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ON THE KNOT ASSOCIATED WITH THE SOLID TORUS

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We have some methods to compose a knot from some knots, for instance composition (or product, sum) [7], union [5], linked sum [11], etc.. Recently, F. Hosokawa defined in [4] a fusion of a link as an extension of composition. And further, we have some kinds of knots associated with the torus already proved of their non-triviality, for example torus knots, parallel knots, doubled knots [12], Schlingknoten [8], etc..

The purpose of the present note is to define a complete-fusion of a link and an associated knot with a solid torus T_μ , and to prove the non-trivialities of them.

In §1, the concept of complete-fusion is described as extensions of composition and fusion, and in §2 we give a standard model of the solid torus T_μ of genus μ and investigate the relation between complete-fusion and solid torus T_μ . In §3, we consider a generalization of knots associated with the so-called torus T_1 .

By the Dehn's Lemma [2], [3], [6], we have the following well-known unknotting theorem: *a knot k is of trivial type if and only if $\pi_1(S^3 - k) \cong \mathbb{Z}$, infinite cyclic group.* Implicit use of this theorem will be made throughout this work.

The following considerations are based upon the semi-linear point of view.

The author wishes to express his hearty thanks to Professor H. Terasaka for leading him to this work.

1. A complete-fusion of a link

Let $L = (k_1, k_2, \dots, k_\mu)$ be an oriented link with μ components ($\mu \geq 2$) in a 3-sphere S^3 , and let $B_1, \dots, B_{\mu-1}$ be narrow bands that span L and that are oriented coherently with L . Suppose that $L + \partial B_1 + \dots + \partial B_{\mu-1}$ is a knot¹⁾, denote $k(L; B_1, \dots, B_{\mu-1})$ or simply $k(L)$, where $+$ means homological addition. We say that the knot $k(L)$ is a *complete-fusion* of L , see Fig. 1. Note that if the link L is split, then this definition is a special case of fusion in [4, §2], while this definition can be meant to extend the concept of fusion to of general links. It

1) ∂ =boundary, \mathcal{I} =interior.

will be noticed that a complete-fusion is not determined uniquely, i.e. the knot type of $k(L)$ depends upon the link type of L and joined bands $B_1, \dots, B_{\mu-1}$.

It is easily shown that the composition (or product, sum) [7] of k_1, k_2, \dots, k_μ is the special case of complete-fusion, and we denote this by $k^*(L) = k_1 \# k_2 \# \dots \# k_\mu$. The composition $k^*(L)$ of L is uniquely determined.

Theorem 1.1. *Let $L = (k_1, k_2, \dots, k_\mu)$ be a split link with μ components ($\mu \geq 2$) in the 3-sphere S^3 . If at least one of components k_1, k_2, \dots, k_μ of L , say k_1 , is of non-trivial knot type, then any complete-fusion $k(L)$ of L is of non-trivial knot type.*

Moreover, let $g(k)$ and $\Delta_k(t)$ be genus and Alexander polynomial of k respectively, then by Terasaka-Yonebayashi's arguments [10], [13], we have:

$$(1.2) \quad g(k(L)) \geq g(k_1) + \dots + g(k_\mu).$$

$$(1.3) \quad \Delta_{k(L)}(t) \equiv F(t)F(t^{-1}) \Delta_{k_1}(t) \cdots \Delta_{k_\mu}(t) \pmod{\pm t},$$

where $F(t)$ is an integral polynomial determined by the crossings of B_i with k_j .

Proof. By the definition at least one of $B_1, \dots, B_{\mu-1}$ attaches to k_μ . So, sliding the ends of $B_1, \dots, B_{\mu-1}$ along k_1, \dots, k_μ and sides of other bands we may assume that each band B_i connects k_i and k_μ , $i = 1, \dots, \mu - 1$, as illustrated in Fig. 1.

