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STRUCTURES OF THE HAKEN MANIFOLDS WITH HEEGAARD SPLITTINGS OF GENUS TWO

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

In this paper we will give a complete list of the closed, orientable 3-manifolds with Heegaard splittings of genus two and admitting non-trivial torus decompositions. We use the following notations.

$D(n)$ ($A(n)$, $M\ddot{o}(n)$ resp.): *the collection of the Seifert fibered manifolds the orbit manifold of each of which is a disk (annulus, Möbius band resp.) with n exceptional fibers.*

M_K (M_L resp.): *the collection of the exteriors of the two bridge knots (links resp.).*

L_K : *the collection of the exteriors of the one bridge knots in lens spaces each of which admits a complete hyperbolic structure or admits a Seifert fibration whose regular fiber is not a meridian loop.*

For the definitions of the one bridge knots in lens spaces see section 5. Then our main result is

Theorem. *Let M be a closed, connected Haken manifold with a Heegaard splitting of genus two. If M has a nontrivial torus decomposition then either*

- (i) *M is obtained from $M_1 \in D(2)$ and $M_2 \in L_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2 ,*
- (ii) *M is obtained from $M_1 \in M\ddot{o}(n)$ ($n=0, 1$ or 2) and $M_2 \in M_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2 ,*
- (iii) *M is obtained from $M_1 \in D(n)$ ($n=2$ or 3) and $M_2 \in M_K$ by identifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2 ,*
- (iv) *M is obtained from M_1 , $M_2 \in D(2)$ and $M_3 \in M_L$ by identifying their boundaries where the regular fiber of M_i ($i=1, 2$) is identified with the meridian loop of M_3 or*
- (v) *M is obtained from $M_1 \in A(n)$ ($n=0, 1$ or 2) and $M_2 \in M_L$ by ident-*

ifying their boundaries where the regular fiber of M_1 is identified with the meridian loop of M_2 .

Conversely if a 3-manifold has a decomposition as in (i)~(v) then it has a Heegaard splitting of genus two.

For the structures of the elements of L_K , M_K or M_L see Lemma 4.2, 4.4, 5.2.

In [9] Thurston listed eight 3-dimensional geometries with compact stabilizers and conjectured that every closed 3-manifold admits a geometric decomposition. Thurston's recent result [10] asserts that every closed, orientable 3-manifold with a Heegaard splitting of genus two has a geometric decomposition. Then our Theorem together with this result implies

Corollary. *If M is a closed, orientable 3-manifold with a Heegaard splitting of genus two then either*

- (i) *M admits one of the eight geometric structures stated in [9], or*
- (ii) *M is one of (i)~(v) in the above theorem.*

We note that for each of the eight geometric structures there is a 3-manifold which has a Heegaard splitting of genus two and admits the geometric structure. See section 7.

2. Preliminaries

Throughout this paper we will work in the piecewise linear category.

For the definitions of *irreducible 3-manifolds*, *incompressible surfaces* we refer to [1]. For the definitions of *Haken manifolds* we refer to [4].

Let M be a closed, connected 3-manifold. $(V_1, V_2; F)$ is called a *Heegaard splitting* of M if each V_i is a 3-dimensional handlebody, $M = V_1 \cup V_2$ and $V_1 \cap V_2 = \partial V_1 = \partial V_2 = F$. Then F is called a *Heegaard surface* of M . The first Betti number of V_i is called the *genus* of the Heegaard splitting.

For the definitions of *Seifert fibered manifolds*, *orbit manifold*, *an isotopy of type A*, *hierarchy* for a surface, *an essential arc* in a surface and other definitions of standard terms in three dimensional topology we refer to [4]. The 3-manifold M is *simple* if every incompressible torus in M is boundary parallel.

By [4] every closed Haken manifold contains a unique, maximal, perfectly embedded Seifert fibered manifold Σ which is called a characteristic Seifert pair for M . The components of the closure of $M - \Sigma$ are simple. The boundary of Σ consists of tori in M . If some components of them are parallel in M then we eliminate one of them from the system of tori. By proceeding

this step we get a system of tori in M which are mutually non-parallel. We get simple manifolds and Seifert fibered manifolds by cutting M along these tori. In this paper, we call this decomposition a *torus decomposition* of M .

3. Essential annuli in genus two handlebody

Let F be a 2-sided surface properly embedded in a 3-manifold M . F is *essential* if it is incompressible and not parallel to a surface in ∂M . Let M' be a 3-manifold obtained by cutting M along F . Then there are copies of F on $\partial M'$ and we denote the component of the copies also by F .

In this section we will classify the system of essential annuli in the genus two handlebody.

Lemma 3.1 *If A is an incompressible annulus properly embedded in the solid torus V , the genus one handlebody, then A is boundary parallel.*

Proof. First, we claim that A cuts V into two solid tori. ∂A cuts ∂V into two annuli A_1, A_2 . Then $A \cup A_i$ ($i=1, 2$) is a torus in V . Since $\pi_1(V) \cong \mathbf{Z}$, $A \cup A_i$ is compressible in V . By the loop theorem [1] and the irreducibility of V we see that $A \cup A_i$ bounds a solid torus V_i . Let p_i ($i=1, 2$) be a positive integer such that $\text{Im}(i_*; \pi_1(A) \rightarrow \pi_1(V_i)) = \langle a_i^{p_i} \rangle$, where a_i is a generator of $\pi_1(V_i)$. Then $\pi_1(V) \cong \langle a_1, a_2 : a_1^{p_1} = a_2^{p_2} \rangle$. Then $p_1=1$ or $p_2=1$ for $\pi_1(V) \cong \mathbf{Z}$. If $p_i=1$ then A is parallel to A_i .

This completes the proof of Lemma 3.1.

Let D be a disk properly embedded in a handlebody V . D is a *meridian disk* of V if D does not separate V . Let $\{D_1, \dots, D_n\}$ be a system of mutually disjoint properly embedded disks in V . $\{D_1, \dots, D_n\}$ is a *complete system of meridian disks* of V if $\bigcup_{i=1}^n D_i$ cuts V into a 3-cell.

Lemma 3.2 *If A is an essential annulus in a genus two handlebody V then either*

- (i) *A cuts V into a solid torus V_1 and a genus two handlebody V_2 and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_1 \cap A = \emptyset$ and $D_2 \cap A$ is an essential arc of A , or*
- (ii) *A cuts V into a genus two handlebody V' and there is a complete system of meridian disks $\{D_1, D_2\}$ of V' such that $D_1 \cap A$ is an essential arc of A .*

See Fig. 1.

