathcal{S}, D) \) be a \( \partial \)-surface of which interior \( S \) satisfies that \( \kappa(S) = p_g(S) = 1 \) and \( \rho_g(S) = 1 \). Suppose that a component \( C_1 \) of \( D \) is not rational. Then \( \kappa(S) = 0 \). Next, assume that \( D \) consists of rational curves. If \( \kappa(S) = 0 \), then there exists an open subset \( S_1 \) of \( S \) such that \( \kappa(S_1) = 0 \) and \( \kappa(S_1) = 1 \). Furthermore, if \( \kappa(S) = 1 \), then there exists an open subset \( S_2 \) of \( S \) such that \( \kappa(S_2) = 0 \) and \( \kappa(S_2) = 2 \).

**Theorem 5.** Let \( S \) be a surface with \( \kappa(S) = p_g(S) = 0 \) and \( \rho_g(S) = 1 \). Then there exists an algebraic pencil \( \{C_u\} \) on \( S \) whose general member \( C_u \) is isomorphic to \( \mathbb{C}^* \). Hence, \( S \) is not measure-hyperbolic. Moreover, the connected component of \( \text{Aut}(S) \) is \( \{1\} \) or \( \mathbb{C}^* \) or \( \mathbb{C}^* \). Further,

\[
\dim \text{Aut}(S) \leq \kappa(S).
\]

**Theorem 6.** Let \( (\mathcal{S}, D) \) be a \( \partial \)-surface whose interior \( S \) satisfies that \( \kappa(S) = 0 \) and \( \rho_g(S) = 1 \). Then, there exists a proper birational morphism \( \rho: \mathcal{S} \to \mathcal{S}_* \) such that i) \( \mathcal{S}_* \) is relatively minimal, ii) \( P_m(\mathcal{S}_* - \rho_*(D)) = 1 \) for any \( m \geq 1 \), iii) \( \rho_*(D) = \Delta + Y \) has only normal crossings with \( K(\mathcal{S}_*) + \Delta \sim 0 \), \( Y \) being a curve of Dynkin type.

\( (\mathcal{S}_*, \rho_*(D)) \) might be called a supermodel of \( S \) (or of \( (\mathcal{S}, D) \)). In the study of non-complete surfaces, minimal model (and even \( \partial \)-minimal model) is not helpful. Instead, supermodel will play the important role. For full discussion of the classification theory of surfaces of non-complete surfaces, see Kawamata's recent article [18].

**Example 1.** Let \( \mathcal{S} \) be a non-singular cubic surface in \( \mathbb{P}^3 \). Let \( E \) be a general hyperplane section on \( \mathcal{S} \). Then \( \mathcal{S} - E \) is a logarithmic \( K3 \) surface of type \( \Pi_a \) and the fundamental group \( \pi_1(\mathcal{S} - E) \cong \{1\} \). Contracting exceptional curves of the first kind, we obtain a proper birational morphism \( \rho: \mathcal{S} \to \mathcal{S}_* \) in which \( \mathcal{S}_* = \mathbb{P}^2 \). \( E_1 = \rho(E) \) is a non-singular elliptic curve on \( \mathbb{P}^2 \). Then \( \pi_1(\mathcal{S}_* - E_1) \cong \mathbb{Z}/(3) \) and \( \mathcal{S} - E \supset \mathcal{S}_* - E_1 \).

**Example 2.** Let \( \varphi(y) \) be a polynomial of degree \( n+1 \) such that \( \varphi(0) \neq 0 \). Let \( \Gamma \) be the graph (\( \subset \mathbb{C}^2 \)) of a rational function \( \varphi(y)/y^{n-m} \) (\( 0 < m < n \)). By
C we denote the closure of $\Gamma$ in $P^2$. Then $P^2-\Gamma$ is a logarithmic $K3$ surface of type $\Pi_b$.

**Figure 1.**

**EXAMPLE 3.** Let $\Phi: C[x,y] \rightarrow C[x,y]$ be a $C$-automorphism. Put $X(x,y) = \Phi(x)$ and $Y(x,y) = \Phi(y)$. Let $F(x,y) = Y(x,y)^n - X(x,y) - \varphi(Y(x,y))$, $\varphi$ being as in Example 3. Then the closure $C_\varphi$ of $V(F) = \text{Spec } k[x,y]/(F)$ in $P^2$ is a complement of a logarithmic $K3$ surface of type $\Pi_b$ if $C_\varphi$ has an analytically reducible (i.e., non-cusp) singular point.

For instance, let $\varphi(y) = y^3 + 1$ and $\Phi(x) = x$, $\Phi(y) = y + x^2$. Then $F = (y + x^2)x - (y + x^3)^3 - 1$. Thus letting $\Gamma$ be the closure of $V(F)$ in $P^2$, $P^2-\Gamma$ is a logarithmic $K3$ surface of type $\Pi_b$.

**EXAMPLE 4.** Let $C = V((y-x^2)^2 - xy^2)$ in $C^2$. Denote by $\Gamma$ the closure of $C$ in $P^2$. Then $S = P^2-C$ has the following numerical characters: $\overline{p}_g=0$, $\overline{P}_2=1$, $\overline{q}=1$, and $\overline{q}=0$.

### 1. Basic notions, notations and conventions

i) $\partial$-manifold and 1/2-point attachment. A pair $(\mathcal{V}, D)$ consisting of a complete non-singular algebraic variety $\mathcal{V}$ and a divisor $D$ with normal crossings on $\mathcal{V}$ is called a $\partial$-manifold. The dimension of $(\mathcal{V}, D)$ is understood as the dimension of $\mathcal{V}$. A 2-dimensional $\partial$-manifold is called a $\partial$-surface. We have a theory of minimal models for $\partial$-manifolds (see [12]). Let $(\overline{S}, D)$ be a $\partial$-surface. Then $D$ is not a minimal boundary if and only if there is an irreducible component $E$ of $D$ which is an exceptional curve of the first kind such that $(E, D') = 1$ or 2, $D'$ being defined by $D = D' + E$. We say that $(\overline{S}, D)$ is relatively $\partial$-minimal if $S-D$ has no exceptional curves of the first kind and if $D$ is a minimal boundary.

Let $(\mathcal{V}_1, D_1)$ and $(\mathcal{V}_2, D_2)$ be $\partial$-manifolds. We say that a morphism $f: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is a $\partial$-morphism when $f^{-1}D_2 \subseteq D_1$. Here $f^{-1}(D_2)$ is the reduced divisor of the pull back $f^*D_2$.

Let $(\overline{S}, D)$ be a $\partial$-surface and take a point $p \in D$. By $\lambda: \overline{S} = Q_p(\overline{S}) \rightarrow \overline{S}$ denote the blowing up at $p$. Defining $D' = \lambda^{-1}(D)$, we have a $\partial$-morphism $\lambda: (\overline{S}', D') \rightarrow (\overline{S}, D)$. If $p$ is a double point of $D$, $\lambda$ is called a canonical blowing
up. Then we have the linear equivalence:

$$K(\bar{S}) + D \sim_\text{line} \lambda^*(K(\bar{S}) + D),$$

where $K(\bar{S})$ and $K(S)$ denote canonical divisors on $\bar{S}$ and $S$, respectively. If $p$ is a simple point of $D$, define $D^*$ by $D^1 = \lambda^{-1}(p) + D^*$. $S^* = \bar{S} - D^*$ contains $S$ as an open subset. $S^*$ is called a 1/2-point attachment to $S$ at $p$. Conversely, $S$ is called a 1/2-point detachment from $S^*$. To make things clear, we may say that $(\bar{S}, D^*)$ is obtained from $(\bar{S}, D)$ by attaching a 1/2-point $\lambda^{-1}(p) - D^*([10])$. It is easy to check that

$$K(\bar{S}) + D^* \sim_\text{line} \lambda^*(K(\bar{S}) + D).$$

Hence, $K(\bar{S}) + D$ modulo linear equivalence is invariant under canonical blowing ups and 1/2-point attachments.

In general letting $(\bar{S}, D)$ be a $\partial$-surface, we consider an irreducible curve $E$ on $\bar{S}$ satisfying that $E$ is an exceptional curve of the first kind, $E \not\subset D$, and $(E, D) = 1$. Such an $E$ is called a $D$-exceptional curve of the first kind. Note that $E - D \cong A^1$, which is called a 1/2-point. $S - D$ is a 1/2-point attachment to $\bar{S} - D - E$.

ii) logarithmic genera. Let $V$ be an algebraic variety. Then there exists a non-singular algebraic variety $V^*$ such that there exists a proper birational morphism $\mu: V^* \to V$. Let $(\bar{V}^*, D^*)$ be a $\partial$-manifold such that $\bar{V}^* = V^* - D^*$. Then we say that $\bar{V}^*$ is a completion of $V^*$ with ordinary boundary $D^*$. According to Deligne [3], we have a sheaf $\Omega^1(\log D^*)$ of logarithmic 1-forms on $\bar{V}^*$. We have the spaces of logarithmic forms:

$$T_i(V^*) = H^0(\bar{V}^*, \Omega^i(\log D^*)), \quad 1 \leq i \leq n;$$

and

$$H^0(\bar{V}^*, (\Omega^* \log D^*)^m) \quad \text{for } m = 1, 2, \ldots,$$

where $\Omega^i(\log D^*) = \bigwedge^i (\Omega^1 \log D^*)$ and $n = \dim V$. These spaces depend only on $V$. Hence, define

$$\bar{q}_i(V) = \dim T_i(V^*)$$

and

$$\bar{P}_m(V) = \dim H^0(\bar{V}^*, (\Omega^* \log D^*)^m).$$

We call $\bar{q}_i(V)$ the logarithmic $i$-th irregularity of $V$ and call $\bar{P}_m(V)$ the logarithmic $m$-genus of $V$. Writing $\bar{q}(V) := \bar{q}_1(V)$ and $\bar{P}_m(V) := \bar{q}_m(V) = \bar{P}_m(V)$, we call them the logarithmic irregularity and the logarithmic geometric genus of $V$, respectively (see [4], [5]).

iii) $D$-dimension and logarithmic Kodaira dimension. In general, let $\bar{V}$ be a normal complete algebraic variety and $D$ a divisor on $\bar{V}$. By $\Phi_m$ we denote
the rational map associated with \(|mD|\) under the assumption that \(|mD| \neq \phi\). We define

\[ \kappa(D, V) = \max \{ \dim \Phi_m(V); \text{when } |mD| \neq \phi \}, \]

which is said to be the \(D\)-dimension of \(V\). If \(|mD|\) is empty for any \(m \geq 1\), we put \(\kappa(D, V) = -\infty\). The following two facts ([6]) are very useful in the study of varieties and divisors.

1) If \(\kappa(D_1, V) \geq 0, \ldots, \kappa(D_n, V) \geq 0\), then for any \(\alpha_1 > 0, \ldots, \alpha_n > 0\), we have

\[ \kappa(\sum D_j, V) = \kappa(\sum \alpha_j D_j, V). \]

2) Let \(f: V \to W\) be a surjective morphism of \(V\) onto a normal complete variety \(W\). For a divisor \(D\) on \(W\) and an effective divisor \(E\) which is \(f\)-exceptional (i.e., \(\text{codim}(f(E)) \geq 2\)), we have

\[ \kappa(f^{-1}(D + E), V) = \kappa(D, W). \]

When \(V\) is non-singular, we denote by \(K(V)\) a canonical divisor on \(V\). The \textit{Kodaira dimension} \(\kappa(V)\) of \(V\) is defined to be \(\kappa(K(V), V)\).

Let \((V, D)\) be a \(\partial\)-manifold of dimension \(n\). \(V = V - D\) is called the \textit{interior} of \((V, D)\). We see that

\[ P_m(V) = \dim H^0(V, O(m(K(V) + D))). \]

The \textit{logarithmic Kodaira dimension} of \(V\) is defined to be

\[ \bar{\kappa}(V) = \kappa(K(V) + D, V), \]

which does not depend on the choice of the smooth completion \(V\) of \(V\) with ordinary boundary \(D\).

\textbf{iv) \(W^2PB\)-equivalence.} If there exists a proper birational morphism \(f: V_1 \to V_2\), then \(\bar{P}_m(V_1) = \bar{P}_m(V_2)\) and \(\bar{q}_i(V_1) = \bar{q}_i(V_2)\). A \textit{proper birational map} is by definition a composition of a proper birational morphism and an inverse of a proper birational morphism. If there is a proper birational map \(f: V_1 \to V_2\), then we say that \(V_1\) is \textit{proper birationally equivalent} to \(V_2\). In this case, \(\bar{P}_m(V_1) = \bar{P}_m(V_2)\) and \(\bar{q}_i(V_1) = \bar{q}_i(V_2)\).

