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List of Publications

Journal Articles

1. **António Oliveira Nzinga René**, Nobuyuki Ueno, and Koji Okuhara, Robust Fuzzy-Coalitional Game Model Applied to Water Resource Management, *European Journal of Operational Research.* (in preparation).

2. **António Oliveira Nzinga René**, Nobuyuki Ueno, Yuki Taguchi, and Koji Okuhara, An Available Solution for Multi-period Production Planning with Constraints Based on Shapley Value, *International Journal of Japan Association for Management Systems*, vol. 8, No. 1, pp. 47-56, 2016.

3. **António Oliveira Nzinga René**, Nobuyuki Ueno, Yuki Taguchi, and Koji Okuhara, Multi-Period Production Planning Using Shapley Value with Constraints, *ICIC Express Letters*, vol. 10, No. 2, pp. 355-362, 2016.

4. **António Oliveira Nzinga René**, Eri Domoto and Koji Okuhara, Shapley Solution Obtained via LP Using the Minimax Characteristic Function, *ICIC Express Letters*, Vol. 9, No. 5, pp. 1341-1348, 2015.

5. **António Oliveira Nzinga René**, Koji Okuhara and Eri Domoto, Allocation of Weights by Linear Solvable Process in a Decision Making Game, *ICIC Express Letters*, Vol. 8, No. 3, pp. 907-914. 2014.

Book Chapter

1. **António Oliveira Nzinga René**, Eri Domoto, Yu Ichifuji, and Koji Okuhara, A Social Network Analysis Based on Linear Programming-Shapley Value Approach for Information Mapping, *Multidisciplinary Social Networks Research* (Eds. L. Wang, S. Uesugi, I-H. Ting, K. Okuhara, and K. Wang), pp. 470-482, Springer, ISBN 978-3-662-48319-0, 2015 (Post-Conference Proceedings of the 2nd Multidisciplinary International Social Networks Conference)

Refereed Conference Publications

1. **António Oliveira Nzinga René**, Eri Domoto, Yu Ichifuji, and Koji Okuhara, A Social Network Analysis Based on Linear Programming-Shapley Value Approach for Information Mapping, *Proceedings of the 2nd Multidisciplinary International Social Networks Conference*, pp. 41, Matsuyama, Japan, September 2015, (Abstract).

2. **António Oliveira Nzinga René**, Nobuyuki Ueno, Yuki Taguchi, and Koji Okuahra, A Proposal for Multi-Period Production Planning Based on Linear Programming-Shapley Value Approach, *Proceedings of the 10th International Conference on Innovative Computing, Information and Control*, p. 59, Dalian, China, August 2015.

3. **António Oliveira Nzinga René**, Nobuyuki Ueno, Yuki Taguchi, and Koji Okuhara, Order Quantity Derivation Based on Risk Allocation in Production Planning, *Proceedings of the 4th International Conference on Informatics and Application Computing*, CD-ROM, Takamatsu, Japan, July 2015.

4. Eri Domoto, **António Oliveira Nzinga René** and Koji Okuhara, Quality of Education Founding on Students' GPA, PBL and Entrance Examination, *Proceedings of International Conference on Electronics and Software Science*, CD-ROM, Takamatsu, Japan, July 2015.

5. **António Oliveira Nzinga René**, Eri Domoto, and Koji Okuhara, Shapley Solution Obtained via LP Using the Minimax Characteristic Function, *Proceedings of the 9th International Conference on Innovative Computing, Information and Control*, CD-ROM, Busan, Korea, June 2014.

6. Eri Domoto, **António Oliveira Nzinga René** and Koji Okuhara, Analysis for Employment Support Using Student Data, *Proceedings of the 1st Multidisciplinary International Social Networks Conference*, CD-ROM, Kaohsiung, Taiwan, September 2014.

7. **António Oliveira Nzinga René**, Koji Okuhara, and Eri Domoto, Allocation of Weights by Linear Solvable Process in a Decision Making Game, *Proceedings of the 8th International Conference on Innovative Computing, Information and Control*, CD-ROM, Kumamoto, Japan, August 2013.

Domestic Conferences

1. **António Oliveira Nzinga René** and Koji Okuhara, Fuzzy-Coalitional Game Theoretic Model Applied to Water Resource Management, *Proceedings of the 29th Annual Conference of Biomedical Fuzzy Systems Association*, CD-ROM, Kochi, Japan, November, 2016.

