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**SOME REMARKS ON STRONGLY INVARIANT RINGS**

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Introduction. Adopting the terminology of [1], a ring $A$ is called $n$-invariant (or strongly $n$-invariant) if $A$ satisfies the condition: Given a ring $B$ and indeterminates $X_1, \ldots, X_n; Y_1, \ldots, Y_n$, if $A[X_1, \ldots, X_n]$ is isomorphic to $B[Y_1, \ldots, Y_n]$, then $A$ is isomorphic to $B$ (or if any ring $B$ and any isomorphism $\phi: A[X_1, \ldots, X_n] \to B[Y_1, \ldots, Y_n]$ are given, then $\phi(A) = B$). If a ring $A$ is $n$-invariant (or strongly $n$-invariant) for every integer $n \geq 1$, $A$ is said to be invariant (or strongly invariant).

Several types of rings are known to be invariant or strongly invariant respectively (cf. [1], [2], [8] etc.). However we have not any good criteria for a ring to be invariant or strongly invariant, and it is tempting to look for criteria of this kind. A purpose of the present paper is to give sufficient or necessary conditions for a ring to be strongly invariant in terms of locally finite (or locally finite iterative) higher derivations.

The present paper consists of three parts. In the first section, definitions of locally finite (or locally finite iterative) higher derivations are recalled, and several results which follow easily from definitions are given. In the second section, sufficient or necessary conditions for strong invariance are given. In the final section we shall see how well these conditions work in giving examples and counter-examples. In the appendix we shall prove a Lemma on a ring which has a locally finite iterative higher derivation.

Our terminology is essentially the same as that of [1].

1. **Preliminaries**

We shall begin with

**Definition 1.1.** Let $A$ be a ring (or an algebra over a ring $R$). A locally finite higher derivation on $A$ is a set of endomorphisms $D = \{D_0, D_1, \ldots\}$ of the abelian group $A$ satisfying the following conditions:

1. $D_0 = \text{identity}, \quad D_i(ab) = \sum_j D_j(a)D_k(b)$ for any $a, b$ of $A$.
2. For any $a$ of $A$, there exists an integer $n > 0$ such that $D_n(a) = 0$ for every...
m \geq n.

When $A$ is an $R$-algebra and $D_n$ is $R$-linear for all $n \geq 0$, $D$ is called $R$-trivial. $D$ is called iterative if $D$ satisfies the additional conditions:

(3) $D_i D_j = \binom{i+j}{i} D_{i+j}$ for all $i, j \geq 0$.

The concept of locally finite (or locally finite iterative) higher derivations has its own geometric meanings. Namely we have

**Lemma 1.2.** Let $A$ be a ring (or an algebra over a ring $R$). Then the following conditions are equivalent to each other:

1. $D$ is a locally finite higher derivation (or a locally finite $R$-trivial higher derivation) on $A$.

2. The mapping $\varphi: A \to A[t]$ given by $\varphi(a) = \sum_{n=0}^{\infty} D_n(a)t^n$ is a homomorphism of rings (or $R$-algebras), where $t$ is an indeterminate.

Similarly the following conditions are equivalent to each other:

1. $D$ is a locally finite iterative higher derivation (or a locally finite iterative $R$-trivial higher derivation) on $A$.

2. $\varphi: A \to A[t]$ defined in the above condition (2) is a homomorphism of rings (or $R$-algebras) such that $(\varphi \otimes \text{id.})\varphi = (\text{id.} \otimes \Delta)\varphi$, where $\Delta: \mathbb{Z}[t] \to \mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ (or $\Delta: R[t] \to R[t] \otimes_{R} R[t]$) is a homomorphism of $\mathbb{Z}$-algebras (or $R$-algebras) defined by $\Delta(t) = t \otimes 1 + 1 \otimes t$ (cf. the following diagram);

$$
\begin{array}{ccc}
A & \xrightarrow{\varphi} & A[t] = A \otimes_{R} R[t] \\
\varphi & \downarrow & \downarrow (\varphi \otimes \text{id}) \\
A \otimes_{R} R[t] & \xrightarrow{\text{id} \otimes \Delta} & A \otimes_{R} R[t] \otimes_{R} R[t]
\end{array}
$$

3. $^a \varphi: \text{Spec}(A) \times_{\mathbb{Z}} G_{a,\mathbb{Z}} \to \text{Spec}(A)$ (or $^a \varphi: \text{Spec}(A) \times_{R} G_{a,R} \to \text{Spec}(A)$) is an action of the additive group scheme $G_{a,\mathbb{Z}}$ (or $G_{a,R}$) on $\text{Spec}(A)$.

Proof. The equivalence between (1) and (2) is nothing but a reformulation of the definition. The equivalence of (1'), (2') and (3') will be found in [7].

Let $D$ be a locally finite higher derivation on a ring $A$. An element $a$ of $A$ is called a $D$-constant if $D_i(a) = 0$ for all $i > 0$. An easy remark on $D$-constants is the following

**Lemma 3.** Let $A$ be an integral domain and let $D$ be a locally finite higher derivation on $A$. Then any invertible element of $A$ is a $D$-constant.

Proof. Let $\varphi: A \to A[t]$ be the homomorphism defined as above. If $a$ is an invertible element of $A$, $\varphi(a)$ is invertible in $A[t]$. Hence $\varphi(a) \in A$. q.e.d.
The set of all $D$-constants form a subring $A_0$ of $A$, and it is easy to see that $A_0$ is algebraically closed in $A$. If $D$ is iterative, we can say more about $A_0$. Namely,

**Lemma 1.4.** Let $A$ be a ring, let $D$ be a locally finite iterative higher derivation on $A$ and let $A_0$ be the subring of $D$-constants. If there is an element $t$ of $A$ such that $D_i(t) = 1$ and $D_i(t) = 0$ for all $i > 1$, then $A = A_0[t]$ and $t$ is algebraically independent over $A_0$. If $A$ is an $R$-algebra and $D$ is $R$-trivial, $A_0$ is an $R$-subalgebra of $A$.

For the proof, we refer to [7].

When an integral domain $A$ has a locally finite higher derivation $D$, we can ask if the quotient field $K$ of $A$ is a simple transcendental extension of a subfield of $K$. (If this is the case, we say that $A$ is birationally ruled.) A partial answer is given in

**Lemma 1.5.** Let $k$ be a field, let $A$ be a finitely generated $k$-domain, let $K$ be the quotient field of $A$ and let $D$ be a non-trivial (i.e., $A 
eq A_0$) locally finite higher derivation on $A$. Assume one of the following conditions:

1. $D$ is iterative.
2. $\dim A = 1$ and the set of $k$-rational points of $\text{Spec}(A)$, $\text{Spec}(A)(k)$, is a dense subset of $\text{Spec}(A)$.
3. The $k$-algebra homomorphism $\varphi: A \to A[t]$ defined by $D$ gives $K(t) = \varphi(K)(t)$.

Then $A$ is birationally ruled.

Proof. (1) When $D$ is iterative, the additive group scheme $G_{a,k}$ acts non-trivially on $\text{Spec}(A)$. Our assertion then follows from ([7], Theorems 2.1, 2.2 and the remark at P. 205). Moreover, $K$ is a simple transcendental extension of the quotient field $K_0$ of $A_0$ (loc. cit.).