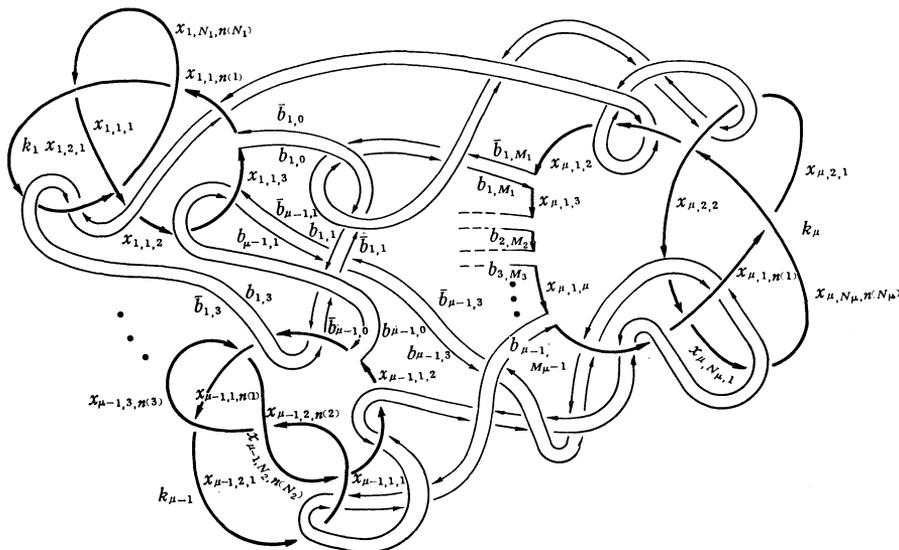


Fig. 1

Let $x_{i,1,1}, x_{i,1,2}, \dots, x_{i,1,m_1}, x_{i,2,1}, \dots, x_{i,N_i, n(N_i)}$ be the generators of the knot group $G(k(L)) = \pi_1(S^3 - k(L))$ corresponding to the arcs of k_i , $i = 1, \dots, \mu$, with the same letter in this order. Let $b_{i,0}, b_{i,1}, \dots, b_{i,M_i}$ and $\bar{b}_{i,0}, \bar{b}_{i,1}, \dots, \bar{b}_{i,M_i}$ be the generators of the knot group corresponding to the boundary of B_i , $i = 1, \dots, \mu - 1$.

Then we have a presentation²⁾ of $G(k(L))$ as follows:

2) See [1], Chapter VI.

$$\left(\begin{array}{l} \text{generators:} \\ x_{i,1,1}, x_{i,1,2}, \dots, x_{i,1,n_1}, x_{i,2,1}, \dots, x_{i,N_i,n(N_i)}, (i=1, \dots, \mu), \\ b_{i,0}, b_{i,1}, \dots, b_{i,M_i}, \bar{b}_{i,0}, \bar{b}_{i,1}, \dots, \bar{b}_{i,M_i}, (i=1, \dots, \mu-1). \\ \text{relations:} \\ r_{i,1} = \dots = r_{i,N_i} = 1, (i=1, \dots, \mu), \\ s_{i,j} = 1, t_{*,*,j} = 1, u_{i,j} = 1, \bar{u}_{i,j} = 1, \\ b_{i,0} = x_{i,1,*}, \bar{b}_{i,\mu} = x_{i,1,*}, (i=1, \dots, \mu-1), \\ b_{i,M_i} = x_{\mu,1,*}, \bar{b}_{i,M_i} = x_{\mu,1,*}, (i=1, \dots, \mu-1). \end{array} \right),$$

where $r_{i,j} = x_{i,j,n_j} x_{i,k,l}^{\varepsilon} x_{i,j+1,1}^{-\varepsilon} x_{i,k,l}^{-\varepsilon}$, $\varepsilon = \pm 1$, which corresponds to a crossing point as shown in Fig. 2,

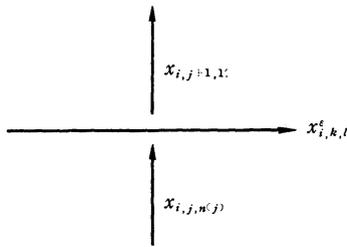


Fig. 2

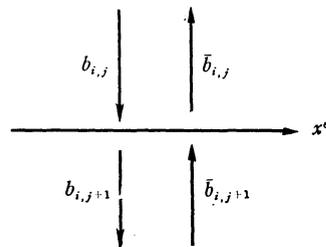


Fig. 3

$$s_{i,j} = x^{\varepsilon} b_{i,j} \bar{b}_{i,j}^{-1} x^{-\varepsilon} \bar{b}_{i,j+1} b_{i,j+1}^{-1}, \varepsilon = \pm 1, \\ x \in \{x_{1,1,1}, \dots, x_{\mu,N_{\mu},n(N_{\mu})}\},$$

