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PROPOSALS FOR UNKNOTTED SURFACES IN FOUR-SPACES

Dedicated to Professor A. Komatu on his 70th birthday

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In this paper we will propose a concept of unknotted surfaces in the Euclidean 4-space R^4 and discuss primary topics related to it. *Throughout this paper, spaces and maps will be considered in the piecewise-linear category, unless otherwise stated.* One result of this paper is as follows: *A locally flat orientable closed connected surface F in R^4 satisfies that $\pi_1(R^4\text{-}F)$ is an infinite cyclic group if and only if an unknotted surface can be obtained from F by hyperboloidal transformations along trivial l-handles* (See Theorem 2.10.). In other words, *$\pi_1(R^4\text{-}F)$ is infinite cyclic if and only if F is stably unknotted in R^4.* As a corollary of this, *if $\pi_1(R^4\text{-}F)$ is infinite cyclic, then the complement $R^4\text{-}F$ is homotopy equivalent to a bouquet of one 1-sphere, 2n 2-spheres and one 3-sphere, where n is the genus of F.* We will denote by $R^3[t_0]$ the hyperplane of R^4 whose fourth coordinate t is t_0, and for a subspace A of $R^3[0]$, $A[a\leq t\leq b]$ means the subspace $\{(x, t)\in R^4 | (x, 0)\in A, a\leq t\leq b\}$ of R^4. The configuration of a surface in R^4 will be described by adopting the *motion picture method.* (cf. R.H. Fox[4], F. Hosokawa[8], A. Kawauchi-T. Shibuya[13] or S. Suzuki[21].)

1. A concept of unknottedness

We consider a closed, connected and oriented[1] surface F_n of genus $n(n\geq 0)$ in the oriented 4-space R^4. We will assume that F_n is *locally flat* in R^4. Before stating our definition of unknotted surfaces, we note the following known basic fact: *Every surface F_n bounds a compact, connected orinetable 3-manifold in R^4.* (cf. H. Gluck[6], A. Kawauchi-T. Shibuya[13], Chapter II.) We will define an unknotted surface as the boundary of a solid torus in R^4. Precisely.

DEFINITION 1.1 F_n is said to be *unknotted* in R^4, if there exists a solid tours T_n of genus n in R^4 whose boundary ∂T_n is F_n. If such a solid torus does not exist, then F_n is said to be *knotted* in R^4.

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1) A non-orientable version will be described in the final section.

In the case of 2-spheres (i.e., surfaces of genus zero), Definition 1.1 is the usual definition of unknotted 2-spheres in R^4 and it is well-known that any unknotted 2-sphere is ambient isotopic to the boundary of a 3-cell in the hyperplane $R^3[0]$.

The following theorem seems to justify Definition 1.1 for arbitrary unknotted surfaces.

Theorem 1.2. *F_n is unknotted in R^4 if and only if F_n is ambient isotopic to the boundary of a regular neighborhood of an n-leafed rose L_n in $R^3[0]$.*

A 0-leafed rose L_0 in $R^3[0]$ is understood as a point in $R^3[0]$. For $n \geq 1$ and n-leafed rose L_n in $R^3[0]$ is a bouquet of n 1-spheres imbedded in a plane in $R^3[0]$.

For example, the surface F genus one in Fig. 1 is unknotted, since it bounds a solid torus of genus one that is shown in Fig. 2.

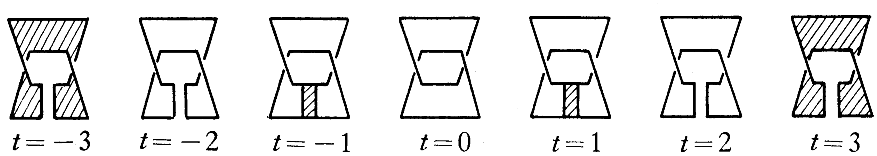


Fig. 1

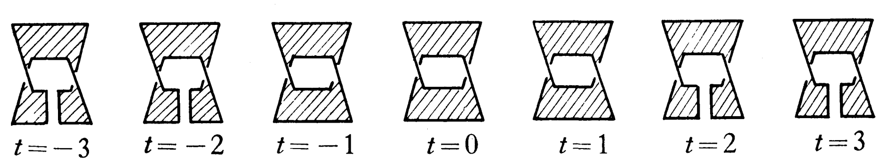


Fig. 2

1.3. Proof of Theorem 1.2. It suffices to prove Theorem 1.2 for the case $n \geq 1$. Assume F_n is unknotted. By definition, F_n bounds a solid torus T_n of genus n. Let a system $\{B_1, \cdots, B_n\}$ be mutually disjoint n 3-cells in T_n, obtained by thickenning a system of meridian disks of T_n, such that $B = cl(T_n - B_1 \cup \cdots \cup B_n)$ is a 3-cell. B is ambient isotopic to a 3-cell in $R^3[0]$; so we assume that B is contained in $R^3[0]$. Let L_n be a bouquet of n 1-spheres in $\mathrm{Int}(T_n)$ at a base point $v \in B$ which is a spine of T_n, i.e., to which T_n collapses. Choose a sufficiently small, compact and connected neighborhood $U(v)$ of v in L_n so that $U(v)$ contains no vertices of L_n except for v. We may consider that $U(v) = L_n \cap B$ and $B[-1 \leq t \leq 1] \cap (L_n - U(v)) = \emptyset$. It is not hard to see that L_n is ambient isotopic to an n-leafed rose in $R^3[0]$ by an ambient isotopy of R^4 keeping $B[-1 \leq t \leq 1]$ fixed. So, we regard L_n as an n-leafed rose in $R^3[0]$. Let $R_0^4 = cl(R^4 - B[-1 \leq t \leq 1])$ and $cl(L_n - U(v)) = l_1 \cup \cdots \cup l_n$, where l_i is a simple arc properly imbedded in B_i, $i = 1, 2, \cdots, n$. Note that $cl(T_n - B) = B_1 \cup \cdots \cup B_n$. We shall show that there exist mutually disjoint regular neighborhoods H_i of l_i in R_0^4 that meet the boundary ∂R_0^4 regularly and such that the pairs $(B_i \subset H_i)$ are proper, i.e., $\partial B_t = (\partial H_t) \cap B_t$. To prove this, triangulate R_0^4 so that $B_1 \cup \cdots \cup B_n$ is a subcomplex of R_0^4 and so that $l_1 \cup \cdots \cup l_n$ is a subcomplex of $B_1 \cup \cdots \cup B_n$. Let X and H' be the barycentric second derived neighborhoods of $l_1 \cup \cdots \cup l_n$ in $B_1 \cup \cdots \cup B_n$ and in R_0^4, respectively. It is easily seen that the pair $(X \subset H')$ is proper. Since $cl(B_1 \cup \cdots \cup B_n - X)$ is homeomorphic to $cl(F_n - \partial B) \times [0,1]$, $B_1 \cup \cdots \cup B_n$ is ambient isotopic to X by an ambient isotopy of R_0^4. Using this ambient isotopy, the desired pair $(B_1 \cup \cdots \cup B_n \subset H_1 \cup \cdots \cup H_n)$ is obtained.