Moreover, when \(V\) is non-singular and \(F\) a closed subset of \(V\) of \(\text{codim} \geq 2\), \(\bar{P}_m(V - F) = \bar{P}_m(V)\) and \(\bar{q}_i(V - F) = \bar{q}_i(V)\). In such a case, we say that \(i: V - F \hookrightarrow V\) is a \textit{strict open immersion}.

A \textit{WPB-map} \(f: V_1 \to V_2\) is by definition a birational map which is a composition of proper birational maps, strict open immersions, and inverses of strict open immersions. If there exists a \(WPB\)-map \(f: V_1 \to V_2\), we say that \(V_1\) is \textit{WPB-equivalent to} \(V_2\).
Now define $\mathcal{W} = \{ f: V_1 \to V_2 \text{ birational morphism}; \text{there exist a morphism } g: V_2 \to V_3 \text{ such that } g \circ f \text{ is a } \mathcal{WPB}\text{-map or a morphism } h: U \to V_1 \text{ such that } f \circ h \text{ is a } \mathcal{WPB}\text{-map} \}$. A birational map which is a composition $f_1 f_2 \cdots f_n$, $f_i \in \mathcal{W}$, is called a $\mathcal{WPB}$-map. If there is a $\mathcal{WPB}$-map $f: V_1 \to V_2$, then we say that $V_1$ is $\mathcal{WPB}$-equivalent to $V_2$ and $\mathcal{P}_m(V_1) = \mathcal{P}_m(V_2)$. \[ \mathcal{q}_i(V_1) = \mathcal{q}_i(V_2). \]

Recall that a surface $S$ is $\mathcal{WPB}$-equivalent to a quasi-abelian surface if and only if $\mathcal{p}(S) = 0$ and $\mathcal{q}(S) = 2$ ([10]).

v) circular boundary. Let $(S, D)$ be a $\partial$-surface. We say that $D$ is a circular boundary if $D$ is a rational curve with only one ordinary double point $p$ such that $D - \{ p \}$ is non-singular or if $D$ is a sum of non-singular rational curves $C_1, C_2, \ldots, C_r$ such that when $r = 2$, we have $(C_1, C_2) = 2$ and when $r \geq 3$, we have $(C_i, C_j) = 1$ for $i - j \equiv \pm 1 \mod r$, and $(C_i, C_j) = 0$ for $i - j \equiv 0, \pm 1 \mod r$.

![Figures 3.](image_url)

vi) curve of Dynkin type. Let $(S, Y)$ be a $\partial$-surface. We say that $Y$ is a curve of Dynkin type $ADE$ if $Y$ is a sum of non-singular rational curves $Y_j$, such that $Y_j^2 = -2$ and the intersection matrix $[(Y_i, Y_j)]$ corresponds to a direct sum of Dynkin diagrams $A_n, D_n, E_l$. Similarly, we can define a curve of extended Dynkin type $ADE_e$, (which are not necessarily reduced divisors).

2. Logarithmic K3 surfaces of type I

Let $S$ be a logarithmic K3 surface, i.e., $\mathcal{p}_e(S) = 1$. $\mathcal{q}(S) = \mathcal{e}(S) = 0$. Let $(S, D)$ be a $\partial$-surface of which interior is $S$. Then $\mathcal{e}(S) = \mathcal{q}(S) = 0, \mathcal{p}_e(S) \leq \mathcal{p}_e(S) = 1$. Hence, $\mathcal{p}_e(S) = 1$ or 0.

First, assume that $\mathcal{p}_e(S) = 1$. Combining this with $\mathcal{e}(S) \leq \mathcal{e}(S) = 0, \mathcal{q}(S) = \mathcal{q}(S) = 0$, we see that $S$ is a K3 surface which may not be minimal. By contracting exceptional curves of the first kind on $S$ successively, we obtain a minimal K3 surface $\tilde{S}_*$ and a birational morphism $\mu: \tilde{S} \to \tilde{S}_*$. If $\mu(D)$ is a finite set of points, then, putting $\tilde{S}_0 = \tilde{S} - \mu^{-1}(\mu(D))$ and $S_* = \tilde{S}_* - \mu(D)$, we have a proper birational morphism $\mu_0 = \mu|_{\tilde{S}_0}: S_0 \to \tilde{S}_*$. We obtain the following commutative diagram:
Hence, by definition (see § 1 iv)) $S_0 \subset S$ and $S \subset \tilde{S}$ are both $W^2PB$-morphisms. Hence $S$ is $W^2PB$-equivalent to $\tilde{S}_*$. If $\mu(D)$ contains a curve, we let $D_*$ be a purely 1-dimensional part of $\mu(D)$. Then by the previous argument, we see that $S$ is $W^2PB$-equivalent to $\tilde{S} - \mu^{-1}(D_*) \cap D$. Thus we may assume $D_*=\mu(D)$.

**Lemma 1.** Let $V$ be a complete non-singular algebraic variety and $D$ a reduced divisor on $V$. Let $\mu : V^* \to V$ be a birational morphism such that $(V^*, \mu^{-1}(D))$ is a $\delta$-manifold. Denote by $D^*$ the proper transform of $D$ by $\mu^{-1}$. Suppose that $\kappa(V) \geq 0$. Then

$$\kappa(V^* - D^*) = \kappa(V^* - \mu^{-1}(D)) = \kappa(V - D) = \kappa(K(V) + D, V).$$

For a proof, see [6]. A generalization of this is the following Lemma 6, whose proof will be given there. By the above lemma, we get

$$0 = \kappa(S) = \kappa(\tilde{S} - D) = \kappa(\tilde{S}_* - D_*),$$

$$= \kappa(K(\tilde{S}_*) + D_*, \tilde{S}_*) = \kappa(D_*, S_*).$$

**Proposition 1.** Let $\tilde{S}$ be a minimal K3 surface and $Y$ a reduced divisor on $\tilde{S}$ such that $\kappa(Y, \tilde{S})=0$. Then $Y$ turns out to be a curve of Dynkin type ADE. Moreover, $P_m(\tilde{S} - Y)=1$ for any $m \geq 1$ and $q(S - Y)=0$. Hence $\tilde{S} - Y$ is a logarithmic K3 surface.

**Proof.** Let $\sum Y_j$ be the irreducible decomposition of $Y$. Then for any $m_j \geq 0$, we have $\kappa(\sum m_j Y_j, \tilde{S})=0$ by the fact 1) in § 1 iii). By making use of Riemann Roch Theorem on $\tilde{S}$ we have

$$0 = \dim |(\sum m_j Y_j)| \geq (\sum m_j Y_j)^2/2 + 1$$

except for $m_1=\cdots=m_s=0$. Hence

$$(\sum m_j Y_j)^2 \leq -2.$$  

In particular, $Y_j^2 \leq -2$. In view of the adjunction formula, we have

$$-2 \leq 2\pi(Y_j) - 2 = Y_j^2.$$


Here \( \pi(Y) \) denotes the virtual genus of \( Y \). Thus \( Y_j^2 = -2 \) and \( \pi(Y_j) = 0 \).

More generally, letting \( \mathcal{Y} \) be a connected reduced curve in \( Y \), we have the exact sequences

\[
0 \to \mathcal{O}(-\mathcal{Y}) \to \mathcal{O} \to \mathcal{O}_\mathcal{Y} \to 0
\]

and

\[
0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_\mathcal{Y}) \to H^1(\mathcal{O}(-\mathcal{Y})) \to H^1(\mathcal{O}) \\
\to H^1(\mathcal{O}_\mathcal{Y}) \to H^2(\mathcal{O}(-\mathcal{Y})) \to H^2(\mathcal{O}) \to 0.
\]

From this, it follows that \( H^1(\mathcal{O}(-\mathcal{Y})) = 0 \) and

\[
\dim H^0(\mathcal{O}(\mathcal{Y})) = \dim H^2(\mathcal{O}(-\mathcal{Y})) = \dim H^1(\mathcal{O}_\mathcal{Y}) + 1 = \pi(\mathcal{Y}) + 1 = \mathcal{Y}^2/2 + 2.
\]

Hence \( \mathcal{Y}^2 = -2 \). In particular, if \( Y_i \neq Y_j \), we have \( (Y_i, Y_j) = 0 \) or 1. It is easy to see that the intersection-matrix \( [(Y_i, Y_j)] (Y_i \leq \mathcal{Y}) \) corresponds to the Dynkin diagram of type \( A_n, D_m, E_r \). Hence, \( Y \) is a curve of Dynkin type \( ADE \).

Therefore,

\[
\kappa(\mathcal{S} - Y) = \kappa(K(\mathcal{S}) + Y, \mathcal{S}) = 0
\]

and \( \mathbf{p}_m(\mathcal{S} - Y) \geq \mathbf{p}(\mathcal{S} - Y) = 1 \) for any \( m \geq 1 \).

Since \( [(Y_i, Y_j)] \) is negative-definite, \( Y_1, \ldots, Y_s \) are linearly independent in \( \text{Pic}(\mathcal{S}) \). We make use of the following

**Lemma 2.** Let \( V \) be a non-singular complete algebraic variety with \( q(V) = 0 \) and \( Y \) a reduced divisor on \( V \). Let \( \sum Y_j \) be the irreducible decomposition of \( Y \). Then, putting \( V = V - Y \), we get

\[
q(V) = \dim \ker (\oplus \mathcal{Y}_j \to \text{Pic}(V) \otimes \mathbb{Q}).
\]

**Proof.** We have the exact sequence:

\[
0 = H^1(V, \mathbb{Q}) \to H^1(V, \mathbb{Q}) \to \oplus \mathcal{Y}_j \to H^2(V, \mathbb{Q}).
\]

Since \( q(V) = 0 \), it follows that \( \text{Im} \delta \subset \text{Pic}(V) \otimes \mathbb{Q} \subset H^2(V, \mathbb{Q}) \). Thus we obtain

\[
q(V) = \dim \ker (\oplus \mathcal{Y}_j \to \text{Pic}(V) \otimes \mathbb{Q}).
\]

Q.E.D.

We proceed with the proof of Proposition 1. By the lemma above we conclude that \( q(\mathcal{S} - Y) = 0 \).

Q.E.D.

Thus we obtain the following

**Theorem 1.** Let \( (\mathcal{S}, D) \) be a \( 0 \)-surface whose interior is a logarithmic K3 surface \( S \) of type I. Then there exists a birational morphism \( \mu : \mathcal{S} \to \mathcal{S}_K \) such that
\( S_* \) is a minimal K3 surface and such that \( \mu(D) \) is a union of a curve \( Y \) of Dynkin type and a finite set \( F \), and hence

\[ S_0 = \bar{S} - \mu^{-1}(Y) - \mu^{-1}(F) \subset S \subset \bar{S} \, . \]

In other words, \( S \) is \( W^2 \) PB-equivalent to \( \bar{S}_* - Y \).

Note that \( D \) and \( Y \) may be empty.

Table I. \( S_* \) being a minimal compact K3 surface

<table>
<thead>
<tr>
<th>class</th>
<th>( D )</th>
<th>( \bar{S}_* - D )</th>
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<tr>
<td>i)</td>
<td>( \phi )</td>
<td>compact</td>
</tr>
<tr>
<td>i)*</td>
<td>curve of Dynkin type ADE</td>
<td>non-compact</td>
</tr>
</tbody>
</table>

3. Logarithmic K3 surfaces of type II. We begin by recalling the elementary result, called the \( g_p \)-formula.

Lemma 3. Let \((\bar{S}, D)\) be a \( \partial \)-surface with \( q(\bar{S})=0 \). Let \( \sum C_j \) be the irreducible decomposition of \( D \). Then

\[ \beta_*(\bar{S}-D) = \beta_*(\bar{S}) + \sum g(C_j) + h(\Gamma(D)) , \]

where \( \Gamma(D) \) is the (dual) graph of the intersection of \( D=\sum C_j \), \( h(\Gamma) \) is the cyclotomic number of the graph \( \Gamma \), and the \( g(C_j) \) denote the genera of the \( C_j \).

For a proof see ([7], the Appendix).

With the notation being in Lemma 3, we further assume that \( S \) is a logarithmic K3 surface of type II. Hence \( \beta_*(\bar{S})=0 \) and \( \beta_*(S)=1 \). By the formula in Lemma 3, we have

\[ 1 = \beta_*(S) = \sum g(C_j) + h(\Gamma(D)) . \]

Hence, there are the following two types;

Type II\(_a\); \( g(C_1) = 1, g(C_2) = \cdots = g(C_s) = 0 \) and \( h(\Gamma(D)) = 0 \).

Type II\(_b\); \( g(C_1)) = g(C_2) = \cdots = g(C_s) = 0 \) and \( h(\Gamma(D)) = 1 \).

Proposition 2. If \( S \) is a logarithmic K3 surface of type II, then \( S \) is a rational surface.