which corresponds to crossing points as shown in Fig. 3,

$$t_{*,*,j} = x_{*,*,j}^{\varepsilon} b \bar{b}^{-1} x_{*,*,j+1}^{-\varepsilon} \bar{b} b^{-1}, \varepsilon = \pm 1, \\ (b, \bar{b}) \in \{(b_{0,1}, \bar{b}_{0,1}), \dots, (b_{\mu-1,M_{\mu-1}}, \bar{b}_{\mu-1,M_{\mu-1}})\},$$

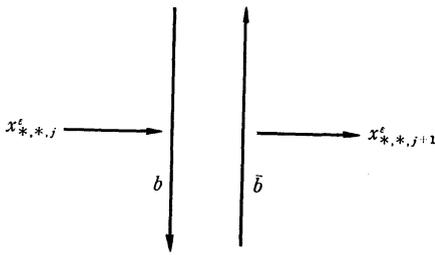


Fig. 4

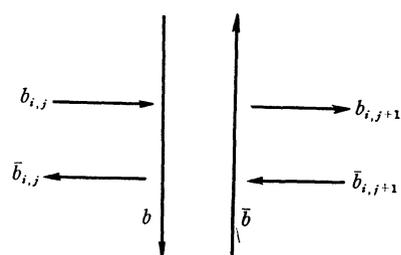


Fig. 5

which corresponds to crossing points as shown in Fig. 4,

$$\begin{aligned}
 u_{i,j} &= b_{i,j} b \bar{b}^{-1} b_{i,j+1}^{-1} \bar{b} b^{-1}, \\
 \bar{u}_{i,j} &= b_{i,j+1} \bar{b} b^{-1} b_{i,j}^{-1} b \bar{b}^{-1}
 \end{aligned}$$

which corresponds to crossing points as shown in Fig. 5.

On the other hand, we have a presentation of the knot group $G(k^*(L)) = \pi_1(S^3 - k^*(L))$ as follows:

$$\left(\begin{array}{l} \text{generators:} \\ x_{i,1}, \dots, x_{i,N_i}, \quad (i = 1, \dots, \mu). \\ \text{relations:} \\ r_{i,1} = \dots = r_{i,N_i} = 1, \quad (i = 1, \dots, \mu), \\ x_{i,1} = x_{\mu,1}, \quad (i = 1, \dots, \mu - 1). \end{array} \right).$$

Let $h: G(k(L)) \rightarrow G(k^*(L))$ be a correspondence defined by

$$\begin{aligned}
 h(x_{i,j,k}) &= x_{i,j}, \quad i = 1, \dots, \mu, \quad j = 1, \dots, N_i, \\
 h(b_{i,j}) &= h(\bar{b}_{i,j}) = x_{\mu,1}, \quad i = 1, \dots, \mu - 1, \quad j = 0, 1, \dots, M_i.
 \end{aligned}$$

Then h is a homomorphism onto. Because $G(k^*(L))$ is not isomorphic to Z , $G(k(L))$ is not isomorphic to Z and then $k(L)$ is not of trivial type.

(1.2) and (1.3) are immediately from [10] and [13].

This completes the proof.

The following is a direct consequence of Theorem 1.1.

Corollary 1.4. *Let $L = (k_1, \dots, k_\mu)$ be a split link with μ components. If a complete-fusion $k(L)$ is of trivial knot type, then L is of trivial link type, i.e. k_1, \dots, k_μ are of trivial knot type.*

REMARK 1.5. The knot group $G(k_1) = \pi_1(S^3 - k_1)$ of k_1 has the following presentation:

$$\left(\begin{array}{l|l} \text{generators:} & \text{relations:} \\ x_{1,1}, \dots, x_{1,N_1} & r_{1,1} = \dots = r_{1,N_1} = 1. \end{array} \right).$$

Let $h': G(k(L)) \rightarrow G(k_1)$ be a correspondence defined by

$$\begin{aligned}
 h'(x_{1,j,k}) &= x_{1,j}, & j &= 1, \dots, N_1, \\
 h'(x_{i,j,k}) &= x_{1,1}, & i &= 2, \dots, \mu, \quad j = 1, \dots, N_i, \\
 h'(b_{i,j}) &= h'(\bar{b}_{i,j}) = x_{1,1}, & i &= 1, \dots, \mu - 1, \quad j = 0, 1, \dots, M_i.
 \end{aligned}$$

Then h' is a homomorphism onto. So we can prove the non-triviality of Theorem 1.1 by using this homomorphism h' .

