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DIRICHLET FORMS PERTURBATED BY ADDITIVE FUNCTIONALS OF EXTENDED KATO CLASS

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1. Introduction

In recent years there were many authors in theory of Dirichlet forms and related fields who studied the so-called Feynman-Kac semigroups, Schrödinger operators and the corresponding bilinear forms. Particularly, the multiplicative functionals in consideration are not necessarily the exponential of classical positive continuous additive functionals or abbreviated as PCAF's. In a series of papers by Albeverio and Ma ([1], [2] and references therein) they investigated the perturbation of Dirichlet forms by signed smooth measures $\mathcal{E}^{\mu}=\mathcal{E}+Q_{\mu}$, where μ is a signed smooth measure and $Q_{\mu}(f,g)=\mu(f\cdot g)$, and found necessary and sufficient conditions for $\mathcal{E}^{\mu}$ to be a lower semi-bounded closed quadratic form. In [15] the author studied the killing transformation by general decreasing multiplicative functionals and perturbation of Dirichlet forms by bivariate smooth measures: $\mathcal{E}^{\nu}=\mathcal{E}+Q_{\nu}$, where ν is a bivariate smooth measure and $Q_{\nu}(f,g)=\nu(f\otimes g)$, and proved the generalized Feynman-Kac formula. He also proved that the killing transformation in theory of Markov processes is equivalent in some sense to the notion of subordination in theory of Dirichlet forms in [17]. In [13] the author also studied the additive functionals in the form of $A_t=A_t^{\mu}+\sum_{s\leq t}F(X_{s-},X_s)$, where μ is a signed smooth measure, A^{μ} the difference of two PCAF's associated with μ and F a bounded Borel function vanishing on the diagonal, but his base processes are symmetric stable processes on $\mathbb{R}^d$. He found the conditions for the Feynman-Kac semigroup $Q_tf(x):=P^x(e^{-A_t}f(X_t))$ to be strongly continuous and the bilinear form corresponding to it. In quite different approach, Albeverio and Song [3] studied the perturbation caused by

$$\mathcal{E}^{\nu}(u,u):=\mathcal{E}(u,u)+\int(u(x)-u(y))^2\nu(dxdy).$$

They gave a necessary and sufficient condition for the form to be closable and constructed the corresponding resolvent which is not the killing type. Very recently Stollmann and Voigt [14] made a thorough investigation on perturbation by a

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signed smooth measure, however their approach is rather analytic and even made no assumption of existence of the associated process. As I am about to send out this paper, I received a preprint from R.K.Getoor. In this paper [7] the generalized Schrödinger equation (attached with a measure) was investigated in context of right Borel Markov processes, but contrary to [14] the approach are totally probabilistic.

In this paper we are going to investigate the perturbation of a symmetric Markov process by a general increasing additive functional. More precisely let X be an m-symmetric Markov process on state space E and A an (increasing, symmetric) additive functional of X. Let $(\mathrm{Exp}A)$ be the Stieltjes exponential of A and set $P_t^{-A}f(x) := P^x[(\mathrm{Exp}A)_t f(X_t)]$ for measurable function f on E and $x \in E$. Then (P_t^{-A}) is a semigroup of kernels which is not Markovian in general. We shall introduce so called additive functionals of extended Kato class as analogous to the notion in [7] and prove that if A belongs to this class, then (P_t^{-A}) may be extended into a strongly continuous semigroup of bounded operators on $L^2(E, m)$. Our approach is very different from the one employed in [7]. We shall also characterize the bilinear form associated with (P_t^{-A}).

The paper is organized as follows. In section 2 we settle down the notations and terminologies used in the sequel, introduce additive functionals of extended Kato class and discuss the properties of the perturbation semigroup. In section 3 we will discuss the relationship between Feynman-Kac semigroup and the corresponding bilinear form. The main theorem extends Prop 3.1 in [2]. In section 4 we will take the symmetric Lévy processes as examples to explain some of results.

I would like to thank P.J. Fitzsimmons and R.K. Getoor for many iluminating discussions and suggestions which, in particular, shape up the right form of the key Lemma 2.1.

Notations and Conventions. We use ':=' as a way of definition, which is always read as 'is defined to be'. For a class $\mathcal{F}$ of functions, we denote by $b\mathcal{F}$ (resp. $p\mathcal{F}(=\mathcal{F}^+)$) the set of bounded (resp. nonnegative) functions in $\mathcal{F}$. We won't distinguish 'nonnegative' and 'positive'. When a number $a > 0$ or a function $f > 0$ everywhere, we say they are strictly positive. For a measure μ and a function f, $\mu(f) := \int f d\mu$. We sometimes write L^p or $L^p(m)$ for $L^p(E, m)$ and $(\cdot, \cdot)$ for the inner product in $L^2(m)$. For f, $g \in \mathcal{B}(E)$, $f \otimes g(x, y) := f(x)g(y)$, x, $y \in E$. Finally we shall use exclusively P^x for both probability measure and expectation.

2. Additive functionals of extended Kato class.

Throughout this paper $(\mathcal{E}, \mathcal{D})$ is a quasi-regular symmetric Dirichlet form on $L^2(E, m)$, where E is a Lusin space and m a Borel measure on E. Let $\mathcal{S}$ be the set of all smooth measures on E and $\mathcal{S}_0$ the subset of $\mathcal{S}$ consisting of all Borel measures of finite energy integral. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ be a Borel right process on E with transition semigroup (P_t) which is m-symmetric and associated with $(\mathcal{E}, \mathcal{D})$.

Let ζ be the life time of X. For a Borel function f on E (write $f \in \mathcal{B}(E)$ sometimes) we set

$$(2.1) \qquad ||f||_Q = \inf_{\mathrm{Cap}(N)=0} \sup_{x \notin N} |f(x)|,$$

where $\mathrm{Cap}(N)$ denotes the 1-capacity of N with respect to $(\mathcal{E}, \mathcal{D})$. When f is quasi-continuous, $||f||_Q$ is the same as $||f||_\infty$ the usual L^∞-norm. In fact it is clear that $||f||_\infty \leq ||f||_Q$. Suppose that $||f||_\infty < ||f||_Q$. Then there exists an m-null set K such that $\sup_{x \notin K} |f(x)| < ||f||_Q$. We may pick r with $\sup_{x \notin K} |f(x)| < r < ||f||_Q$ and set $K_1 := \{x \in E : |f(x)| > r\}$. Then $K_1 \subset K$ and K_1 is finely open. Thus $\mathrm{Cap}(K_1) = 0$ since K_1 is also an m-null set. We have $\sup_{x \notin K_1} |f(x)| \leq r < ||f||_Q$, which is a contradiction.

A subset N of E is called an exceptional set if $\mathrm{Cap}(N) = 0$. A subset Λ of Ω is called an Ω-equivalent set if there exists an exceptional set N such that $P^x(\Lambda) = 1$ for all $x \notin N$. We say that A is an additive functional of X if $A = (A_t)_{t \geq 0}$ is a $[0, \infty]$-valued adapted process on Ω and there exists an Ω-equivalent set Λ such that for all $\omega \in \Lambda$, (i) $A_t(\omega) < \infty$ for $t < \zeta(\omega)$; (ii) $t \mapsto A_t(\omega)$ is right continuous; (iii) $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$ for all t, $s \geq 0$. Let $\mathcal{A}$ be the set of all additive functionals of X. Therefore all additive functionals talked in this paper is assumed to be increasing.

