



Title	On Boardman's generating sets of the unoriented bordism ring
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Citation	Osaka Journal of Mathematics. 1971, 8(2), p. 219-232
Version Type	VoR
URL	https://doi.org/10.18910/6176
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ON BOARDMAN'S GENERATING SETS OF THE UNORIENTED BORDISM RING

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(Received September 24, 1970)

Introduction

For a pointed finite CW pair (X, A) , define as usual the k -dimensional un-oriented cobordism group $\mathfrak{R}^k(X, A)$ of (X, A) by

$$\mathfrak{R}^k(X, A) = \varinjlim_n [S^{n-k}(X/A), MO(n)],$$

and denote $\sum_{-\infty < k < \infty} \mathfrak{R}^k(X, A)$ by $\mathfrak{R}^*(X, A)$.

We identify the coefficient ring \mathfrak{R}^* with the unoriented bordism ring \mathfrak{R}_* by the Atiyah-Poincaré duality [2]

$$D: \mathfrak{R}_k \rightarrow \mathfrak{R}^{-k}.$$

Let P_n be the n -dimensional real projective space and η_n be the canonical line bundle over P_n . Define

$$\mathfrak{R}^*(BO(1)) = \varprojlim_n \mathfrak{R}^*(P_n) \cong \mathfrak{R}_*[[W_1]],$$

where $W_1 = \varprojlim_n W_1(\eta_n)$ is the cobordism first Stiefel-Whitney class [4]. On account of the Kunneth formula, the homomorphism

$$\mu_{m,n}^*: \mathfrak{R}^*(P_{m+n}) \rightarrow \mathfrak{R}^*(P_m \times P_n)$$

induced by a continuous map $\mu_{m,n}$ satisfying $\mu_{m,n}^* \gamma_{m+n} \cong \pi_1^* \gamma_m \otimes \pi_2^* \gamma_n$ gives rise to the comultiplication

$$\mu^*: \mathfrak{R}^*(BO(1)) \rightarrow \mathfrak{R}^*(BO(1)) \otimes_{\mathfrak{R}_*} \mathfrak{R}^*(BO(1)).$$

Let

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_6 W_1^7 + z_8 W_1^9 + \cdots \quad (z_i \in \mathfrak{R}_i)$$

be a primitive element in $\mathfrak{R}^*(BO(1))$ with respect to this comultiplication. Such

elements exist ([3]). Fix once and for all a primitive element P of such kind.

Following Novikov [8, appendix II], we define in section 1 a cobordism stable operation Φ_P which is a multiplicative projection characterised by the formula

$$\Phi_P(W_1) = P.$$

The restriction of the natural transformation

$$\mu | \text{Image } \Phi_P : \text{Image } \Phi_P \rightarrow H^*(X, A; Z_2)$$

is a natural ring isomorphism in the category of finite CW pairs. And this induces a natural \mathfrak{R}_* -algebra isomorphism

$$\mathfrak{R}^*(X, A) \cong \mathfrak{R}_* \hat{\otimes} H^*(X, A; Z_2).$$

Conversely, any such natural isomorphism, commuting with suspensions, is induced by Φ_P for some choice of a primitive element P .

In section 2, we study the relation between the operations S_ω and \bar{S}_ω defined in [8]. The result is applied in section 3 to prove that the coefficient z_{2k} of a primitive element P is the bordism class $[P_{2k}]$ of the real projective space for each $k \geq 0$.

And the coefficient z_{4k+1} is shown to be the class $[P(1, 2k)]$ of Dold manifold [5] in section 4.

The coefficients z_i of dimensions i other than $2k$ and $4k+1$ are expressed as very complicated polynomials in the generators of Dold [5] or of Milnor [7].

The present paper is motivated by the following classification theorem stated in the proof of Theorem 8.1 in [3].

Theorem. P. (Boardman [3])

For an arbitrary family of decomposable elements $\{y_{2^i-1}; y_{2^i-1} \in \mathfrak{R}_{2^i-1}, i \geq 1\}$, there exists one and the only one primitive element

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots.$$

in $\mathfrak{R}^*(BO(1))$, satisfying

$$z_{2^i-1} = y_{2^i-1} \quad (i \geq 1).$$

The coefficients z_{k-1} with k not a power of 2 are a set of polynomial generators for \mathfrak{R}_ .*

Moreover, if $z_{2^i-1} = z'_{2^i-1}$ for $1 \leq i \leq n$ for primitive elements

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \cdots$$

and

$$P' = W_1 + z'_2 W_1^3 + z'_4 W_1^5 + z'_5 W_1^6 + z'_6 W_1^7 + z'_7 W_1^8 + \cdots$$

then $z_{k-1}=z'_{k-1}$ for k not a multiple of 2^{n+1} .

The author wishes to thank Professors M. Nakaoka and F. Uchida for their advices and encouragement.

1. Operation Φ_P

Let $\mathcal{A}^*(0) = \sum_{-\infty < i < \infty} \mathcal{A}^i(0)$ denote the ring of stable operations in the un-oriented cobordism theory. There is an isomorphism of \mathfrak{R}_* -modules ([6], [8])

$$\Psi : \mathcal{A}^*(0) \rightarrow \mathfrak{R}_* \hat{\otimes} Z_2[[W_1, W_2, \dots, W_k, \dots]],$$

where \mathfrak{R}_* is identified with \mathfrak{R}^* by the duality and $\hat{\otimes}$ denotes the complete tensor product.