From this Remark 1.5, we can get the following theorem:

Theorem 1.6. *Let $L=(k_1, k_2, \dots, k_\mu)$ be a link such that k_i is of non-trivial knot type, and k_1 and (k_2, \dots, k_μ) are split. Then any complete-fusion $k(L)$ of L is of non-trivial type.*

Moreover,

$$g(k(L)) \geq g(k_1)$$

and $\Delta_{k(L)}(t)$ contains $\Delta_{k_1}(t)$ as its factor.

In preparation for next section, we will give an useful fashion of $k(L)=k(L; B_1, \dots, B_{\mu-1})$ of $L=(k_1, \dots, k_\mu)$.

In the proof of Theorem 1.1, we assumed that each band B_i connects k_i with k_μ , $i=1, \dots, \mu-1$, at only one arc $x_{\mu-1}$ of a projection of k_μ , (see Fig. 6(a)). Then we can take a band B_μ such that $B_\mu \cap L = \partial B_\mu \cap k_\mu$ and $k_1 + \dots + k_\mu + \partial B_\mu$ is a link, denote $L'=(k_0^0 | k_1, \dots, k_\mu)$, with $\mu+1$ components, (see Fig. 6(b)). Especially, we may assume that $L'-k_0^0$ is the same link type of L and k_0^0 bounds a 2-cell D in S^3-L such that $D \cap (B_1 \cup \dots \cup B_\mu) = \partial D \cap (\partial B_1 \cup \dots \cup \partial B_\mu)$, (see Fig. 6(c)).

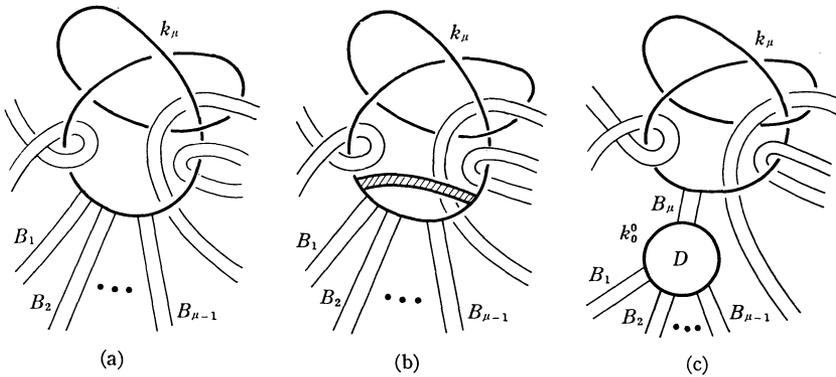


Fig. 6

We summarize the above:

Proposition 1.7. *As a complete-fusion $k(L)$ of $L=(k_1, \dots, k_\mu)$, we consider always the complete-fusion $k(L')=k(L'; B_1, \dots, B_\mu)$ of $L'=(k_0^0 | k_1, \dots, k_\mu)$ such that*

- (i) k_0^0 is the boundary of a 2-cell $D \subset S^3 - L$,
- (ii) B_i connects k_i with k_0^0 , $i = 1, \dots, \mu$,
- (iii) $B_i \cap \mathcal{G}D = \phi$, $i = 1, \dots, \mu$.

2. A standard model of the solid torus T_μ in S^3

The solid torus T_μ of genus μ (Henkelkörper vom Geschlechte μ [9, p. 219]) is a 3-cell with μ solid handles. We now construct a standard model of T_μ in S^3 .