By using the uniqueness theorem of regular neighborhoods, we may assume that $H_i = N(l_i, R_0^3)[-1 \leq t \leq 1], i=1,2,\cdots,n$, where $R_0^3 = cl(R^3[0] - B)$ and $N(l_i, R_0^3)$ is a regular neighborhood of l_t in R_0^3 meeting the boundary ∂R_0^3 regularly. More precisely, we can assume that $(\partial R_0^3) \cap N(l_i, R_0^3) = (\partial B) \cap B_t$.

We need the following lemma:

Lemma 1.4. *Let a* 1*-sphere* S^1 *be contained in a* 2*-sphere* S^2 *and consider a proper surface* Y *in* $S^2 \times [0,1]$, *(absolutely) homeomorphic to* $S^1 \times [0,1]$. *If* $Y \cap S^2 \times 0 = S^1 \times 0$ *and* $Y \cap S^2 \times 1 = S^1 \times 1$[2], *then* Y *is ambient isotopic to* $S^1 \times [0,1]$ *by an ambient isotopy of* $S^2 \times [0,1]$ *keeping* $S^2 \times 0 \cup S^2 \times 1$ *fixed.*

By using Lemma 1.4, $cl(\partial B_t - B)$ is ambient isotopic to $cl(\partial N(l_i, R_0^3) - \partial B)$ by an ambient isotopy of $cl[\partial H_t - (\partial B)[-1 \leq t \leq 1]]$ keeping the boundary fixed. Hence by using a collar neighborhood of $cl[\partial H_t - (\partial B)[-1 \leq t \leq 1]]$ in R_0^4, we obtain that $cl(\partial B_t - \partial B)$ is ambient isotopic to $cl(\partial N(l_t, R_0^3) - \partial B)$ by an ambient isotopy of R_0^4 keeping ∂R_0^4 fixed. This implies that F_n is ambient isotopic to the boundary of a regular neighborhood of L_n in $R^3[0]$. Since the converse is obvious, we complete the proof.

1.5. Proof of Lemma 1.4. Let $D \subset S^2$ be a 2-cell with $\partial D = S^1$. The 2-sphere $Y \cup D \times 0 \cup D \times 1$ bounds the 3-cell E in $S^2 \times [0,1]$, since $S^2 \times [0,1] \subset S^3$. Let $p \in \mathrm{Int}(D)$ and choose a proper simple arc α in E to which E collapses and such that $\alpha \cap S^2 \times 0 = p \times 0$ and $\alpha \cap S^2 \times 1 = p \times 1$. Since there is an ambient isotopy of $S^2 \times [0,1]$ keeping $S^2 \times 0 \cup S^2 \times 1$ fixed and carrying α to $p \times [0,1]$, it follows from the uniqueness theorem of regular neighborhoods that E is ambient isotopic to $D \times [0,1]$ by an ambient isotopy of $S^2 \times [0,1]$ keeping $S^2 \times 0 \cup S^2 \times 1$ fixed. This proves Lemma 1.4.

Corollary 1.6. *For any unknotted surface* F_n *in* R^4, *the bounding solid torus* T_n *is unique up to ambient isotopies of* R^4.

Proof. Let T_n be a solid trous in R^4 with $\partial T_n = F_n$. It suffices to construct an ambient isotopy $\{h_s\}$ of R^4 such that $h_1(T_n)$ is a regular neighborhood of an n-leafed rose in $R^3[0]$. By Theorem 1.2 we can assume that F_n is the boundary of a regular neighborhood of an n-leafed rose in $R^3[0]$. Let $N(F_n)$ be a

2) Here, the equality symbol "=" means "equals with the orientations of ∂Y and $\partial(S^1 \times [0,1])$ associated with some orientations of Y and $S^1 \times [0,1]$".

sufficiently thin regular neighborhood of F_n in $R^3[0]$. Then we may consider that the union of T_n and one component $C(F_n)$ of $N(F_n)-F_n$ is a solid torus T_n'. (Note that $C(F_n)$ is homeomorphic to $F_n\times(0,1]$.) Let T_n'' be a regular neighborhood of an n-leafed rose in $C(F_n)$ such that $cl(T_n'-T_n'')$ is homeomorphic to $F_n\times[0,1]$. Since T_n is ambient isotopic to T_n' and T_n' is ambient isotopic to T_n'', the desired ambient isotopy is obtained. This completes the proof.

One may note that for $n\geq 1$ the bounding solid torus T_n is not unique up to ambient isotopies of R^4 keeping F_n setwise fixed, because, for example, F_n is contained in a 3-sphere S^3 in R^4 so that S^3 is the union of two solid tori with common boundary F_n.

Here is another characterization of unknotted surfaces. (cf. M. Klingmann [14].)

Theorem 1.7. *F_n is ambient isotopic to a surface in $R^3[0]$ if and only if F_n is unknotted in R^4.*

We will give this proof at the last of §2, since it is convenient to use a terminology defined in §2.

2. Hyperboloidal transformations

Let F be a (possibly disconnected) closed and oriented surface in R^4. An oriented 3-cell B in R^4 is said to *span F as a 1-handle,* if $B\cap F=(\partial B)\cap F$ and this intersection is the union of disjoint two 2-cells, and the surface $F\cup\partial B-\mathrm{Int}[(\partial B)\cap F]$ can have an orientation compatible with both the orientations of $F-(\partial B)\cap F$ (induced from F) and $\partial B-(\partial B)\cap F$ (induced from B). Also, an oriented 3-cell B in R^4 *spans F as a 2-handle*, if $B\cap F=(\partial B)\cap F$ and this intersection is homeomorphic to the annulus $S^1\times[0,1]$, and the surface $F\cup\partial B$-Int $[(\partial B)\cap F]$ can have an orientation compatible with both the orientations of $F-(\partial B)\cap F$ and $\partial B-(\partial B)\cap F$.