It is well-known (see §73 in [11]) that there exists a positive continuous additive functional (abbreviated as PCAF) H of X having bounded 1-potential and a kernel N on $(E, \mathcal{B}(E))$ such that (i) $N(x, \{x\}) = 0$ for all $x \in E$; (ii) for every non-negative Borel function f on $E \times E$, $(\int_0^t Nf(X_s) dH_s)_{t>0}$, where $Nf := \int_E N(\cdot, dy) f(\cdot, y)$, is the dual predictable prejection of the random measure

$$\kappa(\omega, dt) := \sum_{s>0} f(X_{s-}(\omega), X_s(\omega)) 1_{\{X_{s-}(\omega) \neq X_s(\omega)\}} \epsilon_s(dt).$$

We call the pair (N, H) a Lévy system of X. Let $J(dx, dy) := N * \rho(dx, dy) := N(x, dy)\rho(dx)$ (noting that the second equality gives a way getting a bivariate measure via a kernel and a measure), where ρ is the Revuz measure of H with respect to m. Then J is the canonical measure of X with respect to m or sometimes called Lévy (also, jumping) measure of $(\mathcal{E}, \mathcal{D})$ and J is symmetric; i.e., $J(dx, dy) = J(dy, dx)$.

Let $\mathcal{B}(E \times E)$ be the set of Borel functions on $E \times E$. The bivariate Revuz measure of $A \in \mathcal{A}$ with respect to m is defined by

$$(2.2) \qquad \nu_A(f) := \uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \int_{]0,t]} f(X_{s-}, X_s) dA_s$$

for $f \in p\mathcal{B}(E \times E)$, it follows from [16] that there exist a smooth measure μ and a nonnegative function F on $E \times E$ vanishing on diagonal such that

$$(2.3) \qquad \nu_A(dx, dy) = \delta * \mu(dx, dy) + F(x, y) J(dx, dy)$$

where δ denotes the unit kernel $\delta(x, B) := 1_B(x)$, and $A_t = A_t^\mu + \sum_{s\le t} F(X_{s-}, X_s)$ a.e. P^m for any $t > 0$, where A^μ is the PCAF determined by μ. For simplicity we write this as $A = A^{\mu+F}$. Actually μ and F are uniquely determined by A. We say that A is symmetric (with respect to m) if $F \cdot J$ is symmetric as a measure on $E \times E$. Since J is symmetric, we may (and do) choose F as a symmetric function on $E \times E$ with $A = A^{\mu+F}$.

Let L be a right continuous increasing function on $[0, \infty[$ with $L_0 = 0$ which may take infinite value. The unique solution Z of the equation

$$Z_t = 1 + \int_{]0,t]} Z_{s-} dL_s, \quad t > 0, \tag{2.4}$$

is usually called the Stieltjes exponential function of L, denoted by $(\mathrm{Exp}L)_t$, which coincides with the usual exponential function if L is continuous. The reason we write $(\mathrm{Exp}L)_t$ instead of $\mathrm{Exp}L_t$ is that the Stieltjes exponential is really defined by paths. It is known (see [11]) that if L^c is the continuous part of L, then

$$(\mathrm{Exp}L)_t = e^{L_t^c} \prod_{s\le t} (1 + \Delta L_s), \tag{2.5}$$

where $\Delta L_s = L_s - L_{s-}$. Clearly $\exp(L_t) \ge (\mathrm{Exp}L)_t$ and the equality holds only if L is continuous.

A smooth measure μ is said to belong to the Kato class, which extends the classical notion of Kato class for Brownian motion, if $\lim_{t\downarrow 0} ||P^\cdot A_t^\mu||_Q = 0$. However inspired by Getoor [7] and Stollmann-Voigt [14], we may actually go a little further as introducing the so-called additive functionals of extended Kato class, which seems more natural to work with in this context. Given an additive functional A, define

$$\begin{aligned} k_t(A) &:= ||P^\cdot A_t||_Q; \\ k(A) &:= \inf_{t>0} k_t(A). \end{aligned} \tag{2.6}$$

Khas'minskii's lemma says that if A is continuous and $P^x A_t \le a < 1$ for all $x \in E$ and a fixed $t > 0$, then

$$P^x e^{A_t} \le \frac{1}{1-a}.$$

This is not true when A is not continuous as shown in an example in §4. However the following lemma shows that it is true if we replace the usual exponential with the Stieltjes exponential.

Lemma 2.1.

(a) *Let $A \in \mathcal{A}$. If there exist $t > 0$ and $\lambda < 1$ such that for all $s < t$ and $x \in E$, $P^x(A_t) \leq \lambda < 1$, then for $x \in E$,*

$$P^x(\mathrm{Exp}A)_t \leq \frac{1}{1-\lambda}. \tag{2.7}$$

(b) *Let $A \in \mathcal{A}$. If $k(A) < 1$, there exist positive constants c and β such that for all $t > 0$,*

$$||P^{\cdot}(\mathrm{Exp}A)_t||_Q \leq c \cdot e^{\beta t}. \tag{2.8}$$

Proof. (a) From §3 of [4], Exp$A$ may be developped as

$$(\mathrm{Exp}A)_t = \sum_{n\geq 0} \int_{]0,t]} dA_{t_n} \int_{]0,t_n[} dA_{t_{n-1}} \cdots \int_{]0,t_2[} dA_{t_1}. \tag{2.9}$$

Reordering the integrations, we have

$$(\mathrm{Exp}A)_t = \sum_{n\geq 0} \int_{0<t_1<\cdots<t_n\leq t} dA_{t_1} \cdots dA_{t_n}. \tag{2.10}$$

Now taking the expectation of the integration and noting that $P^x(A_t - A_s|\mathcal{F}_s) = P^{X_s}(A_{t-s}) \leq \lambda$ for $s < t$, we get

$$\begin{aligned} &P^x \int_{0<t_1<\cdots<t_n\leq t} dA_{t_1} \cdots dA_{t_n} \\ &= P^x \int_{0<t_1<\cdots<t_{n-1}<t} dA_{t_1} \cdots dA_{t_{n-1}} (A_t - A_{t_{n-1}}) \\ &= P^x \int_{0<t_1<\cdots<t_{n-1}<t} dA_{t_1} \cdots dA_{t_{n-1}} P^x(A_t - A_{t_{n-1}}|\mathcal{F}_{t_{n-1}}) \\ &\leq \lambda P^x \int_{0<t_1<\cdots<t_{n-1}\leq t} dA_{t_1} \cdots dA_{t_{n-1}} \\ &\leq \lambda^n. \end{aligned}$$

Then (2.7) follows easily.

(b) Since $k(A) < 1$, we may choose an exceptional set $N \subset E$, $\lambda < 1$ and $T > 0$ such that if starting from any point in $E - N$, the process X never reaches N, and $P^x A_T \leq \lambda$ for all $x \notin N$. By (a) it holds that $P^x \mathrm{Exp}(A_T) \leq 1/(1-\lambda)$ for $x \notin N$.