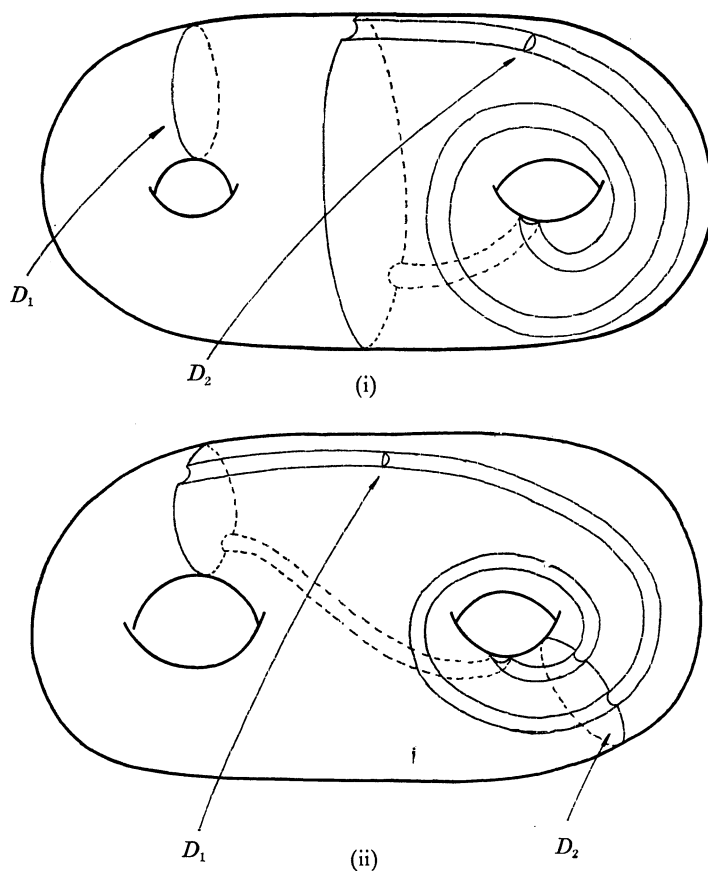


Fig. 1

Proof. Since A is incompressible in V , by using the complete system of meridian disks of V we can find a disk Δ in V such that $\Delta \cap A = a$ is an essential arc of A , $\Delta \cap \partial V = b$ is an arc such that $\partial a = \partial b$, $a \cup b = \partial \Delta$. Then we can perform a surgery on A along Δ to get a disk D properly embedded in V . Since A is essential, D is essential, say D is a meridian disk of V or D cuts V into two solid tori.

If D cuts V into two solid tori V' , V'' then there are copies Δ' , Δ'' of Δ on $\partial V'$. Then there is a meridian disk D_1 of V' such that $D_1 \cap (\Delta' \cup \Delta'') = \emptyset$. Since Δ' and Δ'' are identified in V cut along A , A cuts V into a solid torus V_1 and a genus two handlebody V_2 , where Δ is a meridian disk of V_2 such that $\Delta \cap A$ is an essential arc of A . Then we set $D_2 = \Delta$.

If D is a meridian disk of V then D cuts V into a solid torus V_1 . There are copies Δ' , Δ'' of Δ on ∂V_1 . Since Δ' and Δ'' are identified in V cut along A , A cuts V into a genus two handlebody V' . Then we set $D_1 = \Delta$ and we

have the conclusion (ii).

This completes the proof of Lemma 3.2.

Let M be a 3-manifold and S be a 2-manifold contained in ∂M . Let F be a surface properly embedded in M . Then F is *essential* in (M, S) if F is incompressible, $\partial F \subset S$ and F is not parallel to a surface in S .

Lemma 3.3 *Let V be a genus two handlebody and A_1, A_2 be a system of mutually disjoint annuli in ∂V such that there is a complete system of meridian disks $\{D_1, D_2\}$ of V which satisfies $D_i \cap A_j = \phi$ ($i \neq j$) and $D_i \cap A_i$ is an essential arc of A_i ($i=1, 2$). If A is an essential annulus in $(V, cl(\partial V - (A_1 \cup A_2)))$ then A is parallel to A_1 or A_2 .*

Proof. Since A is incompressible in V , by using $\{D_1, D_2\}$ we can find a disk Δ in V such that $\Delta \cap A = a$ is an essential arc of A , $\Delta \cap cl(\partial V - (A_1 \cup A_2)) = b$ is an arc such that $\partial a = \partial b$, $a \cup b = \partial \Delta$. Then we can perform a surgery on A along Δ to get an essential disk D such that $D \cap (A_1 \cup A_2) = \phi$. Since $D \cap (A_1 \cup A_2) = \phi$, D cuts V into two solid tori V_1, V_2 . We may suppose that $A_i \subset \partial V_i$. By assumption there is a meridian disk D'_i of V_i such that $D'_i \cap A_i$ is an essential arc of A_i . Then by the proof of Lemma 3.2 A cuts V into a genus two handlebody V'_1 and a solid torus V'_2 . We may suppose that $A_2 \subset \partial V'_2$. Then $\text{Im}(i_* : \pi_1(A_2) \rightarrow \pi_1(V_2)) = \pi_1(V_2)$ and $A_2 \cap A = \phi$. Hence A is parallel to A_2 .

This completes the proof of Lemma 3.3.

For the two essential annuli in the genus two handlebody we have

Lemma 3.4 *Let $\{A_1, A_2\}$ be a system of mutually disjoint, non-parallel, essential annuli in the genus two handlebody V . Then either*

- (i) $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2 . Then $A_1 \cup A_2 \subset \partial V_1$, $A_1 \cup A_2 \subset \partial V_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_i \cap A_j = \phi$ ($i \neq j$) and $D_i \cap A_i$ ($i=1, 2$) is an essential arc of A_i ,
- (ii) $A_1 \cup A_2$ cuts V into two solid tori V_1, V_2 and a genus two handlebody V_3 . Then $A_1 \subset \partial V_1$, $A_2 \subset \partial V_2$, $A_1 \cup A_2 \subset \partial V_3$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_3 such that $D_i \cap A_j = \phi$ ($i \neq j$) and $D_i \cap A_i$ ($i=1, 2$) is an essential arc of A_i or
- (iii) $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2 . Then $A_i \subset \partial V_1$ ($i=1$ or 2 , say 1), $A_2 \cap V_1 = \phi$, $A_1 \subset \partial V_2$ and there is a complete system of meridian disks $\{D_1, D_2\}$ of V_2 such that $D_1 \cap A_2$ is an essential arc of A_2 and $D_2 \cap A_i$ ($i=1, 2$) is an essential arc of A_i .

See Fig. 2.