2. **António Oliveira Nzinga René**, Nobuyuki Ueno and Koji Okuhara, Coalitional Model Applied to Production Planning Based on CVaR Approach, *Proceedings of the 28th Annual Conference of Biomedical Fuzzy Systems Association*, pp. 95-96, Kumamoto, Japan, November, 2015.

3. Koji Okuhara, **Ant'onio Oliveira Nzinga René**,Yu Ichifuji, and Noboru Sonehara, Sightseeing Information Recommendation, evacuation and Location for Disaster Prevention Considering Personal Attributes and Characteristics of Route, NICT, *Tech. Rep.*, pp. 51-56, December 2015.

4. **António Oliveira Nzinga René**, Nobuyuki Ueno, Yuki Taguchi and Koji Okuhara, Multi-period Production Planning Considering Resilience Based on Constrained Shapley Value, *Proceedings of the 54th Annual Conference of Japan Management System Association*, pp. 86-87, Gunma, Japan, May 2015.

5. **António Oliveira Nzinga René** and Koji Okuhara, Shapley Solution Obtained Via LP Using the Minimax Characteristic Function, *59th Annual Meeting of the Japan Operations Research Society, Students Presentation*, Tokyo, Japan, June 2014.

Preface

Making decisions is a commom and vital activity among human beings, indeed "Our lives are the sum of our decisions - whether in business or in personal spheres" as Thomas L. Saaty aptly puts it (Saaty, 1980). In the process, decision makers are subject to several factors which might influence their decisions, i.e., a personal preference may influence when deciding between different alternatives as regards to a set of criteria. Thus, implementing such policies may produce positive or negative outcomes consider, for instance, a company under a process of production planning. A possible solution for this particular problem is to find a generalized and systematic approach that is analytically adequate to subjective factors.

In this research, we made use of the great capacity of game theory, particularly cooperative game theory, to strategically analyze multicriteria decision making problems. The objective of this study can be outlined as follows *i)* to propose an efficient mathematical model to support decisions made under uncertainty, that is, a support tool to the group of deciders in order to achieve a certain level of group agreement opposite to an individual benefit; *ii)* to extend the concept of Shapley value in cases where ambiguity prevails factors not considered when using the original technique; *iii)* to demonstrate the applicability of game theoretic techniques to real-world situations. Firstly, the study combines game and risk theories by defining a typical characteristic function which incorporates elements of risk. This makes it possible to study the flow of risk parametrically in order to describe the numerical advantages or disadvantages of chosen policies during the process. Secondly, in order to deal with ambiguity we employ fuzzy theory concepts under coalitional game framework by proposing a minimax optimization model which is implemented to water resources allocation. Through numerical examples a multiperiod production planning problem with demand uncertainty is analyzed by using risk measures such as value-at-risk (VaR) and conditional value-at-risk (CVaR) and, moreover, a case study on the equitable sharing of international water is also considered.

Shapley value is a strong solution concept in cooperative game theory, but with some application in noncooperative too because of the possibility of bringing player to cooperate despite of their self interest. Thus connecting fuzziness to Shapley value would be interesting just to mention that ambiguity often is not analyzed when comes to emplying Shapley value or other solution concept. Therefore, a probable contribution of this research would be the possibility of offering an optimization model to support decisions-makers in uncertainty enviroment. Accordingly, the results from this study may be beneficial to the field of water resource management in the sense that decision-makers can get accurate information regarding their expectation, as well as to forecast economical benefits regarding the process of sharing the river water from the viewpoint of different sectors, and likewise it is possible to evaluate how the population and ecosystem around the basin are positively or negatively affected.

António Oliveira Nzinga René
Osaka University, December 2016

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Chapter 1

Introduction

We human beings are essentially decision makers (DMs). In fact, our lives are based on the decisions we make, good or bad for we make decisions all the time consciously or unconsciously. This thesis, for instance, is a result of a decision made a few years in the past. Thus, in every sector of life, individuals, companies, academics, politics, etc., decisions are very important element to be considered due to the implications their results may produce. This fact is more evident especially in environments where a certain group of people play the role of DM. This group has to face several factors which might influence the performance of their decisions. For instance, a personal preference may influence with regard to decisions on different alternatives based on a set of criteria [59]. Hence, implementing such policies may produce positive or negative outcomes if considering, for instance, a company under the process of production planning.