For a partition $\omega = (i_1, i_2, \dots, i_r)$, denote W_ω the symmetrized monomial of the W_k and the operation $S_\omega \in \mathcal{A}^*(0)$ is defined by $S_\omega = \Psi^{-1}(W_\omega)$.

For a primitive element

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \dots$$

in $\mathfrak{R}^*(BO(1))$ and for a partition $\omega = (i_1, i_2, \dots, i_r)$, we denote the product $z_{i_1} \cdot z_{i_2} \cdots z_{i_r}$ as $z_\omega^{(P)}$.

Following the line of Novikov [8; appendix II], we define an operation $\Phi_P \in \mathcal{A}^0(0)$ by

$$\Phi_P = \sum_{\omega} z_\omega^{(P)} S_\omega,$$

where the summation runs through all the partitions.

Lemma 1.1.

- (1) $\Phi_P(x \cdot y) = \Phi_P(x) \cdot \Phi_P(y)$.
- (2) $\Phi_P(z_0) = z_0$ for $z_0 \in \mathfrak{R}_0$ and $\Phi_P(y) = 0$ for $y \in \mathfrak{R}_i$ ($i > 0$).
- (3) $(\Phi_P)^2 = \Phi_P$.

Proof.

(1). By the definition of Φ_P and from the Cartan formula for S_ω ([6], [8]), part (1) is easily derived.

(2). It is obvious by definition that $\Phi_P(z_0) = z_0$ for $z_0 \in \mathfrak{R}_0$.

It is known that $S_\omega(W_1) = W_1^{k+1}$ if $\omega = (k)$ for some $k \geq 0$ and that $S_\omega(W_1) = 0$ otherwise ([6], [8]). Thus $\Phi_P(W_1) = P$. By the naturality of Φ_P , $(\Phi_P)^2(W_1) = \Phi_P(P)$ is also a primitive element with the leading term W_1 . So it follows from Theorem P in the introduction together with the fact that $\mathfrak{R}_1 \cong \mathfrak{R}_3 \cong \{0\}$ that

$$(\Phi_P)^2(W_1) - \Phi_P(W_1) = \sum_{j \geq 1} y_{8j-1} W_1^{8j}$$

for some decomposable elements $y_{8j-1} \in \mathfrak{N}_{8j-1}$.

On the other hand,

$$\begin{aligned} (\Phi_P)^2(W_1) - \Phi_P(W_1) &= \Phi_P(W_1 + \sum_{k \geq 3} z_{k-1} W_1^k) - \Phi_P(W_1) \\ &= \sum_{k \geq 3} \Phi_P(z_{k-1})(W_1 + \sum_{l \geq 3} z_{l-1} W_1^l)^k. \end{aligned}$$

Comparing both formulas, we see that $\Phi_P(z_{k-1}) = 0$ for $k \leq 7$. So $\Phi_P(z_{8-1}) = 0$ since z_7 is decomposable. So $y_7 = 0$ and it follows Theorem *P* that $y_{16j+7} = 0$ for all $j \geq 0$. Repeting this procedure, we can inductively deduce that $\Phi_P(z_{k-1}) = 0$ for all $k \geq 3$. At the same time we have proved that $(\Phi_P)^2(W_1) = \Phi_P(W_1)$.

Now $(\Phi_P)^2$ is also a multiplicative operation. As in the weakly complex case ([8]), a multiplicative operation of the unoriented cobordism theory is easily seen to be uniquely determined by its value on W_1 . Therefore $(\Phi_P)^2 = \Phi_P$. This completes the proof of Lemma 1.1.

Notation. For a partition $\omega = (i_1, i_2, \dots, i_r)$, let $||\omega|| = i_1 + i_2 + \dots + i_r$ be its degree and $|\omega| = r$ its length. And we call ω *non-dyadic* if none of the component i_k of ω is of the form $2^m - 1$.

Theorem 1.2. *On the category of finite pointed CW pairs and continuous maps, there is a natural direct sum splitting as a graded Z_2 -vector space*

$$\mathfrak{N}^*(X, A) = \bigoplus_{\omega; \text{ non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)),$$

where (1) the restriction

$$\mu | \text{Image } \Phi_P : \Phi_P(\mathfrak{N}^*(X, A)) \rightarrow H^*(X, A; Z_2)$$

is a natural Z_2 -algebra isomorphism, and (2) the scalar multiplication

$$z_{\omega}^{(P)} \cup : \Phi_P(\mathfrak{N}^*(X, A)) \rightarrow z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A))$$

is a graded Z_2 -module isomorphism of degree $-||\omega||$ if ω is non-dyadic.

Therefore we obtain a natural equivalence of graded \mathfrak{N}_* -algebras

$$\mathfrak{N}^*(X, A) \xrightarrow[\cong]{} \mathfrak{N}^* \hat{\otimes} H^*(X, A; Z_2)$$

which commutes with suspension. (Suspension S and a bordism element x act on the right by $S(y \hat{\otimes} a) = y \hat{\otimes} S(a)$ and $x(y \hat{\otimes} a) = x \cdot y \hat{\otimes} a$, respectively.)