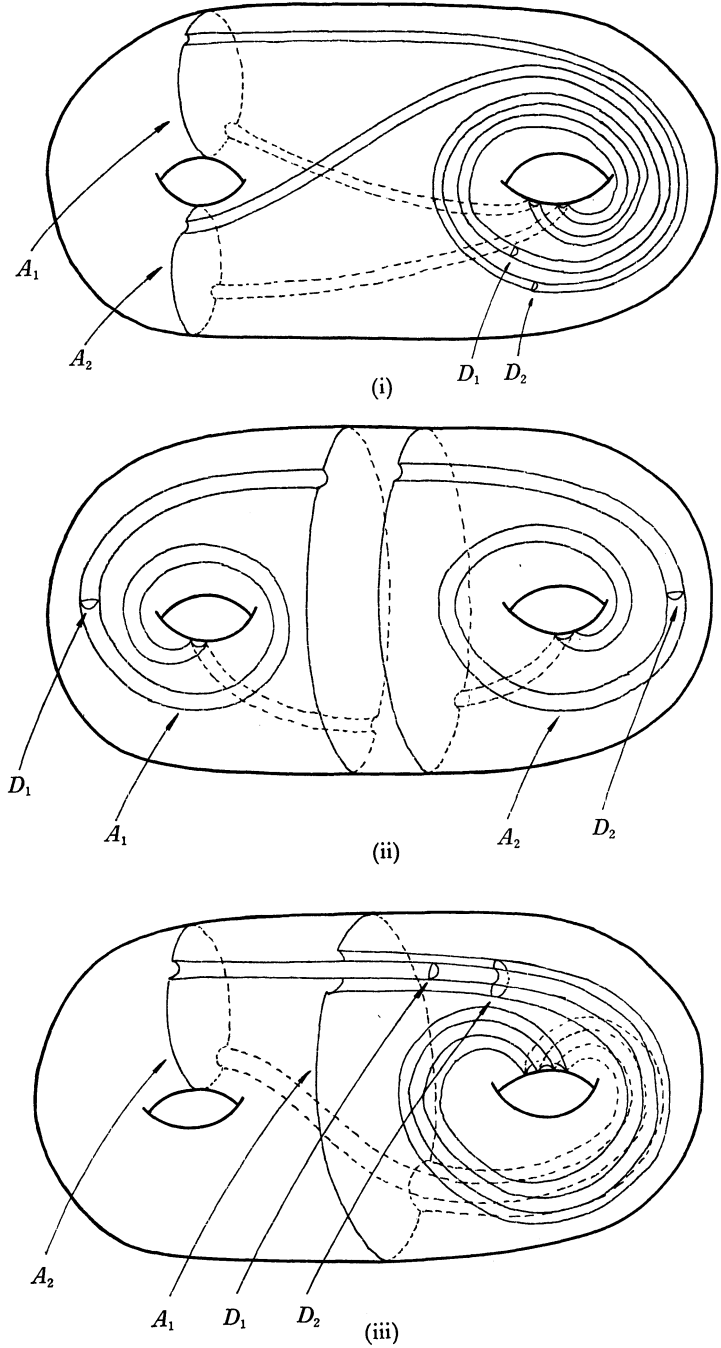


Fig. 2

Proof. There is a disk Δ in V such that $\Delta \cap A_i = \phi$ ($i=1$ or 2 , say 2), $\Delta \cap A_1 = a$ is an essential arc of A_1 , $\Delta \cap \partial V = b$ is an arc in $\partial \Delta$ such that $a \cap b = \partial a = \partial b$, $a \cup b = \partial \Delta$. We can perform a surgery on A_1 along Δ to get an essential disk D' properly embedded in V . Then there is a disk Δ' in V such that $\Delta' \cap D' = \phi$, $\Delta' \cap A_2 = a'$ is an essential arc of A_2 , $\Delta' \cap \partial V = b'$ is an arc in $\partial \Delta'$ such that $a' \cap b' = \partial a' = \partial b'$, $a' \cup b' = \partial \Delta'$. By performing a surgery on A_2 along Δ' we have an essential disk D'' in V , which is disjoint from D' .

We claim that $\{D', D''\}$ is not a complete system of meridian disks of V . Assume that $\{D', D''\}$ is a complete system of meridian disks of V . Then we can move A_2 by a small isotopy into V cut along $D' \cup D''$. This contradicts the fact that A_2 is incompressible in V .

Then we have the following three cases.

Case 1. D' and D'' are parallel and D' (hence, D'') does not separate V . In this case, we have the conclusion (i).

Case 2. D' and D'' are parallel and D' (hence, D'') cuts V into two solid tori. In this case, we have the conclusion (ii).

Case 3. D' and D'' are not parallel. We claim that D' does not separate V . Assume that D' separate V into two solid tori V' and V'' . Then we may suppose that $A_2 \subset V'$. By Lemma 3.1 A_2 is parallel to an annulus A'_2 in $\partial V'$. Then $A'_2 \cap D' = \phi$ for D' and D'' are not parallel. But this contradicts the fact that A_2 is essential.

Then since $\{D', D''\}$ is not a complete system of meridian disks, D'' separates V into two solid tori and we have the conclusion (iii).

This completes the proof of Lemma 3.4.

Lemma 3.5 *Let $\{A_1, A_2, A_3\}$ be a system of pairwise disjoint, non-parallel essential annuli in the genus two handlebody V . Then $A_1 \cup A_2 \cup A_3$ cuts V into two solid tori V_1, V_2 and a genus two handlebody V_3 which satisfies*

1. $A_i \subset \partial V_1$ ($i=1, 2$ or 3 , say 3), $A_1, A_2 \subset \partial V_3$, $A_1, A_2, A_3 \subset \partial V_2$.
2. *there is a complete system of meridian disks $\{D_1, D_2\}$ of V_3 such that $D_i \cap A_j = \phi$ ($i \neq j$) and $D_i \cap A_i$ ($i=1, 2$) is an essential arc of A_i and*
3. *there is a meridian disk D_3 of V_2 such that $D_3 \cap A_i$ ($i=1, 2, 3$) is an essential arc of A_i .*

See Fig. 3.

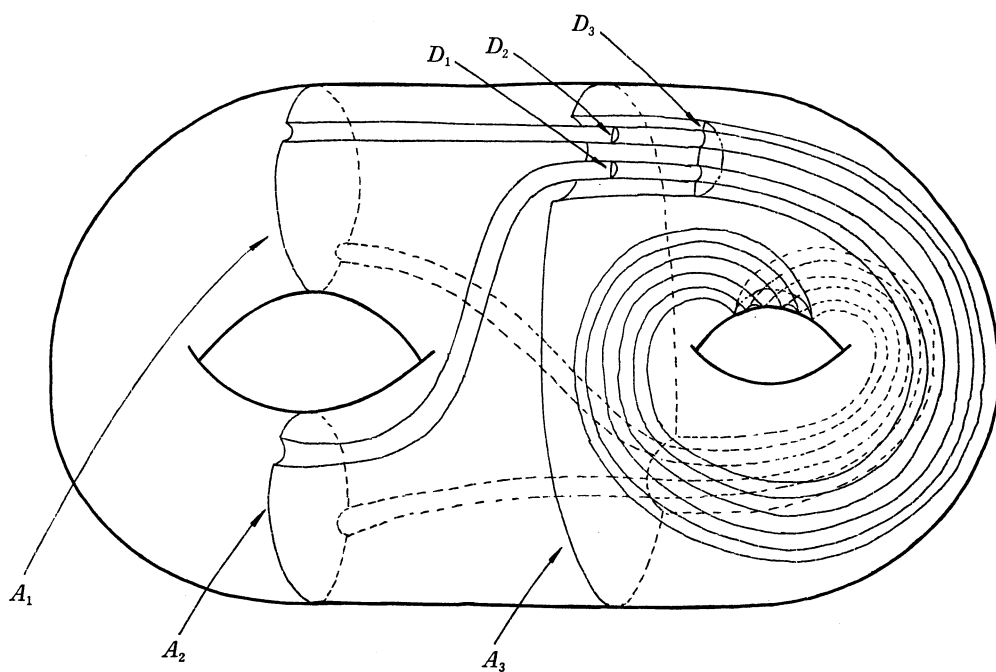


Fig. 3

Proof. $\{A_1, A_2\}$ satisfies one of the conclusions of Lemma 3.4. First, we claim that $\{A_1, A_2\}$ does not satisfy (ii). Assume that $\{A_1, A_2\}$ satisfies (ii). Then $A_1 \cup A_2$ separates V into two solid tori V_1, V_2 and a genus two handlebody V_3 . If $A_3 \subset V_1$ or V_2 then by Lemma 3.1 A_3 is parallel to A_1 or A_2 , which is a contradiction. If $A_3 \subset V_3$ then by Lemma 3.3 A_3 is parallel to A_1 or A_2 , which is a contradiction and the claim is established.