First, assume \( \kappa(S) \) to be 0. Recalling \( \beta_*(\bar{S})=q(\bar{S})=0 \), we see that \( \bar{S} \) is an Enriques surface. Hence, there exists an étale covering \( \pi: \bar{S} \to \bar{S} \) where \( \bar{S} \) is a K3 surface. Let \( \bar{D}=\pi^{-1}(D) \). Since \( \bar{S}-\bar{D} \to \bar{S}-D \) is étale, we have \( \kappa(\bar{S}-\bar{D}) = \kappa(\bar{S}-D) = 0 \) by Theorem 3 [5]. Hence, \( \bar{S}-\bar{D} \) is a logarithmic
K3 surface of type I. By Theorem I, $D$ consists of rational non-singular curves whose intersection matrix is negative-definite. Hence $D$ has the same property as $\bar{D}$. Thus $h(\Gamma(D))=0$. This contradicts the fact that $S$ is of type II. Therefore, it follows that $\kappa(\bar{S})=-\infty$. Recalling Castelnuovo's criterion, $\bar{S}$ is a rational surface, because $g(\bar{S})=0$. Q.E.D.

4. Logarithmic K3 surfaces of type II. Employing the notation in § 3, we assume $S$ to be a logarithmic K3 surface of type $\Pi_a$. Putting $D_A=C_1$ and $D_B=C_2+\cdots+C_s$, we have $D=D_A+D_B$ and $g(D_A)=1$. Hence, $\bar{p}^s(S-D_A)=1$, $\bar{q}(\bar{S}-D_A)\leq \bar{q}(\bar{S}-D)=0$, and $\bar{q}(\bar{S}-D_A)\leq \bar{q}(\bar{S}-D)=0$. These show that $\bar{S}-D_A$ is a logarithmic K3 surface of type $\Pi_a$. Contracting exceptional curves of the first kind in $\bar{S}-D_A$, successively, we have a birational morphism $\mu: \bar{S}\to \bar{S}_*$ such that $\mu(D_A)$ is isomorphic around $D_A$ and $\bar{S}_* - \mu(D_A)$ has no exceptional curves of the first kind, i.e., $(\bar{S}_*, \mu(D_A))$ is a relatively $\partial$-minimal model of $(\bar{S}, D_A)$.

Proposition 3. Let $(\bar{S}, C)$ be a relatively $\partial$-minimal $\partial$-surface such that $C$ is a non-singular elliptic curve with $\kappa(\bar{S}-C)=\bar{q}(\bar{S}-C)=0$. Then $K(\bar{S})+C\sim 0$.

Proof. By Proposition 2, $\bar{S}$ is a rational surface.

If $K(\bar{S})+C$ were linearly equivalent to an effective divisor $\Delta=\sum_{i=1}^{s} \tau_i E_i (\tau_i>0)$, we would derive a contradiction. Since $\kappa(\Delta, \bar{S})=\bar{q}(\bar{S}-C)=0$, we know that the intersection matrix $[(E_i, E_j)]$ is negative semi-definite. In particular $E_j^2\leq 0$ for any $1\leq j\leq s$. If $E_j=C$, then $K=(-K(\bar{S}))\sim \Delta - E_j = \Delta - C_1 \geq 0$. This is a contradiction. Therefore $E_j \not= C$, which implies $(\Delta, C_1) \geq 0$. Since $\Delta^2 \leq 0$, we may assume that $(\Delta, E_1) \leq 0$. Hence, $(K, E_1) \leq -(C, E_j) \leq 0$. By the adjunction formula,

$$-2 \leq 2\pi(E_1) - 2 = E_1^2 + (K, E_1) \leq 0.$$ 

Hence, $\pi(E_1)=0$ or 1. We shall examine various cases, separately.

1) If $\pi(E_1)=1$, we have $E_1^2=(K, E_1)=0$. Hence $(C, E_1)=0$. Thus $C \cap E_1=\emptyset$ and $(\Delta, E_1)=0$.

2) If $\pi(E_1)=0$ and $(C, E_1) \geq 1$, it follows that $(K, E_1) \leq -1$ and $-2=-E_1^2 + (K, E_1) \leq -1$. Hence, $\alpha) E_1^2=(K, E_1)=-1$ or $\beta) E_1^2=0$ and $(K, E_1)=-2$.

In the case of $\alpha)$, we have $1 \leq (C, E_1)=(\Delta, E_1)-(K, E_1) \leq 1$. Hence $(\Delta, E_1)=0$, $-(C, E_1)=(K, E_1)=-1$. This implies that $E_1$ is a $C$-exceptional curve. Hence, we can contract $E_1$. Note that $K+C$ is invariant under 1/2-point detachments (see § 1 i)). Thus we may assume that this case does not occur.

In the case of $\beta)$, we use the following

Lemma 4. Let $\bar{S}$ be a complete surface with $p_*(\bar{S})=q(\bar{S})=0$ and $E$ a curve
on $\bar{S}$ such that $\pi(E)=0$. Then

$$\dim |E| \geq 1 + E^2.$$  

Proof. By Riemann Roch Theorem,

$$\dim |E| \geq (E, E-K)/2, K \text{ being } K(\bar{S}).$$

On the other hand, $(E, E+K)=2\pi(E)-2=-2$. Hence, follows the assertion. Q.E.D.

Therefore letting $S=\bar{S}-C$,

$$0 = \bar{p}_g(S)-1 = \dim |\Delta| \geq \dim |E_i| \geq 1.$$ 

Thus we have arrived at a contradiction.

3) If $\pi(E_i) \neq (C, E_i)=0$, then $E_i^2 \leq -1$ and $(K, E_i)= -1$ or 0. Suppose $(K, E_i) = -1$. We have $E_i^2 = -1$ and $E_i \cap C = \emptyset$. This yields that $E_i$ is an exceptional curve of the first kind on $\bar{S}-C$. This contradicts the hypothesis. Suppose that $(K, E_i)=0$. We have $E_i^2 = -2$. Thus $E_i \cap C = \emptyset$ and $(\Delta, E_i)=0$.

Consequently, after a finite succession of 1/2-point detachments, we have $(\Delta, E_j)=0$, and i) $E_j^2 = 0$, $\pi(E_j)=1$ or ii) $E_j^2 = -2$, $\pi(E_j)=0$. Hence $(K, E_j)=0$ for any irreducible components $E_j$ of $\Delta$. Thus letting $\mathcal{D}_1, \ldots, \mathcal{D}_e$ be the connected components of $\Delta$, we have $\Delta = \sum \mathcal{D}_j$ and $\Delta^2 = \sum |\mathcal{D}_j|^2$. Since $\Delta^2 = 0$ and $|\mathcal{D}_j|^2 \leq 0$ for any $j$, it follows that $|\mathcal{D}_1|^2 = \cdots = |\mathcal{D}_e|^2 = 0$. Recalling that $(K, \mathcal{D}_j)=0$, for any $i$ we have $(K, \mathcal{D}_j)=0$. Therefore, the $\mathcal{D}_j$ are curves of extended Dynkin type $\tilde{A}DE$.

Lemma 5. Let $\bar{S}$ be a complete surface with $p_g(\bar{S})=q(\bar{S})=0$. For an effective divisor $F$ ($\neq 0$) on $S$, we have

$$\dim |F+K| = \dim H^1(\mathcal{O}_F)-1 \geq (F, F+K)/2.$$ 

Moreover, if $\dim H^0(\mathcal{O}_F)=1$, then

$$H^1(\mathcal{O}(F+K)) = 0, \text{ and so } \pi(F) = \dim H^1(\mathcal{O}_F).$$

Hence,

$$\dim |F+K| = (F, F+K)/2.$$ 

Proof. From the exact sequence:

$$0 \rightarrow C = H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}_F) \rightarrow H^1(\mathcal{O}(-F)) \rightarrow 0 = H^1(\mathcal{O}),$$

follows the assertion. Q.E.D.

By this we have
\[
\dim |\mathcal{O}_i + K| \geq (\mathcal{O}_i, \mathcal{O}_i + K)/2 = 0.
\]

But since \( \bar{P}_2(S) - 1 \geq \dim |\Delta + K| \geq \dim |\mathcal{O}_i + K| \), it follows that \( \dim |\mathcal{O}_i + K| = 0 \). Putting \( K(\mathcal{O}_i) = (\mathcal{O}_i + K)|\mathcal{O}_i \), we get the following exact sequence:

\[
0 = H^0(\mathcal{O}(K)) \to H^0(\mathcal{O}(K(\mathcal{O}_i))) \to H^0(\mathcal{O}(K(\mathcal{O}_i))) \to H^1(\mathcal{O}) = 0.
\]

Hence,

\[
\dim |K(\mathcal{O}_i)| = \dim H^0(\mathcal{O}(K(\mathcal{O}_i))) - 1 = \dim H^0(\mathcal{O}(K + \mathcal{O}_i)) - 1 = \dim |K + \mathcal{O}_i| = 0.
\]

Similarly, we have

\[
\dim |K(C)| = 0, \text{ where } K(C) = (K + C)|C,
\]

since \( \bar{P}_2(S - C) - 1 = \dim |K + C| = 0 \). Furthermore,

\[
0 = \bar{P}_2(S - C) - 1 \leq \dim |K + C + \mathcal{O}_i| \leq \dim |2\Delta| = \bar{P}_2(S - C) - 1 = 0.
\]

Hence, \( \dim |K + C + \mathcal{O}_i| = 0 \). Thus,

\[
*) \quad \dim |K(C + \mathcal{O}_i)| = \dim |K + C + \mathcal{O}_i| = 0.
\]

By the way, since \( C \cap \mathcal{O}_i = \phi \), it follows that

\[
K(C + \mathcal{O}_i) = (K + C + \mathcal{O}_i)|(C + \mathcal{O}_i)
= (K + C)|(C + \mathcal{O}_i) |\mathcal{O}_i
= K(C) \oplus (K + \mathcal{O}_i).
\]

Thus, \( \dim |K(C + \mathcal{O}_i)| = \dim |K(C)| + \dim |K(\mathcal{O}_i)| + 1 = 1 \). This contradicts \( *) \).

Q.E.D.

The following lemma is a generalization of Lemma 1.

**Lemma 6.** Let \((V, D)\) be a \( \partial \)-manifold and put \( V = V - D \). Assume that \( \kappa(V) \geq 0 \). Let \( Y \) be a reduced divisor on \( V \) and denote by \( \bar{Y} \) the closure of \( Y \) in \( V \). Take a proper birational morphism \( \rho: \bar{V}^* \to V \) such that \( (V^*, \rho^{-1}(\bar{Y} + D)) \) is a \( \partial \)-manifold. \( \mu = \rho|V^*; V^* = V^* - \rho^{-1}(D) \to V \) is a proper birational morphism. Then letting \( Y^* \) be the proper transform of \( Y \) by \( \mu^{-1} \), we obtain

\[
\kappa(V^* - Y^*) = \kappa(V - Y) = \kappa(K(V) + D + \bar{Y}, V).
\]

Proof. Denoting by \( Z^* \) the closure of \( Z \) in \( V^* \), we have \((\mu^{-1}(Y))^* = Y^* + \mathcal{E}, \mathcal{E} \) being an effective divisor which is \( \rho \)-exceptional. Similarly,
Recall the logarithmic ramification formula ([5]):
\[ K(V^*) + \rho^{-1}(D) = \rho^*(K(V)+D) + \tilde{R}_{\mu}, \]
where \( \tilde{R}_{\mu} \) is the logarithmic ramification divisor for \( \mu \). By definition, we have
\[ \kappa(V - Y) = \kappa(V^* - \mu^{-1}(Y)) = \kappa(V^* - Y^*) \]
\[ = \kappa(K(V^*) + \rho^{-1}(D) + Y^*, V^*) \]
\[ = \kappa(\rho^*(K(V)+D) + \tilde{R}_{\mu} + Y^*, V^*) \]
\[ = \kappa(\rho^*(K(V)+D) + N\tilde{R}_{\mu} + Y^*, V^*), N > 0. \]
This follows from \( \kappa(V) \geq 0 \) by using 2) of § 1. iii). On the other hand, \( \tilde{R}_{\mu} | V^* = R_{\mu} \) and \( \mu^{-1}(Y) \leq Y^* + N_1R_{\mu} \) for some \( N_1 > 0 \). Hence, we have \( (\mu^*Y)^* \leq Y^* + N_2(R_{\mu})^* \) for some \( N_2 > 0 \). Choosing \( N > 0 \), we obtain
\[ \kappa(\rho^*(K(V)+D) + N\tilde{R}_{\mu} + Y^*, V^*) \]
\[ \geq \kappa(\rho^*(K(V)+D) + (\mu^*Y)^*, V^*). \]
We note that
\[ \rho^*(D) + (\mu^*Y)^* = \rho^*(D + \tilde{Y}). \]
Hence,
\[ \kappa(\rho^*(K(V)+D) + (\mu^*Y)^*, V^*) = \kappa(\rho^*(K(V)+D + \tilde{Y}), V^*) \]
\[ = \kappa(K(V)+D + \tilde{Y}, V^*). \]
It is easily seen that
\[ \kappa(K(V)+D + \tilde{Y}, V^*) \geq \kappa(V - Y) \geq \kappa(V^* - Y^*). \]
Thus we obtain the desired equality. Q.E.D.