DEFINITION 2.1. If $B_1,\cdots,B_m$ are mutually disjoint oriented 3-cells in R^4 which span F as 1-handles, then the resulting oriented surface $h^1(F;B_1,\cdots,B_m)=F\cup\partial B_1\cup\cdots\partial B_m\text{-Int}[F\cap(\partial B_1\cup\cdots\cup\partial B_m)]$ with orientation induced from $F-F\cap(B_1\cup\cdots\cup B_m)$ is called *the surface obtained from F be hyperboloidal transformations along 1-handles $B_1,\cdots,B_m$*. Likewise, if $B_1,\cdots,B_m$ span F as 2-handles, the resulting oriented surface $h^2(F;B_1,\cdots,B_m)=F\cup\partial B_1\cup\cdots\cup\partial B_m\text{-Int}[F\cap(\partial B_1\cup\cdots\cup\partial B_m)]$ is called *the surface obtained from F by hyperboloidal transformations along 2-handles $B_1,\cdots,B_m$*.

One may notice that the hyperboloidal transformations along 1-handles and 2-handles, respectively, are dual concepts each other.

We may have the following:

2.2. *For arbitrary integers m and n with* $1 \leq m \leq n$, *if* F_n *is unknotted in* R^4, *then there exist mutaully disjoint m 3-cells* $B_1, \cdots, B_m$ *in* R^4 *which span* F_n *as 2-handles and such that* $h^2(F_n; B_1, \cdots, B_m)$ *is an unknotted surface of genus* $n-m$.

We shall show the following theorem which was partially suggested to the authors by T. Yajima:

Theorem 2.3. *For arbitrary integers m and n with* $1 \leq m \leq n$ *and an unknotted surface* F_n *of genus n in* R^4, *one can find mutually disjoint m 3-cells* $B_1, \cdots, B_m$ *in* R^4 *which span* F_n *as 2-handles and such that* $h^2(F_n; B_1, \cdots, B_m)$ *is a knotted surface of genus* $n-m$. *Further, every knotted surface in* R^4 *is ambient isotopic to a surface* $h^2(F_n; B_1, \cdots, B_m)$ *associated with an unknotted surface* F_n *and certain spanning 2-handles* $B_1, \cdots, B_m$ *for some m and n.*

The proof will be given later.

Combined 2.2 with Theorem 2.3, we conclude that the *knot type*[3] *of the surface* $h^2(F_n; B_1, \cdots, B_m)$ *in* R^4 *depends on the choice of 2-handles* $B_1, \cdots, B_m$, *even if* F_n *is unknotted.* In case F_n is knotted, the same assertion has been obtained by T. Yajima[23]. (See 3.2 later for further topics on this.)

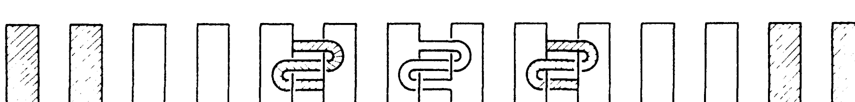
On the other hand, concerning 1-handles, we shall obtain the following:

Theorem 2.4. *Given an unknotted surface* F_n *and mutually disjoint 3-cells* $B_1, \cdots, B_m$ *in* R^4 *which span* F_n *as 1-handles, then the resulting surface* $h^1(F_n; B_1, \cdots, B_m)$ *of genus* $n+m$ *is necessarily unknotted.*

DEFINITION 2.5. A 1-handle B on a surface F in R^4 is said to be *trivial*, if there exists a 4-cell N^4 in R^4 containing B such that $N \cap F = (\partial N) \cap F$ and this intersection is a 2-cell. [Note that the attaching two 2-cells of B to F are contained in the 2-cell $(\partial N) \cap F$, since $(\partial B) \cap F = B \cap F \subset N \cap F = (\partial N) \cap F$.]

From the proof of Theorem 1.2 and trivial observations, one can easily see that $h^1(F; B_1)$ and $h^1(F; B_2)$ belong to the same knot type for arbitrary two trivial 1-handles B_1, B_2 on F in R^4.

REMARK 2.6. In case F_n is a knotted surface, then the knot type of the surface $h_1(F_n; B_1, \cdots, B_m)$ generally depends on the choice of 1-handles $B_1, \cdots, B_m$. For example, let us consider the 2-sphere S illustrated in Fig. 3.



$t=-3$ $t=-2$ $t=-1$ $t=0$ $t=1$ $t=2$ $t=3$

Fig. 3

3) The *knot type* of F in R^4 is the class of imbedded surfaces F' in R^4 such that there exists a homeomorphism $R^4 \to R^4$ sending F onto F' with orientations on R^4 and on F and F' (if F is orientable) preserved.

This 2-sphere S is certainly knotted, since the fundamental group $\pi_1(R^4-S)$ has a presentation $(a, b: aba=bab)$ whose Alexander polynomial is t^2-t+1. [In fact, this 2-sphere has the same knot type as the spun 2-knot of a trefoil.] Let B be a 3-cell that spans S as a 1-handle, as shown in Fig. 4.

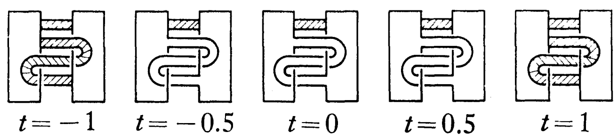


Fig. 4

The surface $F_1=h^1(S; B)$ of genus one is illustrated in Fig. 5.

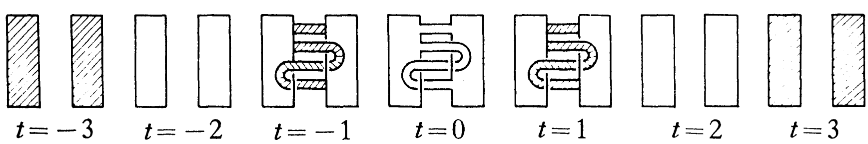


Fig. 5

The fundamental group $\pi_1(R^4-F_1)$ is easily seen to be an infinite cyclic group. [In 2.9 we shall show that this surface F_1 is actually unknotted.] On the other hand, consider a surface F_1' obtained from S by a hyperboloidal transformation along a trivial 1-handle. The fundamental group $\pi_1(R^4-F')$ is isomorphic to the group $\pi_1(R^4-S)$ that is non-abelian. Therefore, the knot types of F_1 and F_1' are distinct.

The following lemma is an important lemma of this paper.