Moreover, the converse holds; such an equivalence is induced by $\bigoplus_{\omega; \text{ non-dyadic}} z_{\omega}^{(P)} \Phi_P$ for some choice of a primitive element P .

For the proof of the above theorem, we need the following operations which

are just the unoriented analogue of those defined in [8].

Lemma 1.3.

For an indecomposable element $y_i \in \mathfrak{R}_i$, define an operation $\Delta_{y_i} = \sum_{k \geq 1} y_i^{k-1} S_{\langle i \rangle}^k$.

$((i)^k = (i, i, \dots, i); \text{ the } k \text{ copies of } i)$

Then

$$\Delta_{y_i}(a \cdot b) = \Delta_{y_i}(a) \cdot b + a \cdot \Delta_{y_i}(b) + y_i \cdot \Delta_{y_i}(a) \cdot \Delta_{y_i}(b)$$

and, in particular,

$$\Delta_{y_i}(y_i \cdot a) = a.$$

The proof of the lemma is straightforward from the definition of Δ_{y_i} and the fact that $S_{\langle i \rangle}(y_i) = 1 \in Z_2$.

Proof of Theorem 1.2.

First we prove property (1). By (2) of Lemma 1.1, property (1) holds for $(X, A) = (S^0, P)$. Since Φ_P commutes with suspensions, (1) also holds for $(X, A) = (S^n, P)$ for $n \geq 1$. Since Φ_P is a projection, $\Phi_P(\mathfrak{R}^*(,))$ is also a cohomology theory. So the general cases are proved by induction on the number of cells in $X - A$, using the five lemma.

Next we prove property (2). The multiplication

$$\mathfrak{z}_{\omega}^{(P)} \cup : \Phi_P(\mathfrak{R}^*(X, A)) \rightarrow \mathfrak{z}_{\omega}^{(P)} \Phi_P(\mathfrak{R}^*(X, A))$$

is obviously a graded Z_2 -module epimorphism of degree $-||\omega||$.

Suppose $\mathfrak{z}_{\omega}^{(P)} \cdot a = 0$ for $a \in \Phi_P(\mathfrak{R}^*(X, A))$ and for a non-dyadic ω . Order the components of $\omega = (i_1, i_2, \dots, i_r)$ as $i_1 \leq i_2 \leq \dots \leq i_r$ and define the operation $\Delta_{z_{\omega}}^{(P)}$ by

$$\Delta_{z_{\omega}}^{(P)} = \Delta_{z_{i_1}} \circ \Delta_{z_{i_2}} \circ \dots \circ \Delta_{z_{i_r}}.$$

Then $a = \Delta_{z_{\omega}}^{(P)}(\mathfrak{z}_{\omega}^{(P)} \cdot a) = \Delta_{z_{\omega}}^{(P)}(0) = 0$ by Lemma 1.3. This proves property (2).

Totally order the set of all non-dyadic partitions by $\omega' < \omega$ if (a) $||\omega'|| < ||\omega||$ or (b) $||\omega'|| = ||\omega||$ and $i_r = j_s, \dots, i_{r-m+1} = j_{s-m+1}, i_{r-m} > j_{s-m}$ for some $m \geq 0$, where $\omega' = (i_1, i_2, \dots, i_r)$ and $\omega = (j_1, j_2, \dots, j_s)$ with $i_1 \leq i_2 \leq \dots \leq i_r$ and $j_1 \leq j_2 \leq \dots \leq j_s$.

We show that

$$\Phi_P \Delta_{z_{\omega'}}^{(P)}(\mathfrak{z}_{\omega}^{(P)} \Phi_P(y)) = 0$$

for any homogeneous element y if $\omega' < \omega$. In case $||\omega'|| < ||\omega||$, Lemma 1.3 implies that

$$\Phi_P \Delta_{z_{\omega'}}^{(P)}(\mathfrak{z}_{\omega}^{(P)} \Phi_P(y)) = \Phi_P(\sum_i u_i \cdot y_i)$$

for some elements $u_i \in \mathfrak{R}_*$ and $y_i \in \Phi_P(\mathfrak{R}^*(X, A))$ with $\dim u_i \geq ||\omega|| - ||\omega'|| > 0$. Thus, by Lemma 1.1 (1), (2),

$$\Phi_P(\sum_i u_i y_i) = \sum_i \Phi_P(u_i) \Phi_P(y_i) = 0.$$

In case $\|\omega'\| = \|\omega\|$ and $i_r = j_s, \dots, i_{r-m} > j_{s-m}$,

$$\begin{aligned} & \Phi_P \Delta_{z_{\omega'}}^{(P)}(z_{\omega}^{(P)} \Phi_P(y)) \\ &= \Phi_P \Delta_{z(i_1, \dots, i_{r-m-1})}^{(P)}(z_{j_1} \cdots z_{j_{s-m}} \Delta_{z_{i_{r-m}}} \Phi_P(y)) = 0. \end{aligned}$$

The last equality follows from the preceding case.

Let $\sum_{\omega' < \omega} z_{\omega'}^{(P)} \Phi_P(\mathfrak{N}^*(X, A))$ be the graded vector space spanned by all $z_{\omega'}^{(P)} \Phi_P(\mathfrak{N}^*(X, A))$ with $\omega' < \omega$.