(2) Assume the second condition. Then there is an element $a$ of $A$ such that $\varphi(a) = a + a_1 t + \cdots + a_n t^n$ and $\varphi(a) \notin A$, where $a_i \in A$ $(1 \leq i \leq n)$ and $a_n \neq 0$. Since $\text{Spec}(A)(k)$ is dense in $\text{Spec}(A)$, there is a $k$-algebra homomorphism $\pi: A \to k$ such that $\pi(a_n) \neq 0$. The composite $\rho = (\pi \otimes \text{id}) \varphi: A \to k[t]$ is a $k$-algebra homomorphism such that $\rho(A) \subset k$. Since $\dim A = 1$, $\text{Ker } \rho = 0$. Hence by Luroth's theorem, $A$ is rational.

(3) Consider the $t^{-1}$-adic valuation of $K(t)$ over $K$ and its valuation ring $\mathfrak{v}^*$. Let us set $\mathfrak{v} = \mathfrak{v}^* \cap \varphi(K)$. Let $\mathfrak{p}^*$ and $\mathfrak{p}$ be the maximal ideals of $\mathfrak{v}^*$ and $\mathfrak{v}$ respectively. Since $D$ is non-trivial, there is an element $a$ of $A$ such that $\varphi(a) \notin A$. Write $\varphi(a) = a + a_1 t + \cdots + a_n t^n$ with $a_n \neq 0$. Then $\varphi(a)^{-1} = t^{-n}(a_n + a_{n-1} t^{-1} + \cdots + a_1 t^{-n})^{-1} \in \mathfrak{v}^* \cap \varphi(K) = \mathfrak{p}$. Hence $\mathfrak{p} \neq 0$. Since $\text{tr. deg}_{\mathfrak{m}} \varphi(K) = \text{tr. deg}_{\mathfrak{m}}(\mathfrak{v}/\mathfrak{p}) + \text{ht } \mathfrak{p}$, the residue field $\mathfrak{v}^*/\mathfrak{p}^*$, which is isomorphic to $K$, is not algebraic over the residue field $\mathfrak{v}/\mathfrak{p}$. Since $\mathfrak{v}^*$ is obviously a discrete valuation
ring, we can conclude by Nagata's theorem [9] that $K$ is a simple transcendental extension of a finite algebraic extension of $\mathfrak{b}/\mathfrak{p}$.

q.e.d.

An easy consequence of the first case of Lemma 1.5 is

**Corollary 1.6.** Let $k$ be a field and let $A$ be a finitely generated $k$-domain of dimension 1. If $A$ has a non-trivial locally finite iterative higher derivation $D$, then $A$ is a polynomial ring over the algebraic closure $k_0$ of $k$ in $A$.

Proof. We have only to show the existence of a $k_0$-rational point in $\text{Spec}(A)$. It is easy to see that $A_0$ is a subfield of $A$ and is in fact the algebraic closure of $k$ in $A$. Then, Lemma 1.5 implies that $\text{Spec}(A)$ is birational to the affine line $A^1$ over $k_0$. Hence $\text{Spec}(A)$ has a $k_0$-rational point. q.e.d.

The assumption that $A$ is a finitely generated $k$-algebra is not necessary in proving (1.6). For the proof we refer to the appendix.

2. Strongly invariant rings

In this section, we are interested in looking for sufficient or necessary conditions for a ring to be strongly invariant. A sufficient condition for strong $1$-invariance is given, making use of Nagata's theorem [9], in the

**Theorem 2.1.** ([1]). Let $k$ be a field and let $A$ be an affine $k$-domain. If $A$ is not birationally ruled, then $A$ is strongly 1-invariant.

If $A$ is not birationally ruled, it seems plausible that $A$ is strongly invariant. However the authors could not determine whether this is true or not. Another sufficient condition for strong invariance is stated as follows:

**Theorem 2.2.** Let $A$ be a ring and assume that $A$ contains a subdomain $C$ of infinite cardinality, whose non-zero elements are non-zero divisors of $A$. If $A$ has no non-trivial locally finite higher derivation, then $A$ is strongly invariant.

Proof. Assume that $A$ is not strongly invariant. Then there is a ring $B$ ($\neq A$) such that $A[X_1, \ldots, X_n]=B[Y_1, \ldots, Y_n]$, where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are algebraically independent over $A$ and $B$ respectively. Since $A \neq B$, there is an element $a \in A$ such that, regarded as an element of $B[Y_1, \ldots, Y_n]$, $a$ is written in the form $a=\sum b_{a_1, \ldots, a_n} Y_1^{a_1}\cdots Y_n^{a_n}$ with $b_{a_1, \ldots, a_n} \in B$. Let $w_1, \ldots, w_n$ be positive integers. Define a locally finite higher derivation $D$ on $B[Y_1, \ldots, Y_n]$ by a homomorphism of rings $\varphi: B[Y_1, \ldots, Y_n] \rightarrow B[Y_1, \ldots, Y_n, t]$, which is given by $\varphi(b)=b$ for all $b \in B$ and $\varphi(Y_i)=Y_i+t^{w_i}$ for $1 \leq i \leq n$. We can choose $w_1, \ldots, w_n$ so that $\varphi(a)=bt^{m}+(a \text{ polynomial in } t \text{ of degree } <m \text{ with coefficients in } B[Y_1, \ldots, Y_n])$, where $b \in B$ and $m>0$. Since $B[Y_1, \ldots, Y_n]=A[X_1, \ldots, X_n]$, we can rewrite $\varphi(a)$ in the form $\varphi(a)=f(X_1, \ldots, X_n)t^{m}+(a \text{ polynomial in } t \text{ of}}$
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degree $<m$ and with coefficients in $A[X, \ldots, X_n])$. Then by virtue of the following easy sublemma, there are elements $c_1, \ldots, c_n$ in $C$ such that $f(c_1, \ldots, c_n) \neq 0$. Let $q: A[X, \ldots, X_n, t] \to A[t]$ be the $A$-algebra homomorphism given by $q(t) = t$ and $q(X_i) = c_i$ for $1 \leq i \leq n$. Let $i$ be the canonical injection $i: A \to A[X, \ldots, X_n]$ and let $\rho = qai$. Then $\rho: A \to A[t]$ is a homomorphism of ring with $\rho(A) \subset A$. Therefore $\rho$ defines a non-trivial locally finite higher derivation on $A$ and this is a contradiction. q.e.d.

**Sublemma.** Let $A$ and $C$ be as in Theorem 2.2. Let $f(X_1, \ldots, X_n)$ be a polynomial over $A$ in $n$-indeterminates $X_1, \ldots, X_n$. If $f(c_1, \ldots, c_n) = 0$ for any set $(c_1, \ldots, c_n)$ of elements of $C$, then $f$ is identically zero.

The proof is standard and we therefore omit it.

The assumption on $A$ in Theorem 2.2 is satisfied if either the characteristic of $A$ is zero or $A$ contains an infinite field.

A criterion of strong invariance given in Theorem 2.2 is rather difficult to use practically, and is well complemented by the following result, which can be proved by the same principle as in Theorem 2.2.

**Theorem 2.3.** Let $k$ be an infinite field and let $A$ be a $k$-domain satisfying the conditions:

1. $\text{Spec}(A)(k)$ is dense in $\text{Spec}(A)$.
2. There is no non-constant $k$-morphism from the affine line to $\text{Spec}(A)$.

Then $A$ is strongly invariant.

The converse of Theorem 2.2 does not hold as it is shown by the following

**Theorem 2.4.** Let $k$ be a field and let $A$ be the affine ring of the affine cone of a smooth projective variety $U$. Assume that there is no non-constant $\overline{k}$-rational mapping from the affine line $A^1$ to $U$, $\overline{k}$ being an algebraic closure of $k$. Then $A$ is strongly invariant, while $A$ has a non-trivial locally finite higher derivation.