We come back to the study of a logarithmic K3 surface \( S \) of type \( \Pi_a \). Writing \( D_A = \mu(D_A) \) and \( Y = \mu^*(D_B) \), we have by Lemma 6
\[ \kappa(S - D_A - Y) = \kappa(S - D) = 0. \]
Since \( K(S^*) + D_A \sim 0 \), we make use of the following proposition.

**Proposition 4.** With the notation being as in Proposition 3, let \( Y \) be a reduced divisor on \( S \) which does not contain \( C \). Suppose that \( \kappa(S - C - Y) = 0 \). Then \( \kappa(Y, S) = 0 \). Moreover, letting \( Q_{j_1}, \ldots, Q_{j_n} \) be the connected components of \( Y \), we have the following assertions, separately.

1) If \( Q_{j_1} \cap C = \phi \), then \( (Q_{j_1}, C) = 1 \) and \( Q_{j_1} \) is an exceptional curve of the first kind in \( S \).
2) If \( Q_{j_1} \cap C = \phi \), then \( Q_{j_1} \) is a curve of Dynkin type ADE.
Proof. Letting $Y_0 = Y \cap S$, $S = \bar{S} - C$, we have $\bar{Y}_0$ (the closure of $Y_0$ in $\bar{S}$) $= Y$. Take a proper birational morphism $\rho: \bar{S}^* \to S$ such that $(\bar{S}^*, \rho^{-1}(C+Y))$ is a $\partial$-surface. By Lemma 6, we have

\[
\kappa(K(\bar{S})+C+Y, \bar{S}) = \kappa(\bar{S}-C-Y) = 0.
\]

Recalling Proposition 3, we get $\kappa(Y, \bar{S}) = 0$. Let $\sum Y_j$ be the irreducible decomposition of $Y$ and let $q_j, \cdots, q_s$ be the connected components of $Y$. By Lemma 5, letting $q_j$ be a connected reduced divisor in $Y$, we have

\[
0 = \dim |q_j| = \dim |K+C+q_j| = \dim H^1(O_{C+q_j}) - 1 \geq (C+q_j, K+C+q_j)/2.
\]

Hence, $(C+q_j, q_j) \leq 0$. If $C+q_j$ is connected,

\[
0 = \dim |K+C+q_j| = (C+q_j, K+C+q_j)/2 = \pi(C+q_j) - 1 = \pi(C) + \pi(q_j) + (C, q_j) - 2 = \pi(q_j) - (C, q_j) - 1 \geq \pi(q_j).
\]

From this, it follows that $\pi(q_j) = 0$ and $(C, q_j) = 1$. If $C+q_j$ is not connected, then

\[
0 = \dim |K+C+q_j| = \dim H^1(O_{C+q_j}) - 1 = \dim H^1(O_C) + \dim H^1(O_{q_j}) = 1 = \pi(q_j) = (q_j, K+q_j)/2 + 1.
\]

On the other hand, $(C, q_j) = 0$ yields $(K, q_j) = 0$, since $K+C \sim 0$. Hence, $q_j = -1$. In particular, if $Y_j \cap C = \phi$, then $Y_j$ is a $C$-exceptional curve, and if $Y_j \cap C = \phi$, then $Y_j = -2$ and $(K, Y_j) = 0$.

For any $m_j \geq 0$, define $Z = \sum m_j Y_j$. We write $Z = \mathbb{Z}_0 + \cdots + \mathbb{Z}_s$ where $\text{Supp} (\mathbb{Z}_0), \cdots, \text{Supp} (\mathbb{Z}_s)$ are the connected components of $\text{Supp} Z$. By Lemma 5,

\[
0 = \dim |Z| = \dim |Z+C+K| = \dim H^1(O_{C+Z}) - 1 \geq (C+Z, C+K+Z)/2 = ((C, Z)+Z^2)/2.
\]

If $(C, Z) > 0$, then $Z^2 \leq -1$. Next, assume $(C, Z) = 0$. Then $(C, \mathbb{Z}_0) = \cdots = (C, \mathbb{Z}_s) = 0$. This implies $(K, \mathbb{Z}_0) = \cdots = (K, \mathbb{Z}_s) = 0$. Hence,

\[
1 = \dim H^1(O_{C+Z}) = \dim H^1(O_C) + \sum \dim H^1(O_{\mathbb{Z}_i}).
\]

Thus $\dim H^1(O_{\mathbb{Z}_i}) = 0$. Recalling Riemann Roch Theorem on $\bar{S}$, we have

\[
(\mathbb{Z}_i, \mathbb{Z}_i+K)/2 = \dim H^1(O_{\mathbb{Z}_i}) - \dim H^0(O_{\mathbb{Z}_i}) \leq -1.
\]

Since $(\mathbb{Z}_i, K) = 0$, we have $\mathbb{Z}_i^2 \leq -2$. Hence $\text{Supp} \mathbb{Z}_i$ is a curve of Dynkin type.
and so the intersection matrix \([(Y_t, Y_j)]\) is negative-definite. Thus we complete the proof of Proposition 4.

**Proposition 5.** Let \(\bar{S}\) be a complete surface and \(C\) a non-singular elliptic curve on \(\bar{S}\). Suppose that \(q(\bar{S})=0\) and \(K(\bar{S})+C\sim 0\). Then \(q(\bar{S}-C)=0\), and (\(\bar{S}, C\)) is obtained from one of the following three \(\delta\)-surfaces by attaching 1/2-points:

a-i) \((P^2, E)\) where \(E\) is a non-singular curve of degree 3,

a-ii) \((P^1 \times P^1, E)\) where \(E\) is a non-singular curve of degree \((2, 2)\),

a-iii) \((\Sigma_n, E)\) where \(\Sigma_n\) is a Hirzebruch surface of degree 2 and \(E\) a non-singular elliptic curve such that \(K(\Sigma_n)+E\sim 0\).

Proof. \(q(\bar{S}-C)=0\) follows from Lemma 2. First assume that \(\bar{S}=P^2\) or \(\Sigma_b=P^1 \times P^1\) or \(\Sigma_b (b\geq 2)\), that is the Hirzebruch surface of degree \(b\).

**Lemma 7.** A Hirzebruch surface \(\Sigma_b (b\geq 1)\) is a non-trivial \(P^1\)-bundle over \(P^1\) on which there exists one and only one irreducible curve \(\Delta_\omega\) with negative self-intersection number \(-b\). \(\Delta_\omega\) is a section of \(\Sigma_b \rightarrow P^1\), whose fiber is denoted by \(F\).

Any section \(C \cap \Delta_\omega\) is linearly equivalent to \(\Delta_\omega + \alpha F\) (\(\alpha \geq b\)). Then \(C^2=2\alpha - b\) and \((C, \Delta_\omega) = \alpha - b\). The smallest \(C^2\) is \(b\). Since \(\dim |\Delta_\omega + bF| = 1 + b\), we have sections \(\Delta_\lambda (\lambda\ being\ a\ point\ of\ C^{1+b})\), which satisfy \(\Delta_\lambda \cap \Delta_\omega = \emptyset\) and \(\Delta_\lambda^2 = b\). Moreover, \(-K(\Sigma_b) \sim \Delta_\omega + \Delta_\lambda + 2F\).

Proof. The verification is easy and omitted.

We continue the proof of Proposition 5. If \(\bar{S}=\Sigma_b\) and \(E\sim -K(\Sigma_b) \sim \Delta_\lambda + \Delta_\omega + 2F\), then \((E, \Delta_\omega) = -b + 2\). By the way, \(E \neq \Delta_\omega\). Hence, \((E, \Delta_\omega) \geq 0\), which implies \(b \leq 2\). We have to show that there exists a non-singular member in \(|-K(\Sigma_b)|\).

**Lemma 8.** Let \(V=P^1 \times P^2\). Then \(\Sigma_b (b\geq 1)\) is isomorphic to a non-singular hypersurface of degree \((b, 1)\) of \(V\).

Proof. Letting \(h\) be a line on \(P^2\), we put \(L=p \times P^2\) and \(M=P^1 \times h\). Then, by the adjunction formula,

\[-K(V) \sim 2L + 3M.\]

Since \(bL+M\) is very ample \((b\geq 1)\), a general member \(W\) of \(|bL+M|\) is non-singular and

\[-K(W) \sim (2L + 3M - M - bL) |W|\]

Hence \(K(W)^2 = 8\). Moreover, the projection \(\pi: V \rightarrow P^1\) induces the fibered surface \(\pi' = \pi | W: W \rightarrow P^1\), whose fiber is linearly equivalent to \(L | W\). Clearly, \((L | W)^2 = 0\) and \(L | W \cong P^1\). Hence, \(\pi | W: W \rightarrow P^1\) is a \(P^1\)-bundle. \(M | W\) is a section which satisfies \((M | W)^2 = b\). Hence \(W \cong \Sigma_b\). Employing the notation in Lemma 7, we see that \(\Delta_\omega \sim (M-bL) | W\) and \(\Delta_\lambda \sim M | W\). Q.E.D.
When \( b=2 \), \(-K(\Sigma_b)\) is linearly equivalent to \( 2M | W \). \((2M | W)^2=8\) and \(2M | W\) has no base points. Therefore a general member of \(|-K(\Sigma_b)\)| is a non-singular elliptic curve. A curve \( E \) on \( P^2 \) or \( P^1 \times P^1 \) which satisfies the condition of Proposition 5 is a non-singular curve of degree 3 or degree \((2, 2)\), respectively.

Recalling that a relatively minimal rational surface \( \tilde{S} \) is isomorphic to \( P^2, P^1 \times P^1 \) or \( \Sigma_b \), we have only to consider the case where there is an exceptional curve \( L \) of the first kind on \( \tilde{S} \). Since \( L+C \) and \( L^2=(K(\tilde{S}), L)=-1 \), we have \((C, L)=-(K(\tilde{S}), L)=1\). Hence, \( L \) is a \( C \)-exceptional curve. Contracting such \( L \) successively, we complete the proof.

With the notation being as in Proposition 5, let \( Y \) be a curve of Dynkin type in \( S=\tilde{S}-C \). Corresponding to the 1/2-point attachments, we have a proper birational morphism \( \mu: \tilde{S} \to \tilde{S}_* \), \( \tilde{S}_*=P^2 \) or \( P^1 \times P^1 \) or \( \Sigma_2 \). By Lemma 6, writing \( Z=\mu(Y) \), we have \( \kappa(\tilde{S}_*-\mu(C)-Z)=\kappa(\tilde{S}_*-C-Y)=\kappa(Y, \tilde{S})=0 \). Hence, \( Z \) is a sum of exceptional curves and a curve of Dynkin type. Since \( \tilde{S}_* \) is relatively minimal, \( Z \) is a curve of Dynkin type such that \( Z \cap \mu(C)=\phi \). Thus, \( Z=\Delta_a \subseteq \Sigma_2 \). Accordingly, \( \mu(Y) \) is a union of a finite set of points in \( \mu(C) \) and \( \Delta_a \subseteq S_*=\Sigma_2 \).

Therefore, \( Y \) is a curve of Dynkin type \( A \). Summarizing the argument above, we obtain the following proposition.

**Proposition 6.** Let \((\tilde{S}, D)\) be a relatively \( \partial \)-minimal surface such that \( S=\tilde{S}-D \) is a logarithmic K3 surface. Suppose that \((\tilde{S}, D_A)\) is relatively \( \partial \)-minimal and that there are no \( D \)-exceptional curves of the first kind on \( \tilde{S} \). Then such \( \partial \)-surfaces \((\tilde{S}, D)\) are classified into the following table. There, \( D=\sum C_i \) is the irreducible decomposition and \( C_i \) is a non-singular elliptic irreducible curve.

<table>
<thead>
<tr>
<th>class ( a-)</th>
<th>( \tilde{S} )</th>
<th>( D ) with the self-intersection numbers</th>
<th>( \pi_1(S) )</th>
<th>( S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a-1) )</td>
<td>( P^2 )</td>
<td>( C_i \subseteq 9 )</td>
<td>( Z/(3) )</td>
<td>affine</td>
</tr>
<tr>
<td>( a-) ii)</td>
<td>( P^1 \times P^1 )</td>
<td>( C_i \subseteq 8 )</td>
<td>( Z/(2) )</td>
<td></td>
</tr>
<tr>
<td>( a-) iii)</td>
<td>( \Sigma_2 )</td>
<td>( C_i \subseteq 8 )</td>
<td>( Z/(2) )</td>
<td>non-affine</td>
</tr>
<tr>
<td>( a-) iii)</td>
<td>( \Sigma_2 )</td>
<td>( C_i \subseteq 8 )</td>
<td>( C_i \subseteq -2 )</td>
<td>?</td>
</tr>
</tbody>
</table>

We have the following

**Theorem II.** Let \((\tilde{S}, D)\) be a \( \partial \)-surface whose interior \( S \) is a logarithmic K3 surface of type \( \Pi_a \). Then there exists a birational morphism \( \mu: \tilde{S} \to \tilde{S}_* \) such that
1) $\tilde{S}_* = P^2$ or $P^1 \times P^1$ or $\sum_2$ 2) $C = \mu(D_A)$ is a non-singular curve, 3) $\mu(D_A)$ is a finite set or a union of a finite set and $Z = \Delta_\infty$ on $\sum_2$. The latter case occurs only when $S_*=\sum_2$.