It follows from the above fact that

$$\sum_{\omega' < \omega} z_{\omega'}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) \cap z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) = 0$$

for each ω , so that there is a direct sum splitting

$$\sum_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) = \bigoplus_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)).$$

Since it can be proved similarly as above that $\text{Image}(\Phi_P \circ \Delta_{z_{\omega}}^{(P)}) = \text{Image} \Phi_P$ for each non-dyadic ω , we have proved that there is a natural linear endomorphism of degree zero

$$\sum_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P \Delta_{z_{\omega}}^{(P)} : \mathfrak{N}^*(X, A) \rightarrow \bigoplus_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) \subset \mathfrak{N}^*(X, A).$$

It is clearly an automorphism for $(X, A) = (S^0, P)$ and therefore an automorphism for every finite CW pair by the effect of suspensions and of the five lemma. Thus

$$\bigoplus_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) = \mathfrak{N}^*(X, A).$$

Since $z_{\omega}^{(P)} \Phi_P(y) \cdot z_{\omega'}^{(P)} \Phi_P(y') = z_{\omega\omega'}^{(P)} \Phi_P(y \cdot y')$, we have obtained a natural equivalence of graded \mathfrak{N}_* -algebras

$$\Theta_P : \mathfrak{N}^*(X, A) \cong \mathfrak{N}^* \hat{\otimes} H^*(X, A; Z_2)$$

which commutes with suspension.

Conversely, each such equivalence Θ induces a natural monomorphism of a graded Z_2 -algebra

$$\lambda = \Theta^{-1} | H^*(X, A; Z_2) : H^*(X, A; Z_2) \rightarrow \mathfrak{N}^*(X, A).$$

Then the composition $\lambda \circ \mu$ is a stable multiplicative operation in $\mathcal{A}^*(0)$ and $\lambda \circ \mu(W_1) = \lambda(w_1) = P$ is a primitive element in $\mathfrak{N}^*(BO(1))$. And the element P has the leading term W_1 since

$$\Theta : \mathfrak{N}_*[[W_1]] \rightarrow \mathfrak{N}_* \hat{\otimes} Z_2[[w_1]]$$

is an \mathfrak{N}_* -algebra isomorphism. Therefore

$$\Theta = \bigoplus_{\omega; \text{non-dyadic}} \{1 \hat{\otimes} (\mu | \text{Image } \Phi_P)\} : \\ \mathfrak{N}^*(X, A) = \bigoplus_{\omega; \text{non-dyadic}} z_{\omega}^{(P)} \Phi_P(\mathfrak{N}^*(X, A)) \rightarrow \bigoplus_{\omega; \text{non-dyadic}} \{z_{\omega}^{(P)} \hat{\otimes} H^*(X, A; Z_2)\}$$

This completes the proof of Theorem 1.2.

2. Operations \bar{S}_{ω}

Let \bar{W}_{ω} denote the symmetrized monomial of the cobordism normal characteristic classes \bar{W}_k . ($\bar{W}_{\omega}(\xi) = W_{\omega}(-\xi)$ for every stable vector bundle ξ .) The operation \bar{S}_{ω} is defined in [8] by $\bar{S}_{\omega} = \Psi^{-1}(\bar{W}_{\omega})$, where Ψ is the additive isomorphism mentioned in section 1.

Notation 2.1. (Landweber [6])

For a partition $\omega = (i_1, \dots, i_r)$ let $r_{\omega}(i)$ denote the occurrences of the integer i in ω . And define

$$\binom{n}{\omega} = \begin{cases} 0 & \text{if } n < |\omega| = r \\ \frac{n!}{r_{\omega}(1)! r_{\omega}(2)! \dots (n - |\omega|)!} & \text{if } n \geq |\omega|. \end{cases}$$

The modulo 2 reduction of $\binom{n}{\omega}$ is denoted by $\binom{n}{\omega}_2$.

Similarly to the weakly complex case [8], we can easily determine the value $\bar{S}_{\omega}[P_k]$.

Lemma 2.2.

- (1) $\bar{S}_{\omega}[P_k] = \binom{k+1}{\omega}_2 [P_{k-|\omega|}]$.
- (2) $S_{\omega}[P_k] = \binom{2^p - k - 1}{\omega}_2 [P_{k-|\omega|}]$ for p such that $2^p > k + 1$.

Proof. By the geometric interpretation of the action of $\mathcal{A}^*(0)$ on \mathfrak{N}_* given in [6], [8], $\bar{S}_{\omega}[P_k] = \varepsilon W_{\omega}(\tau_{P_k}) = \varepsilon \binom{k+1}{\omega}_2 W_1^{|\omega|} = \binom{k+1}{\omega}_2 [P_{k-|\omega|}]$. Part (2) is proved similarly. Now we give some relations between S_{ω} and \bar{S}_{ω} .

Lemma 2.3.