Proof. (1) Write $A = k[Z_0, \ldots, Z_t]/(F_1, \ldots, F_m)$, where $F_1, \ldots, F_m$ are homogeneous polynomials of $k[Z_0, \ldots, Z_t]$. Define a higher derivation $D$ on $k[Z_0, \ldots, Z_t]$ by setting $D_z = \text{id}.$, $D_1(Z_i) = Z_i$ and $D_j(Z_i) = 0$ for $0 \leq i \leq t$ and $j \geq 2$. It is then easy to see that $D$ induces a non-trivial locally finite higher derivation $\tilde{D}$ on $A$.

(2) We shall show next that $A$ is strongly invariant. Assume that we are given a $k$-algebra $B$ such that $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$, where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are algebraically independent over $A$ and $B$ respectively. By virtue of Lemma 2.7 below, we may assume that $k$ is algebraically closed. Set $V = \text{Spec}(A)$ and $W = \text{Spec}(B)$. Noting that $B$ is an affine $k$-domain as well, $V$ and $W$ are the affine varieties defined over $k$. Then we have $V \times A^n = W \times A^n$. 


where $A^n$ stands for the affine $n$-space. First of all we show that for any point $w$ of $W$, the affine space $w=A^n$ (which is a fibre of the canonical fibration $W \times A^r \rightarrow W$) coincides with an affine space $v \times A^n$ for some point $v$ of $V$, $v \times A^n$ being a fibre of the canonical fibration $q: V \times A^n \rightarrow V$. Let $\pi$ be the canonical projection $V - (0, \cdots, 0) \rightarrow U$. The image $\pi q(w \times A^n)$ should be one-point, since there is no non-constant rational mapping from $A^1$ to $U$. Therefore the image $q(w \times A^n)$ is either one-point or an affine rational curve with only one place at infinity. Assume the latter case. Then $q(w \times A^n)$ coincides with a generator of the cone. Hence $q(w \times A^n)$ passes through the vertex $O=(0, \cdots, 0)$ of the cone, which is the only singular point of $V$. This means that $w \times A^n$ has a non-empty intersection with the singular locus $O \times A^n$ of $V \times A^n$. However it is easy to see that $W$ has the only one singular point $w_o$, and the singular locus of $W \times A^n$ must be $w_o \times A^n$. Since the affine space $w \times A^n$ has no non-empty intersection with the singular locus $O \times A^n$ if $w \neq w_o$, the above observation leads to a contradiction. Thus $q(w \times A^n)$ is a one-point $v$ of $V$ and $w \times A^n$ is contained in the affine space $v \times A^n$. Then we have obviously $w \times A^n=v \times A^n$. Actually this is the case for all points $w$ of $W$, taking into account the fact that $O \times A^n=w_o \times A^n$. This means that every maximal ideal of $B$ is vertical relative to $A$. Then we conclude our proof owing to the result ([1], (1.13)). q.e.d.

To state Lemma 2.7 which we used in the above proof, we need

**Definition 2.5.** Let $R$ be a ring and let $A$ be an $R$-algebra. $A$ is called $R$-$n$-invariant (or strongly $R$-$n$-invariant) if $A$ satisfies the condition: Given an $R$-algebra $B$ and indeterminates $X_1, \cdots, X_n, Y_1, \cdots, Y_n$, if $A[X_1, \cdots, X_n]$ and $B[Y_1, \cdots, Y_n]$ are $R$-isomorphic to $B[Y_1, \cdots, Y_n]$, then $A$ is $R$-isomorphic to $B$ (or if any $R$-algebra $B$ and any $R$-isomorphism $\phi: A[X_1, \cdots, X_n] \rightarrow B[Y_1, \cdots, Y_n]$ are given, then $\phi(A)=B$). If an $R$-algebra $A$ is $R$-$n$-invariant (or strongly $R$-$n$-invariant) for all $n \geq 1$, $A$ is simply called $R$-invariant (or strongly $R$-invariant).

**Lemma 2.6.** Let $R$ be a ring and let $A$ be a reduced $R$-algebra. Assume that $A$ satisfies the condition: Given any $R$-algebra $B$ and a relation $A[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n], X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ being algebraically independent over $A$ and $B$ respectively, we have $B[X_1, \cdots, X_n]=B[Y_1, \cdots, Y_n]$. Then $A$ is strongly $R$-$n$-invariant.

For the proof, we refer to [2].

Now the result used in the proof of Theorem 2.4 can be stated as follows:

**Lemma 2.7.** Let $R$ be a ring and let $A$ be a reduced $R$-algebra. Let $R'$ be an $R$-algebra which is a faithfully flat $R$-module. If $A \otimes_R R'$ is strongly $R$-$n$-invariant, $A$ is strongly $R$-$n$-invariant.
Proof. Assume that we are are given an \( R \)-algebra \( B \) such that \( A[X, \ldots, X_n]=B[Y, \ldots, Y_n] \). Then \( (A \otimes_R R')[X, \ldots, X_n]=(B \otimes_R R')[Y, \ldots, Y_n] \). Since \( A \otimes_R R' \) is strongly \( R'-n \)-invariant, we have \( A \otimes_R R'=B \otimes_R R' \). On the other hand, we have the canonical inclusion \( B[X, \ldots, X_n] \subseteq B[Y, \ldots, Y_n] \), and \( (B \otimes_R R')[X, \ldots, X_n]=(B \otimes_R R')[Y, \ldots, Y_n] \). By the (fpqc)-descent theory ([5], IV (2.7.1)), we have \( B[X, \ldots, X_n]=B[Y, \ldots, Y_n] \). Then Lemma 2.6 accomplishes our proof. q.e.d.

One of the concrete examples which satisfy the conditions of Theorem 2.4 is: \( A=\mathbb{k}[X, Y, Z]/(X^3+Y^3+Z^3) \).

Before going to further sufficient conditions for strong invariance, we shall give a necessary condition:

**Theorem 2.8.** Let \( A \) be a ring. If \( A \) is strongly 1-invariant, \( A \) has no non-trivial locally finite iterative higher derivation.

Proof. Assuming that \( A \) has a non-trivial locally finite iterative higher derivation \( D \), we shall show that \( A \) is not strongly 1-invariant. Define a ring homomorphism \( \varphi: A \rightarrow A[t] \) as usual, and let \( B=\varphi(A) \). We claim that \( A[t]=B[t] \). In fact, the inclusion \( B[t] \subseteq A[t] \) is obvious. To show the converse inclusion, we need the notion of length of each element of \( A \) with respect to a given locally finite iterative higher derivation \( D \). The length \( l(a) \) of an element \( a \) of \( A \) is a non-negative integer \( n \) such that \( D^m(a) \neq 0 \) and \( D^m(a)=0 \) for all \( m>n \). By induction on the length \( l(a) \), we prove the inclusion: \( A \subseteq B[t] \). If \( l(a)=0 \), then \( a \in B \). Assume that all elements of \( A \) of length \( <n \) belong to \( B[t] \). Let \( a \) be an element with \( l(a)=n \). Then \( l(D_i(a))<n \) if \( i>1 \). Hence \( D_i(a) \in B[t] \). By the way, \( \varphi(a)=a+D_1(a)t+\cdots+D_n(a)t^n \). Hence \( a \in B[t] \). Thus we have proved \( A[t]=B[t] \). Since \( D \) is non-trivial, \( B \neq A \). Therefore \( A \) is not strongly 1-invariant. q.e.d.