If $\{A_1, A_2\}$ satisfies the conclusion (i) of Lemma 3.4 then $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2 where $A_1, A_2 \subset \partial V_1$, $A_1, A_2 \subset \partial V_2$. By Lemma 3.3 we see that A_3 is not contained in V_2 . Then $A_3 \subset V_1$ and by Lemma 3.1 A_3 is parallel to an annulus A' in ∂V_1 . Since A_3 is essential and is not parallel to A_i ($i=1, 2$), $\partial A_1 \cup \partial A_2$ is contained in A' . Then we easily verify that $\{A_1, A_2, A_3\}$ satisfies the conclusions of Lemma 3.5.

If $\{A_1, A_2\}$ satisfies the conclusion (iii) then $A_1 \cup A_2$ cuts V into a solid torus V_1 and a genus two handlebody V_2 , where $A_1, A_2 \subset \partial V_2$ and $A_i \cap \partial V_1 = \phi$ ($i=1$ or 2 , say 1). By Lemma 3.1 we see that A_3 is contained in V_2 . Since $A_3 \cap (A_1 \cup A_2) = \phi$, by Lemma 3.3 we see that A_3 is parallel to an annulus A' in ∂V_2 . Since A_3 is essential and is not parallel to A_i ($i=1, 2$), $\partial A_1 \cup \partial A_2$ is contained in A' . Then by changing the suffix we easily verify that $\{A_1, A_2, A_3\}$ satisfies the conclusions of Lemma 3.5.

This completes the proof of Lemma 3.5.

4. Two bridge knot, link complements

A *knot* is a simple closed curve in the 3-sphere S^3 . A *link* is a union of mutually disjoint simple closed curves in S^3 with more than one component. For the definitions of the *two bridge knots* and *links* we refer to [8]. A *exterior* $Q(K)$ ($Q(L)$ resp.) of a knot K (link L resp.) is the closure of the complement of a regular neighborhood of K (L resp.). The *meridian* of K (L resp.) is a simple loop in $\partial Q(K)$ ($\partial Q(L)$ resp.) which bounds a meridian disk of the regular neighborhood of K (L resp.). A knot (link resp.) is *simple* if the exterior is a simple 3-manifold.

Lemma 4.1 *Let V_i ($i=1, 2$) be the genus two handlebody and A_1^i, A_2^i ($\subset \partial V_i$) be a system of pairwise disjoint, incompressible annuli such that there is a complete system of meridian disks $\{D_1^i, D_2^i\}$ of V_i which satisfies (i) $D_k^i \cap A_l^i = \emptyset$ ($k \neq l$) and (ii) $D_k^i \cap A_k^i$ is an essential arc of A_k^i ($k=1, 2$). If M is obtained from V_1 and V_2 by identifying their boundaries by a homeomorphism $h: cl(\partial V_1 - (A_1^1 \cup A_2^1)) \rightarrow cl(\partial V_2 - (A_1^2 \cup A_2^2))$ then M is homeomorphic to certain two bridge knot complement or a two bridge link complement, where the component of ∂A_j^i corresponds to a meridian loop.*

Proof. This can be proved by using the similar arguments of the section 4 of [5].

Lemma 4.2 *If K is a non-trivial two bridge knot then $Q(K)$ admits a complete hyperbolic structure or is a Seifert fibered manifold with orbit manifold a disk with two exceptional fibers.*

Proof. Since K is a simple knot [8], by [9] and the torus theorem [4] we see that $Q(K)$ admits a complete hyperbolic structure or is a special Seifert fibered manifold. If $Q(K)$ is a special Seifert fibered manifold then the orbit manifold is a disk or a Möbius band for $\partial Q(K)$ has one component (see 155p. of [4]). If the orbit manifold of $Q(K)$ is a Möbius band then it has no exceptional fibers. Hence $Q(K)$ is the twisted I -bundle over the Klein bottle but this is impossible for $Q(K)$ does not contain the Klein bottle.

This completes the proof of Lemma 4.2.

Let $\{a_1, \dots, a_n\}$ be a system of mutually disjoint, essential arcs in a punctured torus T . We say that a_i is of *type 1* if a_i joins distinct components of ∂T , a_i is of *type 2* if a_i joins one component of ∂T and a_i separates T , a_i is of *type 3* if a_i joins one component of ∂T and a_i does not separate T . We say that a_i is a *d-arc* if a_i is of type 1 and there is a component S of ∂T such that a_i is the only arc that joins S .

The next Lemma is perhaps known but no reference could be found.

Lemma 4.3 *Every two bridge link is a simple link.*

Proof. Let L be a two bridge link. Since L is a union of two trivial tangles with two arcs, $Q(L)$ has a decomposition as in Lemma 4.1 (see section 4 of [5]). Then we use the notations in Lemma 4.1. Let T be an incompressible torus in $Q(L)$. We may suppose that the components of $T \cap V_1$ are all disks and that the number of the components of $T \cap V_1$ is minimum among all tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Since T is incompressible, $T \cap V_1 \neq \emptyset$.

Let $T_2 = T \cap V_2$. Then by using D_1^2, D_2^2 we have a hierarchy $(T_2^{(0)}, a_0), \dots, (T_2^{(m)}, a_m)$ of T_2 and a sequence of isotopies of type A which realizes the hierarchy as in [4]. Let $T^{(1)}$ be the image of T after an isotopy of type A at a_0 and $T^{(k+1)}$ ($k \geq 1$) be the image of $T^{(k)}$ after an isotopy of type A at a_k .

Then we will show that $T \cap V_1$ consists of a disk.

Assume that $T \cap V_1$ consists of $n (\geq 2)$ disks D_1, \dots, D_n . We claim that D_1, \dots, D_n are mutually parallel in V_1 and each D_i cuts V_1 into two solid tori. If some D_i does not separate V_1 then $D_i \cap (A_1^1 \cup A_2^1) \neq \emptyset$ for $\text{Im}(i_*: \pi_1(A_1^1 \cup A_2^1) \rightarrow \pi_1(V_1)) = \pi_1(V_1)$, which is a contradiction. By the minimality of T it follows that each D_i cuts V_1 into two solid tori. Hence D_1, \dots, D_n are mutually parallel and the claim is established.

Then we show

(*) a_0, \dots, a_{n-1} are of type 3 and a_i and a_j joins pairwise distinct components of ∂T_2 if $i \neq j$.

By Lemma 3.3 each essential annulus in $(V_1, cl(\partial V_1 - (A_1^1 \cup A_2^1)))$ is parallel to A_1^1 or A_2^1 . By Lemma 3.3 of [5] we see that a_0, a_1 are of type 3 and we may suppose that a_0, a_1 joins D_1, D_2 respectively. Note that in [5] we considered the non-separating incompressible torus, but in Lemma 3.1, 3.2, 3.3 of [5] which are proved by using the argument of the inverse operation of isotopy of type A in [2] the non-separating property is not essential.