Overcoming uncertainty in decision making is an important issue for research. Once the uncertainty is detected and solved, decisions can be efficient. For instance, a company may have a loss of revenue if production managers or those responsible for decisions do not consider uncertain factors seriously; a group of DMs can perform their activities well if they have the right information in order to avoid subjective factors.

For centuries, probability theory and error calculus has led the research related to uncertainty for being the only methodology accepted to treat uncertainty [24]. However, in recent times new studies have emerged and, consequently, different approaches have been proposed to deal with uncertainty within the decision theory and other fields. These methods are capable to solve problems where classical probability theory cannot succeed. For instance, with regards to intangible elements, the analytic hierarchy process (AHP) and its extension analytic network process (ANP) proposed by Saaty [58, 59] have been applied. Both methodologies are used for measurement through pairwise comparisons and rely on the judgements of experts to derive priority scales. As judgments may be inconsistent, the concern of AHP is to measure inconsistency and improve the judgements when possible to obtain better consistency. Zadeh [77] proposed the Fuzzy Set Theory to handle incomplete numerical and linguistic information, DM's subjectivity, etc. The theory is essentially non-statistical in nature, providing a natural procedure for dealing with problems in which the source of imprecision is evident. The technique has been extended by introducing the concept of fuzzy variables by Kaufmann [30], and explored by others [45, 78]. Stochastic Optimization (SO) and Sensitivity Analysis (SA) are traditional approaches to treating data uncertainty in Optimization. Linear Programming (LP) models [25] can, often, be used to get important information. However uncertainty may reside within the data making the LP model become uncertain. This issue is considered in methods such as Robust Optimization (RO) suggested in [3, 4, 8]. The method consits of detecting data uncertainty, which can heavily be affect the quality of the nominal solution and generate a robust solution [3].

In this research we treat the problem of uncertainty under the framework of Game Theory (GT), particularly cooperative game theory (CGT), also known as coalitional game theory. Within the thesis, we use both terms interchangeably to refer to the same nature of games. Game theory uses mathematical models known as games to capture the key attributes of scenarios in which self-interested players interact. The word *game* is used in a technical sense, that is, the technical sense of game theory. Therefore, obviously, it does not refer to the games in the everyday recreational sense (chess, checkers, poker, etc), although the use of that term (and much of the associated theory) was originally derived from the study of recreational games such as poker [29, 49, 50].

A player is an independent decision making unit with a certain interest regarding its decisions [11, 13]. According to the context, players may represent persons such as consumers, suppliers of certain services, a group of tradesmen, politicians, or subjects, or even a group of people with common interests, a corporate conglomerate, or even a nation. Basically, a player can make decisions, take actions, and have objectives to achieve; that is, he/she has choices over the actions during the game.

1.1 Consensus Decision Making

Consensus decision making is a formal, structured process for making decisions, which leads to a non-violent resolution of conflicts with the cooperation of decision that everyone involved can support. This type of decision making is usually performed in major companies, the North Atlantic Treaty Organization (NATO) [48], the United Nations (UN) [46], and several activists or nonprofit groups [10, 12, 62].

Consensus is achieved when all the members of the group consent to the final decision even when that decision is opposite to their personal preference; the point is to choose the best decision for the group and not the individual. The concept can be understand through the following summary [62].

- Consensus is not unanimity: consensus is simply the process for deciding what is best for a group. It is a decision to which the group consent for representing the best choice for the group.
- Consensus is a cooperative process: it is a process for people who want to work together to seek the good solution.
- Consensus may lead to democracy: gives participants power to express their opinions and make decisions, and at the same time demands complete responsibility for those decisions.
- Skills and the desire to cooperate are requirements for consensus: people in the process have to be committed and open to accountability and willing to help.

Herrera-Viedma et al., [1] proposed a consensus model for multiperson decision making with different structures. Their model evaluates two types of consensus measures, namely: consensus degree, and the linguistic distances. These elements are applied in three acting levels, i.e., level of preference, level of alternative, and level of preference relation.