A question which arises from Theorem 2.8 is: Let \( A \) be a ring. If \( A \) has no non-trivial locally finite iterative higher derivation, is \( A \) strongly invariant (or at least strongly 1-invariant)? When \( A \) is an affine domain of dimension 1 over a field, this is true and was essentially proved in [1], (3.4). We have in fact

**Theorem 2.9.** Let \( k \) be a field and let \( A \) be an affine \( k \)-domain of dimension 1. Then the following conditions are equivalent to each other:

1. \( A \) is strongly invariant.
2. \( A \) is strongly 1-invariant.
3. \( A \) has no non-trivial locally finite iterative higher derivations.
Proof. The implication $(1) \Rightarrow (2)$ is clear, while $(2) \Rightarrow (3)$ follows from (2.8). Therefore it remains to show $(3) \Rightarrow (1)$. For this, we use the result, proved in [1], (3.4), that under the above assumptions $A$ is either strongly invariant or $A$ is a polynomial ring $k[x]$ over the algebraic closure $k_0$ of $k$ in $A$. In the latter case, $A$ has a non-trivial locally finite iterative higher derivation. Therefore, $A$ should be strongly invariant under the condition (3). q.e.d.

When $\dim A = 2$, a partial answer to the above question is stated as follows:

**Theorem 2.10.** Let $k$ be an algebraically closed field of characteristic zero and let $A$ be an irrational smooth $k$-affine domain of dimension 2. Then one of the following three cases takes place:

1. $A$ is strongly 1-invariant.
2. $A$ has a non-trivial locally finite iterative higher derivation.
3. There is a surjective morphism from $\text{Spec}(A)$ to a non-singular complete curve of genus $>0$.

Proof. If $A$ is not strongly 1-invariant, there exists a $k$-algebra $B$ ($\neq A$) such that $A[X] = B[Y]$, where $X$ and $Y$ are algebraically independent over $A$ and $B$ respectively. Then the ruling of $\text{Spec}(A[X])$ given by the fibration $\text{Spec}(A[X]) = \text{Spec}(A) \times A' \rightarrow \text{Spec}(A)$ does not coincide with the ruling of the same variety given by the fibration $\text{Spec}(B[Y]) = \text{Spec}(B) \times A' \rightarrow \text{Spec}(B)$. For, otherwise, all maximal ideals of $B$ are vertical relative to $A$. Hence $A = B$ (cf. [1], (1.13)). Let $V = \text{Spec}(A)$ and $W = \text{Spec}(B)$. Then $V \times A' = W \times A'$. Let $\overline{V}$ be a non-singular completion of $V$, whose existence will be clear and which we may assume that $F = \overline{V} - V$ has no exceptional curves of the first kind. $\overline{V}$ is a ruled surface (cf. (2.1)). Let $C$ be the base curve of $\overline{V}$ and let $\pi: \overline{V} \rightarrow C$ be the canonical projection. By the assumption that $A$ is not rational. $C$ is a non-singular complete curve of genus $>0$. Denote by $l_w$ the affine line $w \times A'$ for any $w \in W$ and by $l'_w$ the image of $l_w$ by the canonical projection $q: V \times A' \rightarrow V$. For a general $w \in W$, $l'_w$ is an affine rational curve with only one place at infinity. Since the genus of $C$ is $>0$, $l'_w$ must be contained in a fibre of $\pi$. Therefore, for a sufficiently general point $w \in W$, $l'_w$ is isomorphic to $A'$ and is of the form:

If $\pi(V) = C$, we are led to the third case in the statement of Theorem. Assume that $\pi(V) \neq C$. Then $\pi(V)$ is an affine open set of $C$. Let $\pi(V) = \text{Spec}(A_0)$. $A_0$ is a $k$-subalgebra of $A$. It is now easy to see that there exists an element $s$ of $A_0$ such that setting $U = (\text{the set of points of } \pi(V))$ where $s$ does not vanish, $\pi^{-1}(U)$ is a trivial $P^3$-bundle, i.e., $\pi^{-1}(U) \cong U \times P^3$, and that for any $u \in U$, $V \cap \pi^{-1}(u)$ is of the form for $l'_w$ some $w \in W$ and is isomorphic to $A'$. Let $F' = F \cap \pi^{-1}(U)$. Since $\pi|_{F'}: F' \rightarrow U$ is generically one to one and the characteristic of $k$ is zero, $\pi|_{F'}$ is birational. Shrinking $U$ to a smaller affine
open set of $C$ if necessary, we may assume that $\pi|_{F'}: F' \to U$ is an isomorphism. Then $F'$ is a section of the trivial $P^1$-bundle $\pi^{-1}(U)$. Therefore $V \cap \pi^{-1}(U) = \pi^{-1}(U) \times A^1$. Let $S$ be the multiplicatively closed set generated by $s$ in $A_0$. Then $S^{-1}A = (S^{-1}A_0)[t]$ for some element $t$ of $A_0$ which is algebraically independent over $A$. We can now define a non-trivial locally finite iterative higher derivation $D$ on $A$ as follows: $D_0 = \text{id}_A$, $D_1$ and $D_i(t) = s^a$ with sufficiently large integer $a$ and $D_i := (1/i!)D_i$. If $a_1, \ldots, a_n$ are generators of $A$ over $k$ and if $a_i = f_i(t)|s^a_i$ with $f_i(t) \in A_0[t]$ for $1 \leq i \leq n$, it is enough to take $a$ as large as $\alpha \geq \max_{1 \leq i \leq n}(\alpha_i)$.

The authors have an impression that the third case is contained in the first case. Further observation on the third case will be given in the final part of this paper. When $\dim A > 2$, we have nothing to say.

3. Examples and counterexamples

1. One of the interesting examples of strongly invariant rings was first given in [6] and later discussed in [3]. This is the following:

Let $R$ be the field of real numbers and let $A_n$ be the affine ring of the real $n$-sphere, i.e., $A_n = R[X_0, \ldots, X_n]/(X_0^2 + \cdots + X_n^2 - 1)$ for $n=1, 2, \ldots$. Then $A_n$ is strongly invariant, a one-parameter polynomial ring $A_n[t]$ is invariant and a polynomial ring $A_n[t_1, \ldots, t_n]$ of dimension $n$ is not 1-invariant if $n \neq 1, 3, 7$.

Looking at this example one might ask if a one-parameter polynomial ring $A[t]$ is invariant (or $A$-invariant) provided that $A$ is strongly invariant. However this is false as it is shown by the following example (cf. [2]): Let $k$ be a field of characteristic $p > 0$ ($p \neq 2$) and let $A = k[t^p, t^{p+1}]$, $t$ being an indeterminate. Let $A[X]$ be a one-parameter polynomial ring. Then $A[X]$ is not invariant, while $A$ is strongly invariant.

2. The converse of Lemma 2.7 is false. An example is a one-dimensional $k$-wound unipotent group: Let $k$ be a non-perfect field of characteristic $p > 0$, and let $A = k[X, Y]/(Y^p - X - a_1X^p - \cdots - a_rX^{p^r})$, where $a_0, \ldots, a_r \in k$ and one of $a_1, \ldots, a_r \notin k^p$. Then $A$ is strongly invariant (use (2.3)). (Note that $A$ is not $k$-rational except when $p = 2$ and $Y^2 = X + aX^2$ with $a \in k$ and $a \notin k^2$.) However if $k'$ is the perfect closure of $k$, $A \otimes_{k} k'$ is a polynomial ring over $k'$. Hence $A \otimes_{k} k'$ is not strongly invariant.