Lemma 2.7. *Consider a surface F in R^4 such that $\pi_1(R^4-F)$ is an infinite cyclic group. Then an arbitrary 1-handle B on F is trivial.*

Proof. Let α be a simple proper arc in B such that the union $F\cup\alpha$ is a spine of the union $F\cup B$. We may assume that $F\cap R^3[0]$ is a link in $R^3[0]$. By sliding α along F and by deforming α itself, we can assume that α is attached to the same component C of the link $F\cap R^3[0]$ and the two attaching points of α to C have compact and connected neighborhoods n^+ and n^- in α which are contained in $R^3[0]$. Let β be one component of C divided by the attaching points of α. Let $\alpha'=cl(\alpha-n^+\cup n^-)$. We join the end points of α' with a simple arc γ such that the loop $\beta\cup n^+\cup n^-\cup\gamma$ bounds a non-singular disk D in $R^3[0]$ with $(D-\beta\cup n^+\cup n^-)\cap(F\cup\alpha)=\emptyset$. We illustrated this situation in Fig. 6.

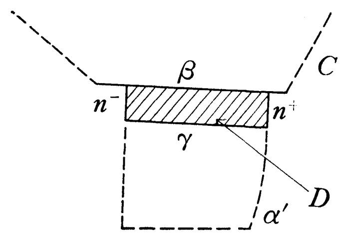


Fig. 6

The simple loop $\gamma \cup \alpha'$ is in general not homologous to zero in R^4-F. However, by twisting γ along C(See for example Fig. 7.), we can assume that the simple loop $\gamma \cup \alpha'$ is homologous to zero in R^4-F.

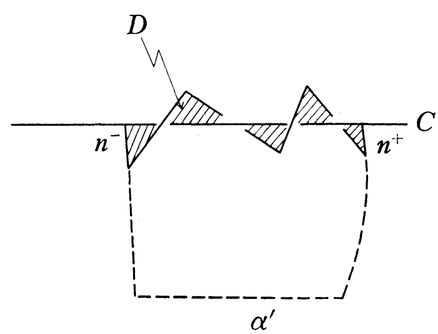


Fig. 7

Since, by the assumption, we have the Hurewicz isomorphism $\pi_1(R^4-F) \approx H_1(R^4-F; Z)$, the simple loop $\gamma \cup \alpha'$ is null-homotopic in R^4-F. Hence by general position and by slight modification, this simple loop can bound a locally flat non-singular 2-cell in R^4-F. Thus, $F \cup \alpha$ is ambient isotopic to F with attaching arc α^0 in the hyperplane $R^3[0]$, as in Fig. 8. Then by using the

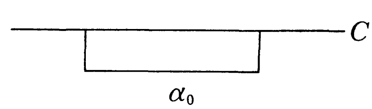


Fig. 8

uniqueness theorem of regular neighborhoods, one can easily find a 4-cell N^4 containing B such that $N \cap F = (\partial N) \cap F$ and this intersection is a 2-cell. That is, B is a trivial 1-handle on F. This completes the proof.

2.8. Proof of Theorem 2.4. For an unknotted surface F_n, $\pi_1(R^4-F_n)$ is an infinite cyclic group. The conclusion follows immediately from Lemma 2.7.

2.9. Proof of Theorem 2.3. We shall show that, for an unknotted surface F_1 of genus one, there exists a 3-cell B_1 in R^4 which spans F_1 as a 2-handle and such that $h^2(F_1; B_1)$ is a knotted 2-sphere with non-abelian fundamental group $\pi_1(R^4 - h^2(F_1; B_1))$. Then for arbitrary m and n with $m \leq n$ it is easy to find mutually disjoint 3-cells $B_1, \cdots, B_m$ which span an unknotted surface F_n as 2-handles and such that $h^2(F_n; B_1, \cdots, B_m)$ is a knotted surface of genus $n-m$ with $\pi_1(R^4 - h^2(F_n; B_1, \cdots, B_m))$ isomorphic to the non-abelian group $\pi_1(R^4 - h^2(F_1; B_1))$. Consider, for example, the surface F_1 in Fig. 5. This surface is actually unknotted. In fact, let $\bar{B}$ be the 3-cell which spans F_1 as a 2-handle, illustrated in Fig. 9. The resulting 2-sphere $S_0 = h^2(F_1; \bar{B})$ is clearly unknotted.

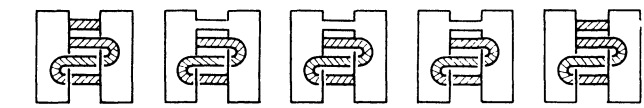


Fig. 9

Then Theorem 2.4 shows that the surface $F_1 = h^1(S_0; \bar{B})$ is unknotted. Consider the 3-cell B in Fig. 4 that spans F_1 as a 2-handle. The resulting 2-sphere $h^2(F_1; B)$ is a knotted 2-sphere with non-abelian fundamental group $\pi_1(R^4 - h^2(F_1; B))$, because $h^2(F_1; B)$ is S in Fig. 3. Secondly, we shall show that any knotted surface F in R^4 is ambient isotopic to a surface $h^2(F_n; B_1, \cdots, B_m)$ associated with an unknotted surface F_n and some spanning 2-handles $B_1, \cdots, B_m$. Consider a compact, connected orientable 3-manifold M in R^4 with $\partial M = F$. We can find mutually disjoint 3-cells $B_1, \cdots, B_m$ in M which span F as 1-handles and such that $T = cl(M - B_1 \cup \cdots \cup B_m)$ is a solid torus with some genus. [In fact, take a 2-complex K that is a spine of M and let $K^{(1)}$ be the 1-skelton of K. Take the regular neighborhood $T' = N(K^{(1)}, M)$ of $K^{(1)}$ in M. We may assume that $cl(K - T')$ consists of m 2-cells $\Delta_1, \Delta_2, \cdots, \Delta_m$ for some m. For each i, let B_i' be a 3-cell thickenning Δ_i in $cl(M - T')$. The union $M' = T' \cup B_1' \cup \cdots \cup B_m'$ is a regular neighborhood of K in M. Using the uniqueness theorem of regular neighborhoods, we obtain that M' is homeomorphic to M. Divide M into a solid torus T and m 3-cells $B_1, \cdots, B_m$ corresponding to T' and $B_1', \cdots, B_m'$ respectively, by utilizing the homeomorphism $M' \to M$. The desired T and $B_1, \cdots, B_m$ are thus obtained.] Let $F_n = \partial T$, where n is the genus of T. By definition, F_n is unknotted. From construction, we have $F = h^2(F_n; B_1, \cdots, B_m)$. This completes the proof.