Let $M := (\mathrm{Exp}A)$, which is an increasing multiplicative functional of X. Then for any integer $n \geq 1$ we have

$$
\begin{aligned}
P^x M_{nT} &= P^x\{M_T \circ \theta_{(n-1)T} M_{(n-1)T}\} \\
&= P^x\{M_{(n-1)T} E^{X_{(n-1)T}} M_T\} \\
&\leq \frac{1}{1-\lambda} P^x M_{(n-1)T} \leq \left(\frac{1}{1-\lambda}\right)^n.
\end{aligned}
$$

For any $t > 0$, take n such that $(n-1)T < t \leq nT$. It follows that

$$
P^x M_t \leq P^x M_{nT} \leq \left(\frac{1}{1-\lambda}\right)^n \leq \left(\frac{1}{1-\lambda}\right)^{\frac{t}{T}+1}.
$$

Now (2.8) holds for $c = 1/(1-\lambda)$ and $\beta = (1/T)\log(1/(1-\lambda))$. □

Remark. The argument in the proof actually proves a little more. If L is a right continuous adapted increasing process on Ω with $L_0 = 0$ and, for $x \in E$ and $s < t$, $P^x(L_t - L_s|\mathcal{F}_s) \leq \lambda < 1$, then

$$
P^x(\mathrm{Exp}L)_t \leq \frac{1}{1-\lambda}.
$$

The readers may compare it to a result of Dellacherie and Meyer : if $P^x(L_t - L_{s-}|\mathcal{F}_s) \leq \lambda < 1$, then

$$
P^x e^{L_t} \leq \frac{1}{1-\lambda},
$$

from which the Khasminskii's lemma follows easily. We may also feel the difference between two exponentials.

Let $\mathcal{P}$ be the set of all symmetric additive functionals of X and define

$$
\begin{aligned}
\mathcal{P}_0 &:= \{A \in \mathcal{P} : \rho_A \text{ is smooth}\} \\
\mathcal{P}_K &:= \{A \in \mathcal{P} : k(A) < 1\}.
\end{aligned} \tag{2.11}
$$

The element in $\mathcal{P}_0$ is called quasi-integrable and the element in $\mathcal{P}_K$ is called an additive functionals of the extended Kato class. A smooth measure μ (resp. $F \in p\mathcal{B}(E \times E)$) with $A^\mu \in \mathcal{P}_K$ (resp. $A^F \in \mathcal{P}_K$) is said to belong to the extended Kato class. Let $H_t^F := \int_0^t NF(X_s)dH_s$ and $A^* := A^\mu + H^F$. Clearly $P^x A_t = P^x A_t^*$ for all $t > 0$ and $x \in E$, and if $A \in \mathcal{P}_0$, A^* is a PCAF of the extended Kato class.

Suppose that $A \in \mathcal{P}$. We define the A-perturbation semigroup of (P_t) (or, X) as

$$
P_t^{-A} f(x) := P^x((\mathrm{Exp}A)_t f(X_t)), \quad f \in p\mathcal{B}(E), \quad x \in E, \tag{2.12}
$$

(we use $-A$ to be consistent with the standard notation P_t^q) and the A-perturbation bilinear form $(\mathcal{E}^{-A}, \mathcal{D}^{-A})$ (or, (X,m)) as

$$\begin{aligned} \mathcal{D}^{-A} &:= \mathcal{D} \cap L^2(\rho_A), \\ \mathcal{E}^{-A}(u,u) &:= \mathcal{E}(u,u) - \nu_A(u \otimes u), \quad u \in \mathcal{D}^{-A}, \end{aligned} \tag{2.13}$$

where ρ_A denotes the Revuz measure of A, which equals the marginal measure of ν_A. It follows from the additivity of A that (P_t^{-A}) is a semigroup of kernels on $(E, \mathcal{B}(E))$ and from the Hölder's inequality that $(\mathcal{E}^{-A}, \mathcal{D}^{-A})$ is a well-defined bilinear form on $L^2(m)$.

Lemma 2.2. *Let A be an additive functional of X. If A is symmetric, then (P_t^{-A}) is m-symmetric; i.e., for all f, $g \in p\mathcal{B}(E)$,*

$$(P_t^{-A} f, g) = (f, P_t^{-A} g).$$

Proof. Recall the reversibility of X under P^m. Let (γ_t) be the reversal operators on Ω; namely, for any $\omega \in \Omega$ and $t < \zeta(\omega)$

$$X_s(\gamma_t \omega) := \begin{cases} X_{(t-s)-}(\omega), & s \leq t; \\ \Delta, & s > t, \end{cases}$$

where Δ is the trap point of X. Since X is m-symmetric, it is reversable under P^m; more precisely, for any $t > 0$ and a nonnegative $\mathcal{F}_t$-measurable random variable G,

$$P^m(G; t < \zeta) = P^m(G \circ \gamma_t; t < \zeta). \tag{2.14}$$

Since F is symmetric, it is easy to check that $(\mathrm{Exp} A)_t \circ \gamma_t = (\mathrm{Exp} A)_t$ and hence we have

$$\begin{aligned} P^m((\mathrm{Exp}A)_t f(X_t) g(X_0)) &= P^m((\mathrm{Exp}A)_t f(X_t) g(X_0)) \circ \gamma_t) \\ &= P^m((\mathrm{Exp}A)_t f(X_0) g(X_t)). \end{aligned}$$

That completes the proof. $\square$

We are now going to show that if $A \in \mathcal{P}_K$, the perturbation semigroup is actually a strongly continuous semigroup of bounded operators on $L^2(E,m)$. When A is continuous, Getoor [7] proved a much more general and stronger result for perturbation semigroup. Unfortunately his argument does not apply in our situation because the Stieltjes exponential behaves very differently from the usual one in some way. Thus we shall use a rather different approach.

Assume that $A \in \mathcal{P}_0$. Then $A - A^*$ is a local martingale and its Doleans-Dade exponential may be written as

$$M := \mathrm{Exp}(A - A^*) = \{e^{-A_t^*}(\mathrm{Exp}A)_t\}_t,$$

which is a supermartingale multiplicative functional of X. Let Y be the subprocess of X transformed by M, which is m-symmetric, and (P_t^Y) the corresponding transition semigroup, which is also a strongly continuous contraction semigroup of bounded operators on $L^2(E,m)$. Let $(\mathcal{E}^Y, \mathcal{D}^Y)$ be the Dirichlet form on $L^2(E,m)$ associated with Y and set

$$\begin{aligned} \mathcal{D}^F &:= \mathcal{D} \cap \{u \in \mathcal{D} : \int (u(y) - u(x))^2 \nu_A(dx,dy) < \infty\}, \\ \mathcal{E}^F(u,u) &:= \mathcal{E}(u,u) + \frac{1}{2}\int (u(y) - u(x))^2 \nu_A(dx,dy), \quad u \in \mathcal{D}'. \end{aligned} \tag{2.15}$$

Lemma 2.3. *Suppose $A \in \mathcal{P}_0$. Then*
(a) *$\mathcal{D} \cap L^2(\rho_A)$ is densely contained in $\mathcal{D}^Y$, $\mathcal{D} \cap L^2(\rho_A) = \mathcal{D}^Y \cap L^2(\rho_A)$ and for $u \in \mathcal{D} \cap L^2(\rho_A)$, $\mathcal{E}^Y(u,u) = \mathcal{E}'(u,u)$;*
(b) *$\mathcal{D}^Y \subset \mathcal{D}'$ and for $u \in \mathcal{D}^Y$, $\mathcal{E}'(u,u) \le \mathcal{E}^Y(u,u)$.*