Structure of logarithmic $K3$ surfaces of type $\Pi_\alpha$ is studied precisely by examining each class of a-i) through a-iii)* separately. We use the following notion: Let $S$ be a surface and let $\mu$ be a proper birational morphism: $S^* \rightarrow S$ such that there exists a dominant morphism $f: S^* \rightarrow J$, $J$ being a curve, whose general fiber $f^{-1}(u)$ is $C_*$. Then we say that $S$ is a $C^*$-fibered surface or $S$ has the structure of $C^*$-fibered surface.

**Proposition 7.** Every surface of the class a-ii) or a-iii) has a structure of $C^*$-fibered surface.

Proof is easy.

**Proposition 8.** Let $S$ be a surface of the class a-i) or a-iii)*. Then $S$ does not admit the structure of $C^*$-fibered surface.

Proof. First we let $S$ be a surface of the class a-iii)*. Suppose that there exist a proper birational morphism $\mu: S^* \rightarrow S$ and a dominant morphism $f: S^* \rightarrow J$, $J$ being a complete curve, whose general fiber is $C_*$. Choosing a suitable completion $\bar{S}^*$ of $S^*$ with smooth boundary $\bar{D}^*$, we assume that $\mu$ defines a morphism $\mu_\bar{}: \bar{S}^* \rightarrow \sum_\alpha$ and $\bar{D}^* = \mu^{-1}(C_1 + C_2)$ and that $f$ defines a morphism $\tilde{f}: \bar{S}^* \rightarrow J$. By $C^*_f$ we denote the proper transform of $C_1$ by $\mu^{-1}$, which is a non-singular elliptic curve. Since a general fiber of $\tilde{f}$ is $P^1$, $C^*_f$ is not contained in a fiber of $\tilde{f}$. Hence $f(C^*_f) = J$. Since $\bar{S}^*$ is rational, $J$ is $P^1$. This implies that $\tilde{f}|C^*_f: C^*_f \rightarrow P^1$ is a two-sheeted covering. Hence, $f(C^*_f)$ is a point, because $f^{-1}(u) \cap \bar{D}^* = \{p_1, p_2\}$ for a general point $u \in J$. Therefore, $g = f \cdot \mu_\bar{}^{-1}: \sum_2 \rightarrow J$ turns out to be a morphism. Moreover, $g(C_2)$ is a point $a$. Hence, $C_2$ is a part of the singular fiber $g^{-1}(a)$. Since $C_2^2 = -2$, there is another component $C_3$ in $g^{-1}(a)$ such that $C_3^2 = -1$. This contradicts the fact that $\sum_2$ is a relatively minimal surface. It is easier to prove the same result for surfaces of the class a-i).

Q.E.D.

**Proposition 9.** There exists an algebraic pencil $\{C_u\}$ on each surface of the classes a-i) and a-iii)* whose general member $C_u$ is $C^*$.

Here, an algebraic pencil $\{C_u\}$ on $S$ is understood as follows: there exist an algebraic surface $S^*$ and a proper birational morphism $\rho: S^* \rightarrow S$ in which $\psi: S^* \rightarrow J$ is a fibered surface whose general fiber $C^*_u$. $\{C_u = \rho(C^*_u)\}$ is the algebraic pencil on $S$.

We omit the proof of Proposition 9.
If there is a proper birational map $f: S_1 \to S_2$ then the existence of the algebraic pencil $\{C_u\}$, $C_u \cong C^*$, on $S_1$, induces the existence of the same thing on $S_2$. Moreover, when $S_1$ is an open set of $S_2$ with $\kappa(S_2) \geq 0$, the existence of an algebraic pencil of $C_u \cong C^*$ on $S_1$ implies the existence of the same thing on $S_2$. In fact, there are a proper birational morphism $\rho: S^* \to S_1$ and a morphism $\psi: S^* \to J$ with $C_u = \rho(\psi^{-1}(u)) \cong C^*$ for a general $u \in J$. Let $\Gamma_u$ be the closure of $C_u$ in $S_2$. Then $\kappa(\Gamma_u) \leq 0$. If $\kappa(\Gamma_u) = -\infty$, it would imply that $\kappa(S) = -\infty$, a contradiction.

Accordingly we get

Proposition 10. There is an algebraic pencil $\{C_u\}$ with the general member $C_u \cong C^*$ on any logarithmic K3 surface of type $\Pi_a$.

Corollary. A logarithmic K3 surface of type $\Pi_a$ is not measure-hyperbolic.

Proof follows from the fact that $C^*$ is not measure-hyperbolic.

Proposition 11. Let $S$ be a surface in the TABLE $\Pi_a$. Then, Aut $(S)$ is a finite group.

Proof. We give a proof for a surface of the class $a$-$iii^*$. Let $\varphi \in \text{Aut}(S)$. Then $\varphi$ extends to an isomorphism of $\tilde{S} = \Sigma_2$, since $g(C) = 1$ and $C^2 = -2 \leq -2$ ([12]). Thus $\text{Aut}(S) \subset \text{Aut}(\Sigma_2) = \{\varphi \in \text{Aut}(\Sigma_2); \varphi(D) = D\}$. Let $\pi: \Sigma_2 \to P^1$ be the $P^1$-bundle structure of $\Sigma_2$. We have the group extension:

$$1 \to G_1 \to \text{Aut}(\Sigma_2) \to PGL(1, k) = \text{Aut}(P^1) \to 1.$$ 

It is well known that $G_1$ is an algebraic group of dimension 4. Moreover, $G_1$ is an affine group. Hence $\text{Aut}(\Sigma_2)$ is an affine algebraic group. And so is $\text{Aut}_0(\Sigma_2)$. Furthermore, we have the group homomorphism $\gamma: \text{Aut}_0(\Sigma_2) \to \text{Aut}(C)$ which is the restriction, i.e., $\gamma(\varphi) = \varphi|C$. Therefore, $\text{Im} \gamma$ is finite, since $\text{Aut}(C)$ is a finite union of elliptic curves. Put $G_2 = \text{Ker} \gamma$, which turns out to be a finite group. Thus $\text{Aut}_0(\Sigma_2)$ is finite and so is $\text{Aut}(S)$. Q.E.D.

Proposition 12. Let $\tilde{S}$ be a rational surface and $C$ a non-singular elliptic curve on $\tilde{S}$. Let $Y$ be a reduced divisor on $S$ such that $\kappa(\tilde{S}-(C \cup Y)) = 0$. Then $\tilde{q}(\tilde{S}-(C \cup Y)) = 0$, i.e., $\tilde{S}-(C \cup Y)$ is a logarithmic K3 surface of type $\Pi_a$.

A proof follows from the arguments in the proofs of Propositions 3 and 4. Actually, the intersection matrix of $Y$ is negative-definite and hence we can use Lemma 2.

Proposition 13. Let $(\tilde{S}, D)$ be a $\partial$-surface whose interior $S$ is a logarithmic K3 surface of type $\Pi_a$. Suppose that 1) $(\tilde{S}, D)$ is relatively $\partial$-minimal, 2) $S$ has no 1/2-points, and 3) $D$ is connected. Then $(\tilde{S}, D)$ is one of $a$-$i$) $\sim$ $a$-$iii$ in Proposition 5.
Proof. At the beginning of §4 we have had the decomposition: \( D = D_A + D_B \). Suppose that there exists an irreducible exceptional curve \( E \) of the first kind on \( \tilde{S} - D_A \). In view of Proposition 4, by contracting \( E \) we have a proper birational \( \partial \)-morphism \( \lambda: (S, D) \rightarrow (S_1, D_1) \). We have the following cases:
1) If \( E \subset D_B \) or \( E \cap D_B = \phi \), this contradicts the hypothesis. 2) If \( E \cap D_B \neq \phi \), then \( \lambda: (\tilde{S}, D + E) \rightarrow (S_1, D_1) \) is a non-canonical blowing up. In fact if \( \lambda \) were canonical, \( D \) would be disconnected. Thus \( E - D_B \subset S \) is a 1/2-point. This is also a contradiction. Accordingly, we conclude that \( \tilde{S} - D_A \) is relatively minimal. By Proposition 4, \( D_B \) is a union of exceptional curves of the first kind. Hence \( D_B = \phi \). Since, there are no \( D \)-exceptional curves, it follows that \( \tilde{S} \) is a relatively minimal surface. Q.E.D.

5. Logarithmic K3 surfaces of type II\(_b\). In §5, let \( S \) be a logarithmic K3 surface and let \( (\tilde{S}, D) \) be a \( \partial \)-surface such that \( S = \tilde{S} - D \). By \( C_1, \ldots, C_s \) we denote the irreducible components of \( D \). Since \( h(\Gamma(D)) = 1 \), there is a circular boundary \( D_A = C_1 + \cdots + C_s \leq D \). \( \bar{p}_i(\tilde{S} - D_A) = 1 \) induces that \( \tilde{S} - D_A \) is also a logarithmic K3 surface of type \( \Pi_b \). Contracting exceptional curves of the first kind in \( \tilde{S} - D_A \) successively, we have a non-singular complete surface \( \tilde{S}_* \) and a birational morphism \( \mu: \tilde{S} \rightarrow \tilde{S}_* \) such that \( \mu \) is isomorphic around \( D_A = \mu(D_A) \) and such that \( \tilde{S} - \mu(D_A) \) has no exceptional curves of the first kind. After choosing \( D \) to be a minimal boundary, we have a minimal boundary \( D_A = \mu(D_A) \). Then \( (\tilde{S}_*, D_A) \) is a relatively \( \partial \)-minimal \( \partial \)-surface.

We write \( D = D_A + D_B \) and \( Y = \mu_*(D_B) \). By Lemma 6 we have

\[
0 = \kappa(\tilde{S} - D) = \kappa(\tilde{S}_* - D_A - Y).
\]

From the condition \( h(\Gamma(D_A)) = 1 \), we infer readily that \( \bar{p}_i(\tilde{S}_* - D_A) = 1 \). Hence, \( \bar{p}_i(\tilde{S}_* - D_A) = 1 \) for any \( i \geq 1 \). However, \( \bar{q}(\tilde{S}_* - D_A) \geq 0 \).

Proposition 14. Let \( (\tilde{S}, D) \) be a circular \( \partial \)-surface (i.e., \( D \) is circular) which is relatively \( \partial \)-minimal. Suppose that \( \kappa(\tilde{S} - D) = 0 \). Then \( K(\tilde{S}) + D \sim 0 \).

Proof. It is easy to check that \( \tilde{S} \) is a rational surface. Assuming that \( |K(\tilde{S}) + D| \) has a non-trivial member \( \Delta = \sum r_i E_i \) \( (r_i > 0) \) we shall derive a contradiction.

Now, \( 0 = \kappa(\tilde{S} - D) = \kappa(K(\tilde{S}) + D, \tilde{S}) = \kappa(\Delta, \tilde{S}) = \kappa(\sum E_i, \tilde{S}) \) implies that the intersection matrix \( [(E_i, E_j)] \) is negative semi-definite. We assume \( (\Delta, E_i) \leq 0 \) and \( E_i \not\subset D \). Then by the same reasoning as in the proof of Proposition 4, we have the following cases:

Case 1: \( \pi(E_i) = 1 \). Then \( E_i \cap D = \phi \) and \( E_i = (K, E_i) = 0 \).

Case 2: \( \pi(E_i) = 0 \) and \( (D, E_i) \geq 1 \). Then \( E_i = (K, E_i) = -1 \) and \( (E_i, D) = 1 \). Hence \( E_i \) is \( D \)-exceptional. By detaching 1/2-points, we may assume that this case does not occur.
Case 3: \( \pi(E_1) = 0 \) and \( (D, E_1) = 0 \). Then \( E_1 \cap D = \emptyset \) and \( E_1^2 = -2 \), \((K, E_1) = 0\).
In all cases we have \( (\Delta_1, E_1) = 0 \). If \( E_1 \subset D \) and \( r \geq 2 \), we have \( D' + E_1 = D \), \( E_1 = \mathbb{P}^1 \) and \( (D', E_1) = 2 \). Hence
\[
\dim |K + D'| = \mathbb{P}(\mathcal{O} - D') - 1 = k(\mathcal{O}(D')) - 1.
\]
On the other hand, \( |K + D'| \ni (r_1 - 1)E_1 + r_2E_2 + \cdots \). This is a contradiction.
Thus, \( \Delta^2 = \sum r_i(\Delta, E_i) \geq 0 \). Since \( \kappa(\Delta, \mathcal{S}) = 0 \), we have \( \Delta^2 = 0 \). By the similar argument to the proof of Proposition 4, we derive a contradiction. Q.E.D.