1.2 Motivation and Purpose of the Research

This research explores the great capacity of game theory to strategically analyze multicriteria decision making problems. The motivation behind this study is that very often, solution concepts in cooperative game theory, in particular, do not treat uncertainty directly. A related study performed by Aghassi and Bertsimas [2] treats robustness in noncooperative game theory (NGT) by using concepts of RO, while this work takes the direction of fuziness to treat robustness in CGT. We incorporate Shapley value to situations where ambiguity is prevalent. To extend the large possibility of applications of game theory, we also analyse production planning problem under the framework of CGT. In general, through this research we aim to propose mathematical models to support DMs while dealing with uncertainty.

1.3 Thesis Structure

Starting with this introduction through which we present an overview of the problem we attempt to solve, the thesis is divided in five chapters described as follows.

The focus of Chapter 2 is Shapley value. The chapter starts by describing some concepts related to GT in Subsection 2.1. Important concepts to distinguish the two branches of GT, i.e., cooperative games and noncooperative games are described. In Section 2.2 Shapley value is introduced and the theory behind the value; the next part of the chapter, that is, Section 2.3 is reserved to the extension of Shapley value to fuzzy concepts in order to deal with ambiguity existent in the process of decision making. A numerical illustration regarding the computation of Shapley values in fuzzy framework is then considered in Section 2.4.

In Chapter 3 we combine cooperative games with production planning considering risk measures, such as value-at-risk (VaR) and conditional value-at-risk (CVaR) to propose models to solve production planning problems in a multi-period time. Three models for inventory management are proposed. These models may serve as tools to support production managers to forecasting. Several cases are considered in Section 3.3 followed by a discussion on the results, respectively.

Chapter 4 extends models of Fuzzy Shapley value in Chapter 2 by introducing a minimax model. The the characteristic functions as in the other cases represent fuzzy numbers. Through the proposed models DMs may detect the level of ambiguity while the model generates a robust solution. Two case studies related to water resources management are considered as numerical examples.

The thesis is concluded in Chapter 5 through a summary with regard to the results of this research, followed by a description on main contribution of the work. Future directions are also pointed.

Chapter 2

Foundations of Shapley Value

Game Theory is a branch of Mathematics used to solve problems where players (or DMs) interact strategically in situation where conflict and cooperation can be considered. Applications of this approach to solve real-world problems are vast in engineering field, politics, economics, etc. This chapter, starts by describing some theoretic concepts of game theory in Section 2.1. Shapley value, probably the most known solution concept in CGT is introduced in Section 2.2. The following section is dedicated to the extension of the value to an LP model, followed by a fuzzy Shapley value model to support the process of decision making when uncertaint factors have to be considered. The chapter concludes in Section 2.4 where a numerical example is considered.

2.1 Basic Concepts of Cooperative n-Person Games

Denote the finite set of players N, e.g., a game with three persons, or simply a 3-person game, will be denoted as the set of players $N = \{1, 2, 3\}$, and for a 6-person game one just need to add three more elements into the later set, i.e., $N = \{1, 2, 3, 4, 5, 6\}$ [29, 71].

While players are autonomous, they can make agreement to coordinate how they play through joint decisions which, probably, could not be guaranteed in case of acting independently. This agreement or cooperation is called a *coalition.*

A coalition, mathematically, is a subset of the set of players N hereinafter defined by $\mathcal{S}$. The coalition $\mathcal{S}$ is formed relying on the agreement between every player in the set and by no player not in $\mathcal{S}$, that is, $N - \mathcal{S}$. Additionally, it is not allowed agreement between any player of $\mathcal{S}$ and those of $N - \mathcal{S}$. Hence, the agreement plays an important role to the coalition formation.

The *grand coalition*, the coalition of all n players, is referred to as a coalition N. A *coalition structure* describes how players in N divide into common exclusive and complete coalitions. Any partition of the players can be described by the set $\mathcal{L} = \{S_1, S_2, \ldots, S_m\}$ of the m coalitions that formed. Basically, $\mathcal{L}$ is a partition of N satisfying the following three conditions:

- $\mathcal{S}_j \neq \emptyset$, $j = 1, 2, \ldots,$ m,
- $\mathcal{S}_i \cup \mathcal{S}_j = \emptyset$ for all $i \neq j$, and
- $\bigcup_{\mathcal{S}_j \in \mathcal{L}} \mathcal{S}_j = N$

This conditions indicate that each player is in one and only one of the m nonempty coalitions within the coalition structure, and pointing out that all elements of a coalition are associated each other, yet not to anyone not in the coalition.