Theorem 2.10. *A surface F in R^4 satisfies that $\pi_1(R^4 - F)$ is an infinite cyclic group if and only if an unknotted surface can be obtained from F by hyperboloidal transformations along trivial* 1*-handles.*

Proof. The hyperboloidal transformation along a trivial 1-handle does not alter the fundamental groups of the complements of surfaces in R^4. Hence if one produce an unknotted surface from F by hyperboloidal transformations along trivial 1-handles, then we obtain that $\pi_1(R^4 - F)$ is an infinite cyclic group.

Conversely, assume that $\pi_1(R^4-F)$ is an infinite cyclic group. By Theorem 2.3, there are 1-handles $B_1, \cdots, B_m$ on F such that $h^1(F; B_1, \cdots, B_m)$ is unknotted in R^4. But by Lemma 2.7 these 1-handles $B_1, \cdots, B_m$ are all trivial, since $\pi_1(R^4-F)$ is an infinite cyclic group. This completes the proof.

As a corollary of Theorem 2.10, we obtain the following:

Corollary 2.11. *The complement R^4-F_n is homotopy equivalent to a bouquet of one 1-sphere, $2n$ 2-spheres and one 3-sphere for an arbitrary surface F_n of genus n (≥ 0) in R^4 such that $\pi_1(R^4-F_n)$ is an infinite cyclic group.*

Proof. Let $\pi_1(R^4-F_n)$ be an infinite cyclic group. By Theorem 2.10 there are trivial 1-handles $B_1^0, \cdots, B_m^0$ on F_n such that $F_{n+m}=h^1(F_n; B_1^0, \cdots, B_m^0)$ is unknotted in R^4. It is convenient to consider that the surfaces F_n and F_{n+m} are centained in the 4-sphere $R^4 \cup \{\infty\}=S^4$. Identify $\pi_1(S^4-F_{n+m})$ with the infinite cyclic group I. It is easily calculated that $H_2(\widetilde{S^4-F_{n+m}}; Z) \approx \oplus Z[I]^{2(n+m)} \approx H_2(\widetilde{S^4-F_n}; Z) \oplus Z[I]^{2m}$ by using the Mayer-Vietoris sequence, where$\sim$denotes the universal cover, which is obviously an infinite cyclic cover and $Z[I]$ denotes the integral group ring of I. By a result of D. Quillen[19], $H_2(\widetilde{S^4-F_n}; Z)$ is a free $Z[I]$-module of rank n.[D. Quillen showed precisely that a finitely generated projective module over a polynomial ring with coefficients in a principal ideal domain is free. Our variant is easily follows from his argument. See R.G. Swan [24].] Next, we shall show that $H_3(\widetilde{S^4-F_n}; Z)=0$. Let M^4 be the manifold obtained from S^4 by removing the interior of a regular neighborhood of F in S^4. Since $H_3(M; Q)=0$, it follows that $H_3(\tilde{M}; Q)$ is finitely generated over Q. Using $H_4(\tilde{M}; Z)=0$, from the partial Poincaré duality[10], Theorem 2.3, Case(5) we obtain a duality $H^3(\tilde{M}; Q) \approx H_0(\tilde{M}, \partial\tilde{M}; Q)$. $\partial\tilde{M}$ is connected, for the homomorphism $H_1(\partial M; Z) \to H_1(M; Z)$ induced by inclusion is onto. Hence $H_3(\tilde{M}; Q) = H_0(\tilde{M}, \partial\tilde{M}; Q)=0$. But $H_3(\tilde{M}; Z)$ is a torsion-free abelian group. Therefore $H_3(\widetilde{S^4-F_n}; Z)=H_3(\tilde{M}; Z)=0$. Let $f_1, f_2, \cdots, f_{2n}: (S^2, p) \to (S^4-F_n, x_0)$ be maps representing a $Z[I]$-basis for $\pi_2(S^4-F_n, x_0)=H_2(\widetilde{S^4-F_n}; Z)$ and let $f: (S^1, p) \to (S^4-F_n, x_0)$ be a map representing a generator of $\pi_1(S^4-F_n, x_0)$. The one-point-union map $f \vee f_1 \vee \cdots \vee f_{2n}: (S^1 \vee S_1^2 \vee \cdots \vee S_{2n}^2, p) \to (S^4-F_n, x_0)$ clearly gives a homotopy equivalence. Therefore, $R^4-F_n=S^4-F_n \cup \{\infty\}$ is homotopy equivalent to a bouquet $S^1 \vee S_1^2 \vee \cdots \vee S_{2n}^2 \vee S^3$. This completes the proof.

2.12. Proof of Theorem 1.7. It sufficies to prove that if $F_n \subset R^3$, then there exists a solid torus T_n of genus n in R^4 with $\partial T_n=F_n$, since the converse follows from Theorem 1.2. By a result of R.H. Fox[5] or S. Suzuki[20], Proposition 1.3, $F_n(\subset R^3[0])$ can be obtained from the union $\tilde{S}=S_1 \cup \cdots \cup S_s$ of mutually disjoint 2-spheres S_j in $R^3[0]$ by performing one by one hyperboloidal transformations along 1-handles $B_1, \cdots, B_{n+s-1}$ in $R^3[0]$. Push one by one these 1-handles $B_{n+s-1}, \cdots, B_1$ into $R^3[0\leq t<+\infty)$ so that the resulting 1-handles $B'_{n+s-1}, \cdots, B'_1$ are mutually disjoint and for each i, $\tilde{S}\cap B'_i$ consists of the attaching two 2-cells of B'_i to $\tilde{S}$ and for each u with $0\leq u\leq 1$ $B'_i\cap R^3[u]=(\tilde{S}\cap B'_i)[t=u]$. By changing the index j of S_j, if necessary, we may assume that for each j, $j=1,2,\cdots,s$, the 2-sphere S_j is innermost in the 2-spheres $S_1,\cdots,S_j$. Let $0=t_0<t_1<\cdots<t_s=1$ and $\tilde{B}'=B'_1\cup\cdots\cup B'_{n+s-1}$. Remove for each j the part $(S_j\cap\tilde{B}')[0\leq t\leq t_j]\cup S_j$ from $\tilde{B}'\cup\tilde{S}$ and then replace it by $S_j[t=t_j]$. Let $S'_j=S_j[t=t_j]$ and $\tilde{S}'=S'_1\cup\cdots\cup S'_s$. Denote by B''_i the 3-cell attaching to $\tilde{S}'$ as a 1-handle that is obtained from B'_i by this subtraction. Let $\tilde{B}''=B''_1\cup\cdots\cup B''_{n+s-1}$. Take the 3-cell E_j in $R^3[t_j]$ bounded by S'_j and let $\tilde{E}=E_1\cup\cdots\cup E_s$. From construction the union $\tilde{E}\cup\tilde{B}''$ is a solid torus of genus n. Since the deformation of F_n into $h^1(\tilde{S}'; B''_1, \cdots, B''_{n+s-1})$ is certainly realized by an ambient isotopy of R^4 and the surface $h^1(\tilde{S}'; B''_1,\cdots, B''_{n+s-1})$ bounds the solid torus $\tilde{E}\cup\tilde{B}''$, the original surface F_n bounds a solid torus T_n. This completes the proof.