- (1) If the occurrence $r_{\omega}(i) \leq 1$ in ω for all i , then $S_{\omega} = \bar{S}_{\omega}$.

$$(2) \quad S_{(i)^k} = \sum_{||\omega||=k} \bar{S}_{i*\omega} \quad \text{and dually} \\ \bar{S}_{(i)^k} = \sum_{||\omega||=k} S_{i*\omega},$$

where $i*\omega$ is meant a partition $(i \cdot j_1, i \cdot j_2, \dots, i \cdot j_r)$ for $\omega = (j_1, j_2, \dots, j_r)$.

After Landweber [6] we denote the partition $(i)^k$ by $k\Delta_i$ and the totality of linear combinations of the S_ω by $A^*(0)$. $A^*(0)$ is proved a Hopf algebra over \mathbb{Z}_2 ([6], [8]).

Theorem 2.4. (Landweber [6])

The set $\{S_{2^k\Delta_1}, S_{2^k\Delta_2}; k \geq 0\}$ provides a minimal set of generators of $A^*(0)$.

Corollary 2.5.

The set $\{\bar{S}_{2^k\Delta_1}, \bar{S}_{2^k\Delta_2}; k \geq 0\}$ provides a minimal set of generators of $A^*(0)$.

Proof of Lemma 2.3.

By the Whitney product formula, it follows that $\sum_{\omega=\omega_1+\omega_2} W_{\omega_1} \cdot \bar{W}_{\omega_2} = 0$ if $\omega \neq (0)$.

Therefore $W_{(i)} = \bar{W}_{(i)}$ for all $i \geq 1$ and we see by induction on the lengths of partitions that $W_\omega = \bar{W}_\omega$ if $r_\omega(i) \leq 1$ for all i . Part (1) follows from this and from the definition of S_ω and \bar{S}_ω .

Put

$$\sum_{0 \leq i \leq s} \bar{W}_i x^i = \prod_{1 \leq j \leq s} (1 + u_j x)$$

for a sufficiently large s .

Then part (2) of the lemma is proved by induction on k as follows ;

$$\begin{aligned} W_{(i)^k} &= \sum_{0 \leq l \leq k-1} W_{(i)^l} \bar{W}_{(i)^{k-l}} = \sum_{0 \leq l \leq k-1} \left(\sum_{||\omega||=l} \bar{W}_{i*\omega} \right) \cdot \bar{W}_{(i)^{k-l}} \\ &= \sum_{0 \leq l \leq k-1} \left\{ \sum_{j_1 + \dots + j_m = l} (\sum (u_1^t)^{j_1} \dots (u_m^t)^{j_m}) \right\} \left\{ \sum (u_1^t) \dots (u_{k-l}^t) \right\} \\ &= \sum_{i_1 + \dots + i_n = k} (\sum (u_1^t)^{i_1} \dots (u_n^t)^{i_n}) \left(\sum_{0 \leq l \leq k-1} \binom{n}{k-l}_2 \right) \\ &= \sum_{||\omega||=k} \bar{W}_{i*\omega} \binom{|\omega|}{0}_2 = \sum_{||\omega||=k} \bar{W}_{i*\omega}. \end{aligned}$$

Part (2) follows from this.

Proof of Corollary 2.5.

It follows from Lemma 2.3 and Theorem 2.4 that

$$\begin{aligned} \bar{S}_{\Delta_1} &= S_{\Delta_1}, \\ \bar{S}_{2^k\Delta_1} &= S_{2^k\Delta_1} + S_{2^{k-1}\Delta_2} + \text{decomposables in } A^*(0), \text{ and} \\ \bar{S}_{2^k\Delta_2} &= S_{2^k\Delta_2} + \text{decomposables in } A^*(0). \end{aligned}$$

Thus the corollary follows from Theorem 2.4.

3. Even dimensional coefficients

Following suit of Novikov [8, appendix I], we obtain the following. We omit the proof.

Lemma 3.1.

For a partition ω and for a positive integer $k=2^p(2q+1)$ ($p \geq 0, q \geq 1$), the following formula holds if $||\omega|| \geq 2^p$;

$$\sum_{\omega=\omega_1\omega_2} S_{\omega_1}(z_{k-1-||\omega_2||}) \binom{k-||\omega_2||}{\omega_2}_2 = 0,$$

where the z_i denote the coefficients of a fixed primitive element P as in the introduction.

Now we prove the following theorem.

Theorem 3.2.

The coefficient z_{2k} of a primitive element

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \dots$$

in $\mathcal{R}^*(BO(1))$ is equal to the bordism class $[P_{2k}]$ for all $k \geq 1$.

Proof. For $k=1$, the theorem is clear since z_2 is indecomposable from Theorem P in the introduction.

Assume that the theorem holds up to dimension $2(k-1) \geq 2$.

In order to show that $S_{\omega}(z_{2k} + [P_{2k}]) = \bar{W}_{\omega}(z_{2k} + [P_{2k}]) = 0$ for all ω with $||\omega|| = 2k$, it suffices from Theorem 2.4 to prove

$$S_{2^s \Delta_i}(z_{2k} + [P_{2k}]) = 0 \quad (i = 1, 2).$$

To prove this, we see from Lemma 3.1 and the induction assumption that it is sufficient to show

$$\sum_{m+n=2^s} S_{m \Delta_i}[P_{2k-n_i}] \binom{2k+1-n_i}{n}_2 = 0 \quad (i = 1, 2).$$

This is obvious in case $2^s i > 2k$ or $s=0$ since

$$S_{m \Delta_i}[P_{2k-n_i}] = \left\{ \sum_{||\omega||=m} \binom{2k+1-n_i}{\omega}_2 \right\} [P_{2k-2^s i}]$$

by Lemmas 2.2 (1) and 2.3 (2).