An *outcome* constitutes the main goal of the game, while a quantitative representation of the outcome of a player in the game is called *payoff*, i.e., each player i at the end of the game receives a payoff, which is denoted by x_i. The set of payoffs to all players may be expressed as the row vector $X =< x_1, x_2, \ldots, x_n >$ of each player's payoff. X is called as *payoff vector*. Thus a payoff is a number representing the worth to a player of an outcome of a game.

2.1.1 Cooperative vs. Noncooperative Games

The absence of pure conflicts between players implies that there exists the possibility of agreement among players. This fact helps to distinguish cooperative to noncooperative games. Basically, in the former players may embark to mutual agreements; it is expected that:

- Players can negotiate before the game starts.
- All negotiations are clearly known by each player and their intended targets.
- All agreements are determined and completed to the rules of the game.
- The set of outcomes does not depend on the rules from the negotiations.

The expression *cooperative* implies that players may team up to their common profit; however, this profit is not always guaranteed. In the later type of games, i.e., *noncooperative*, a prior agreement among the players is not allowed. Mathematically, cooperative games represent a subset of the most common noncooperative games [29].

Representation of Games

Games in Normal Form

The normal form, or the strategic or matrix form, is the most used representation of strategic interactions in game theory. This form of defining a game seeks to represent every player's utility for every action, specially those cases where set of actions depends only on the players' combined actions. As described in Brown and Shoam [11], since most of representations of interest can be reduced to this typical representation of a game, the normal form turns to be the most fundamental in game theory.

Definition 2.1.1 (**Normal-form game**). *A finite, n-person normal-form game is a tuple* (A, N, u)*, where:*

- N is a finite set of players.
- $A = \{A_1 \times A_2 \times \cdots \times A_n\}$, with A_i denoting a finite set of *actions* available to player i. Each vector $a =< a_1, a_2, \cdots, a_n >\in A$ is considered to be an *action profile.*
- $u = \{u_1, \cdots, u_n\}$ where, $u_1 \colon A \to \mathbb{R}$ is real-valued utility (or payoff) function for player i.

Usually, an n-dimanesional matrix is used to represent the games. For instance, Fig.2.1.1 shows a two dimensional game matrix well known among game theorists.

Games in Extensive Form

The extensive or tree form is also a way of representing a game. Two groups define this class of games, namely, perfect-information game and imperfect-information game.

Brown and Shoham in [11] offer the formal definition for both.

Definition 2.1.2 (Perfect-information game). *A perfect-information game is a tuple* $G = (N, A, D, Z, \zeta, \delta, \lambda, u)$*, with:*

- N *is a finite set of* n *players;*
- A *is a set of actions;*
- D *is a set of nonterminal choice nodes;*
- Z *is a set of terminal nodes, disjoint from* H*;*
- $\zeta : D \to 2^A$ *is the action function, which assigns to each choice node a set of possible actions;*
- $\delta : D \to N$ *is the player function, which assigns to each nonterminal node a player* $i \in N$ *who chooses an action at that node;*
- $\lambda : D \times A \to D \cup Z$ *is the sucessor function, which maps a choice node and an action to a new choice node or terminal node such that for all* $h_1, d_2 \in H$ *and* $a_1, a_2 \in A$*, if* $\lambda(d_1, a_1) = \lambda(d_2, a_2)$ *then* $d_1 = d_2$ *and* $a_1 = a_2$*; and*
- $l = \{(l_1, \cdots, l_n)\}$ *where* $l_1 : Z \to \mathbb{R}$ *is real-valued utility function for player* i *on the terminal nodes* Z*.*

Definition 2.1.3 (Imperfect-information game). *An imperfect-information game is a tuple* $(N, A, D, Z, \zeta, \delta, \lambda, u, I)$*, where: A perfect-information game is a tuple* $G = (N, A, D, Z, \zeta, \chi, \lambda, u)$*, where:*

- $(N, A, D, Z, \zeta, \delta, \lambda, u, I)$ *is a perfect-information extensive game; and*
- $I = (I_1, \cdots, I_n)$ *with* $I_j = (I_{j,1}, \cdots, I_{j,k_j})$ *is an equivalence relation on (i.e., a partition of)* $h \in D : \delta(d) = j$ *with the property that* $\zeta(d) = \zeta(d')$ *and* $\delta(d) = \delta(d')$ *whenever there exists a* j *for which* $h \in I_{j,r}$ *and* $d' \in I_{j,r}$*.*

Regarding the representation of games, consider a typical illustration set of games starting with the Prisoner's Dilemma [11, 29, 49] which, probably, might be the most famous example of game theory problems.