3. Further topics and related problems

3.1. Unknotting problems. The unknotting problem asks *whether a surface F in R^4 with the infinite cyclic fundamental group $\pi_1(R^4-F)$ is necessarily unknotted.* [Notice that if $\pi_1(R^4-F)$ is infinite cyclic, then the homotopy type of R^4-F is completely determined by Corollary 2.11.] A somewhat special problem of this is as follows: *Is a surface F_n of genus n in R^4 unknotted, if F_n has $2n+2$ critical points associated with parallel hyperplanes $R^3[t]$, $-\infty<t<+\infty$?* Note that $2n+2$ is the least number of critical points which F_n can admit by the Morse's inequality. Further, note that $\pi_1(R^4-F_n)$ is certainly infinite cyclic, since F_n has just one maximal point and one minimal point. [Apply the van Kampen theorem for, for example, a normal form of F_n in A. Kawauchi-T. Shibuya[13].] This problem in the case $n=1$ corresponds to Problem 4.30 of R. Kirby[15]. *A trivial m-link* of surfaces is the union of m connected surfaces which is the boundary of the union of mutually disjoint m solid tori in R^4. Then one can find mutually disjoint m 4-cells each of which contains one of these m solid tori. For disconnected surfaces, the corresponding problem on the least critical points is in general false. For example, consider the 2-link F of a surface of genus one and a 2-sphere illustrated in Fig. 10, using critical bands instead of critical points.

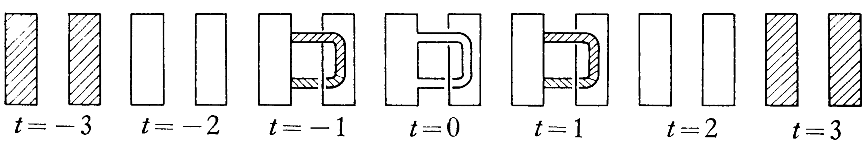


Fig. 10

The corresponding problem asks whether this 2-link F with $4+2=6$ critical bands is trivial. In fact, this 2-link F is non-trivial, since $\pi_1(R^4-F)$ is not a free group, but a free abelian group. However, we can notice that *an m-link L^m of 2-spheres in R^4 is trivial, if L^m has 2m critical points.* [To see this, first modify L^m so that L^m has only critical bands (See[13].) and then deform L^m such that all of the maximal bands of L^m are in the level $R^3[1]$ and all of the minimal bands of L^m are in the level $R^3[0]$. By using the isotopy extension theorem, we can assume that $L^m \cap R^3[0]=D_1 \cup \cdots \cup D_m$, the union of mutually disjoint 2-cells and for each s, $0<s<1$, $L^m \cap R^3[s]=[\partial D_1 \cup \partial D_2 \cup \cdots \cup \partial D_m]\,[t=s]$ and $L^m \cap R^3[1]$ is the union of mutually disjoint m 2-cells bounded by the link $[\partial D_1 \cup \cdots \cup \partial D_m]\,[t=1]$. (See A. Kawauchi-T. Shibuya [13] sublemma 2.8.1) Then the Horibe and Yanagawa's lemma in [13] assures that the replacement of 2-cells of $L^m \cap R^3[1]$ by new ones in $R^3[1]$ deos not alter the knot type of L^m. Hence L^m belongs to the knot type of the boundary of $[D_1 \cup \cdots \cup D_m]\,[0 \leq t \leq 1]$. That is, L^m is trivial (See, also, S. Suzuki [21], Lemma 5.5 for a quick proof of this assertion.)]

Another approach of the unknotting problem is to know *when the surface obtained from a trivial link of surfaces by hyperboloidal transformations is unknotted.* The problem on 1-handles asks *whether the (connected) surface F obtained from a trivial m-link of surfaces by hyperboloidal transformations along m-1 1-handles is unknotted if $\pi_1(R^4-F)$ is infinite cyclic.* In the case $m=2$ this is affirmative. The proof is essentially parallel to Y. Marumoto's proof which shows a special case that the 2-sphere S obtained from a trivial 2-link of 2-spheres by a hyperboloidal transformation along a 1-handle is unknotted if $\pi_1(R^4-S)$ is infinite cyclic (See [16].) and omitted. As a consequence, a somewhat weaker assertion of the main theorem in F. Hosokawa[8][4] follows. That is, the 2-sphere S with one minimal point and one saddle point and two maximal points is equivalent[5] to an unknotted 2-sphere by an auto-homeomorphism of R^4 with the standard piecewise-linear structure of R^4 destroyed at a finite number of points. [The proof is mainly due to S. Suzuki. Note that the knot sum $\bar{S}$ of the 2-sphere S and the reflected inverse of S is unknotted, since it is the 2-sphere obtained from a trivial 2-link of 2-spheres by a hyperboloidal transformation along a 1-handle and $\pi_1(R^4-\bar{S})$ is an infinite cyclic group. Then by the inverse theorem of B. Mazur[18], S is equivalent to an unknotted 2-sphere by a desired homeomorphism.] The problem on 2-handles asks *whether for an unknotted surface F_n of genus n and a 2-handle B on F_n, $h^2(F_n; B)$ is unknotted if $h^2(F_n; B)$ is a surface of genus n-1 and $\pi_1(R^4-h^2(F_n; B))$ is infinite cyclic.* It seems that this problem is difficult even in the simplest case $n=1$.

3.2. Knotted surfaces and 2-handles. Our first problem was *whether therc*

4) The proof of Lemma 2 in [8] contains a gap and hence the main theorem of [8] remains open.