Assume that (*) does not hold then there is such $i (\geq 3)$ that a_i is not of type 3 or a_i is of type 3 and a_i joins D_k that some a_l ($l < i$) joins. Then we may suppose that a_j ($j < i$) is of type 3 and joins D_j . Then $T^{(i-1)} \cap V_1 = A_1 \cup \dots \cup A_{i-1} \cup D_i \cup \dots \cup D_n$, where each A_i is an essential annulus in $(V_1, cl(\partial V_1 - (A_1^1 \cup A_2^1)))$.

Assume that a_i is of type 1. If a_i joins some A_k and D_l ($l \geq i$) or D_k and D_l ($k, l \geq i$) as an arc on $T^{(i-1)} \cap V_2$ then $T^{(i)} \cap V_1$ consists of $i-1$ annuli and $n-i$ disks. Then by performing a sequence of isotopies of type A on $T^{(i)}$ we have such a torus T' that $T' \cap V_1$ consists of $n-1$ disks, which contradicts the minimality of T . If a_i joins some A_k and A_l then A_k is parallel to A_l in V_1 for D_i separates V_1 into two solid tori. Then $T^{(i)} \cap V_1$ consists of $i-2$

annuli, $n-i$ disks and one disk with two holes B . Some component l of ∂B bounds a disk on ∂V . Since T is incompressible and $Q(L)$ is irreducible, we see that l bounds a disk on $T^{(i)}$ and there is an ambient isotopy h_t ($0 \leq t \leq 1$) of $Q(L)$ such that $h_1(T^{(i)}) \cap V_1$ consists of $i-1$ annuli and $n-i$ disks. Then we have a contradiction as above.

Assume that a_i is of type 2. Then there is an arc a in ∂T_2 such that $a \cap a_i = \partial a = \partial a_i$, $a \cup a_i$ bounds a planar surface P in T_2 . We easily see that some a_j ($\subset P$) is a d -arc. Hence by Lemma 3.1 of [5] T is ambient isotopic to such a torus T' that $T' \cap V_1$ consists of $n-1$ disks, which is a contradiction.

Assume that a_i is of type 3 and a_i joins D_j ($j < i$). Then there are two arcs b_1, b_2 in ∂T_2 such that $a_j \cup b_1 \cup a_i \cup b_2$ is a simple loop in T_2 and $a_j \cup b_1 \cup a_i \cup b_2$ bounds a planar surface P in T_2 . Then see that some a_k ($\subset P$) is a d -arc and we have a contradiction as above.

Hence (*) is established.

Then $T^{(n)} \cap V_1$ ($T^{(n)} \cap V_2$ resp.) consists of n essential annuli A_1, \dots, A_n (A'_1, \dots, A'_n resp.). By Lemma 3.3 each A_i is parallel to either A_1^1 or A_2^1 . We may suppose that A_n is outermost in $(V_1, cl(\partial V_1 - (A_1^1 \cup A_2^1)))$ and is parallel to A_1^1 . Then some A'_j is parallel to A_k^2 ($k=1$ or 2) and $\partial A_n = \partial A'_j$. This contradicts the fact that $n \geq 2$.

Hence $T \cap V_1$ consists of a disk. Then $T^{(1)} \cap V_i$ consists of an annulus A^i which is parallel to A_j^i ($j=1$ or 2). Hence $T^{(1)}$ is parallel to a component of $\partial Q(L)$.

This completes the proof of Lemma 4.3.

Lemma 4.4 *If L is a two bridge link then $Q(L)$ admits a complete hyperbolic structure or is a Seifert fibered manifold with orbit manifold an annulus with at most one exceptional fiber.*

Proof. By Lemma 4.3 together with [4] and [9] $Q(L)$ is a hyperbolic manifold or a special Seifert fibered manifold. If $Q(L)$ is a special Seifert fibered manifold then the orbit manifold of $Q(L)$ is an annulus and it has at most one exceptional fiber for $\partial Q(L)$ has two components.

5. One bridge knots in lens spaces

Let us give the definition of the *one bridge knot in a lens space*. For the definition of lens spaces we refer to 20p. of [1]. In this paper we think that $S^3, S^2 \times S^1$ are lens spaces. Let V be a solid torus and let a be an arc properly embedded in V . We say that a is trivially embedded in V if there is a disk D in V such that $D \cap \partial V = b$ an arc, $cl(\partial D - b) = a$. It is easily seen that if a' is another trivially embedded arc in V then there is an ambient isotopy h_t ($0 \leq t \leq 1$) of V such that $h_1(a) = a'$. Let K be a knot in a lens space L_n . We say that K is a one bridge knot in L_n if there is a Heegaard splitting $(V_1, V_2;$

F) of L_n of genus one such that $V_i \cap K$ ($i=1, 2$) is an arc trivially embedded in V_i . We denote the exterior of K also by $Q(K)$. Then we can naturally define a meridian loop on $Q(K)$.

Lemma 5.1 *Let V_i ($i=1, 2$) be a genus two handlebody and A_i ($\subset \partial V_i$) be an incompressible annulus such that there is a complete system of meridian disks $\{D_1^i, D_2^i\}$ of V_i which satisfies (i) $D_1^i \cap A_i = \emptyset$ and (ii) $D_2^i \cap A_i$ is an essential arc of A_i . If M is obtained from V_1 and V_2 by identifying their boundaries by a homeomorphism $h: cl(\partial V_1 - A_1) \rightarrow cl(\partial V_2 - A_2)$ then M is homeomorphic to certain one bridge knot complement in lens space, where the component of ∂A_i corresponds to a meridian loop.*

Proof. This is proved by using the similar arguments of the proof of Lemma 4.1.

Lemma 5.2 *Let K be a one bridge knot in a lens space L_n . If $Q(K)$ is a Seifert fibered manifold with incompressible boundary, whose regular fiber is not a meridian loop then either*

- (i) $Q(K) \in D(2)$ where the regular fiber in $\partial Q(K)$ intersects their meridian loop transversely in a single point,
- (ii) $Q(K) \in M\ddot{o}(1)$ where the regular fiber in $\partial Q(K)$ intersects the meridian loop transversely in a single point or
- (iii) $Q(K)$ is homeomorphic to the twisted I -bundle over the Klein bottle.

Proof. We fix the fiber structure of $Q(K)$. Since an incompressible torus in $Q(K)$ is separating, the orbit manifold of $Q(K)$ is a disk or a Möbius band.

Suppose that the orbit manifold of $Q(K)$ is a disk. First we claim that L_n does not admit such a Seifert fibration that the orbit manifold is a 2-sphere with n (≥ 3) exceptional fibers. $n \geq 4$ implies that L_n contains an incompressible torus, which is a contradiction. By Theorem 12.2 of [1] $n=3$ implies that there is an epimorphism from $\pi_1(L_n)$ to the group

$$G = \langle a, b; a^p = b^q = (ab)^r = 1 \rangle \quad (p, q, r > 1).$$

This is impossible for G is not a cyclic group [7] and the claim is established.

Assume that $Q(K)$ contains m (≥ 3) exceptional fibers. Then since the regular fiber of $Q(K)$ is not a meridian loop, L_n admits such a Seifert fibration that the orbit manifold is a 2-sphere with m or $m+1$ exceptional fibers, which contradicts the above claim. Hence $Q(K)$ contains two exceptional fibers. Then if the regular fiber is not isotopic to a loop which intersects the meridian loop transversely in a single point then L_n admits such a Seifert fibration that the orbit manifold is a 2-sphere with three exceptional fibers, which contradicts the above claim.