Proof. (a) Let

$$\begin{aligned} M_t^+ &:= (\mathrm{Exp}A^F)_t = \prod_{s \le t}(1 + F(X_{s-}, X_s)), \\ M_t^- &:= \exp(-H_t^F) = \exp\left(-\int_0^t NF(X_s)dH_s\right), \end{aligned}$$

and Z be the subprocess of X transformed by M^- with the associated Dirichlet form $(\mathcal{E}^Z, \mathcal{D}^Z)$ on $L^2(E,m)$, which is given exactly by

$$\begin{aligned} \mathcal{D}^Z &= \mathcal{D} \cap L^2(\rho_{A^*}); \\ \mathcal{E}^Z(u,u) &= \mathcal{E}(u,u) + \rho_{A^*}(u^2), \quad u \in \mathcal{D}^Z. \end{aligned} \tag{2.16}$$

Clearly Z coincides with the subprocess of Y transformed by

$$\frac{1}{M^+} = \left\{\prod_{s \le t}\left(1 - \frac{F(X_{s-}, X_s)}{1 + F(X_{s-}, X_s)}\right)\right\}_t.$$

It is known from [9] that the $((1+F)N, H)$, where $((1+F)N)(x,dy) = (1 + F(x,y))N(x,dy)$, is a Lévy system of Y. Hence the jumping measure of Y equals

$(1+F)J$ and the bivariate Revuz measure of $1/M^+$ computed with respect to (Y,m) equals ν_A. It follows from [16] that

$$\begin{aligned} \mathcal{D}^Z &= \mathcal{D}^Y \cap L^2(\rho_A); \\ \mathcal{E}^Z(u,u) &= \mathcal{E}(u,u) + J(F\cdot(u\otimes u)), u\in\mathcal{D}^Z. \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17), (a) follows.

(b) Assume that $u\in\mathcal{D}^Y$. We may choose a sequence $\{u_n\}\subset\mathcal{D}\cap L^2(\rho_A)$ such that $u_n\longrightarrow u$ in $\mathcal{E}_q^Y$-norm. Then $\{u_n\}$ is an $\mathcal{E}^Y$-Cauchy sequence and by the result above it is also an $\mathcal{E}$-Cauchy sequence. Therefore $u\in\mathcal{D}$ and $u_n\longrightarrow u$ in $\mathcal{E}_q$-norm and quasi-everywhere (at least for a subsequence). Invoking the Fatou's lemma we have

$$\mathcal{E}'(u,u)\le\lim_n\mathcal{E}'(u_n,u_n)=\lim_n\mathcal{E}^Y(u_n,u_n)=\mathcal{E}^Y(u,u)<\infty.$$

Therefore (b) follows. □

A bilinear form $(b,D(b))$ on $L^2(m)$ is lower semi-bounded if there exists $q>0$ such that $b(u,u)+q(u,u)\ge 0$ for all $u\in D(b)$. Theorem 4.1 of [2] says that if A is a PCAF, then the A-perturbation semigroup of X is a strongly continuous semigroup on $L^2(m)$ if and only if the A-perturbation bilinear form of X is lower semibounded. The part (a) of the following result generalizes this theorem slightly.

Theorem 2.4. *Suppose that* $A\in\mathcal{P}_0$.

(a) *The* A*-perturbation semigroup of* (P_t) *is a strongly continuous semigroup on* $L^2(m)$ *if and only if the* A*-perturbation bilinear form of* $(\mathcal{E},\mathcal{D})$ *is lower semibounded.*

(b) *If the* A^**-perturbation semigroup of* (P_t) *is a strongly continuous semigroup of bounded operators on* $L^2(E,m)$*, so is the* A*-perturbation semigroup of* (P_t).

Proof. (a) It is obvious that the A-perturbation semigroup of (P_t) is exactly the same as the A^*-perturbation semigroup of (P_t^Y). We denote by $(\mathcal{E}^*,\mathcal{D}^*)$ the A^*-perturbation bilinear form of $(\mathcal{E}^Y,\mathcal{D}^Y)$. By the definition and Lemma 2.3 $\mathcal{D}^*=\mathcal{D}^Y\cap L^2(\rho_{A^*})=\mathcal{D}\cap L^2(\rho_A)=\mathcal{D}^{-A}$ and for $u\in\mathcal{D}^*$,

$$\mathcal{E}^*(u,u)=\mathcal{E}^Y(u,u)-\rho(u^2)=\mathcal{E}(u,u)-\nu_A(u\otimes u)=\mathcal{E}^{-A}(u,u).$$

It means that the A-perturbation bilinear form of $(\mathcal{E},\mathcal{D})$ is exactly the same as the A^*-perturbation bilinear form of $(\mathcal{E}^Y,\mathcal{D}^Y)$. Now (a) follows from Theorem 4.1 of [2] applying to A^* and Y.

(b) Given the condition, we know that the A^*-perturbation bilinear form of $(\mathcal{E},\mathcal{D})$ is lower semi-bounded, that is, there exists $q>0$ such that for $u\in\mathcal{D}^{-A^*}$,

$$\mathcal{E}_q(u,u)-\rho_{A^*}(u^2)\ge 0.$$

By Lemma 2.3, $\mathcal{D}^{-A} = \mathcal{D} \cap L^2(\rho_A) \subset \mathcal{D}^Y$ and for $u \in \mathcal{D}^{-A}$,

$$\rho_{A^*}(u^2) \leq \mathcal{E}_q(u,u) \leq \mathcal{E}_q^Y(u,u);$$

i.e., $\mathcal{E}_q(u,u) - \nu_A(u \otimes u) \geq 0$. It means that the A-perturbation bilinear form of $(\mathcal{E}, \mathcal{D})$ is lower semi-bounded and therefore by (a) it follows that the A-perturbation semigroup of (P_t) is strongly continuous on $L^2(m)$. □

Now comes our main theorem of this section.

Theorem 2.5. *Suppose that $A \in \mathcal{P}_K$. Then the A-perturbation semigroup of (P_t) is a strongly continuous semigroup of symmetric bounded operators on $L^2(E,m)$.*

Proof. That $A \in \mathcal{P}_K$ implies that $A^* \in \mathcal{P}_K$. By Theorem (4.15) of [7], the A^*-perturbation semigroup of (P_t) is a strongly continuous semigroup of bounded operators on $L^2(E,m)$. Hence the conclusion follows from Lemma 2.4(b). □

3. Perturbation bilinear forms.

In this section we are going to further characterize the relationship between the A-perturbation semigroup and A-perturbation bilinear form of X. First we introduce the resolvent corresponding the perturbation semigroup.