3. The remaining part of this paper is devoted to an investigation of the third case of Theorem 2.10. In the following $k$ is an algebraically closed field of characteristic $p$ and $C$ is a non-singular complete curve of genus $g > 0$. We use the term "vector bundle" synonymously with "locally free $O_C$-module". A section of a projective bundle $P(E)$ is a curve on $P(E)$, on which the restriction of the canonical projection $P(E) \to C$ is an isomorphism.
We have in the first place

**Lemma 3.1.** Let $L$ be an ample line bundle on $C$ and let $E$ be a non-trivial extension of $L$ by $\mathcal{O}_C$. Let $s$ be the section of $P(E)$ corresponding to $L$ and let $X=P(E)-s$. Assume that the characteristic of $k$ is zero or $\deg L>2g$. Then $X$ is an affine surface such that the restriction onto $X$ of the projection $P(E)\to C$ is a surjective morphism onto $C$. Conversely if an affine surface $X$ is $\mathbb{P}^1$-bundle $P(E)$ on $C$ deleted a section, then $X$ is isomorphic to an affine surface constructed in the above fashion.

Proof. Let $L$ be an ample line bundle on $C$ and let $E$ be a non-trivial extension of $L$ by $\mathcal{O}_C$. Assume that the characteristic of $k$ is zero or $\deg L>2g$. Then it is known (cf. [4]) that $E$ is an ample vector bundle on $C$ and the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(E)}(s)$. Therefore $s$ is an ample divisor on a non-singular projective surface $P(E)$ and $X=P(E)-s$ is affine. It is obvious that the restriction onto $X$ of the projection $P(E)\to C$ is a surjective morphism onto $C$.

Conversely, let $E$ be a vector bundle on $C$, let $P(E)$ be the $\mathbb{P}^1$-bundle associated with $E$ and let $X$ be the $P(E)$ deleted a section $s$. Let $L'$ be the quotient line bundle of $E$ corresponding to $s$ and let $L$ be the kernel of $E\to L'$; $\mathcal{O}\to L\to E\to L'\to \mathcal{O}$. We shall show that $L'\otimes L^{-1}$ is ample and that $X$ is isomorphic to $P(E\otimes L^{-1})$—the section $s'$ corresponding to $L'\otimes L^{-1}$. Since $X$ is affine, $s$ is irreducible and $P(E)$ is non-singular, the section $s$ regarded as a divisor on $P(E)$ must be ample. Let $i: P(E)\to P(E\otimes L^{-1})$ be the canonical isomorphism. Then the section $s$ is transformed to the section $s'$ by $i$ and $X$ to the affine surface $P(E\otimes L^{-1})-s'$. Hence $s'$ is an ample divisor on $P(E\otimes L^{-1})$. Let $j: C\to P(E)$ be the isomorphism sending $C$ to $s$. Then $i\cdot j$ is an imbedding. Taking account of the facts that $\mathcal{O}_{\mathbb{P}(E\otimes L^{-1})}(1)\cong\mathcal{O}_{\mathbb{P}(E\otimes L^{-1})}(s')$ and $(i\cdot j)^*\mathcal{O}_{\mathbb{P}(E\otimes L^{-1})}(1)^*\cong j^*\mathcal{O}_{\mathbb{P}(E\otimes L^{-1})}(1)$, we know that $L'\otimes L^{-1}$ is ample on $C$.

The affine ring of an affine surface constructed in Lemma 3.1 has no non-trivial locally finite iterative higher derivation, though it has a non-trivial locally finite higher derivation. This is an immediate consequence of

**Lemma 3.2.** Let $V$ be a variety defined over $k$, let $L$ be a line bundle over $V$ and let $E$ be an extension of $L$ by $\mathcal{O}_V$. Let $X$ be the $P(E)$ minus the section corresponding to $L$. If $H^0(V, L^{-1})\neq 0$, $X$ has a non-trivial $G_a$-action. If there is no non-constant morphism from $A^1$ to $V$, then $H^0(V, L^{-1})\neq 0$ provided that $X$ has a non-trivial $G_a$-action.

Proof. Let $U=\{U_i\}_{i\in I}$ be an affine open covering of $V$ such that $E|U_i$ is trivial for any $i\in I$ and let $\begin{pmatrix} a_{ij} & b_{ij} \\ 0 & 1 \end{pmatrix}$ be the transition matrices of $E$ relative to
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U, where \( \{a_{ij}\} \) is the transition functions of \( L \). \( X \) is in fact an \( A^4 \)-bundle on \( V \) with affine coordinates \( \{x_i\} \) which are subject to \( x_j=a_{jix_i+b_{ji}} \) for any \( i,j \in I \).

If \( H(V^0, L^{-1})=0 \), we may assume that there is a set of functions \( \{s_i\} \) on \( V \) such that \( s_i \in \Gamma(U_i, \mathcal{O}_V) \) and \( s_j=a_{ij}s_i \) for any \( i,j \in I \). Define a non-trivial locally finite iterative higher derivation \( D=\{D_0, D_1, \cdots \} \) on \( \Gamma(U_i, \mathcal{O}_V)[x_j] \), which is the affine ring of \( \pi^{-1}(U_i) \), \( \pi \) being the canonical projection from \( X \) to \( V \).

Then \( D \) gives rise to a non-trivial \( G_\alpha \)-action in \( \{\pi^{-1}(U_i)\} \) patch each other on \( \pi^{-1}(U_i \cup U_j) \) to give a nontrivial \( G_\alpha \)-action on \( X \).

Assume next that there is no non-constant morphism from \( A^1 \) to \( V \). If \( X \) has a non-trivial \( G_\alpha \)-action, then \( G_\alpha \) should act on \( X \) along fibres of \( \pi \), and the \( G_\alpha \)-invariant subfield in \( k(X) \) is \( k(V) \) (cf. [7]). For each \( i \in I \), the \( G_\alpha \)-action restricted on \( \pi^{-1}(U_i) \) gives rise to a \( \Gamma(U_i, \mathcal{O}_V)-\)homomorphism \( \phi_i : \Gamma(U_i, \mathcal{O}_V)[x_j] \rightarrow \Gamma(U_i, \mathcal{O}_V)[x_i, t] \), \( t \) being an indeterminate. Write \( \phi_i(x_j)=s_j t^n+(\text{terms of lower degree in } t) \) with coefficients in \( \Gamma(U_i, \mathcal{O}_V) \), where \( n \geq 1 \) and \( s_i \in \Gamma(U_i, \mathcal{O}_V)[x_j] \).

Since the \( G_\alpha \)-invariant subfield of \( k(X) \) is \( k(V) \) and since \( s_i \) is \( G_\alpha \)-invariant, \( s_i \in \Gamma(U_i, \mathcal{O}_V)[x_i] \cap k(V)=\Gamma(U_i, \mathcal{O}_V) \). Moreover it is easy to see that \( n \) is independent of \( i \) and \( s_j=a_{ij}s_i \) for any \( i,j \in I \). Then \( \{s_i\}_{i \in I} \) give a non-zero section of \( H^n(V, L^{-1}) \). Hence \( H^n(V, L^{-1})=0 \). q.e.d.