**Proposition 15.** With the notation being as in Proposition 14, let \( Y \) be a reduced divisor on \( \mathcal{S} \) which does not contain any components of \( D \). Suppose that \( \kappa(\mathcal{S} - D - Y) = 0 \). Then \( \kappa(Y, \mathcal{S}) = 0 \). By \( \mathcal{O} \), \( \cdots \), \( \mathcal{O}_n \), we denote the connected components of \( Y \). If \( \mathcal{O}_j \cap D = \emptyset \), then \( (\mathcal{O}_j, D) = 1 \) and \( \mathcal{O}_j \) is an exceptional curve of the first kind. If \( \mathcal{O}_j \cap D = \emptyset \), then \( \mathcal{O}_j \) is a curve of Dynkin type \( A \).

The proof of Proposition 4 can be used again here.

**Proposition 16.** Let \((\mathcal{S}, D)\) be a circular \( \partial \)-surface such that \( K(\mathcal{S}) + D \sim 0 \). Then \((\mathcal{S}, D)\) is obtained from one of the following \( \partial \)-surfaces by attaching several \( 1/2 \)-points and canonical blowings ups.

- a-i) \( \mathcal{S} = \mathbb{P}^2 \), \( D = H_1 + H_2 + H_3 \) where each \( H_i \) is a line on \( \mathbb{P}^2 \),
- b-i) \( \mathcal{S} = \mathbb{P}^1 \times \mathbb{P}^1 \), \( D = H_1 + H_2 + G_1 + G_2 \) where each \( H_i \) is a line of degree \((1, 0)\) and each \( G_i \) is a line of degree \((0, 1)\),
- a-ii) \( \mathcal{S} = \mathbb{P}^1 \), \( D = H + C \) where \( H \) is a line and \( C \) is a conic,
- b-i) \( \mathcal{S} = \mathbb{P}^1 \times \mathbb{P}^1 \), \( D = C_1 + C_2 \), where each \( C_i \) is a curve of degree \((1, 1)\),
- b-ii) \( \mathcal{S} = \mathbb{P}^1 \), \( D = F + \Delta \) where the \( \Delta \) is a section which is different from \( \Delta_\infty \),
- b-iii) \( \mathcal{S} = \mathbb{P}^1 \), \( D = F + \Delta_+ + C_3 \) where \( C_3 \) is a non-singular rational curve which is linearly equivalent to \( \Delta_0 + F \),
- b-iv) \( \mathcal{S} = \mathbb{P}^1 \times \mathbb{P}^1 \), \( D = H_1 + G_1 + C \) where \( H \) is a line of degree \((1, 0)\), \( G_1 \) is a line of degree \((0, 1)\), and \( C \) is a curve of degree \((1, 1)\),
- b-v) \( \mathcal{S} = \mathbb{P}^2 \), \( D = C \) where \( C \) is a cubic curve with one ordinary double point,
- b-vi) \( \mathcal{S} = \mathbb{P}^2 \), \( D = C \) where \( C \) is a curve of degree \((2, 2)\) which has one ordinary double point,
- b-vii) \( \mathcal{S} = \mathbb{P}^2 \), \( D = C \) where \( C \) is a rational curve with only one ordinary double point which is linearly equivalent to \( \Delta_0 + 2F \),
- b-viii) \( \mathcal{S} = \mathbb{P}^1 \times \mathbb{P}^1 \), \( D = G + C \) where \( G \) is a line of degree \((0, 1)\) and \( C \) is a curve of degree \((2, 1)\),
- b-ix) \( \mathcal{S} = \mathbb{P}^1 \), \( D = \Delta_\infty + C \) where \( C \) is a curve which is linearly equivalent to \( \Delta_0 + 2F \).
Proof is easy and left to the reader.

In the following Table II, we exhibit $q$ and configurations of components of $D$ of b-i) $\sim$ b-xiii).

**Proposition 17.** Let $(\bar{S}, D)$ be a circular $\partial$-surface whose interior $S$ is a logarithmic $K3$ surface or a surface satisfying the following conditions: 1) $\bar{S}$ is rational, 2) $\kappa(S)=0$, 3) $\bar{p}_1(S)=1$, and 4) $q(S)=1$ or 2. Suppose that i) $(\bar{S}, D)$ is relatively $\partial$-minimal, ii) $D$ is connected, and iii) $S$ has no $1/2$-points. Then $(\bar{S}, D)$ is one of b-i) $\sim$ b-xiii) in TABLE II.

Proof is similar to that of Proposition 13.

<table>
<thead>
<tr>
<th>$q$</th>
<th>class</th>
<th>$\bar{S}$</th>
<th>configuration of $D$</th>
<th>$\pi_1(S)$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>b-i)</td>
<td>$P^2$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b-ii)</td>
<td>$P^1 \times P^1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z^2$</td>
<td>$C^*$</td>
</tr>
<tr>
<td></td>
<td>b-iii)</td>
<td>$\Sigma_\beta$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z^2$</td>
<td>$\Sigma_\beta$</td>
</tr>
<tr>
<td></td>
<td>b-iv)</td>
<td>$P^2$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b-v)</td>
<td>$P^1 \times P^1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>b-vi)</td>
<td>$\Sigma_\alpha$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b-vii)</td>
<td>$\Sigma_\beta$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>b-viii)</td>
<td>$P^1 \times P^1$</td>
<td><img src="image" alt="Diagram" /></td>
<td>$Z$</td>
<td></td>
</tr>
</tbody>
</table>
Next we treat the 3-surface \((S, D)\) whose boundary is not connected. As in § 4, we have to look for a curve \(Z\) of Dynkin type on \(\bar{S} - D\) where \((\bar{S}, D)\) is one of \(b-i)\) through \(b-xiii)\) in TABLE II. Such \(Z\) exists only in the cases \(b-vi)\) and \(b-xi)\). Then \(Z\) turns out to be \(\Delta_\omega\) of \(\Sigma_2\). We write \(b-vi)*\) or \(b-xi)*\) in the case of disconnected boundaries. Therefore we obtain the following

**Theorem II\(_b\).** Let \((\bar{S}, D)\) be a 3-surface whose interior \(S\) is a logarithmic K3 surface of type II\(_b\). Then, there exists a birational morphism \(\mu: \bar{S} \to \bar{S}_*\) such that \((\bar{S}_*, \mu(D_A))\) is one of \(b-i)\) through \(b-xiii)\) in TABLE II\(_b\). Moreover, \(\mu(D_B)\) is a finite set or a union of a finite set and \(Z=\Delta_\omega\) on \(\Sigma_2\). The latter case occurs only when \((\bar{S}, \mu_*(D) - Z)\) is the class \(b-vi)\) or \(b-xi)\).

**Remark.** In the above theorem the hypothesis that \(S\) is a logarithmic K3 surface of type II\(_b\) is replaced by the following condition that 1) \(\bar{F}_S(S) = 1\) and \(\varphi(S) = 0\), 2) \(\bar{S}\) is rational, 3) \(D\) consists of rational curves.

In order to prove the generalized Theorem II\(_b\), we have only to note that

<table>
<thead>
<tr>
<th>(\bar{q})</th>
<th>class</th>
<th>(\bar{S})</th>
<th>configuration of (D)</th>
<th>(\pi_1(S))</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(b-x))</td>
<td>(P^1 \times P^1)</td>
<td>(\bigcirc^8)</td>
<td>(Z/(2))</td>
<td>(\bigcirc^8)</td>
</tr>
<tr>
<td>0</td>
<td>(b-xi))</td>
<td>(\Sigma_2)</td>
<td>(\bigcirc^8)</td>
<td>(Z/(2))</td>
<td>(\bigcirc^8)</td>
</tr>
<tr>
<td>0</td>
<td>(b-xii))</td>
<td>(P^1 \times P^1)</td>
<td>(\bigcirc^{4+\alpha})</td>
<td>(Z/(2))</td>
<td>(\bigcirc^{4+\alpha})</td>
</tr>
<tr>
<td>0</td>
<td>(b-xiii)_{\beta})</td>
<td>(\Sigma_\beta)</td>
<td>(\bigcirc^{4+\beta})</td>
<td>(\bigcirc^{4+\beta})</td>
<td>(\bigcirc^{4+\beta})</td>
</tr>
<tr>
<td>1</td>
<td>(b-vi)*)</td>
<td>(\Sigma_2)</td>
<td>(\bigcirc^{2-2})</td>
<td>(\bigcirc^{2-2})</td>
<td>(\bigcirc^{2-2})</td>
</tr>
<tr>
<td>0</td>
<td>(b-xi)*)</td>
<td>(\Sigma_8)</td>
<td>(\bigcirc^{8-2})</td>
<td>(\bigcirc^{8-2})</td>
<td>(\bigcirc^{8-2})</td>
</tr>
</tbody>
</table>
Propositions 14, 15 and 16 were proved without the logarithmic irregularity condition to the effect $\bar{q}=0$.

6. Surfaces with $\bar{\kappa}=0$ and $\bar{p}_g=1$. In general, let $(\bar{S}, D)$ be a $\partial$-surface such that the interior $S$ satisfies $\bar{p}_e(S)=1$ and $\kappa(S)=0$. Then $\bar{p}_e(\bar{S})=1$ and $\kappa(\bar{S})\leq 0$.

Proposition 18. If $\bar{p}_e(\bar{S})=0$, then $\kappa(\bar{S})=-\infty$. Hence, $\bar{S}$ is a ruled surface.

Proof. In view of Proposition 2, it suffices to derive a contradiction from the hypothesis that $\kappa(\bar{S})=0$, $\bar{p}_e(\bar{S})=0$, and $\bar{q}(\bar{S})\geq 1$. Such a surface $\bar{S}$ is birationally equivalent to a hyperelliptic surface, whose universal covering surface is an abelian surface. Namely, contracting exceptional curves of the first kind on $\bar{S}$ successively, we get a hyperelliptic surface $\bar{S}_*$ and a birational morphism $\mu: \bar{S} \to \bar{S}_*$. Then by Lemma 1,

$$0 = \kappa(S) = \kappa(K(\bar{S})+D, \bar{S}) = \kappa(K(\bar{S}_*+\mu_*D), \bar{S}_*)$$

This implies that $\mu_*D=0$. Thus

$$H^0(\mathcal{O}(K(\bar{S})+D)) = H^0(\mathcal{O}(\mu_*(K(\bar{S}_*))+\mu_*D))$$

is a contradiction to $H^0(\mathcal{O}(K(S)+D))=1$. Q.E.D.

Consequently, we have the following cases to examine separately.

1) If $\bar{p}_e(\bar{S})=0$ and $\bar{q}(\bar{S})=0$, then $\bar{S}$ is a rational surface. Hence, letting $\sum_{j=1}^\infty C_j$ be the irreducible decomposition of $D$,

\(\alpha\) if $g(C_i)=1$, then put $D_A=C_i$,

\(\beta\) if $g(C_i)=\cdots=g(C_j)=0$, then there is a circular boundary $D_A=C_1+\cdots+C_r\subset D$.

2) If $\bar{p}_e(\bar{S})=0$ and $\bar{q}(\bar{S})=1$, then $\bar{S}$ is a ruled surface of genus 1. Let $f: \bar{S} \to J$ be the Albanese map of $S$, $J$ being an elliptic curve, since $\bar{p}_e(\bar{S})=0$. For a general point $y\in J$, $f^{-1}(y)$ turns out to be a non-singular rational curve. Define $C_y=f^{-1}(y)-D\cap f^{-1}(y)$. Then by Kawamata's Theorem ([14]), we obtain

$$0 = \kappa(S) \geq \kappa(C_y)+\kappa(J) = \kappa(C_y).$$

Hence, $\kappa(C_y)=0$ follows. This implies that $C_y\subset C^*$ and $(D, f^{-1}(y))=2$. Hence, the horizontal component $D_A$ defined to be $\{\sum C_j; f(C_j)=J\}$ satisfies that $(D_A, f^{-1}(y))=2$. Referring to the following lemma, we have

$$\dim |K(\bar{S})+D_A|=0,$$

i.e., $\bar{p}_e(\bar{S}-D_A)=1$. 

Lemma 9. Let $V$ be a complete normal variety and let $A$, $B$ be divisors on $V$ such that $\kappa(A, V)^0, |A+B|\neq \phi$, $B$ is effective, and $\kappa(A+B, V)=0$. Then $|A|\neq \phi$.