We know that if $A \in \mathcal{A}$, then $\{1/(\mathrm{Exp}A)_t\}_{t>0}$, is a decreasing multiplicative functional of X, which we denote by $(\mathrm{Exp}A)^-$. It is easy to see that $(\mathrm{Exp}A)^-$ does not vanish before ζ and it is the unique solution of the equation

$$Z_t = 1 - \int_0^t Z_s dA_s. \tag{3.1}$$

Let $A^1, \cdots A^a$, $B^1, \cdots B^b$, $K^1, \cdots K^k$, $L^1, \cdots L^l \in \mathcal{A}$ and introduce notations as follows.

$$\begin{aligned}
\mathcal{L} &:= [A^1, \cdots A^a, -B^1, \cdots - B^b, K^1_-, \cdots K^k_-, -L^1_-, \cdots - L^l_-];\\
(\mathrm{Exp}\mathcal{L})_t &:= \prod_{1\leq i\leq a} (\mathrm{Exp}A^i)_t \prod_{1\leq i\leq b} (\mathrm{Exp}B^i)^-_t \prod_{1\leq i\leq k} (\mathrm{Exp}K^i)_{t-} \prod_{1\leq i\leq l} (\mathrm{Exp}L^i)^-_{t-}.
\end{aligned}$$

Clearly $\mathrm{Exp}\mathcal{L}$ is still a multiplicative functional of X which does not vanish before ζ. Note that the order in $\mathcal{L}$ is not relevent and if some elements in $\mathcal{L}$ vanish, they can be simply removed. Then we define

$$\begin{aligned}
P_t^{-\mathcal{L}} f(x) &:= P^x((\mathrm{Exp}\mathcal{L})_t f(X_t)),\\
U_L^{q-\mathcal{L}} f(x) &:= P^x \int_{]0,\infty[} e^{-qt} (\mathrm{Exp}\mathcal{L})_t f(X_t) dL_t,
\end{aligned} \tag{3.2}$$

where $f \in p\mathcal{B}(E)$, $x \in E$, $q \geq 0$ and $L \in \mathcal{A}$. Obviuously P_t^{-A} defined in §2 coincides with $P_t^{-[A]}$. Thus we also write U_L^{q-A} (resp. U_L^{q-A-}) for $U_L^{q-[A]}$ (resp. $U_L^{q-[A-]}$). The following lemma gives a few formulas similar to the resolvent equation.

Lemma 3.1. *Let A^1, A^2, B^1, B^2, $L \in \mathcal{A}$, $q \geq 0$ and $f \in p\mathcal{B}(E)$.*
(a) *If $U^{q-[A^1,-A^2]}f(x) < \infty$, then*

$$\begin{aligned} &U_L^{q-[A^1,-A^2]}f(x) + U_{B^1}^{q-[A^1,-A^2,B^1_-,-B^2_-]}U_L^{q-[A^1,-A^2]}f(x) \\ &= U_L^{q-[A^1,-A^2,B^1_-,-B^2_-]}f(x) + U_{B^2}^{q-[A^1,-A^2,B^1,-B^2]}U_L^{q-[A^1,-A^2]}f(x). \end{aligned}$$

(b) *If B^1, B^2 are continuous and $U_L^{q-[A^1,-A^2]}f(x) < \infty$, then*

$$\begin{aligned} &U_L^{q-[A^1,-A^2]}f(x) + U_{B^1}^{q-[A^1,-A^2]}U_L^{q-[A^1,-A^2,B^1,-B^2]}f(x) \\ &= U_L^{q-[A^1,-A^2,B^1,-B^2]}f(x) + U_{B^2}^{q-[A^1,-A^2]}U_L^{q-[A^1,-A^2,B^1,-B^2]}f(x). \end{aligned}$$

(c) *If $A \in \mathcal{A}$ and $U_L^q f(x) < \infty$, then*

$$U_L^{q-A-}f(x) = U_L^q f(x) + U_A^q U_L^{q-A-}f(x).$$

Proof. (a) By (2.4), (3.1) and using the Markovian property,

$$\begin{aligned} &U_{B^1}^{q-[A^1,-A^2,B^1_-,-B^2_-]}U_L^{q-[A^1,-A^2]}f(x) \\ &= P^x \int_{]0,\infty[} e^{-qt}(\mathrm{Exp}[A^1,-A^2])_t(\mathrm{Exp}B^2)_{t-}^{-}d(\mathrm{Exp}B^1)_t \\ &\quad \cdot \left(\int_{]0,\infty[} e^{-qs}(\mathrm{Exp}[A^1,-A^2])_s f(X_s)dL_s \right) \circ \theta_t \\ &= P^x \int_{]0,\infty[} (\mathrm{Exp}B^2)_{t-}^{-} \int_{]t,\infty[} e^{-qs}(\mathrm{Exp}[A^1,-A^2])_s f(X_s)dL_s d(\mathrm{Exp}B^1)_t \\ &= -U_L^{q-[A^1,-A^2]}f(x) \\ &\quad + P^x \int_{]0,\infty[} (\mathrm{Exp}[B^1_-,-B^2_-])_t d\left(-\int_{]t,\infty[} e^{-qs}(\mathrm{Exp}[A^1,-A^2])_s f(X_s)dL_s \right) \\ &\quad - P^x \int_{]0,\infty[} (\mathrm{Exp}B^1)_t \int_{]t,\infty[} e^{-qs}(\mathrm{Exp}[A^1,-A^2])_s f(X_s)dL_s d(\mathrm{Exp}B^2)_t^{-} \\ &= -U_L^{q-[A^1,-A^2]}f(x) + U_L^{q-[A^1,-A^2,B^1_-,-B^2_-]}f(x) \\ &\quad + U_{B^2}^{q-[A^1,-A^2,B^1,-B^2]}U_L^{q-[A^1,-A^2]}f(x). \end{aligned}$$

The proof of (b) is similar.

(c) By a direct computation, we have

$$\begin{aligned} U_A^q U_L^{q-A-} f(x) &= P^x \int_{]0,\infty[} e^{-qt} dA_t \left(\int_{]0,\infty[} e^{-qs+A_{s-}} f(X_s) dL_s \right) \circ \theta_t \\ &= -P^x \int_{]0,\infty[} d(\mathrm{Exp}A)_t^- \int_{]t,\infty[} e^{-qs} (\mathrm{Exp}A)_{s-} f(X_s) dL_s \\ &= U_L^{q-A-} f(x) - P^x \int_{]0,\infty[} (\mathrm{Exp}A)_{t-}^- e^{-qt} (\mathrm{Exp}A)_{t-} f(X_t) dL_t \\ &= U_L^{q-A-} f(x) - U_L^q f(x). \end{aligned}$$

That completes the proof. □

Now we assume that $A \in \mathcal{P}_K$ and define $\beta(A)$ to be the minimum β such that

$$||P^{\cdot}(\mathrm{Exp}A)_t||_Q \leq c \cdot e^{\beta t}$$

holds for a constant c and all $t > 0$. Clearly $\beta(A) < \infty$.

Theorem 3.2. *Let $A \in \mathcal{P}_K$ and $q > \beta(A)$.*

(a) *For all $f \in L^2(m)$, $U^{q-A} f \in \mathcal{D}^{-A}$.*

(b) *For all $f \in L^2(m)$ and $u \in \mathcal{D}$,*

$$(f, u) = \mathcal{E}_q(U^{q-A} f, u) - \nu_A(U^{q-A} f \otimes u).$$

Hence for $u \in \mathcal{D}^{-A}$, $(f, u) = \mathcal{E}_q^{-A}(U^{q-A} f, u)$.

(c) *$(\mathcal{E}^{-A}, \mathcal{D}^{-A})$ is a closable lower semibounded bilinear form on $L^2(m)$.*

(d) *$\mathcal{D}^{-A} = \mathcal{D}$.*

(e) *If $2k(A) < 1$, then $(\mathcal{E}^{-A}, \mathcal{D}^{-A})$ is closed.*

Before proving this theorem, we will present a few lemmas first. We should also mention that many ideas and approaches come directly from [2].