Then we have the conclusion (i).

Suppose that the orbit manifold of $Q(K)$ is a Möbius band. Since L_n does not contain an incompressible torus $Q(K)$ contains at most one exceptional fibers. If $Q(K)$ contains one exceptional fiber then the regular fiber intersects the meridian loop transversely in a single point and we have the conclusion (ii). If $Q(K)$ contains no exceptional fibers then we have the conclusion (iii).

This completes the proof of Lemma 5.2.

6. Proof of Theorem

Lemma 6.1 *Let M be a simple manifold whose boundary components are all tori. If M contains an essential annulus then M is a Seifert fibered manifold.*

Proof. This is a consequence of the characteristic Seifert pair theorem [4].

We shall divide the proof of Theorem into several cases.

Case 1. M contains a non-separating incompressible torus. In this case by Theorem 2 of [5] we have the conclusion (v) of the Theorem.

Hereafter, we will suppose that each incompressible torus in M is separating.

Case 2. M is decomposed into two components M_1, M_2 by the torus decomposition. Let T be the torus which cuts M into M_1, M_2 and $(V_1, V_2; F)$ be a genus two Heegaard splitting of M . We may suppose that the components of $T \cap V_1$ are all disks and that the number of the components of $T \cap V_1$ is minimum among all tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Let $T_2 = T \cap V_2$. Then as in [4] we have a hierarchy $(T_2^{(0)}, a_0), \dots, (T_2^{(m)}, a_m)$ of T_2 and a sequence of isotopies of type A which realizes the hierarchy. Let $T^{(1)}$ be the image of T after an isotopy of type A at a_0 and $T^{(k+1)}$ ($k \geq 1$) be the image of $T^{(k)}$ after an isotopy of type A at a_k .

Then we claim that $T \cap V_1$ consists of at most two components. Assume that $T \cap V_1$ consists of n (≥ 3) disks D_1, \dots, D_n . Then by [5] a_0, a_1 are of type 3 and, hence, $T^{(1)} \cap V_1 = A_1 \cup D_2 \cup \dots \cup D_n$, $T^{(2)} \cap V_1 = A_1 \cup A_2 \cup D_3 \cup \dots \cup D_n$, where A_i ($i=1, 2$) is an essential annulus in V_1 . If D_1, D_2 are separating in V_1 and A_1, A_2 are parallel in V_1 then there are two annuli A', A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2) = A' \cap A_1 = \partial A' = \partial A_1$, $A'' \cap (A_1 \cup A_2) = \partial A''$, $A' \cap A''$ is a component of ∂A_1 . We may suppose that $A' \subset M_1$ and $A'' \subset M_2$. See Fig. 4. By the minimality of T , A' (A'' resp.) is an essential annulus in M_1 (M_2 resp.). Hence by Lemma 6.1 and Theorem VI. 34 of [4] M_1 and M_2 admit such Seifert fibrations that the component of ∂A_1 is a regular fiber. Hence M admits a Seifert fibration, which is a contradiction. If D_1, D_2 are separating in V_1 and A_1 is not parallel to A_2 in V_1 then D_1, \dots, D_n are parallel in V_1 . See Fig.

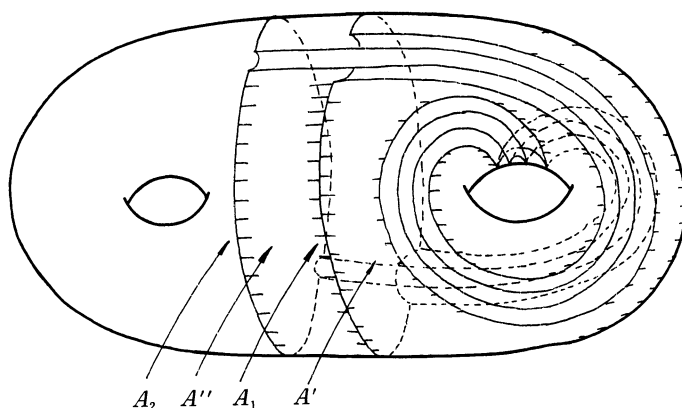


Fig. 4

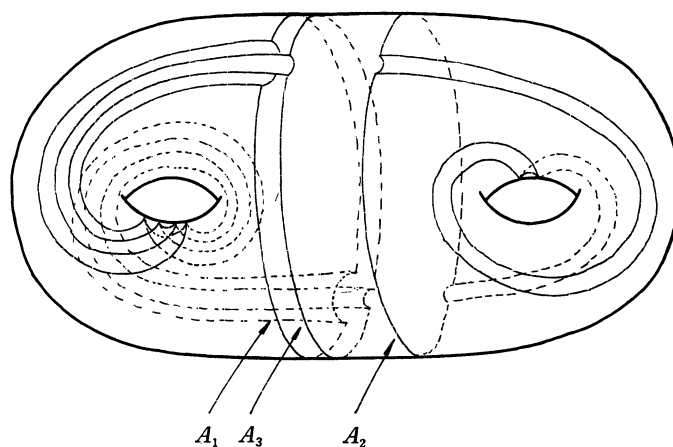


Fig. 5

5. Since each a_i is not a d -arc [5], a_2 is of type 3 and we may suppose that $T^{(3)} \cap V_1 = A_1 \cup A_2 \cup A_3 \cup \cdots \cup D_n$ where A_3 is an essential annulus in V_1 . Then A_3 is parallel to A_1 or A_2 (Lemma 3.3) and we have a contradiction as above. If D_1 is separating and D_2 is non-separating in V_1 then there exists annuli A', A'' as above and we have a contradiction. Since A_1 is incompressible, the case of D_1 being non-separating and D_2 being separating cannot occur. If D_1, D_2 are non-separating in V_1 then D_1, \dots, D_n are mutually parallel in V_1 . Since each a_i is not a d -arc, a_2 is of type 3 and we may suppose that $T^{(3)} \cap V_1 = A_1 \cup A_2 \cup A_3 \cup \cdots \cup D_n$, where A_3 is an essential annulus in V_1 . Then there exists annuli A', A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2 \cup A_3) = \partial A'$, $A'' \cap (A_1 \cup A_2 \cup A_3) = \partial A''$ and $A' \cap A''$ is a component of $\partial A'$. Then we have a contradiction as above and we establish the claim.

Now, we have two subcases.

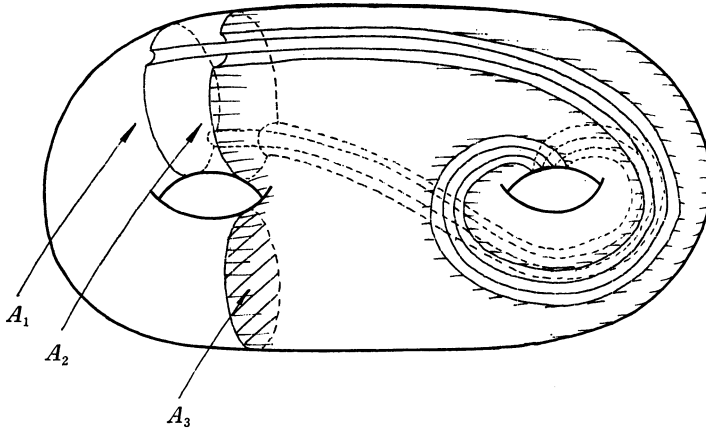


Fig. 6

Case 2.1. $T \cap V_1$ consists of a disk D_1 . Since T separates M , D_1 cuts V_1 into two solid tori. Let $A_1 = V_1 \cap T^{(1)}$, $A_2 = V_2 \cap T^{(1)}$. Then by Lemma 3.2 A_i ($i=1, 2$) cuts V_i into a solid torus V_i^1 and a genus two handlebody V_i^2 . By attaching V_1^1 and V_2^1 along $cl(\partial V_i^1 - A_i)$ we have M_1 ($\in D(2)$) and by attaching V_1^2 and V_2^2 along $cl(\partial V_i^2 - A_i)$ we have M_2 ($\in L_K$) (Lemma 5.1). Then we have the conclusion (i) of the Theorem.