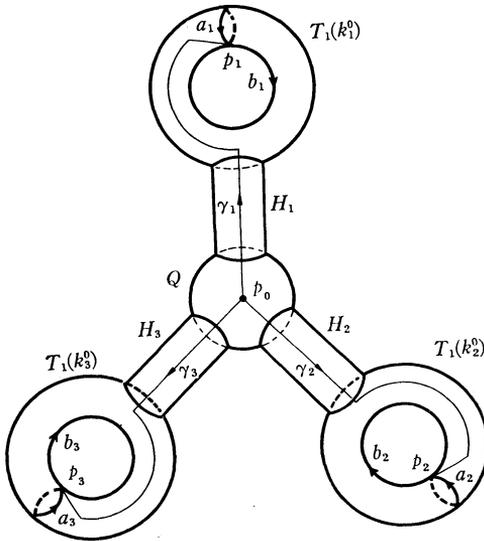


Fig. 7

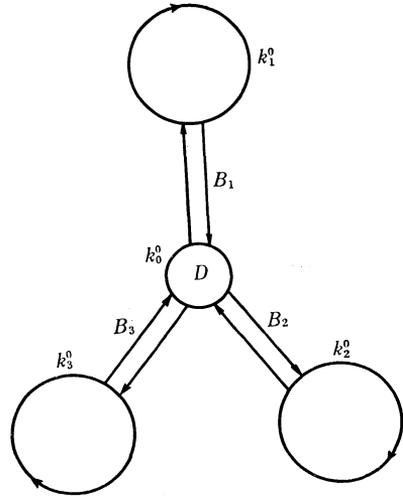


Fig. 8

It is an analogue of that sketched in Fig. 7 for $\mu=3$, and a description of T_μ is as follows.

Let D, D_1, \dots, D_μ be $\mu+1$ mutually disjoint 2-cells in a hyperplane $S_0^2 = R_0^2 \cup \{\infty\}$ of S^3 and let B_1, \dots, B_μ be mutually disjoint narrow bands (=2-cells) in S_0^2 such that B_i connects D with D_i as indicated in Fig. 8.

Let $\partial D_i = k_i^0, i=1, \dots, \mu$. Then we have a representative $L^0 = (k_1^0, \dots, k_\mu^0)$ of the trivial link with μ components in S^3 and the composition $k^*(L^0) = k^*(L^0; B_1, \dots, B_\mu)$. We will call a regular neighborhood $N(K; S^3)$ of the complex $K = D \cup B_1 \cup \dots \cup B_\mu \cup k_1^0 \cup \dots \cup k_\mu^0$ in S^3 a standard model of the solid torus of genus μ and denote this by T_μ . T_μ consists of a 3-cell $Q = N(D; S^3)$, μ solid cylinders $H_i = N(B_i; S^3)$ and μ solid tori (of genus 1) $T_1(k_1^0) = N(k_1^0; S^3), i=1, \dots, \mu$, united as indicated.

We take simple oriented loops $\mathbf{a}_i, \mathbf{b}_i$ and simple oriented arcs γ_i on ∂T_μ as Fig. 7. Let $p_0 = \bigcap_{i=1}^\mu \gamma_i$ and $p_i = \mathbf{a}_i \cap \mathbf{b}_i, i=1, \dots, \mu$. We regard \mathbf{a}_i and \mathbf{b}_i as p_0 -based loops $\gamma_i \mathbf{a}_i \gamma_i^{-1}$ and $\gamma_i \mathbf{b}_i \gamma_i^{-1}$, respectively, in $\pi_1(T_\mu; p_0)$ and $\pi_1(\partial T_\mu; p_0)$.

It is obvious that

(2.1) $\mathbf{b}_1, \dots, \mathbf{b}_\mu$ form a free basis for $\pi_1(T_\mu; p_0)$.

(2.2) $\mathbf{a}_1, \dots, \mathbf{a}_\mu, \mathbf{b}_1, \dots, \mathbf{b}_\mu$ form a basis for $\pi_1(\partial T_\mu; p_0)$, and subject to the single relation³⁾

$$\prod_{i=1}^\mu \mathbf{a}_i \mathbf{b}_i \mathbf{a}_i^{-1} \mathbf{b}_i^{-1} \simeq 1 \text{ on } \partial T_\mu.$$

(2.3) $\mathbf{a}_i \simeq 1$ in $T_\mu, i = 1, \dots, \mu$.