Proof. Choose $i>0$ such that $|iA|\neq \phi$ and take $X \in |A+B|$ and $Z \in |iA|$. Then $Z+iB \sim iX$. By $\kappa(X, V)=0$, we have $Z+iB=iX$. Hence, $Z=i(X-B)$ is effective. This implies that $X-B$ is effective. Q.E.D.

3. If $p^*(\bar{S})=1$, then put $D_A=0$.

In all cases above, we define $D_B$ by $D=D_A+D_B$

Theorem III. With the notation being as above, we suppose that $\bar{S}-D_A$ has no exceptional curves of the first kind. Then $K(S)+D_A \sim 0$.

Proof. Recalling Propositions 3 and 14, it suffices to prove under the assumption that $\bar{S}$ is a ruled surface with $q(S)=1$. Take $E \in |K+D_A|$ to be the irreducible decomposition of $\Delta$. [(E_i, E_j)] is negative semi-definite. In particular, $E_i^2 \leq 0$. First assume that $(\Delta, E_i)=0$, since $\Delta^2 \leq 0$. If $E_i\subset D_A$, then, putting $D_A=E_i+D'$, we would have $(f^{-1}(y), D') \leq 1$. This would imply $\kappa(\bar{S}-D')=\kappa(\bar{S})=1$ while $\kappa(S-D')=\kappa(K(S)+D-A-E_i, S)=\kappa(\Delta-E_i, \bar{S})=\kappa((r_i-1)E_i+, \cdots, \bar{S})=0$. Therefore, $E_i \notin D_A$. Hence $(D_A, E_i) \geq 0$. Since $(\Delta, E_i)=(K, E_i)+(D_A, E_i) \leq 0$, we have $E_i^2 \leq 0$ and $(K, E_i) \leq 0$.

As in the proof of Proposition 3 we have the following cases to examine separately.

1) If $E_i^2=-2$, $(K, E_i)=0$, then $\pi(E_i)=0$ and $(D_A, E_i)=0$.

2) If $E_i^2=-1$, $(K, E_i)=-1$, then $(D_A, E_i)=0$ or 1. In this case, $(D_A, E_i)=0$ contradicts the hypothesis that $\bar{S}-D_A$ has no exceptional curves of the first kind. In the case when $(D_A, E_i)=1$, contracting $E_i$ corresponds to a 1/2-point detachment.

3) If $E_i^2=0$, $(K, E_i)=-2$, then $(D_A, E_i)=2$. Since $\pi(E_i)=0, f(E_i)=p \in J$. Hence, $E_i=f^{-1}(p)$. Therefore, by Kawamata's Theorem (14), $\pi(S-E_i) \geq \pi(C_j)+\pi(f^{-1}(p))=1$. On the other hand, $\kappa(K(\bar{S})+D_A+E_i, \bar{S}) \geq \kappa(S-E_i) \geq 1$. Since $E_i \leq \Delta \in |K(\bar{S})+D_A|$, we have

$$\kappa(K(\bar{S})+D_A+E_i, \bar{S})=0.$$ 

This is a contradiction. Hence, we conclude that the case 3) does not occur.

4) If $E_i^2=0$ and $(K, E_i)=0$, then $\pi(E_i)=1$ and $(D_A, E_i)=0$. In all cases, we have $(D_A, E_i)=0$ and $(\Delta, E_i)=0$. Therefore, $(\Delta, E_i)=0$ for all $j$, hence $\Delta^2=\Sigma r_j(\Delta, E_j)=0$. Letting $\mathcal{D}_1, \cdots, \mathcal{D}_s$ be the connected components of $\Delta$, we can easily see that these are curves of extended Dynkin type $\overline{A\overline{D}\overline{E}}$. In particular, $\mathcal{D}_1^2=\cdots=\mathcal{D}_s^2=0$.

$\alpha$) If $\mathcal{D}_1$ consists of one irreducible component, then $\mathcal{D}_1$ is an elliptic curve. Hence $f(\mathcal{D}_1)=J$, and so $(\mathcal{D}_1+D_A, f^{-1}(y)) \geq 3$. This implies $\pi(S-\mathcal{D}_1) \geq 1$ by
Kawamata's Theorem. By the way,
\[ \kappa(K(S)+D_A+\mathcal{D}_1, S) \geq \kappa(S-\mathcal{D}_1) \geq 1 \]
and
\[ \kappa(K(S)+D_A+\mathcal{D}_1, S) = \kappa(\Delta+\mathcal{D}_1, S) = 0. \]
This is a contradiction.

\( \beta) \) If \( \mathcal{D}_1 \) has more than 1 irreducible components, \( f(\mathcal{D}_1) \) is a point. Hence \( \mathcal{D}_1 \) is a reducible member of \( |f^*(\mathcal{Y})| \). This implies \( h(\Gamma(\mathcal{D}_1))=0 \), a contradiction.

Next, we shall consider the counterparts of Propositions 4 and 15 in the case of \( q(S)=1 \).

**Proposition 19.** Let \( \bar{S} \) be a ruled surface of \( q(\bar{S})=1 \) with the Albanese fibered surface \( f: \bar{S} \rightarrow J \). Let \( D_A \) be a divisor with normal crossings consisting of horizontal components such that \( K(\bar{S})+D_A \sim 0 \). Suppose that a reduced divisor \( Y \) on \( \bar{S} \), each component of which is not contained in \( D_A \), satisfies the condition that \( \kappa(\bar{S}-D_A-Y)=0 \). Then \( \kappa(Y, \bar{S})=0 \). Moreover, letting \( \mathcal{Q}_1, \cdots, \mathcal{Q}_n \) be the connected components, we see that if \( \mathcal{Q}_j \cap D_A \neq \emptyset \), \( \mathcal{Q}_j \) is an exceptional curve of the first kind such that \( (\mathcal{Q}_j, D_A)=1 \) and that if \( \mathcal{Q}_j \cap D_A=\emptyset \), then \( \mathcal{Q}_j \) is a curve of Dynkin type \( A \).

Proof. Let \( \sum \mathcal{Y}_j \) be the irreducible decomposition of \( Y \). If \( \mathcal{Y}_j \) is horizontal with respect to \( f \), then \( (\mathcal{Y}_j+D_A, f^{-1}(u)) \geq 3 \) for a general \( u \in J \). By Kawamata's Theorem, we get
\[ \kappa(S-\mathcal{Y}_j) \geq \kappa(f^{-1}(u)-\mathcal{Y}_j-D_A)+\kappa(J) = 1, \]
where \( S=\bar{S}-D_A \).
This contradicts \( \kappa(S-Y)=0 \). Hence, \( f(Y) \) is a finite set of points. For a connected reduced curve \( \mathcal{Q} \subset Y \), we have a point \( p=f(\mathcal{Q}) \), and so \( \mathcal{Q} \subset f^{-1}(p) \). In view of \( \kappa(S-\mathcal{Q})=1 \), we see that \( \mathcal{Q} \subset f^{-1}(p) \). Therefore, \( \mathcal{Q} \) consists of non-singular rational curves \( Y_j \) with negative-definite intersection matrix \([ (Y_i, Y_j)]\), \( Y_i \subset \mathcal{Q} \). If \( Y_j \cap D_A = \emptyset \), then \( (D_A, Y_j)=0 \) and so \( (K, Y_j)=-(D_A, Y_j)=0 \). Combining this with \( Y_j^2 \leq -1 \), we have \( Y_j^2=-2 \) and \( \kappa(Y_j)=0 \). If \( Y_j \cap D_A \neq \emptyset \), then \( (Y_j, D_A)=-(Y_j, K)>0 \). Hence \( Y_j \) is an exceptional curve of the first kind and \( (Y_j, D_A)=1 \).

**Proposition 20.** With the same notation as in Proposition 19, we further assume that \( \bar{S} \) is relatively minimal. Then
\[
c-i) \quad \bar{S}=P^1 \times J, \quad D_A=p_1 \times J+p_2 \times J, \]
or
\[
c-ii) \quad \bar{S} \rightarrow J \text{ is a } C^*\text{-bundle of degree 0 which is not } P^1 \times J, \text{ and } D_A=\Gamma_0+\Gamma_\infty, \]
$\Gamma_0$ and $\Gamma_\infty$ being sections with $\Gamma_0^2=\Gamma_\infty^2=(\Gamma_0, \Gamma_\infty)=0$. Note that $\Gamma_0$ is cohomologically equivalent to $\Gamma_\infty$.

Further,

c-iii) $\tilde{S}\to J$ is a $C^*$-bundle of degree $m>0$ and $D_A=\Gamma_0+\Gamma_\infty$, $\Gamma_0$ and $\Gamma_\infty$ being sections with $\Gamma_0^2=m$ and $\Gamma_\infty^2=-m$.

In order to prove this, we need the following lemma.

**Lemma 10.** Let $f: \tilde{S}\to J$ be a $P^1$-bundle over an elliptic curve $J$. Then we have the following table.

| class | $\tilde{S}\to J$ | dim $| -K(\tilde{S})|$ | a member of $| -K(\tilde{S})|$ | $\tilde{q}(\tilde{S}-D_A)$ |
|-------|-------------------|---------------------|---------------------|---------------------|
| i)    | $P^1\times J$     | 2                   | $D_A=p_1\times J+p_2\times J$ | 2                   |
| ii)   | $C^*$-bundle of degree 0 | 0     | $D_A=\Gamma_0+\Gamma_\infty$ $(\Gamma_0^2=\Gamma_\infty^2=(\Gamma_0, \Gamma_\infty)=0)$ | 2                   |
| iii)  | $C^*$-bundle of degree $m$, $m\geq 1$ | $m$ | $D_A=\Gamma_0+\Gamma_\infty$ $(\Gamma_0^2=m, \Gamma_\infty^2=-m, (\Gamma_0, \Gamma_\infty)=0)$ | 1                   |
| iv)   | affine bundle $A_0$ | 0     | $2\Gamma_\infty$ $(\Gamma_\infty^2=0)$ | $D_A$ does not exist. |
| v)    | affine bundle $A_{-1}$ | $-\infty$ | $\phi$ | $\tilde{q}(\tilde{S}-D_A)$ |

For the notation used above, we refer the reader to [2] and [18]. Explicit constructions of $\tilde{S}$ in [18] are used to compute dim $| -K(\tilde{S})|$ and to find a normal crossing divisor in $| -K(\tilde{S})|$. We omit the details.

Proposition 20 follows from the lemma above. In the case of the class c-i) or c-ii), $\tilde{S}-D_A$ is a quasi-abelian surface. Attaching several $1/2$-points to $\tilde{S}-D_A$ at points of $D_A$, we have surfaces with $\kappa=0$ and $q=q=1$.

**Proposition 21.** Let $(\tilde{S}, D)$ be a $\partial$-surface with the interior $S$. Suppose that $p_\kappa(\tilde{S})=1$, $\kappa(S)=0$, and $q(\tilde{S})=1$. Then $\tilde{S}$ is a ruled surface of genus 1. Moreover, $D$ is disconnected. $D_A$ consists of two sections of the Albanese fibered surface $f: \tilde{S}\to J$ of $\tilde{S}$. In particular, $S$ cannot be affine.

Proof. If $\kappa(S)=0$, it would follow that $p_\kappa(\tilde{S})=0$ from the classification theory of projective surfaces. Combined with Proposition 18, this would imply $\kappa(\tilde{S})=-\infty$, a contradiction. Thus, $\tilde{S}$ turns out to be a ruled surface of genus 1. In view of Lemma 6, by contracting exceptional curves of the first kind on $\tilde{S}-D_A$, we may assume that $K(\tilde{S})+D_A\sim 0$. Then we contract
successively connected exceptional curves \( Q \) of the first kind \( \leq D_b \) such that \((Q, D_A) = 1\). Thus we arrive at the situation that \( D_b \cap D_A = \emptyset \). Detaching several half-points in \( \tilde{S} - D_A \), we have a relatively minimal surface \( \tilde{S}_* \) and a proper birational map \( \mu : \tilde{S} \to \tilde{S}_* \). By Lemma 6, \( \pi(\tilde{S} - \mu(D_A) - \mu(\pi(D_b)), S) = 0 \). Hence \( \mu(*)(D_b) \subseteq \mu(D_A) \). Thus we can apply Proposition 21. Especially \( D \) and \( D_A \) are disconnected. Q.E.D.

**Proposition 22.** Let \((\tilde{S}, D)\) be a \( \partial \)-surface whose interior \( S \) satisfies that \( \bar{\pi}(S) = 1 \), \( \bar{\pi}(S) = 0 \), \( \bar{\pi}(\tilde{S}) = 0 \), and \( q(\tilde{S}) = 1 \). Suppose that \( q(S) = 2 \). Then there are a relatively minimal ruled surface \( \tilde{S}_* \) and a birational morphism \( \mu : \tilde{S} \to \tilde{S}_* \) such that \( \mu(D_b) \) is a finite set and \((\tilde{S}_*, \mu(D_A))\) is c-i) or c-ii) in Proposition 20. Moreover, if \( \mu(D_b) \subseteq \mu(D_A) \), \( S \) is proper birationally equivalent to a quasi-abelian surface.