Lemma 3.3. *Let $A \in \mathcal{P}_K$ and $q > \beta(A)$. Then*

$$||U^{q-A} 1||_Q + ||U_A^{q-A-} 1||_Q + ||U_A^q 1||_Q < \infty. \tag{3.3}$$

Proof. It is obvious that since $q > \beta(A)$, $a := ||P^x \int_0^\infty e^{-qt} (\mathrm{Exp}A)_t dt||_Q < \infty$. Now it is easily seen that $||U^{q-A} 1||_Q \leq a$ and

$$\begin{aligned} U_A^{q-A-} 1 &\leq P^{\cdot} \int_0^\infty e^{-qt} (\mathrm{Exp}A)_{t-} dA_t \\ &= P^{\cdot} \int_0^\infty e^{-qt} d(\mathrm{Exp}A)_t = -1 + qP^{\cdot} \int_0^\infty e^{-qt} (\mathrm{Exp}A)_t dt. \end{aligned}$$

Hence $||U_A^{q-A-}1||_Q \leq qa - 1 < \infty$. Finally $U_A^q 1 \leq U_A^{q-A-}1$. That completes the proof. $\square$

Lemma 3.4. *Let $A \in \mathcal{P}_K$ and $q > \beta(A)$.*
(a) *If $g \in L^2(m + \rho_A)$ and $U_A^q g \in L^2(\rho_A)$, then $U_A^q g \in \mathcal{D}$.*
(b) *For any $\alpha > 0$, $U^\alpha(L^2(m)) \subset L^2(\rho_A)$.*
(c) *If $f \in L^2(m)$ and $U^{q-A}f \in L^2(\rho_A)$, then $U^{q-A}f \in \mathcal{D}$.*

Proof. (a) Taking the approximating form $\mathcal{E}_q^{(p)}$ of $\mathcal{E}_q$, we have for $g \geq 0$,

$$\begin{aligned}\mathcal{E}_q^{(p)}(U_A^q g, U_A^q g) &= p(U_A^q g, U_A^q g - pU^{q+p}U_A^q g)\\ &= p(U_A^q g, U_A^{q+p} g)\\ &= \nu_A(pU^{q+p}U_A^q g \otimes g).\end{aligned}$$

The last equality follows from the Revuz formula (see [8]): for any $u, v \in p\mathcal{B}(E)$,

$$(u, U_A^\alpha v) = \nu_A(U^\alpha u \otimes v). \tag{3.4}$$

Since $U_A^q g$ is q-excessive,

$$\sup_p \mathcal{E}_q^{(p)}(U_A^q g, U_A^q) = \nu_A(U_A^q g \otimes g) < \infty,$$

by the conditions. Hence $U_A^q g \in \mathcal{D}$.

(b) By the Revuz formula (3.4) again, for $f \in L^2(m)$,

$$\rho_A[(U^\alpha f)^2] = \nu_A((U^\alpha f)^2 \otimes 1) \leq \frac{1}{\alpha}\nu_A(U^\alpha f^2 \otimes 1) \leq (f^2, U_A^\alpha 1).$$

Thus (b) follows from (3.3).

(c) By Lemma 3.1(c) it also suffices to show that $U_A^q U^{q-A} f \in \mathcal{D}$ for $f \geq 0$. We know by (b) and Lemma 3.1(c) that $U_A^q U^{q-A} f = U^{q-A} f - U^q f \in L^2(\rho_A)$. Hence it follows from (a) that $U_A^q U^{q-A} f \in \mathcal{D}$. $\square$

Remark. Suppose that ξ is a smooth measure of extended Kato class. We may easily see from (b) that $U^\alpha(U^2(m)) \subset L^2(\xi)$ for $\alpha > 0$.

The key to prove that U^{q-A} carries $L^2(m)$ into $\mathcal{D}$ is to prove that it carries $L^2(m)$ into $L^2(\rho_A)$. We need a genaralized Revuz formula. Let $\mathcal{L}^q$ be the q-energy functional of X which is m-symmetric. We list two properties of $\mathcal{L}^q$ which may be checked easily by using the properties of energy functional (see [6]).

L-1. $\mathcal{L}^q$ is m-symmetric in the sense that $\mathcal{L}^q(h_1 m, h_2) = \mathcal{L}^q(h_2 m, h_1)$ for q-excessive functions h_1 and h_2.

L-2. If $A \in \mathcal{A}$, $f \in p\mathcal{B}(E)$ and h is q-excessive, then $\mathcal{L}^q(hm, U^q_A f) = \nu_A(h \otimes f)$.
The Revuz formula (3.4) follows easily from L-2.

Lemma 3.5. *Let $A \in \mathcal{P}_K$ and $q > \beta(A)$. Suppose that A^1, $A^2 \in \mathcal{P}_0$. Then we have a so-called generalized Revuz formula*

$$\nu_{A^1}(U^{q-A-}_{A^2} f_2 \otimes f_1) = \nu_{A^2}(U^{q-A-}_{A^1} f_1 \otimes f_2), \tag{3.5}$$

for f_1, $f_2 \in p\mathcal{B}(E)$.

Proof. It follows from Lemma 3.1(c) that $U^{q-A-}_{A^i} f_i$, $i = 1, 2$ are q-excessive for X. Hence by (L-2) we have

$$\begin{aligned}
&\mathcal{L}^q((U^{q-A-}_{A^1} f_1) \cdot m, U^{q-A-}_{A^2} f_2) \\
&= \mathcal{L}^q((U^{q-A-}_{A^1} f_1) \cdot m, U^q_{A^2} f_2) + \mathcal{L}^q((U^{q-A-}_{A^1} f_1) \cdot m, U^q_A U^{q-A-}_{A^2} f_2) \\
&= \nu_{A^2}(U^{q-A-}_{A^1} f_1 \otimes f_2) + \nu_A(U^{q-A-}_{A^1} f_1 \otimes U^{q-A-}_{A^2} f_2).
\end{aligned}$$

Switching A^1 and A^2 respectively, we also have

$$\begin{aligned}
&\mathcal{L}^q((U^{q-A-}_{A^2} f_2) \cdot m, U^{q-A-}_{A^1} f_1) \\
&= \nu_{A^1}(U^{q-A-}_{A^2} f_2 \otimes f_1) + \nu_A(U^{q-A-}_{A^2} f_2 \otimes U^{q-A-}_{A^1} f_1).
\end{aligned}$$

By symmetry (L-1), we see that (3.5) holds as soon as

$$\nu_A(U^{q-A-}_{A^1} f_1 \otimes U^{q-A-}_{A^2} f_2) < \infty. \tag{3.6}$$

We first assume $k(A^1) = k(A^2) = 0$. Then $L := A + A^1 \in \mathcal{P}_K$. Let $s > \beta(L)$. Then by Lemma 3.5

$$||U^{s-A-}_{A^1} 1||_Q \leq ||U^{s-L-}_L 1||_Q < \infty.$$

But mimicking the proof of Lemma 3.1(b) and taking a special case, we have

$$U^{q-A-}_{A^1} 1 = U^{s-A-}_{A^1} 1 + (s-q) U^{q-A} U^{s-A-}_{A^1} 1.$$

which is similar to the resolvent equation. Hence $||U^{q-A-}_{A^1} 1||_Q < \infty$ and similarly $||U^{q-A-}_{A^2} 1||_Q < \infty$.