The affine ring of an affine surface constructed in Lemma 3.1 is strongly 1-invariant if the characteristic of \( k \) is zero. To prove this result, we need

**Lemma 3.3.** Let \( k \) be a field of characteristic zero and let \( \phi \) be a \( k \)-automorphism of the polynomial ring \( k[x, y] \) in two variables given by \( \phi(x)=f, \phi(y)=g \). Then \( f \) has the following form unless \( f \) is a polynomial in \( x \) or \( y \) alone.

\[
(*) \quad f = ax^m + by^n + \sum_{m+j \geq n} c_{ij}x^iy^j
\]

where \( a, b \) and \( c \)'s are elements of \( k \) and \( ab \neq 0 \). The same holds for \( g \).

**Proof.** (1) First we shall treat the case where one of \( f \) and \( g \), say \( f \), is a polynomial in, say, \( y \) alone. By the assumption \( \phi \) is an automorphism of \( k[x, y] \). Hence we must have that \( \partial f/\partial y \neq 0 \) and \( \partial g/\partial x \neq 0 \). From this we get easily \( f=ay+e, g=bx+h(y) \).

Now assume \( f \) has the form \((*) \). We shall show that if \( g \) is not a polynomial in \( x \) or \( y \) alone, then \( g \) has also the form of \((*) \). Let us set first

\[
g = \alpha_0(y)x^u + \alpha_1(y)x^{u-1} + \cdots + \alpha_u(y) \quad (\alpha_0(y) \neq 0, u>0)
\]

where \( \alpha_i(y) \in k[y] \). From the fact that \( \partial(f, g)/\partial(x, y) \) is a non-zero constant
we easily get \( \alpha_0(y)=0 \). Hence \( \alpha_0(y) \) is a non-zero element of \( k \). Similarly if we rewrite \( g \) in the form

\[
g = \beta_0(x)y^n + \beta_1(x)y^{n-1} + \cdots + \beta_n(x) \quad (\beta_0(x) \neq 0, n > 0)
\]

We easily see that \( \beta_0 \) is an element of \( k \). These two results imply that \( g \) has the form \( (*) \).

(II) It is well known (cf. [10]) that any \( k \)-automorphism of \( k[x, y] \) is written as a composite of linear automorphisms; \( (x, y) \mapsto (ax + \beta y + c, \gamma x + \delta y + d) \) with \( \alpha \delta - \beta \gamma \neq 0 \), and Jonquière automorphisms; \( (x, y) \mapsto (x, y + h(x)) \) with \( h(x) \in k[x] \). Using this result, we shall show that any \( k \)-automorphism of \( k[x, y] \) is a composite of automorphisms, each of which is an automorphism \( \rho \) such that \( \rho(x) \) or \( \rho(y) \) coincides with one of \( x \) and \( y \). (We shall say such an automorphism to be of type \((P)\). Since Jonquière automorphisms are of type \((P)\), it suffices to show that a linear automorphism is a composite of two linear automorphisms of type \((P)\). In fact, a linear automorphism \( (x, y) \mapsto (ax + \beta y + c, \gamma x + \delta y + d) \) is decomposed as follows: If \( \alpha \delta - \beta \gamma \neq 0 \), \( (x', y') = (x, y) \mapsto (x', (\gamma/\alpha)x' + ((\alpha \delta - \beta \gamma)/\alpha)y' + (d - (\gamma c/\alpha))) \). If \( \alpha = 0 \), \( (x, y) \mapsto (x', y') = (y, \gamma x + \delta y + d) \), \( (x', y') = (x, (\gamma/\alpha)x + (\alpha \delta - \beta \gamma)/\alpha)y + (d - (\gamma c/\alpha)) \).

(III) Write the given automorphism \( \varphi \) as \( \varphi = \varphi_r \circ \varphi_{r-1} \circ \cdots \circ \varphi_1 \), where \( \varphi_1, \ldots, \varphi_r \) are automorphisms of type \((P)\). We shall prove our assertion by induction on \( r \). If \( r = 1 \), \( \varphi \) is one of the following forms \( (x, y) \mapsto (ax + h(y), y) \), \( (x, y) \mapsto (y, ax + h(y)) \), \( (x, y) \mapsto (x, by + l(x), y) \) or \( (x, y) \mapsto (b, y + l(x), y) \), where \( a, b, l \in k \) and \( h(y), l(y) \in k[y] \) and \( l(x) \in k[x] \). Hence the assertion is clear. Assuming next that the assertion is true when \( \varphi \) is a composite of less than \( r \) automorphisms of type \((P)\), we consider the case where \( \varphi = \varphi_r \circ \varphi_{r-1} \circ \cdots \circ \varphi_1 \). Let \( \psi = \varphi_{r-1} \circ \cdots \circ \varphi_1 \) and let \( \psi(x, y) = (f_1, g_1) \) with \( f_1, g_1 \in k[x, y] \).

By induction assumption \( f_1, g_1 \) have the form \( (*) \) unless they are polynomials in \( x \) or \( y \) alone. Since \( \varphi_r \) is an automorphism of type \((P)\) we have one of the following four cases.

(i) \( \varphi(x) = f_1 \),  
(ii) \( \varphi(x) = g_1 \),  
(iii) \( \varphi(y) = f_1 \),  
(iv) \( \varphi(y) = g_1 \).

In any case we easily get the assertion from the results in the step (I). q.e.d.

Now we can prove

**Theorem 3.4.** Let \( k \) be an algebraically closed field of characteristic zero, let \( C \) be a non-singular complete curve of genus \( \geq 0 \) defined over \( k \), let \( L \) be an ample line bundle over \( C \) and let \( E \) be a non-trivial extension of \( L \) by \( \mathcal{O}_C \). Let \( X \) be the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(E) \) minus the section \( s \) corresponding to \( L \) and let \( A \) be the affine ring of \( X \). Then \( A \) is strongly \( 1 \)-invariant.

**Proof.** Our proof consists of several steps.
(1) Let $B$ be a $k$-algebra such that $A[T]=B[V]$, $T$ and $V$ being algebraically independent over $A$ and $B$ respectively. Let $Y=\text{Spec}(B)$ and let $\pi: X\to C$ be the canonical projection. By the composite of projections $Y\times A^1\to X\times A^1\xrightarrow{\phi_1} X\to C$, each line $(y)\times A^1$ with $y\in Y$ is sent to a point of $C$. Hence $\pi\cdot\phi_1$ factors as $Y\times A^1\xrightarrow{\pi^*} Y\to C$ and $Y$ is regarded as a $C$-scheme by $q$. Hence $q$ is surjective. Let $U_i=\{U_i\}\subseteq I$ be an affine open covering of $C$ such that $E\mid U_i$ is trivial for any $i\in I$. Let $\{x_i\}\subseteq I$ be an affine coordinate system of $X$ relative to $U_i$. Then $\{x_i\}$ is subject to $x_j=a_{ji}x_i+b_{ji}$ for any $i,j\in I$. Let $U_i=\text{Spec}(R_i)$ and let $q^{-1}(U_i)=\text{Spec}(B_i)$. Then $R_i[x_i, T]=B_i[V]$. Since $B_i$ is an $R_i$-algebra and $R_i$ is regular, there is an element $y_i\in B_i$ such that $B_i=R_i[y_i]$ (cf. [1]). For any $i,j\in I$, we have: $y_j=a_{ji}y_i+b_{ji}$ with $a_{ji}\in R_i^*$ and $b_{ji}\in R_i$, where $U_i\cap U_j=\text{Spec}(R_{ij})$. Then, it is easy to see that $\begin{pmatrix} a_{ji} & b_{ji} \\ 0 & 1 \end{pmatrix}$ are transition matrices of a vector bundle $E'$, which is an extension of a line bundle $\mathcal{L}'$ by $\mathcal{O}_C$, $\mathcal{L}'$ being defined by transition functions $\{a_{ji}\}$ relative to $U_i$. Moreover $Y$ is the $\mathbb{P}^*$-bundle $\mathbb{P}(E')$ deleted the section $s$ corresponding to $\mathcal{L}'$.