For the remaining cases, it suffices to prove the following lemma.

Lemma 3.3.

$$(1) \quad \sum_{m+n=2^s} \left(\sum_{||\omega||=m} \binom{k-n}{\omega} \right) \binom{k-n}{n} \equiv 0 \pmod{2} \quad \text{for } k \geq s \geq 2.$$

$$(2) \quad \sum_{m+n=s} \left(\sum_{||\omega||=m} \binom{k-2n}{\omega} \right) \binom{k-2n}{n} \equiv 0 \pmod{2} \quad \text{for } k \geq 2s \geq 2.$$

Proof.

(1) Put

$$A(k, s) = \sum_{m+n=s} \left(\sum_{||\omega||=m} \binom{k-n}{\omega} \right) \binom{k-n}{n} \quad (k \geq 0, s \geq 0), \quad \text{and}$$

$$B(k, s) = \sum_{m+n=s} \left(\sum_{||\omega||=m} \binom{k-2n}{\omega} \right) \binom{k-2n}{n} \quad (k \geq 0, s \geq 0).$$

Then it holds in general that

$$\begin{aligned} \binom{k-n}{n} &= \binom{k-n-1}{n} + \binom{k-n-1}{n-1} \quad \text{and} \\ \sum_{||\omega||=m} \binom{k-n}{\omega} &= \sum_{0 \leq ||\omega|| \leq m} \binom{k-n-1}{\omega}. \end{aligned}$$

So we obtain that

$$(*) \quad A(k, s) = \sum_{0 \leq s' \leq s} A(k-1, s') + \sum_{0 \leq s'' \leq s-1} A(k-2, s'') \quad \text{and}$$

$$(**) \quad B(k, s) = \sum_{0 \leq s' \leq s} B(k-1, s') + \sum_{0 \leq s'' \leq s-1} B(k-3, s'').$$

Part (1) clearly holds when $k=s=2$.

Assume, by induction, that (1) holds for such (k, s) that $k_0 > k \geq 2$ and $k \geq s \geq 2$.

Thus, for (k_0, s_0) with $k_0 > s_0 \geq 2$,

$$A(k_0, s_0) \equiv \sum_{s'=0,1} A(k_0-1, s') + \sum_{s''=0,1} A(k_0-2, s'') \equiv 0 \pmod{2}$$

by the induction hypothesis and by the fact that $A(k, s) \equiv 1$ for $k \geq s$ and $s=0, 1$.

And for (k_0, k_0) , the iterated application of (*) shows that

$$\begin{aligned} A(k_0, k_0) &\equiv A(k_0-1, k_0) + A(k_0-2, k_0-1) \\ &\equiv A(1, k_0) + \sum_{0 \leq s'' \leq k_0-1} A(0, s'') \equiv 0 \pmod{2}. \end{aligned}$$

Part (2) of the lemma is proved similarly, using the formula (**) repeatedly. This completes the proof of Lemma 3.3 and Theorem 3.2.

REMARK 3.4. Theorem 3.2 has been proved independently by F. Uchida [9] by a geometric method.

4. The coefficients of dimensions $4k+1$

A. Dold has defined in [5] manifolds $P(m, n)$ which are the identification

spaces of $S^m \times CP_n$ with $(x, z) = (-x, z)$. He proved that, for $2^p(2q+1)-1$ ($p \geq 1, q \geq 1$), the bordism class $[P(2^p-1, 2^p q)]$ provides a polynomial generator of \mathfrak{N}_* in the corresponding dimension.

Theorem 4.1.

The coefficient z_{4k+1} of a primitive element

$$P = W_1 + z_2 W_1^3 + z_4 W_1^5 + z_5 W_1^6 + z_6 W_1^7 + z_7 W_1^8 + \dots$$

in $\mathfrak{N}^*(BO(1))$ is equal to the bordism class $[P(1, 2k)]$ for all $k \geq 1$.

For the proof of this theorem, we need the following notations.

Notation 4.2.

- (1) Let $c_p(m)$ denote the coefficient of 2^p in the dyadic expansion of the integer m ;

$$m = c_0(m) + c_1(m) \cdot 2 + c_2(m) \cdot 2^2 + \dots, \quad c_i(m) = 0, 1.$$

- (2) For a partition ω , we denote by $\omega(c_p)$ the partition determined by $r_{\omega(c_p)}(i) = c_p(r_\omega(i))$ for all $i \geq 1$. Thus $\omega = \prod_{0 \leq p} (\omega(c_p))^{2^p}$. For brevity, $\prod_{2 \leq p} (\omega(c_p))^{2^{p-2}}$ and $\omega(c_1)^2 \cdot \omega(c_0)$ are denoted as $\bar{\omega}$ and $\bar{\bar{\omega}}$, respectively; $\omega = (\bar{\omega})^4 \bar{\bar{\omega}}$.