Example 2.1.1 (The Prisoner's Dilemma). *Two criminals, A and B, are arrested. They are suspected of having robbed a bank. Because there is very little evidence, the two can only be sentenced to a year of imprisonment on the basis of what evidence there is. For this reason, the two are held in separate cells, with no way of meeting or communicating, and no way to make binding agreements. A deal is offered to each of them:*

- *If one confess and the other does not, the confessor will be freed, and the other will be jailed for twelve years;*
- *If both confess, then each will be jailed for six years.*

Table 2.1: The Prisoner's Dilemma in matrix form

Prisoner B		Prisoner A: Confess	Prisoner A: Keep quiet
	Confess	6, 6	0, 12
	Keep quiet	12, 0	1, 1

This information can be modeled by using a 2×2 matrix as shown in Table 2.1 The pair (x, y) at the intersection of row i and column j means that the row player gets x and the column player gets y. Intuitively, the prisoners have to decide whether to **cooperate**, which is equal to (*keep their mouth shut*, i.e., *not to confess to the crime*) and **not cooperate**, which implies (*to confess to the crime*). How should one choose rationally between these two strategies? For this purpose, consider the following line of reasoning, from the point of view of player **A**:

- Suppose **B** confesses; then, if I confess, my prison term would be six years, and if I keep quiet, it would be twelve years. Thus, my best choice would be to confess.
- If **B** keeps quiet; then, if I confess, I would walk free and if I keep quiet, I would spend a year in jail. Again, my best course of action is to confess.

The Prisoner's Dilemma is a *symmetric* game for prisoner **B** will reason in the same way about his opponent, and conclude that his best choice is also to confess. The conclusion is that they both confess, and the overall outcome of the game is that both prisoners will serve 6 years in jail.

From the common sense it seems that this outcome is not the best that could be done, i.e., why both players do not cooperate by keeping quiet? This would lead to the outcome where both players would serve 1 year in jail. Such mutual cooperation would be strictly preferred over mutual confession by both prisoners, and that is a very strong solution concept known as *dominant strategy equilibrium*, which does not always exist in games, yet when they do, it is very hard to imagine any other outcome occurring through rational choice. The reason on considering this game as a dilemma lays on the fact that a unique rational outcome, according to dominant strategy equilibrium, is strictly worse for both players than another outcome. Thus, both prisoners have a sub-optimal rational outcome, that is, this is a typical game in which gains comes through cooperation-the best outcome for both players is that neither confesses. Independent to what one player does, the other will prefer to *confes* to *keep quiet* and, consequently the game has a unique Nash equilibrium (*confess*, *confess*) [49].

The Prisoner Dilemma can also be depicted in a game of extensive form, such as shown in Fig. 2.1, where A and B represent player A and player B, respectively, and their actions C (confess) and D (deny).

2.1.2 Characteristic Functions and Payoff Configurations

As stated previously, players in the game can form coalitions which result in payoffs defined as money. An important element related to cooperative games is the concept of characteristic function, which is described in the next lines.

Definition 2.1.4 (Cooperative n-person game in characteristic function form). *A cooperative n-person game in characteristic function is a pair (N, v), with $N = \{1, 2, \cdots, n\}$ denoting the set of players and v represents a real valued function defined on the subsets of N called a* characteristic function*, whose main role is to assign a real value $v(\mathbb{S})$ to* *each subset $\mathcal{S}$ of players. The number $v(\mathcal{S})$ represents the value of $\mathcal{S}$, i.e., it is the money which coalition $\mathcal{S}$ can obtain while its members cooperate. The empty set, $\varnothing$, has always $v(\varnothing) = 0$.*

Implicitly, a set of assumptions [29] arises from this definition, namely:

(i) The value of any coalition of players, i.e., money is always preferred to be no less.

(ii) A set of players formed by agreement a coalition on the way that the value of the coalition is shared among its members.

(iii) Actions on $N - \mathcal{S}$ does not affect the amount $v(\mathcal{S})$. Thus none of the amount of $v(\mathcal{S})$ is given to members of $N - \mathcal{S}$, in the same way amount from $N - \mathcal{S}$ can be given to any element of $\mathcal{S}$.