Case 2.2. $T \cap V_1$ consists of two disks D_1, D_2 . In this case $T^{(2)} \cap V_1$ ($T^{(2)} \cap V_2$ resp.) consists of two essential annuli A_1, A_2 (A'_1, A'_2 resp.).

We claim that if A_1 is parallel to A_2 then A_1, A_2 satisfies the conclusion (i) of Lemma 3.4. First, we show that A_1 is non-separating in V_1 . If A_1 is separating in V_1 then there are annuli A', A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2) = A' \cap A_1 = \partial A' = \partial A_1$, $A'' \cap (A_1 \cup A_2) = \partial A''$, $A' \cap A''$ is a component of $\partial A'$. Hence we have a contradiction as in Case 2. Then by Lemma 3.2 A_1 cuts V_1 into a genus two handlebody V' . Let A'_1, A'_1' be the copies of A_1 on V' . By the proof of Lemma 3.4 we can show that there is a complete system of meridian disks $\{D'_1, D'_2\}$ of V' such that $(D'_1 \cup D'_2) \cap A'_1 = D'_1 \cap A'_1$ is an essential arc of A'_1 , $(D'_1 \cup D'_2) \cap A'_1' = D'_2 \cap A'_1'$ and each component of $D'_2 \cap A'_1'$ is an essential arc of A'_1' . If needed by exchanging A_1 and A_2 we may suppose that A_2 is parallel to A'_1' in V' . We will show that D'_2 can be taken so that $D'_2 \cap A'_1'$ is an essential arc of A'_1' . If this is done then the claim is established. Since A_1 is parallel to A_2 there is an annulus A''' in ∂V_1 such that $A''' \cap A_i$ ($i=1, 2$) is a component of $\partial A'''$. We may suppose that $A''' \subset M_1$. Then A''' is an essential annulus in M_1 and by Lemma 6.1 and [4] M_1 admits such a Seifert fibration that A''' is a union of regular fibers. If the meridian disk D'_2 as above cannot be taken then there is an essential annulus A_3 in V' such that $A_3 \cap T^{(2)} = A_3 \cap A_2 = \partial A_3 = \partial A_2$, $A_3 \cap D'_1 = \emptyset$, A_3 is not parallel to A_2 and $A_2 \cup A_3$ bounds a solid torus T' in V' . See Fig. 6. Then $M'_1 = M_1 \cup T'$

admits a Seifert fibration and M'_1 is not homotopic into M_1 . This contradicts the maximal property of the characteristic Seifert pair.

By the above claim and Lemma 3.4 we see that $\{A_1, A_2\}$ ($\{A'_1, A'_2\}$ resp.) satisfies one of the conclusions of Lemma 3.4.

Note that $\{A_1, A_2\}$ ($\{A'_1, A'_2\}$ resp.) does not satisfy the conclusion (iii) of Lemma 3.4 for $T^{(2)}$ is separating in M .

We claim that either $\{A_1, A_2\}$ or $\{A'_1, A'_2\}$ does not satisfy the conclusion (ii) of Lemma 3.4. Assume that both $\{A_1, A_2\}$ and $\{A'_1, A'_2\}$ satisfy (ii) of Lemma 3.4. Then $A_1 \cup A_2$ ($A'_1 \cup A'_2$ resp.) cuts V_1 (V_2 resp.) into two solid tori and a genus two handlebody, but this contradicts the fact that $T^{(2)}$ is connected.

Then we have two subcases.

Case 2.2.1. Both $\{A_1, A_2\}$ and $\{A'_1, A'_2\}$ satisfy the conclusion (i) of Lemma 3.4. In this case $A_1 \cup A_2$ ($A'_1 \cup A'_2$ resp.) cuts V_1 (V_2 resp.) into a solid torus $V_1^{(1)}$ ($V_1^{(2)}$ resp.) and a genus two handlebody $V_2^{(1)}$ ($V_2^{(2)}$ resp.) where $A_1 \cup A_2 \subset \partial V_1^{(1)}$, $A_1 \cup A_2 \subset \partial V_2^{(1)}$ ($A'_1 \cup A'_2 \subset \partial V_1^{(2)}$, $A'_1 \cup A'_2 \subset \partial V_2^{(2)}$ resp.). By attaching $V_1^{(1)}$ and $V_1^{(2)}$ along $cl(\partial V_1^{(i)} - (A_1 \cup A_2))$ we get M_1 ($\in Mo(n)$, $n=0, 1$ or 2) and by attaching $V_2^{(1)}$ and $V_2^{(2)}$ along $cl(\partial V_2^{(i)} - (A_1 \cup A_2))$ we get M_2 ($\in M_K$) (Lemma 4.1). Then we have the conclusion (ii) of the Theorem.

Case 2.2.2. $\{A_1, A_2\}$ satisfies the conclusion (i) and $\{A'_1, A'_2\}$ satisfies the conclusion (ii) of Lemma 3.4. In this case $A_1 \cup A_2$ cuts V_1 into a solid torus $V_1^{(1)}$ and a genus two handlebody $V_2^{(1)}$, $A'_1 \cup A'_2$ cuts V_2 into two solid tori $V_1^{(2)}$, $V_2^{(2)}$ and a genus two handlebody $V_3^{(2)}$. By attaching $V_1^{(1)}$ and $V_1^{(2)} \cup V_2^{(2)}$ along $cl(\partial V_1^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_1^{(2)} - A_1 \cup cl(\partial V_2^{(2)} - A_2))$ we get M_1 ($\in D(n)$, $n=2$ or 3) and by attaching $V_2^{(1)}$ and $V_3^{(2)}$ along $cl(\partial V_2^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_3^{(2)} - (A_1 \cup A_2))$ we get M_2 ($\in M_K$) (Lemma 4.1).

Then we have the conclusion (iii) of the Theorem.

Case 3. M is decomposed into three components M_1 , M_2 and M_3 by the torus decomposition. Let T_1, T_2 be the pair of tori which cuts M into M_1, M_2 and M_3 and let $T = T_1 \cup T_2$. Then we may suppose that $T_1 \subset \partial M_1$, $T_2 \subset \partial M_3$ and $T \subset \partial M_2$. Let $(V_1, V_2; F)$ be a genus two Heegaard splitting of M . Then we may suppose that the components of $T \cap V_1$ are all disks and that the number of the components of $T \cap V_1$ is minimum among all the pair of tori which are isotopic to T and the components of the intersection of each of which with V_1 are all disks. Let $T' = T \cap V_2$. Then we have a hierarchy $(T^{(0)}, a_0), \dots, (T^{(m)}, a_m)$ of T' and a sequence of isotopies of type A which realizes the hierarchy.