Lemma 4.3.

$$\binom{n}{\omega}_2 = \prod_{0 \leq p} \binom{c_p(n)}{\omega(c_p)}_2. \quad \text{Thus } \binom{n}{\omega}_2 = \left(\frac{n - c_1(n) \cdot 2 - c_0(n)}{4} \right)_2 \binom{c_1(n) \cdot 2 + c_0(n)}{\bar{\bar{\omega}}}_2.$$

Proof. By definition,

$$\binom{n}{\omega}_2 = \binom{n}{r_\omega(1)}_2 \binom{n - r_\omega(1)}{r_\omega(2)}_2 \dots \binom{n - \sum_{1 \leq i \leq k-1} r_\omega(i)}{r_\omega(k)}_2 \dots$$

Then, by Lucas' theorem [1],

$$\begin{aligned} \prod_{1 \leq k} \binom{n - \sum_{1 \leq i \leq k-1} r_\omega(i)}{r_\omega(k)}_2 &= \prod_{1 \leq k} \binom{c_p(n - \sum_{1 \leq i \leq k-1} r_\omega(i))}{c_p(r_\omega(k))}_2 \\ &= \prod_{0 \leq p} \left(\prod_{1 \leq k} \binom{c_p(n) - \sum_{1 \leq i \leq k-1} c_p(r_\omega(i))}{c_p(r_\omega(k))}_2 \right) = \prod_{0 \leq p} \binom{c_p(n)}{\omega(c_p)}_2. \end{aligned}$$

This completes the proof.

Now we calculate all the normal Stiefel-Whitney numbers of $P(1, 2k)$. It is easily seen that the cobordism Stiefel-Whitney numbers of manifolds agree with

the cohomological ones ([6], [8]). So, by abuse of a notation, we denote both Stiefel-Whitney numbers by \bar{W}_ω (and the normal ones by \bar{W}_ω).

Lemma 4.4.

$$\bar{W}_\omega[P(1, 2k)] = \begin{cases} 0 & \text{if } |\bar{\omega}| \geq 3 \text{ and } \bar{\omega} \neq 3\Delta_1 \text{ or } \bar{\omega} = (1), \\ \binom{2^p-1-k}{\bar{\omega}}_2 & \text{if } \bar{\omega} = 3\Delta_1 \text{ or} \\ & 2 \geq |\bar{\omega}| \geq 1 \text{ and } \bar{\omega} \neq (1), \end{cases}$$

where p is any integer with $2^p > k+1$.

Proof.

According to Dold [5].

$$H^*(P(1, 2k); Z_2) \cong H^*(P_1 \times CP_{2k}; Z_2)$$

as a ring. Let c and d denote the 1- and 2-dimensional generators of $H^*(P(1, 2k); Z_2)$. The total Whitney class is given in [5] by

$$w_*P(1, 2k) = (1+c)(1+c+d)^{2k+1},$$

and thus

$$\bar{w}_*P(1, 2k) = (1+c)(1+t)^{4(2^p-k-1)}(1+t_1)(1+t_2),$$

where p is any integer with $2^p > k+1$ and $t^2 = t_1 \cdot t_2 = d$ and $t_1 + t_2 = c$.

By formula (26) in [5],

$$t_1^{2i} + t_2^{2i} = 0 \quad \text{and} \quad t_1^{2i+1} + t_2^{2i+1} = cd^i.$$

The lemma follows from these facts and the preceding lemma.

Proof of Theorem 4.1.

Theorem P in the introduction asserts that $z_{4+1} = [P(1, 2)]$. Assume, by induction, that $z_{4k'+1} = [P(1, 2k')]$ for $k' \leq k-1$.

By Lemma 3.1 and Theorem 3.2, together with Lemma 2.2 (2), 4.3 and 4.4,

$$\begin{aligned} S_\omega(z_{4k+1}) &= \sum_{\substack{\omega = \omega_1 \omega_2 \\ ||\omega_2|| = 4m \neq 0}} S_{\omega_1}(z_{4(k-m)+1}) \binom{k-m}{\bar{\omega}}_2 \binom{2}{\bar{\omega}_2}_2 \\ &\quad + \sum_{\substack{\omega = \omega_1 \omega_2 \\ ||\omega_2|| = 2n+1}} \binom{2^p-1-4||\bar{\omega}_1||-||\bar{\omega}_1||}{(\bar{\omega}_1)^4 \bar{\omega}_1}_2 \binom{4||\bar{\omega}_1||+||\bar{\omega}_1||+1}{(\bar{\omega}_2)^4 \bar{\omega}_2}_2 \end{aligned}$$

for ω such that $||\omega|| = 4k+1$. (The terms with $||\omega_2|| \equiv 2 \pmod{4}$ vanish by Lemma 4.3.)