(2) Let $\Omega_{X/C}^1$ be the $\mathcal{O}_X$-module of 1-differential forms of $X$ over $C$. Since $\Omega_{X/C}^1\subseteq \mathcal{O}_X$ and $dx_j=a_{ji}dx_i$, we have $\Omega_{X/C}^1=L\otimes_{\mathcal{O}_X}\mathcal{O}_X$. The relation $A[T]=B[V]$ implies that $L\otimes_{\mathcal{O}_X}\mathcal{O}_X[T] \cong \mathcal{O}_X[T]$. Hence $L\otimes_{\mathcal{O}_X}\mathcal{O}_X[T] \cong \mathcal{O}_X[T]$ and $(L\otimes_{\mathcal{O}_X}\mathcal{O}_X[T])\otimes_{\mathcal{O}_X}\mathcal{O}_X[T] \cong \mathcal{O}_X[T]$. Then reducing $\mathcal{O}_X[T]$ by $T\mathcal{O}_X[T]$, we have $(L\otimes_{\mathcal{O}_X}\mathcal{O}_X[T])\otimes_{\mathcal{O}_X}\mathcal{O}_X[T] \cong \mathcal{O}_X$. Let $D$ be a divisor on $C$ such that $L\otimes\mathcal{O}_C(D)$. Then there exists a rational function $h$ of $\mathcal{O}(X)$ such that $\varpi^{-1}(D)=(h)$. Let $\varpi$ be the canonical projection $\mathcal{P}(E)\to C$. Then, looking upon $h$ as an element of $h(\mathcal{P}(E))$, we have $\varpi^{-1}(D)+ms=(h)$, where $m\in\mathbb{Z}$ and $s$ is the section of $\mathcal{P}(E)$ corresponding to $L$. If $m\neq 0$, $\varpi^{-1}(D)+ms\cdot l=m$ for a general fibre $l$ of $\varpi$. This is absurd unless $m=0$. Then restricting $\varpi^{-1}(D)=(h)$ on the section $s$, we know that $D\sim 0$ on $C$. Therefore $L\sim \mathcal{L}'$. Shrinking the affine open covering to a finer one, we may assume that $a_{ji}=a_{ji}^*$ for any $i,j\in I$.

(3) We have $R_i[x_i, T]=R_i[y_i, V]$ for any $i\in I$. Therefore we can write: $y_i=f_{i0}(x_i)+f_{i1}(x_i)T+\cdots+f_{in}(x_i)T^n$ with $f_{i0}(x_i), \ldots, f_{in}(x_i)\in R_i[x_i]$. We shall prove that $n=0$. Otherwise, $y_i$ is a generator of the two-dimensional polynomial ring $K[x_i, T]$, $K$ being the function field $h(C)$ of $C$. Then, Lemma 3.3 implies that $f_{i0}(x_i)\in K$. Hence $f_{i0}(x_i)\in R_i[x_i]\cap K=R_i$. On the other hand, it is easy to see that $f_{jm}(x_i)T^n$ is the term of the highest degree in $T$ of $y_i$ and that $f_{jm}(x_i)=a_{ji}f_{i0}(x_i)$ for any $i,j\in I$. Let $\alpha_i=f_{i0}(x_i)$. Then $\{\alpha_i\}_{i=1}^n$ defines a non-zero section of $H^0(C, L^{-1})$. This is absurd since $L$ is an ample divisor on $C$. Therefore we know that $n=0$. Then $y_i\in R_i[x_i]$ for any $i\in I$. It is easy to see that $B\subseteq A$. Changing the roles of $x_i$ and $y_i$ in the above argument, we have $A\subseteq B$. Therefore $A=B$, and $A$ is thus strongly 1-invariant. q.e.d.
We shall next give a brief indication of constructing more complicated affine surfaces from the affine surface given in Lemma 3.1. We use the same notations as in Lemma 3.1. Let \( l \) be a general fibre of the projection \( \pi: P(E) \rightarrow C \), and let \( P_1, \cdots, P_m \) be points on \( l \) other than the point of intersection \( P_0 = l \cdot s \). Blow up points \( P_1, \cdots, P_m \) and let \( E_{P_1}, \cdots, E_{P_m} \) be the exceptional curves. Let \( l' \) be the proper transform of \( l \). Then it is not difficult to show that \( ns + l' \) is an ample divisor on the surface \( \text{Dil}_{P_1,\cdots,P_m}(P(E)) \), for sufficiently large \( n \). Therefore \( X' = \text{Dil}_{P_1,\cdots,P_m}(P(E)) \) is an affine surface which is isomorphic to \( X \) with \( m \) affine lines inserted in place of one fibre deleted. More complicatedly, we can blow up as many infinitely near points at \( P_1, \cdots, P_m \) successively as we like. Let \( \rho: S \rightarrow P(E) \) be the surface obtained by these blowings up. Then a divisor on \( S \) whose components are irreducible components of \( \rho^{-1}(s \cup l') \) with appropriate multiplicities is an ample divisor if we require the following conditions satisfied:

1. \( D \) contains \( \rho^{-1}(s) \) with sufficiently high multiplicity.
2. \( D \) is connected and \( S - D \) contains no complete curves.

Then \( S - D \) is affine. The operation of this kind can be made on a finite number of fibres of \( \pi \). A question which arises naturally is to ask whether these affine surfaces are strongly invariant or not. In simpler cases, we can prove that they are strongly 1-invariant.

An affine version of our Theorem 3.4 was given in [3]. To state the results, we need

**Definition 3.5.** Let \( R \) be a ring and let \( A \) be an \( R \)-algebra. \( A \) is said a local polynomial ring over \( R \) if for any prime ideal \( \mathfrak{p} \) of \( R \), \( A_{\mathfrak{p}} \) is a polynomial ring over \( R_{\mathfrak{p}} \).

Then we can prove the following two results, for whose proofs the reader can refer to [3].

**Theorem 3.6.** Let \( R \) be a reduced ring and let \( A \) be a finitely generated local polynomial ring over \( R \) of relative dimension 1. Then there exists a projective module \( P \) of rank 1 such that \( A \) is \( R \)-isomorphic to the symmetric \( R \)-algebra \( S'(P) \) generated by \( P \).

**Theorem 3.7.** Let \( R \) be a normal domain and let \( P \) be a projective \( R \)-module of rank 1. Then the symmetric \( R \)-algebra \( S'(P) \) generated by \( P \) is \( R \)-invariant, but it is not strongly \( R \)-invariant.

In the proof of the last theorem, we use the fact that if \( R \) is a normal domain, then a polynomial ring over \( R \) of dimension 1 is \( R \)-invariant (cf. [2]).

**Appendix**

In this appendix we shall determine the structure of an integral domain.
which has a locally finite iterative higher derivation (abbreviated as "lfihd").

**Theorem.** Let \( A \) be an integral domain and let \( D=\{D_i, i=0, 1, \ldots \} \) be a lfihd on \( A \) and let \( A_0 \) be the ring of \( D \)-constant, i.e. the set of elements \( a \) such that \( D_i(a)=0 \) for \( i \geq 1 \). Then there exists an element \( u \) of \( A_0 \) such that we have \( A[u^{-1}]=A_0[u^{-1}][x] \) where \( x \) is a variable element over \( A_0 \). Conversely assume that \( A \) is finitely generated over a subring \( A_0 \). Then the existence of an element \( u \) satisfying the above condition implies that \( A \) has a lfihd.