We will show that we may suppose that a_0 and a_1 are of type 3 and a_1 joins distinct component of $\partial T'$ that a_0 joins. By the argument of section 3 of [5] both a_0 and a_1 are of type 3. Suppose that a_1 joins the same component of

$\partial T'$ that a_0 joins. We may suppose that $a_0, a_1 \subset T_1$. Then $T_1 \cap V_1$ consists of a disk D_1 for if $T_1 \cap V_1$ has more than one component then $a_0 \cup a_1$ cuts $T_1 \cap V_2$ into a planar surface and hence some a_i ($\subset T_1 \cap V_2$) is a d -arc, which contradicts the minimality of T (see Lemma 3.1 of [5]). Let T'_1 be the image of T_1 after an isotopy of type A at a_0 . Then $T'_1 \cap V_1 = A_1$ ($T'_1 \cap V_2 = A'_1$ resp.) is an essential annulus in V_1 (V_2 resp.). Since T_1 is separating in M , A'_1 cuts V_2 into a solid torus $V_1^{(2)}$ and a genus two handlebody $V_2^{(2)}$ where there is a complete system of meridian disks $\{D_1, D_2\}$ of $V_2^{(2)}$ such that $D_1 \cap A'_1 = \phi$, $D_2 \cap A'_1$ is an essential arc of A'_1 . Since $T_2 \cap V_2$ is incompressible in V_2 , $(T_2 \cap V_2) \subset V_2^{(2)}$. Then by using $\{D_1, D_2\}$ we can define an isotopy of type A at some essential arc b in $T_2 \cap V_2$. Then by the minimality of T , b is of type 3. Hence by taking b as a_1 we may suppose that a_0, a_1 are of type 3 and a_1 joins distinct component of $\partial T'$ that a_0 joins.

Then by the argument of Case 2 we see that $T \cap V_1$ consists of two disks. Let T^1 be the image of T after an isotopy of type A at a_0 , T^2 be the image of T^1 after an isotopy of type A at a_1 . Then $T^2 \cap V_1$ ($T^2 \cap V_2$ resp.) consists of two essential annuli A_1, A_2 (A'_1, A'_2 resp.) where $\partial A_1 = \partial A'_1$ and $\partial A_2 = \partial A'_2$. By the argument of Case 2.2 $\{A_1, A_2\}$ ($\{A'_1, A'_2\}$ resp.) satisfies one of the conclusions of Lemma 3.4.

Since T_1 and T_2 are separating in M , each A_i (A'_i resp.) is separating in V_1 (V_2 resp.). Hence both $\{A_1, A_2\}$ and $\{A'_1, A'_2\}$ satisfy the conclusion (ii) of Lemma 3.4. $A_1 \cup A_2$ ($A'_1 \cup A'_2$ resp.) cuts V_1 (V_2 resp.) into two solid tori $V_1^{(1)}, V_2^{(1)}$ ($V_1^{(2)}, V_2^{(2)}$ resp.) and a genus two handlebody $V_3^{(1)}$ ($V_3^{(2)}$ resp.) where $A_i \subset \partial V_i^{(1)}$ ($A'_i \subset \partial V_i^{(2)}$ resp.) ($i=1, 2$). By attaching $V_1^{(1)}$ and $V_1^{(2)}$ ($V_2^{(1)}$ and $V_2^{(2)}$ resp.) along $cl(\partial V_1^{(1)} - A_1)$ and $cl(\partial V_1^{(2)} - A'_1)$ ($cl(\partial V_2^{(1)} - A_2)$ and $cl(\partial V_2^{(2)} - A'_2)$ resp.) we get M_1 ($\in D(2)$) ($M_2 \in D(2)$ resp.). By attaching $V_3^{(1)}$ and $V_3^{(2)}$ along $cl(\partial V_3^{(1)} - (A_1 \cup A_2))$ and $cl(\partial V_3^{(2)} - (A'_1 \cup A'_2))$ we get M_3 ($\in M_L$) (Lemma 4.1).

Then we have the conclusion (iv) of the Theorem.

Note that M does not have such a torus decomposition that M decomposed into more than three components. Assume that M has such a decomposition. Let T_1, \dots, T_n ($n \geq 3$) be a system of tori which gives the decomposition. We may suppose that each component of $(T_1 \cup \dots \cup T_n) \cap V_1$ is a disk. Note that $(T_1 \cup \dots \cup T_n) \cap V_1$ has more than two components and we can derive a contradiction by using the arguments of Case 2.

If M admits a decomposition as in (i)~(v) of Theorem then by tracing the above arguments conversely we can show that M has a Heegaard splitting of genus two.

This completes the proof of Theorem.

7. Geometric structures of the 3-manifolds with Heegaard splittings of genus two

In this section we show that for each of eight geometric structures in [9] there exists a 3-manifold M which has a Heegaard splitting of genus two and admits the geometric structure.

Lemma 7.1 *If M is a Seifert fibered manifold with orbit manifold a 2-sphere with three exceptional fibers then M has a Heegaard splitting of genus two.*

Proof. Let f be an exceptional fiber in M and Q be the closure of the complement of a regular neighborhood of f . Then Q contains such an essential annulus A that A cuts Q into two solid tori. Let a be an essential arc in A and V_1 be a regular neighborhood of $N \cup a$ in M . Then V_1 is a genus two handlebody. We easily see that $cl(M - V_1)$ is also a genus two handlebody.

This completes the proof of Lemma 7.1.

Let M be a Seifert fibered manifold as in Lemma 7.1. Then by Theorem 12.1 of [1] $\pi_1(M)$ has the presentation

$$\langle a, b, c, t; [a, t] = [b, t] = [c, t] = 1, a^p = t^{p'}, b^q = t^{q'}, c^r = t^{r'}, abc = 1 \rangle$$

where $p > 1, q > 1, r > 1$. Then for the geometric structure of M the following theorem is given by Kojima [6].

Theorem. *If M is a Seifert fibered manifold as above then M admits a geometric structure according to the table:*

$\begin{matrix} 1/p+1/q+1/r \\ p'/p+q'/q+r'/r \end{matrix}$	> 1	$= 1$	< 1
$= 0$	ϕ	type 2	type 5
$\neq 0$	type 1	type 7	type 6

where the type of geometries appears in [9].

Then by Lemma 7.1 we see that for each of the geometries that appeared in the above Theorem there is a 3-manifold with a Heegaard splitting of genus two, which has the geometric structure.

The examples of the 3-manifolds with Heegaard splittings of genus two in the hyperbolic geometry (type 3) are obtained by Dehn surgery on the figure eight knot [11].

The closed 3-manifolds in the type 4 geometry are only either $S^2 \times S^1$ or $P^3 \# P^3$, each of which has a Heegaard splitting of genus two.

Any torus bundle over S^1 with a hyperbolic monodromy has type 8 geometric structure [9]. Then the torus bundle whose monodromy is $\begin{pmatrix} 0 & 1 \\ -1 & m \end{pmatrix}$

($m \geq 3$) has type 8 geometric structure and by [3] it has a Heegaard splitting of genus two.

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