5) B. Mazur [18] called it "*-equivalent".

is a connected surface F_n of genus $n\geq 1$ such that there is no 2-handle B on F_n satisfying that $h^2(F_n;B)$ is a connected surface of genus $n-1$. Certainly, for each $n\geq 1$, infinitely many such examples of surfaces of genus n exist. In fact, K. Asano [1] constructs infinitely many examples of surfaces F_n in R^4 such that a simple closed curve α in F_n which is null-homotopic in $(R^4-F_n)\cup\alpha$ is necessarily null-homologous in F_n. Let F_n be a connected surface of genus n such that there is a 2-handle B on F_n satisfying that $h^2(F_n;B)$ is a connected surface of genus $n-1$. Our second problem is *whether one can necessarily find a 2-handle B' on F_n such that $\pi_1(R^4-h^2(F_n;B'))$ is isomorphic to $\pi_1(R^4-F_n)$.* For $n=1$ there is a counter-example to this. The surface F_1 of genus one illustrated in Fig. 11 is such a counter-example.

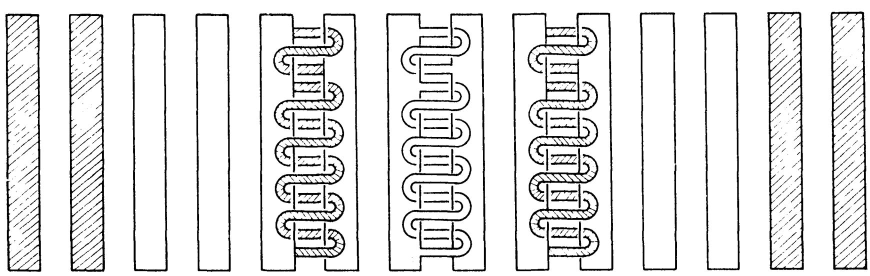


Fig. 11

In fact, it is easy to obtain a 2-handle B on the surface F_1 such that $h^2(F_1;B)$ is a knotted 2-sphere. However, for any 2-handle B' on F_1, $\pi_1(R^4-h^2(F_1;B'))$ is never isomorphic to $\pi_1(R^4-F_1)$, because the presentation of $\pi=\pi_1(R^4-F_1)$ is $(a, b\,|\,ab=ba^2, ba^5=a^5b)$[6], which cannot be the group of a knotted 2-sphere in R^4.[To see this, consider the abelianized commutator subgroup π'/π'' of $\pi=\pi_1(R^4-F)$. Let π/π' be identified with the infinite cyclic group $\langle t\rangle$ with a specified generator t. By sending b to t, π'/π'' is isomorphic to $Z_5\langle t\rangle/(2t-1)$ as $Z\langle t\rangle$-modules. Suppose π is the group of a knotted 2-sphere S in S^4, i.e., $\pi\approx\pi_1(M)$ with $M=cl(S^4-N(S))$ for the regular neighborhood $N(S)$ of S in S^4. We have $H_1(\tilde{M};Z)=Z_5\langle t\rangle/(2t-1)$ for the infinite cyclic connected cover $\tilde{M}$ of M with covering translation group $\langle t\rangle$. Note that $2t$-1 is the characteristic polynomial of $t_*: H_1(\tilde{M};Z_5)\to H_1(\tilde{M};Z_5)$. Since $H^1(\tilde{M};Z_5)=\mathrm{Hom}_{Z_5}[H_1(\tilde{M};Z_5), Z_5]$, it follows that $2t-1$ is the characteristic polynomial of $t^*: H^1(\tilde{M};Z_5)\to H^1(\tilde{M};Z_5)$. Using the duality $\cap\mu: H^1(\tilde{M};Z_5)\approx H_2(\tilde{M},\partial\tilde{M};Z_5)$ (See[10].) with equality $(t^*u)\cap\mu=t_*^{-1}(u\cap\mu)$ for $u\in H^1(\tilde{M};Z_5)$ and the natural isomorphism $H_2(\tilde{M};Z_5)\approx H_2(\tilde{M},\partial\tilde{M};Z_5)$ we obtain that the characteristic polynomal of $t_*: H_2(\tilde{M};Z_5)\to H_2(\tilde{M};Z_5)$ is $t-2$. Note that $H_2(\tilde{M};Z)=0$ because of the

6) The group π with this presentation is the group of a knotted 3-sphere in R^5. (See A. Kawauchi [11] or S. Suzuki [21].)

duality $0=H^1(\tilde{M}; Z)\approx H_2(\tilde{M}, \partial\tilde{M}; Z)$ and the boundary isomorphism $\partial: H_3(\tilde{M}, \partial\tilde{M}; Z)\approx H_2(\partial\tilde{M}; Z)$. Thus, from the universal coefficeint theorem $H_2(\tilde{M}; Z_5)$ is identical with a subgroup $\tau_5(H_1(\tilde{M}; Z))$ of $H_1(\tilde{M}; Z)$ consisting of all elements x in $H_1(\tilde{M}; Z)$ with $5x=0$. Since there is a natural isomorphism $\tau_5(H_1(\tilde{M}; Z))\otimes Z_5\simeq H_1(\tilde{M}; Z_5)$, $t-2$ is the characteristic polynomial of $t_*: H_1(\tilde{M}; Z_5)\to H_1(\tilde{M}; Z_5)$. This implies that $2t-1$ and $t-2$ are equal up to units of Z_5, which is impossible. Therefore, π is not the group of a 2-sphere in S^4. (cf. [9] and M.A. Gutierrez[7].)]