By abelianizing the group we therefore get:

(2.4) $\mathbf{b}_1, \dots, \mathbf{b}_\mu$ form a free abelian basis for $H_1(T_\mu)$; and $\mathbf{a}_1, \dots, \mathbf{a}_\mu, \mathbf{b}_1, \dots, \mathbf{b}_\mu$ form a free abelian basis for $H_1(\partial T_\mu)$.

By the definition of the standard solid torus T_μ , we can think that the composition $k^*(L^0)$ of the trivial link $L^0 = (k_0^0 | k_1^0, \dots, k_\mu^0)$ is contained in T_μ as the core such that k_i^0 is the core of $T_1(k_i^0), i = 1, \dots, \mu$.

Let $L = (k_0^0 | k_1, \dots, k_\mu)$ be a link. Then for any complete-fusion $k(L)$ of L , there exists an embedding φ of T_μ in S^3 such that

- (i) $\varphi(T_\mu) = N(K; S^3)$, a regular neighborhood of the complex $K = D \cup B_1 \cup \dots \cup B_\mu \cup k_1 \cup \dots \cup k_\mu$ in S^3 . We denote $\varphi(T_\mu)$ by $T_\mu(k(L))$.
- (ii) $\varphi(Q) = N(D; S^3), \varphi(H_i) = N(B_i; S^3), \varphi(T_1(k_i^0)) = N(k_i; S^3), i = 1, \dots, \mu$.
- (iii) $\varphi(k_i^0) = k_i, i = 1, \dots, \mu$.
- (iv) $\varphi(\mathbf{a}_i) = m_i, \varphi(\mathbf{b}_i) = l_i, i = 1, \dots, \mu$,
where m_i and l_i are the meridian and longitude respectively, of $\partial N(k_i; S^3)$ as usual meaning.

Note that an embedding φ of T_μ is determined by a complete-fusion $k(L)$ of L and is unique to within twistings of B_1, \dots, B_μ .

3. An associated knot with a torus $T_\mu(k(L))$

In this section we will consider some knots by using the knotted tori $T_\mu(k(L))$ in §2.

DEFINITION 3.1. Let λ be a simple oriented loop (i.e. knot) in $T_\mu \subset S^3$, and let $\varphi: T_\mu \rightarrow T_\mu(k(L)) \subset S^3$ be as before. Then the knot $\varphi(\lambda) \subset S^3$ is called an associated knot with $T_\mu(k(L))$.

Theorem 3.2. Let λ be a knot in $T_\mu \subset S^3$ such that

$$\lambda \sim \beta_1 \mathbf{b}_1 + \beta_2 \mathbf{b}_2 + \dots + \beta_\mu \mathbf{b}_\mu \text{ in } T_\mu.$$

Let $L = (k_1, \dots, k_\mu)$ be a split link. If at least one of components k_1, \dots, k_μ of L , say k_1 , is of non-trivial knot type, then for any complete-fusion $k(L)$ of L the knot $\varphi(\lambda)$

3) \simeq means homotopic to, \sim means homologous to.

$\subset S^3$ associated with $T_\mu(k(L))$ is of non-trivial type provided that $\beta_1 \neq 0$.

Sketch of the Proof. The proof is the same as that of Theorems 1.1 and 1.6 except for obvious modifications. Let $\mathcal{P}k(L)$ be a regular projection of $k(L)$ into a hyperplane $S_0^2 = R_0^2 \cup \{\infty\}$ that is chosen suitably in the sense of knot theory, (see [1] Chapter VI). In particular, we may assume that $\mathcal{P}T_\mu(k(L))$ is a regular neighborhood of $\mathcal{P}k(L)$ in R_0^2 , and that $\mathcal{P}T_\mu(k(L))$ consists of $\sum_{i=1}^\mu n(N_i) + \sum_{i=1}^\mu M_i$ domains $U_{i,j,*}$ and $V_{i,j}$ such that $U_{i,j,*} \supset x_{i,j,*}$ and $V_{i,j} \supset b_{i,j}$ where $x_{i,j,*}$ and $b_{i,j}$ are connected arcs of $\mathcal{P}k(L)$ corresponding to the generators of $G(k(L))$.