**Proof.** We shall denote by \( A_i \) the set of elements defined by

\[
A_i = \{a \in A \mid D_n(a) = 0 \text{ for } n \geq i\}.
\]

\( A_0 \) is the ring of \( D \)-constants, \( A_i \)’s are \( A_0 \)-modules and we have \( A = \bigcup A_i \). An integer \( n \) will be called a jump index if we have \( A_{n-1} \subseteq A_n \). If the first jump index is 1 the proof is immediate. In fact let \( x \) be an element of \( A_1 \) not in \( A_0 \). Then \( u=D(x) \) is a \( D \)-constant. Hence we can extend \( D \) uniquely to the quotient ring \( A[u^{-1}] \) in which the element \( xu^{-1} \) satisfies the condition in (1.4). Hence we have \( A[u^{-1}]=A_0[u^{-1}][x] \). Since the first jump index is 1 if the characteristic of \( A \) is zero and \( D \) is not trivial, we shall hereafter be mainly interested in the case of positive characteristic \( p \). Hence in the following we shall assume that the characteristic of \( A \) is a positive prime \( p \) and the first jump index of \( D \) is larger than 1. We shall prove first the followings:

1. The first jump index \( n \) is a power of \( p \), say, \( n=p^s \).
2. The \( m \)-th jump index is \( mp^s \) \((m=1, 2, \cdots)\).
3. Let \( a \) be an arbitrary element of \( A_n \setminus A_{n-1} \). Then \( \text{Supp}(a) \) consists of powers of \( p \), where we mean by \( \text{Supp}(a) \) the set of integers \( k \) such that \( D_k(a)\neq 0 \). Moreover \( D_k(a) \)'s are \( D \)-constant for any \( k \in \text{Supp}(a) \).

Let \( n \) be the first jump index and let

\[
n = n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s, \quad (0 \leq n_i < p, \ n_s \neq 0)
\]

be its \( p \)-adic expansion. Since \( D \) is iterative we can see easily that we have

\[
D_n = (1/n!)(D_{n_1})^{n_1}(D_{n_2})^{n_2} \cdots (D_{n_s})^{n_s}
\]

where \( n! = n_0! n_1! \cdots n_s! \). Now assume \( n \) is not a power of \( p \). Then we have either

(i) \( n_0 \geq 1 \)

or

(ii) \( n_0 = 0 \) and \( n_1 + \cdots + n_s \geq 2 \).

In case (i) we have \( n \equiv 0 \pmod{p} \). Then \( c=D_{n-1}(a) \) belongs to \( A_1 \) but not to \( A_0 \) since \( D_i(c)=D_i D_{n-1}(a)=nD_{n-1}(a)\neq 0 \) where \( a \in A_n \setminus A_{n-1} \). This is a contradiction to the assumption that the first jump index is \( >1 \). In case (ii) we shall set
Then we can see immediately that any integer larger than \( n - p^s \) \((<n-1)\) cannot be a support of \( a \). Hence \( c \) belongs to \( A_{n-1} = A_q \). On the other hand we see that \( D_{n-p}D_{p^s}(a) = \eta D_{n}(a) \neq 0 \) and \( n - p^s \geq 1 \). This is a contradiction. Thus we have proven that \( n \) is a power of \( p \).

Next we shall prove (2). It is clear that \( mp^s (m = 1, 2, \cdots) \) are jump indices because if \( x \) is in \( A_{p^s} \) and not in \( A_q \), then \( x^m \) is contained in \( A_{mp^s} \) but not in the preceding \( A \)'s. Hence it remains to show that \( mp^s \) exhaust all jump indices. Let \( q \) be the least jump index which is not a multiple of \( p^s \) and assume that we have
\[
mp^s < q < (m + 1)p^s.
\]
Let us set \( q_0 = q - mp^s < p^s \). Let \( x \) be an element of \( A_q \) not in \( A_{q-1} \) and let us set \( c = D_{mp^s}(x) \). Then we have \( D_{q_0}(c) = D_{q_0}D_{mp^s}(x) = D_{q}(x) \neq 0 \). Since \( p^s > q_0 > 0 \), \( c \) is not a \( D \)-constant. On the other hand the support of \( c \) consists of integers less than \( p^s \) because \( x \) is in \( A_q \). This is a contradiction.

The proof of (3) will be carried out by a similar device. In fact if \( x \) is an element of \( A_{p^s} \), which is not a \( D \)-constant and \( m \) be a support of \( x \), i.e., \( D_{m}(x) \neq 0 \). If we set
\[
m = m_0 + m_1 p + \cdots + m_t p^t \quad (m_i \geq 0, \ 0 \leq m_i < p)
\]
and if we assume either \( m_i \geq 2 \) or some of \( m_i (i < t) \) is not zero then we shall set \( c = D_{p^t}(x) \). Then we have \( D_{mp^s}(c) = D_{mp^s}D_{mp^s}(x) = D_{mp^s}(x) \neq 0 \) and this will lead us to a contradiction since \( m \leq p^s \) and \( A = \cdots = A_{p^t-1} \). The rest of assertion in (3) is also immediate.

After these preparation we shall go to the proof of the Theorem. Let, as before, \( n = p^s \) be the first jump index for a \( \mathfrak{m} \)-factor \( D \) and let \( x \) be an elements of \( A_n \) not in \( A_q \). Let \( u \) be the product of non-zero \( D_{p^s}(x) \)'s \((0 < i \leq s) \). Since \( u \) is a \( D \)-constant we can extend \( D \) to the quotient ring \( A[u^{-1}] \) in a unique way. Let \( a \) be an arbitrary element of \( A \). Then there exists an integer \( m \) such that \( a \) belongs to \( A_{mp^s} \). Let us set
\[
a_1 = a - D_{mp^s}(a)D_{p^s}(x)^{-m}\text{x}^m.
\]
Then we have \( D_{mp^s}(a_1) = 0 \). Hence \( a_1 \) is contained in \( A_{(m-1)p^s} \). We can continue this process until we get a polynomial expression in \( x \) with coefficients in \( A_{n-u^{-1}} \). It is easy to see that \( x \) is a variable over \( A_q \).

Conversely assume that there exist an element \( u \) in \( A \) such that we have \( A[u^{-1}] = A_q[x] \) where \( A_q \) is a subring of \( A \) containing \( u \) and \( x \) is a variable element over \( A_q \) contained in \( A \) and \( A \) is finitely generated over \( A_q \). Then we can define a \( \mathfrak{m} \)-factor \( D \) on \( A[u^{-1}] \) in a standard way. Since \( A \) is finitely generated over \( A_q \) there is an index \( f \) such that \( u^f \) transforms \( A \) into itself (see the proof of (2.9)). This proves the Theorem completely.
Corollary. Let $A$ be an integral domain over a field $k$ of transcendence degree 1. Then $A$ has a lfhld if and only if we have $A = k_0[x]$ where $k_0$ is the algebraic closure of $k$ in $A$.

Proof. Let $u$ be an element of $A_0$ such that $A[u^{-1}] = A_0[u^{-1}][x]$. Since $A$ is of transcendence degree 1 over $k$ and $x$ is variable over $A_0$, $A_0$ must be algebraic over $k$, i.e., $A_0$ is contained in $k_0$. Hence $u$ must be a unit of $A$. This implies that $A$ is contained in $k_0[x]$. The converse inclusion is trivial and we have $A = k_0[x]$.

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Bibliography