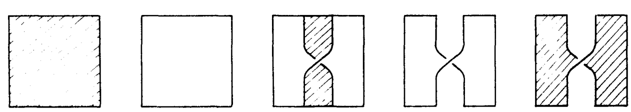
3.3. The non-fibered property of surface exteriors. We show *that for any surface F_n of genus $n\geq 1$ in S^4, S^4-F_n cannot be fibered over a circle. Let $M_n=cl(S^4-N(F_n))$* for a regular neighborhood $N(F_n)$ of F_n in S^4. If S^4-F_n and hence M_n is fibered over a circle, then the infinite cyclic connected cover $\tilde{M}_n$ of M_n can be written as the Cartesian product of a compact connected 3-manifold N and the real line R^1, since we work in the piecewise-linear category. In particular, $H_*(\tilde{M}_n; Q)\approx H_*(N\times R^1; Q)$ is finitely generated over Q. However, we now show *that $H_2(\tilde{M}_n; Q)$ has the rank $2n$ as a $Q\langle t\rangle$-module, where $Q\langle t\rangle$ is the rational group ring of the covering translation group $\langle t\rangle$ of $\tilde{M}_n$. Thus, $H_2(\tilde{M}_n; Q)$* is infinitely generated over Q. Therefore, for $n\geq 1$ M_n and hence S^4-F_n cannot be fibered over a circle. To show that $\mathrm{rank}_{Q\langle t\rangle}H_2(\tilde{M}_n; Q)=2n$, consider the following part of the Wang exact sequence $H_2(\tilde{M}_n; Q)\xrightarrow{t-1}H_2(\tilde{M}_n; Q)\xrightarrow{p_*}H_2(M_n; Q)=\oplus Q^{2n}$, where $p: \tilde{M}_n\to M_n$ is the covering projection. Since $H_1(M_n; Q)=Q$, it follows that $t-1: H_1(\tilde{M}_n; Q)\approx H_1(\tilde{M}_n; Q)$ and hence $p_*: H_2(\tilde{M}_n; Q)\to H_2(\tilde{M}_n; Q)$ is onto. Write $H_2(\tilde{M}_n; Q)\approx\oplus Q\langle t\rangle^m\oplus T$, where T is the $Q\langle t\rangle$-torsion part of $H_2(\tilde{M}_n; Q)$. [Note that $Q\langle t\rangle$ is a principal ideal domain.] Since $H_1(M_n, \partial M_n; Q)=0$, it follows that $H_1(\tilde{M}_n, \partial\tilde{M}_n; Q)$ is a finitely generated $Q\langle t\rangle$-torsion module and $t-1: H_1(\tilde{M}_n, \partial\tilde{M}_n; Q)\approx H_1(\tilde{M}_n, \partial\tilde{M}_n; Q)$. Consider a cyclic decomposition $Q\langle t\rangle/(f_1(t))\oplus\cdots\oplus Q\langle t\rangle/(f_r(t))$ of $H_1(\tilde{M}_n, \partial\tilde{M}_n; Q)$. According to Duality Theorem (II) of A. Kawauchi[12] (See also, R.C. Blanchfield[3].), T is $Q\langle t\rangle$-isomorphic to $Q\langle t\rangle/(f_1(t^{-1}))\oplus\cdots\oplus Q\langle t\rangle/(f_r(t^{-1}))$ and hence $t-1: T\to T$ is a $Q\langle t\rangle$-isomorphism. Therefore we have the following exact sequence:

$$\begin{array}{ccccc} H_2(\tilde{M}_n; Q)/T & \xrightarrow{t-1} & H_2(\tilde{M}_n; Q)/T & \xrightarrow{p_*} & Q^{2n} \to 0\,. \\ \| & & \| & & \\ \oplus Q\langle t\rangle^m & & \oplus Q\langle t\rangle^m & & \end{array}$$

From this we have that $m=2n$, as desired.

3.4. *The asphericity problem.* The asphericity problem asks *whether there is a knotted surface F_n of genus $n\geq 1$ in S^4 such that S^4-F_n is aspherical.*

3.5. *Non-orientable version.* The case of non-orientable surfaces becomes somewhat complicated in comparison with the case of orientable surfaces. For simplicity, we will only treat of a locally flat, connected non-orientable surface F in the oriented 4-space R^4. According to H. Whitney[22], the Euler number $e(F)$ of the disk bundle over F associated with a regular neighborhood of F in R^4 is the invariant of the knot type of $F \subset R^4$. The possible value of $e(F)$ is $2\chi+4$, 2χ, $2\chi+4, \cdots, 4-2\chi$ (See W.S. Massey[17].), where χ is the Euler characteristic of F. Consider the projective plane P illustrated in Fig. 12. We have $e(P)=+2$.



$t=-2$ $t=-1$ $t=0$ $t=1$ $t=2$

Fig. 12

We choose and fix the orientation of the containing 4-space R^4 so that $e(P)=+2$ and denote this P by P_+. Let P_- be the projective plane obtained by the reflection of P_+ on the fourth axis of R^4. We have $e(P_-)=-2$. Since $e(F)=e(F_1)+e(F_2)$ for the knot sum F of non-orientable surfaces F_1, F_2 in R^4 (See W.S. Massey [17].), it follows that the possible value of $e(F)$ can be realized by the knot sum of some copies of P_+ and P_-. Let $F_{i,j}$ denote the knot sum of $i(\geq 0)$ copies of P_+ and $j(\geq 0)$ copies of P_- with $i+j\geq 1$. Note that $e(F_{i,j})=2i-2j$ and $i+j$ is the non-orientable genus of $F_{i,j}$, i.e., the Z_2-rank of $H_1(F_{i,j}; Z_2)$.

DEFINITION 3.5.1. A non-orientable surface F in R^4 is *unknotted*, if F belongs to the knot type of $F_{i,j}$ for some i and j.

It is easy to see that the knot type of an unknotted surface accompanied with the non-orientable genus and the Euler number is unique and that $\pi_1(R^4-K_{i,j})=Z_2$ for all i, j. This also implies that the knot type of $F\subset R^4$ does not determined uniquely by the fundamental group $\pi_1(R^4-F)$ alone. This solves, in a sense, Preblem 30 of R.H. Fox[4] by considering the case $i+j=1$. Now we consider a surface F in R^4 such that the Euler number $e(F)$ is 0. By an analogous method of H. Gluck[6], K. Asano[2] showed that $e(F)=0$ if and only if F bounds a compact 3-manifold in R^4.

As an analogh of Theorem 1.2, we have the following:

3.5.2. *A surface F in R^4 is the boundary of a solid Klein bottle* (i.e., *the disk sum of some copies of $S^1 \underset{\tau}{\times} B^2$*) *in R^4 if and only if F is unknotted with $e(F)=0$.*

We note that the concepts of hyperboloidal transformations along 1-handles and 2-handles are defined as an analogy of the orientable case. Consider a non-orientable surface F in R^4 with $e(F)=0$. F bounds a compact 3-manifold in R^4. Then there exist 1-handles $B_1, \cdots, B_m$ on F such that the surface F_0 obtained from F by hyperboloidal transformations along these 1-handles $B_1, \cdots, B_m$

bounds a solid Klein bottle in R^4(cf. 2.9.). By 3.5.2, this surface F_0 is unknotted with $e(F_0)=0$. Further, suppose $\pi_1(R^4-F)=Z_2$. Then these 1-handles $B_1, \cdots, B_m$ are all trivial by an analogy of the proof of Lemma 2.7. Since for an arbitrary non-orientable surface F in R^4 the knot sum F' of F and $F_{i,j}$ for some i, j satisfies $e(F')=0$, we have the following:

3.5.3. *A non-orientable surface F in R^4 has the fundamental group $\pi_1(R^4-F) \approx Z_2$ and the Euler number e if and only if the knot sum of F and $F_{i,i}$ for some i is unknotted with Euler number e.*

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