Let $y_{i,j,*,k}$ and $c_{i,j,k}$ be generators of $G(\varphi(\lambda))$ corresponding to arcs of $\mathcal{P}\varphi(\lambda)$ such that $y_{i,j,*,k} \subset U_{i,j,*}$ and $c_{i,j,k} \subset V_{i,j}$.

Let $h': G(\varphi(\lambda)) \rightarrow G(k_1)$ be a correspondence defined by

$$\begin{aligned} h'(y_{1,j,*,k}) &= x_{1,j}, \\ h'(y_{i,j,*,k}) &= x_{1,1}, \quad i=2, \dots, \mu, \\ h'(c_{i,j,k}) &= x_{1,1}. \end{aligned}$$

It is easily checked that for every relation s_i having in $U_{i,j,*}$ and $V_{i,j}$, $h'(s_i)$ becomes a trivial relation in $G(k_1)$; hence that h' is a homomorphism onto.

The homomorphism h' of $G(\varphi(\lambda))$ onto $G(k_1)$ enables us to give the following theorem that is a generalization of Theorem 3.2.

Theorem 3.3. *With λ as in 3.2, let $L = (k_1, \dots, k_\mu)$ be a link such that k_1 is of non-trivial knot type, and k_1 and (k_2, \dots, k_μ) are split. Then for any complete-fusion $k(L)$ of L , the knot $\varphi(\lambda) \subset S^3$ associated with $T_\mu(k(L))$ is of non-trivial type provided that $\beta_1 \neq 0$.*

REMARK 3.4. Theorem 1.1 and Theorem 1.6 are special cases of Theorems 3.2 and 3.3, respectively.

Corollary 3.5. (Parallel knots, Cables). *With L and $k(L)$ as in 3.2 or 3.3, and let λ be a knot on $\partial T_\mu \subset S^3$ such that*

$$\lambda \sim \alpha_1 \mathbf{a}_1 + \beta_1 \mathbf{b}_1 + \dots + \alpha_\mu \mathbf{a}_\mu + \beta_\mu \mathbf{b}_\mu \text{ in } \partial T_\mu.$$

Then $\varphi(\lambda)$ associated with $T_\mu(k(L))$ is of non-trivial type provided that $\beta_1 \neq 0$.

REMARK 3.6. Let λ be a knot in T_μ , which may be homotopically trivial in T_μ . Let $\sigma_1 \cup \dots \cup \sigma_n = \lambda \cap T_1(k_1^0)$ and $\tau_1 \cup \dots \cup \tau_n = \lambda \cap (T_\mu - T_1(k_1^0))$, where σ_i and τ_i are simple sub-arcs of λ . We take simple arcs τ'_i in $N(H_1 \cap T_1(k_1^0))$; $T_1(k_1^0)$, a regular neighborhood of $H_1 \cap T_1(k_1^0)$ in $T_1(k_1^0)$, corresponding

to $\tau_i, i=1, \dots, n$, so that $\lambda_1 = \sigma_1 \cup \dots \cup \sigma_n \cup \tau'_1 \cup \dots \cup \tau'_n$ is a knot in $T_1(k_1^0)$. Then, with L and $\varphi: T_\mu \rightarrow T_\mu(k(L))$ as in 3.2 or 3.3, we have an onto homomorphism $f: G(\varphi(\lambda)) \rightarrow G(\varphi(\lambda_1))$ defined by

$$f(x_i) = \begin{cases} x_j & \text{if } x_j \subset (\sigma_1 \cup \dots \cup \sigma_n), \\ \tau'_i & \text{if } x_j \subset \tau_i, i = 1, \dots, n, \end{cases}$$

where x_j and τ'_i are generators of $G(\varphi(\lambda))$ and $G(\varphi(\lambda_1))$ corresponding to arcs of $\mathcal{P}\varphi(\lambda)$ and $\mathcal{P}\varphi(\lambda_1)$. As we have the non-triviality of some kinds of knots associated with a solid torus $T_1(k)$, for example, in [5], [11], [12], we can get the non-triviality of knots $\varphi(\lambda)$ by using this homomorphism f .

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