Since $A \in \mathcal{P}_K$, A^* is a PCAF and we may choose, for the smooth measure ρ_{A^*}, an increasing sequence $\{E_n\}$ of subsets of E such that (i) for any $n \geq 1$, $1_{E_n} \rho_{A^*}$ is a finite measure of Kato class; (ii) $\rho_{A^*}(E - \cup_{n=1}^{\infty} E_n) = 0$; (iii) $\lim_n \mathrm{Cap}(K - E_n) = 0$ for any compact set K. Set $B_n := E_n \times E_n$. Then $1_{B_n} \nu_A(E \times E) \leq \rho_{A^*}(E_n) < \infty$ and $1_{B^n} \nu_A \uparrow \nu_A$. Define

$$H^n_t := \int_0^t 1_{B_n}(X_{s-}, X_s) dA_s, \quad t > 0.$$

Then $H^n \in \mathcal{P}_K$ and $\nu_{H^n} = 1_{B^n}\nu_A$ for each $n \geq 1$. Thus (3.6), and then (3.5), holds for bounded f_1, f_2 by replacing A with H^n. Now the monotone convergence theorem (MCT) implies that (3.5) holds for all f_1, $f_2 \in p\mathcal{B}$ with B^n in place. Let n tend to infinity. Applying MCT again, (3.5) follows under our assumption $k(A^1) = k(A^2) = 0$. To remove this assumption, it suffices to use the arguments above for A^1, A^2 and MCT again. □

Remark. One may prove a slightly more general formula

$$(3.7) \qquad \nu_{A^1}((U_{A^2}^{q-A_-} f_2 \otimes 1) \cdot f_1) = \nu_{A^2}((U_{A^1}^{q-A_-} f_1 \otimes 1) \cdot f_2), \quad f_1, f_2 \in p\mathcal{B}(E \times E)$$

where

$$U_{A^1}^{q-A_-} f_1 := P^{\cdot} \int_0^\infty e^{-qt+A_{t-}} f_1(X_{t-}, X_t) dA_t^1,$$

and similar for the other.

Proof of Theorem 3.2. (a) We need only to show that for $f \in L^2(m)$, $U^{q-A} f \in L^2(\rho_A)$. Let c be the constant of the lefthand side in (3.3). We find by (3.5)

$$\begin{aligned} \rho_A(|U^{q-A} f|^2) &\leq c\rho_A(U^{q-A}|f|^2) \\ &= c\nu_A(U^{q-A}|f|^2 \otimes 1) \\ &= c\nu_m(U_A^{q-A_-} 1 \otimes |f|^2) \\ &\leq c^2 m(|f|^2) < \infty, \end{aligned}$$

where ν_m denotes the bivariate Revuz measure of dt which equals $\delta * m$.

(b) Without loss of generality we assume f, $u \geq 0$. First by Lemma 3.1(c), we have

$$(f, u) = \mathcal{E}_q(U^q f, u) = \mathcal{E}_q(U^{q-A} f, u) - \mathcal{E}_q(U_A^q U^{q-A} f, u).$$

Employing the approximating form, resolvent equation and Revuz formula,

$$\begin{aligned} \mathcal{E}_q(U_A^q U^{q-A} f, u) &= \lim_p p(U_A^{q+p} U^{q-A} f, u) \\ &= \lim_p \nu_A(U^{q-A} f \otimes pU^{q+p} u). \end{aligned}$$

A similar argument as in the proof of (a) shows that the measure $\nu_A(U^{q-A} f \otimes \cdot)$ is finite if f is L^1-integrable. It is clear that $pU^{q+p}u \longrightarrow u$ q.e. as $p \to \infty$. Let $u_n := u \wedge n$ and $f_n := 1_{E_n}(f \wedge n)$, where $\{E_n\}$ is a sequence chosen for m as in the proof of Lemma 3.4. Then $m(f_n) < \infty$ and by the dominated convergence theorem, we have

$$\lim_p \nu_A(U^{q-A} f_n \otimes pU^{q+p} u_l) = \nu_A(U^{q-A} f_n \otimes u_l).$$

Hence

$$(f_n, u_l) + \nu_A(U^{q-A} f_n \otimes u_l) = \mathcal{E}_q(U^{q-A} f_n, u_l). \tag{3.8}$$

First $u_l \uparrow u$ q.e. and in $\mathcal{E}_q$-norm. Thus we may erase l in (3.8) by MCT. Now $f_n \uparrow f$ a.e.-m, also in L^2, and then $U^{q-A} f_n \uparrow U^{q-A} f$ q.e., also in L^2. A little more computation shows that

$$\mathcal{E}_q(U^{q-A} f_n, U^{q-A} f_n) \leq \mathcal{E}_q(U^{q-A} f, U^{q-A} f).$$

Hence $U^{q-A} f_n \longrightarrow U^{q-A} f$ weakly in $(\mathcal{E}, \mathcal{D})$ and (b) follows.

(c) Let $(\bar{H}, D(\bar{H}))$ be the generator of (P_t^{-A}). Since (P_t^{-A}) is strongly continuous and symmetric, $(q - \bar{H}, D(\bar{H}))$ is a positive definite self adjoint operator on $L^2(m)$, and hence it is associated with a closed quadratic form, say $(\bar{\mathcal{E}}_q, \bar{\mathcal{D}})$. It is easy to check that $\mathcal{D} \subset \bar{\mathcal{D}}$ by the approximating form. Hence we have an inclusion chain: $U^{q-A}(L^2(m)) = D(\bar{H}) \subset \mathcal{D}^{-A} \subset \mathcal{D} \subset \bar{\mathcal{D}}$. Now we will show that the restriction of $\bar{\mathcal{E}}_q$ on $\mathcal{D}^{-A}$ is nothing but $\mathcal{E}_q^{-A}$. By Lemma 3.4(b), we know that $U^q(L^2(m)) \subset \mathcal{D} \cap L^2(\rho_A) = \mathcal{D}^{-A} \subset \bar{\mathcal{D}}$. Let $f \in L^2(m)$ nonnegative and $g := U^q f$. A switching order of integration gives for $p > 0$

$$pU^{p+q-A} g = P^{\cdot} \int_0^\infty e^{-qt} f(X_t) \left(p \int_0^t e^{-ps+A_s} ds \right) dt.$$

Hence we conclude that

$$pU^{p+q-A} g \leq U^{q-A} f \quad \text{and} \quad pU^{p+q-A} g \longrightarrow g \text{ q.e.}$$

Now we have

$$\begin{aligned} p(g, g - pU^{p+q-A} g) &= p(g, g - pU^{p+q} g) - p^2(g, U_A^{p+q} U^{p+q-A} g) \\ &= p(g, g - pU^{p+q} g) - \nu_A(pU^{p+q} g \otimes pU^{p+q-A} g). \end{aligned}$$

Since $0 \leq pU^{p+q} g \otimes pU^{p+q-A} g \leq g \otimes U^{q-A} f$ and $g, U^{q-A} f \in L^2(\rho_A)$, it follows from the dominated convergence theorem that $\bar{\mathcal{E}}_q(g, g) = \mathcal{E}_q^{-A}(g, g)$; namely the restriction of $\bar{\mathcal{E}}_q$ on $U^q(L^2(m))$ is $\mathcal{E}_q^{-A}$. A consequence is that for all $g \in U^q(L^2(m))$,

$$\nu_A(|g| \otimes |g|) \leq \mathcal{E}_q(|g|, |g|) \leq \mathcal{E}_q(g, g). \tag{3.9}$$