By these theorem and propositions, we have another proof of Theorem I in [10].

**Theorem IV.** Let \( S \) be a logarithmic abelian surface, i.e., \( \bar{\pi}(S) = 0 \), \( \bar{\pi}(S) = 2 \). Then \( S \) is \( W^2PB \)-equivalent to a quasi-abelian surface.

Proof. Let \( \alpha : S \to A_S \) be a quasi-Albanese map. Let \( J \) be the closure of \( \alpha(S) \) in \( A_S \). Then by Kawamata's Theorem, \( J \) turns out to be a surface \( A_S \). Hence, \( \bar{\pi}(S) = \bar{\pi}(A_S) = 1 \). Therefore, we can apply Theorem III and Propositions 20, 22. We omit the details.

**Corollary 1.** Let \( S \) be an affine normal surface with \( \bar{\pi}(S) = 0 \) and \( \bar{\pi}(S) = 2 \). Then \( S \) is isomorphic to \( \mathbb{C}^* \).

**Corollary 2.** Let \( S \) be a surface with \( \bar{\pi}(S) = q(S) = 0 \) and \( \bar{\pi}(S) = 2 \). Then \( S \) is \( W^2PB \)-equivalent to \( \mathbb{C}^* \).

The above two corollaries are found in [10].

**Proposition 23.** Let \((\tilde{S}, D)\) be any \( \partial \)-surface in TABLE II. If \( \bar{\pi}(S) = 0 \), then there is a reduced divisor \( R \) on \( S \) such that \( \bar{\pi}(S - R) = 0 \) and \( \bar{\pi}(S - R) = 1 \). Similarly, if \( \bar{\pi}(S) = 1 \), then there is \( R' \) on \( S \) such that \( \bar{\pi}(S - R') = 0 \) and \( \bar{\pi}(S - R') = 2 \). Hence \( S - R' \approx \mathbb{C}^* \).

Proof. We use the notation in Proposition 16 and we shall look for \( R \) in each case, separately.

i) If \( S \) is the class b-iv), take a line \( \tilde{R} \) on \( P^2 \) such that \( \tilde{R} \cap C = \{ p \} \) and \( H \cap C = \{ p \} \). Then \( \tilde{S} - D - \tilde{R} \approx \mathbb{C}^* \).

ii) If \( S \) is the class b-v), take two curves \( C_3 \) and \( C_4 \) of degree \((1, 0)\) such that, denoting by \( \{ p_1, p_2 \} \) the intersection \( C_1 \cap C_2 \), \( C_3 \ni p_1 \) and \( C_4 \ni p_2 \). Defining \( \tilde{R} = C_3 + C_4 \), we have \( S - \tilde{R} \approx \mathbb{C}^* \).
iii) If $S$ is the class b-vi), write $C_1 \cap C_2 = \{p_1, p_2\}$. Take two fibres $C_3$ and $C_4$ of $\sum_{\alpha} P^1$ of such that $C_3 \ni p_1$ and $C_4 \ni p_2$. Then defining $\bar{R} = C_3 + C_4$, we have $S - \bar{R} = C^*_{\#2}$.

iv) If $S$ is the class b-vii), write $C_3 \times \{p\}$. Take two fibres $C_3$ and $C_4$ of $\Sigma \frak{P}^1$ of such that $C_3 \ni p$ and $C_4 \ni p$. Then defining $\bar{R} = C_3 + C_4$, we have $S - \bar{R} = C^*_{\#2}$. Moreover, $S - C_1$ is a surface of the class b-vii)$_2$.

v) If $S$ is the class b-viii), write $H_1 \cap C = \{p\}$. Take a curve $\bar{R} = G_2$ of degree $(0, 1)$ passing through $p$. Then $S - \bar{R} = C^*_{\#2}$.

vi) If $S$ is the class b-ix), by $p$ we denote the singular point of $C$. Take two lines $C_3$ and $C_4$ which are tangential to $C$ at $p$. Putting $\bar{R} = C_3 + C_4$, we have $S - \bar{R} = C^*_{\#2}$. Moreover, $S - C_1$ is a surface of the class b-vii)$_2$.

vii) If $S$ is the class b-x), by $p$ we denote the singular point of $C$. Take two curves $C_2$ and $C_3$ of degree $(1, 0)$ and $(0, 1)$, respectively, such that $C_2 \ni p$ and $C_3 \ni p$. Then putting $\bar{R} = C_2 + C_3$, we see $S - \bar{R}$ is a surface of the class b-iv).

viii) If $S$ is the class b-xi), by $p$ we denote the singular point of $C$. Take a fiber $C_2$ passing through $p$. Defining $\bar{R} = C_2 + \Delta_{\infty}$, we see $S - \bar{R}$ is a surface of the class b-iv).

ix) If $S$ is the class b-xii), take a curve $\bar{R}$ of degree $(1, 0)$ passing through a point $\in G \cap C$. Then $S - \bar{R}$ is a surface of class b-iv).

x) If $S$ is the class b-xiii), by $p$ we denote the singular point of $C$. Take a fiber $\bar{R}$ passing through a point $\in \Delta_{\infty} \cap C$. Then $S - \bar{R}$ is a surface of the class b-vii)$_2$.

xi) If $S$ is the class b-xiv), take a fiber $\bar{R}$ passing through a point $\in \Delta_{\infty} \cap C$. Then $S - \bar{R}$ is a surface of the class b-iv).

xii) If $S$ is the class b-xv), take a fiber $\bar{R}$ passing through a point $\in \Delta_{\infty}$ passing through $p$. Then $S - \bar{R} = C^*_{\#2}$.

Therefore, we establish the following

**Proposition 24.** Let $S$ be a surface with $\kappa(S) = 0$, $\bar{p}_g(S) = 1$ and $p_4(S) = q(S) = 0$. Suppose that $S$ is not a logarithmic $K3$ surface of type $\Pi_{a}$. If $\bar{q}(S) = 0$, then there is an open subset $S_1$ of $S$ such that $\bar{q}(S_1) = \bar{q}(S) = 0$ and $q(S_1) = 1$. Moreover if $\bar{q}(S) = 1$, then there is an open subset $S_2$ of $S$ such that $\bar{q}(S_2) = 0$ and $q(S_2) = 2$.

**Corollary.** Let $S$ be a surface in Proposition 24. Then there is a surjective morphism $\psi: S \rightarrow J$ whose general fiber $\psi^{-1}(u) \simeq C^*$. Here $J \simeq \mathbb{P}^1$ or $A^1$, if $\bar{q}(S) = 0$. And $J \simeq C^*$, if $\bar{q}(S) = 1$ or 2.

A proof follows from the fact that $S_2$ with $\bar{q}(S_2) = q(S_2) = 0$ and $q(S_2) = 2$ is $W^2PB$-equivalent to $C^*_{\#2}$.

**Example.** Let $C$ be an irreducible curve with a non-cuspidal singular point. Then $\mathbb{P}^2 - C$ is a logarithmic $K3$ surface, i.e., $\kappa(\mathbb{P}^2 - C) = 0$ if and only if there exist two irreducible curves $C_1$ and $C_2$ such that $\mathbb{P}^2 - C - C_1 - C_2 \simeq C^*_{\#2}$.

**Proposition 25.** Let $C = V(\phi)$, $\phi$ being an irreducible polynomial, be a
curve on $A^2$ and let $S = A^2 - C$. Suppose $\kappa(S) = 0$. Then, choosing an appropriate system of coordinates $(x, y)$ of $A^2$, $\varphi$ is written as follows:

$$\varphi = x'y + a_0 + a_1 x + \cdots + a_s x^s.$$ 

Proof. Since $\overline{q}(S) = 1$ and $\kappa(S) = 0$, it follows that $\overline{f}_g(S) = 1$. Actually, assume that $\overline{f}_g(S) = 0$. Then $C$ (the closure of $C$ in $P^2$) is a rational curve whose singularities are cuspidal. If $C$ were singular, then a general member $C_\lambda$ of the fiber space $\varphi: S \to C^*$ would be of hyperbolic type, i.e., $\kappa(C_\lambda) = 1$. Kawamata’s Theorem would assert that $\kappa(S) \geq \kappa(C_\lambda) + \kappa(G_m) = 1$, a contradiction. Thus $C$ is non-singular and hence $C \cong A^1$. By the imbedding theorem of $A^1$ due to Abhyankar and Moh [1], we know that $S \cong A^1 \times G_m$, which implies that $\kappa(S) = -\infty$.

Accordingly, we conclude that $\overline{f}_g(S) = 1$ and $\kappa(S) = 0$. Applying Proposition 24, we have an irreducible curve $C_3$ such that $P^2 - C_1 \cup C_2 \cup C_3 \cong C^{*2}$, where $C_1 = P^2 - A^2$ and $C_2 = \overline{C}$. Since $\overline{f}_g(S - C_3) = 1$, $C_2$ or $C_3$ has only cuspidal singularities. We may assume that $C_3$ has only cuspidal singularities. Hence, applying Kawamata’s Theorem and Abhyankar and Moh Theorem, we can assume that $A^2 \cap C_3$ is $V(x)$, i.e., the $y$-axis of the affine plane. Therefore

$$\text{Spec } k[x, y, x^{-1}, \varphi^{-1}] \cong C^{*2}. $$

From this it follows that $y \in k[x, y, x^{-1}, \varphi^{-1}] = k[x, x^{-1}, \varphi^{-1}]$. Hence

$$y = f(x, \varphi)|x^n \varphi^n$$

where, $m, n > 0$ and $f(x, Y)$ is a polynomial. Then consider the $y$-derivative $\partial_y = \partial/\partial_y$. Thus,

$$x^n \varphi^n + nx^n \varphi^{n-1} \partial_y \varphi = \partial_y f(x, \varphi) \partial_y \varphi.$$

Hence,

$$x^n \varphi^n = \partial_y \varphi \{\partial_y f(x, \varphi) - nx^n \varphi^{n-1}\}.$$

Since $\varphi$ is irreducible, $\partial_y \varphi = \alpha x^l$ for some $\alpha \neq 0$, $l \geq 0$. This yields that $\varphi = \psi(x) + \alpha x^l y$, $\psi$ being a polynomial. We may assume $\alpha = 1$ and hence

$$\varphi = x'y + a_0 + a_1 x + \cdots + a_s x^s.$$ 

Q.E.D.

In the above, we may assume that $a_0 = 1$ and $a_i \neq 0$. We have the following cases: 1) If $l + 1 \geq s$, then writing $C_1 \cap C_2 = \{p_1, p_2\}$, $C_2$ has the cusp singularity
at \( p_i \) and \( C_1 + C_2 \) has normal crossings at \( p_2 \). 2) If \( 2 + l \leq s \), then \( C_2 \) has two (analytically irreducible) branches at \( p \), the singular point of \( C_2 \). Hence \( P^2 - C_2 \) is a logarithmic K3 surface of type \( II_b \).

**Proposition 26.** If \( S \) satisfies that \( \kappa(S) = 0 \), \( \bar{p}_g(S) = 1 \) and \( p_g(S) = 0 \). Then there exists an algebraic pencil \( \{ C_u \} \) whose general member \( C_u \) is \( C^* \). Hence \( S \) is not measure-hyperbolic.

This follows from Corollary to Proposition 24 and Propositions 9, 21.

**Proposition 27.** Let \((\tilde{S}, D)\) be a \( \theta \)-surface in the TABLE II\(_b\). Define \( \text{Aut}(\tilde{S}, D) = \{ \varphi \in \text{Aut}(\tilde{S}); \varphi D = D \} \). Then \( \text{Aut}(\tilde{S}, D) \) is a finite group if \( \varphi(S) = 0 \).

**Proof.** First assume that \((\tilde{S}, D)\) is the class b-ix). A point \( p \) of inflexion of \( D \) (a nodal cubic curve), is characterized by the existence of a line \( L \) on \( P^2 \) such that \( L \cap D = \{ p \} \). There are three such points. Hence \( \varphi \in \text{Aut}(\tilde{S}, D) \) preserves the set of points of inflexion. Therefore the image of the homomorphism \( \text{Aut}(S, D) \to \text{Aut}(D) \) is a finite group. Using the similar argument to the proof of Proposition 11, we complete the proof. We can check the finiteness of \( \text{Aut}(\tilde{S}, D) \) for the other classes. Q.E.D.

From the above, we infer the following Proposition, whose proof is not given here.

**Proposition 28.** Let \( S \) be a logarithmic K3 surface. Then, \( \text{Aut}(S) \) has at most countably many elements.

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**References**