Therefore, by the induction hypothesis and by Lemma 4.3, together with the fact that $|\bar{\omega}_1| + |\bar{\omega}_2| = |\bar{\omega}| + 4l$ ($l \geq 0$), it can be shown that

$$S_\omega(z_{4k+1}) = \sum 0 + \sum 0 = 0 \quad \text{if} \quad |\bar{\omega}| \geq 5.$$

In case $\bar{\omega} = (2i, 2i, 4j, 4(k - \|\bar{\omega}\| - i - j) + 1)$,

$$S_{\omega}(z_{4k+1}) = \sum_{\bar{\omega} = \bar{\omega}_1 \bar{\omega}_2} \left\{ \binom{2^{p'} - 1 - (k - \|\bar{\omega}_2\| - i)}{\bar{\omega}_1}_2 \binom{k - \|\bar{\omega}_2\| - i}{\bar{\omega}_2}_2 \right. \\ \left. + \binom{2^{p'} - 1 - (\|\bar{\omega}\| + i + j - \|\bar{\omega}_2\|)}{\bar{\omega}_1}_2 \binom{\|\bar{\omega}\| + i + j - \|\bar{\omega}_2\|}{\bar{\omega}_2}_2 \right\}$$

by the induction hypothesis and by Lemma 4.3.

Suppose $\binom{2^{p'} - 1 - (k - \|\bar{\omega}_2\| - i)}{\bar{\omega}_1}_2 \binom{k - \|\bar{\omega}_2\| - i}{\bar{\omega}_2}_2 = 1$ for some separation

$\bar{\omega}_1 \bar{\omega}_2$ of $\bar{\omega}$.

Since $c_p(2^{p'} - 1 - (k - \|\bar{\omega}_2\| - i)) \neq c_p(k - \|\bar{\omega}_2\| - i)$ for each p , there is at most one $i \geq 1$ such that $c_p(r_{\bar{\omega}}(i)) \neq 0$. Let r be the number of such odd integers $2i + 1 \geq 1$ that satisfy $r_{\bar{\omega}}(2i + 1) > 0$. Then, by Lemma 4.3, the numbers of such separations $\bar{\omega}_1 \bar{\omega}_2 = \bar{\omega}$ and $\bar{\omega}_1' \bar{\omega}_2' = \bar{\omega}$ that satisfy

$$\binom{2^{p'} - 1 - (k - \|\bar{\omega}_2\| - i)}{\bar{\omega}_1}_2 \binom{k - \|\bar{\omega}_2\| - i}{\bar{\omega}_2}_2 = 1 \quad \text{and} \\ \binom{2^{p'} - 1 - (\|\bar{\omega}\| + j + k - \|\bar{\omega}_2'\|)}{\bar{\omega}_1'}_2 \binom{\|\bar{\omega}\| + j + k - \|\bar{\omega}_2'\|}{\bar{\omega}_2'}_2 = 1,$$

respectively, are both 2^r .

The situation is the same if we suppose

$$\binom{2^{p'} - 1 - (\|\bar{\omega}\| + j + k - \|\bar{\omega}_2\|)}{\bar{\omega}_1}_2 \binom{\|\bar{\omega}\| + j + k - \|\bar{\omega}_2\|}{\bar{\omega}_2}_2 = 1$$

for some separation $\bar{\omega}_1 \cdot \bar{\omega}_2 = \bar{\omega}$.

Therefore $S_{\omega}(z_{4k+1}) = 0 + 0 = 0$ or $= 1 + 1 = 0$ if $\bar{\omega} = (2j, 2j, 4k, 4(s - \|\bar{\omega}\| - j - k) + 1)$.

We can prove analogously in other cases when $\bar{\omega} = (1)$ or $|\bar{\omega}| \geq 3$ and $\bar{\omega} \neq 3\Delta_1$ that $S_{\omega}(z_{4k+1}) = 0$.

When $|\bar{\omega}| = 2$, from dimensional reasons, $\bar{\omega} = (2j, 4(s - \|\bar{\omega}\|) - 2j + 1)$ for some $j \geq 1$. In this case

$$S_{\omega}(z_{4k+1}) = \sum_{\substack{\bar{\omega} = \bar{\omega}_1 \bar{\omega}_2 \\ \bar{\omega}_2 \neq (0)}} \binom{2^p - 1 - (k - \|\bar{\omega}_2\|)}{\bar{\omega}_1}_2 \binom{k - \|\bar{\omega}_2\|}{\bar{\omega}_2}_2 \\ + \sum_{\bar{\omega} = \bar{\omega}_1 \bar{\omega}_2} \binom{2^p - 1 - (2\|\bar{\omega}\| - 2\|\bar{\omega}_2\| + j)}{\bar{\omega}_1}_2 \binom{2\|\bar{\omega}\| - 2\|\bar{\omega}_2\| + j}{\bar{\omega}_2}_2 \\ = \sum_{\bar{\omega} = \bar{\omega}^*(0)} \binom{2^p - 1 - (k - \|\bar{\omega}_2\|)}{\bar{\omega}_1}_2 \binom{k - \|\bar{\omega}_2\|}{\bar{\omega}_2}_2 \\ = \binom{2^p - 1 - k}{\bar{\omega}}_2$$

as required.

When $\bar{\omega}=3\Delta_1$ or $\bar{\omega}=\Delta_{4n+1}$ ($n\geq 1$), analogous arguments show that $S_{\omega}(z_{4k+1}) = \binom{2^p-1-k}{\bar{\omega}}$.

Comparing these facts with Lemma 4.4, we deduce that $S_{\omega}(z_{4k+1}) = \bar{W}_{\omega}(z_{4k+1}) = \bar{W}_{\omega}(P(1, 2k))$ for all ω with $\|\omega\|=4k+1$. This completes the proof of Theorem 4.1.

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