For any $g \in \mathcal{D}^{-A} \subset \mathcal{D}$, there exists a sequence $\{g_n\} \subset U^q(L^2(m))$ which converges to g in $\mathcal{E}_q$-norm. By (3.9) $\{g_n\}$ is an $\bar{\mathcal{E}}_q$-Cauchy sequence and $\{g_n\}$ converges to g in $\bar{\mathcal{E}}_q$-norm. Consequently $\bar{\mathcal{E}}_q(g,g) = \mathcal{E}_q^{-A}(g,g)$ and the restriction of $\bar{\mathcal{E}}_q$ on $\mathcal{D}^{-A}$ coincides with $\mathcal{E}_q^{-A}$. Hence $(\mathcal{E}^{-A}, \mathcal{D}^{-A})$ is closable and for $u \in \mathcal{D}^{-A}$,

$$\nu_A(|u| \otimes |u|) \leq \mathcal{E}_q(u,u). \tag{3.10}$$

(d) $\mathcal{D}^{-A}$ is dense in $\mathcal{D}$ with $\mathcal{E}_q$-norm since it contains $U^q(L^2(m))$ by Lemma 3.4(b). Let $u \in p\mathcal{D}$. there exists a sequence $\{u_n\} \subset p\mathcal{D}^{-A}$ such that $u_n \to u$ in $\mathcal{E}_q$-norm. By (3.10) and using Fatou's lemma,

$$\nu_A(u \otimes u) \leq \liminf_n \nu_A(u_n \otimes u_n) \leq \mathcal{E}_q(u,u). \tag{3.11}$$

However (3.11) holds for any $A \in \mathcal{P}_K$ (with a different q) and certainly holds for A^*. Hence

$$\rho_A(u^2) = \rho_{A^*}(u^2) = \nu_{A^*}(u \otimes u) \leq \mathcal{E}_q(u,u),$$

i.e., $\mathcal{D}^{-A} = \mathcal{D} \cap L^2(\rho_A) = \mathcal{D}$.

(e) Since $2k(A) < 1$, A, $2A \in \mathcal{P}_K$ and we may choose s large enough such that for $u \in \mathcal{D} = \mathcal{D}^{-A}$

$$\begin{aligned}
0 &\leq \mathcal{E}_q(u,u) - \nu_A(u \otimes u) \\
&\leq \mathcal{E}_q(u,u) \\
&\leq \mathcal{E}_s(u,u) - 2\nu_A(u \otimes u) + \mathcal{E}_q(u,u) \\
&= \mathcal{E}_s^{-A}(u,u) + \mathcal{E}_q^{-A}(u,u) \\
&= \bar{\mathcal{E}}_s(u,u) + \bar{\mathcal{E}}_q(u,u).
\end{aligned}$$

Since $\mathcal{D}$ is dense in $\bar{\mathcal{D}}$ in $\bar{\mathcal{E}}_q$-norm, for any $w \in \bar{\mathcal{D}}$, there exists $\{w_n\} \subset \mathcal{D}$ such that $w_n \to w$ in $\bar{\mathcal{E}}_q$-norm, then in $\bar{\mathcal{E}}_s$-norm. By the inequality above, $\{w_n\}$ is a $\mathcal{E}$-Cauchy sequence and $w_n \to w$ in L^2. Hence $w \in \mathcal{D}$; namely $\mathcal{D} = \bar{\mathcal{D}}$. That completes the proof. $\square$

4. Examples.

In this section we shall use Lévy processes to construct two examples.

EXAMPLE 4.1. Let X be a symmetric Lévy process on $\mathbb{R}^d$ with its Lévy exponent

$$\phi(x) = \frac{1}{2}(Sx,x) + \frac{1}{2}\int (1 - \cos(x,y))J(dy), \tag{4.1}$$

where S is a $d \times d$ nonnegative definite matrix and J a symmetric measure on $\mathbb{R}^d$ carried by $\mathbb{R}^d - \{0\}$ satisfying $\int (1 \wedge |x|^2) J(dx) < \infty$. Then the corresponding Dirichlet form is

$$\begin{aligned} \mathcal{D} &= \left\{ u \in L^2 : \int \phi |\hat{u}(x)|^2 dx < \infty \right\}, \\ \mathcal{E}(u,u) &= \int \phi(x) |\hat{u}(x)|^2 dx. \end{aligned} \tag{4.2}$$

Let h a positive symmetric function on $\mathbb{R}^d$ with $h(0) = 0$, and $A_t := \sum_{s \le t} h(\Delta X_s)$. Then $(\mathrm{Exp} A)_t = \prod_{s \le t} (1 + h(\Delta X_s))$ and $A \in \mathcal{P}_K$ if and only if h is J-integrable. We may see that the form defined below is a lower semibounded closed quadratic form associated with the perturbation by $((\mathrm{Exp} A)_t)$:

$$\begin{aligned} \mathcal{D}^h &:= \mathcal{D}; \\ \mathcal{E}^h(u,u) &:= \mathcal{E}(u,u) - \int \int u(x+y) u(x) dx h(y) J(dy). \end{aligned} \tag{4.3}$$

and the perturbation semigroup is still spatially homogeneous with Lévy exponent

$$\phi^h(x) = \frac{1}{2}(Sx, x) + \frac{1}{2} \int (1 - \cos(x,y))(1 + h(y)) J(dy) - J(h). \tag{4.4}$$

In the case that X is a symmetric stable process of index $\alpha \in]0, 2[$, the condition means

$$\int \frac{h(x)}{|x|^{d+\alpha}} dx < \infty. \tag{4.5}$$

The following example shows that we can not expect the Khas'minskii's lemma holds for the natural exponential function.

EXAMPLE 4.2. Let X be a Lévy process on Z, the set of integers, with convolution semigroup π given by

$$\pi_t = e^{-t} \sum_n \frac{t^n}{n!} J^{*n},$$

where J is a probability measure on Z defined by $J(\{-n\}) = J(\{n\}) := c/(n^2)$ for $n \ge 1$ and $J(\{0\}) := 0$. Let h be a function on Z defined by $h(n) := \log |n|$ for $n \ne 0$ and $h(0) = 0$. Set $A_t := \sum_{s \le t} h(\Delta X_s)$, which is an AF of the Kato class since $J(h) < \infty$. We claim that $Ee^{A_t} = \infty$ for any $t > 0$. (We write P^x as P since $P^x e^{A_t}$ does not depend on x while A is spatially homogeneous.)

Suppose that $Pe^{A_t} < \infty$. Then there exist constants $c, q > 0$ such that for all $s > 0$

$$Pe^{A_s} \leq ce^{qt}.$$

Hence $e^{-qs}Pe^{A_s} \leq c$ and

$$\begin{aligned} P\sum_{s\leq t} e^{-qs}(e^{\Delta A_s} - 1) &= P\int_0^t e^{-qt}\frac{de^{A_s}}{e^{A_{s-}}} \\ &\leq P\int_0^t e^{-qs}dA^{A_s} \\ &= e^{-qt}Ee^{A_t} - 1 + q\int_0^t e^{-qs}Pe^{A_s}ds \leq c - 1 + qtc < \infty. \end{aligned}$$

By Lévy system formula,

$$P\sum_{s\leq t} e^{-qs}(e^{\Delta A_s} - 1) = J(e^h - 1)\int_0^t e^{-qs}ds.$$

This leads to a contradiction since $e^{h(n)} - 1 = |n| - 1$ for $n \neq 0$ and $J(e^h - 1) = \infty$.

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