(iv) All players know the characteristic function v, consequently all spending related to the value of v and agreements concerning the establishment of a coalition is known to all n players.

(v) Only the characteristic function has power over players linking although, other exceptions may be specified.

(vi) Although the concept of all nonempty coalition can also be formed, the characteristic functions may be designed in order to a typical formation of certain coalitions make sits members infeasible.

Logically payoff and coalition allow the rising of the first three assumptions. Assumption (iv) implies that players have a previous knowledge regarding their negotiations. A difference between models, i.e., those in which players are abstract and those in which they quarrel among them is presented in assumption (v). The concept of characteristic function requires assumption (vi), but in terms of practicality it is often pointless.

Symmetry and Desirability

In an n-person game the numerical ability for players to obtain more payoff for themselves is found in the values of coalitions of which they are members, since a known characteristic function represents these players and their respective coalitions worth. *Symmetry* and *desirability* are two basic relationship for such ability. The former can be understood as a representation of equality or substitutability of players, while the later indicates the order of scores in terms of players' ability to reach payoffs [29].
More precisely, given known an n-person game (N, v), two players A and B in (N, v) are considered to be symmetric if, for all coalitions that neither player belongs to, a new coalition in the same value is obtained by adding either player. i.e., players A and B are symmetric if

$$v(\mathcal{S} \cup \{A\}) = v(\mathcal{S} \cup \{B\}) \text{ for all } \mathcal{S} \subset N \text{such that} A, B \in \mathcal{S}. \quad (2.1)$$

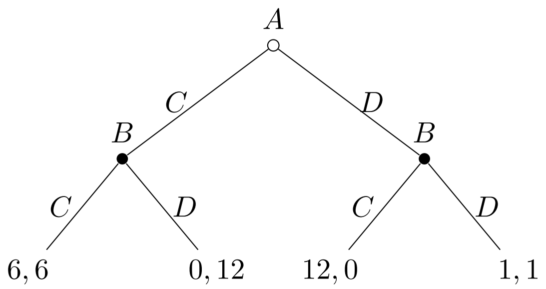

Figure 2.1: The Prisoner's Dilemma in Extensive-form

According to [29], for all theories of coalition formation held that for symmetric players A and B in the same coalition if the payoff vector $X = (x_1, x_2, \cdots, x_n)$ is an element of the solution of the game, then the payoff vector $\hat{X}$ follows the same conclusion, with

$$\widehat{X}_r = x_r, r \neq i, j; \;\; \hat{X}_i = x_j; \;\; \hat{x}_j = x_i.$$

As regard to desirability, player B is more desirable than player A if

$$v(\mathcal{P} \cup \{B\}) \geqslant v(\mathcal{P} \cup \{A\}) \text{ for all } \mathcal{P} \subset N \text{ such that } A, B \notin \mathcal{P}.$$

This inequality is rigorously observed for at least one coalition $\mathcal{P}$, i.e., replacing player B by player A in any coalition, the value of that coalition does not decrease, and it increases in at least one instance.
For the satisfaction of coalition formation [29], a more desirable player of a coalition should not receive less payoff than a less desirable player in the same coalition. Under the point of view of symmetry, this agreement is determined necessarily from the definition of the characteristic function; more desirable elements, having alternative coalitions with larger values, have at least as much strategic benefit as less desirable players.

Example 2.1.2 (**Symmetry and desirability**). *To investigate symmetry and desirability properties, consider the following 3-person game with the characteristic functions for each player and respective coalitions defined as:*

$$\begin{aligned} v(a) &= 3; \\ v(b) = v(c) &= 1; \\ v(bc) &= 8; \\ v(ac) = v(ab) &= 11; \\ v(N) &= 15 \end{aligned}$$

Players b and c are symmetric for $v(b) = v(c)$ and, A is more desirable than B and C since $v(a) \geqslant v(b)$ and $v(ac) \geqslant v(bc)$, as long as the inequality is strict at least once. Thus, desirability is a transitive relationship, i.e., if player a is more desirable than player b, and player b more desirable than player c, then player a is more desirable than player c.

Payoff Configurations

Any outcome of a game can be denoted through a *payoff configuration* ($\mathcal{PC}$), formally defined as a pair $(\mathbf{x}; \mathfrak{K}) = (x_A, x_B, \cdots, x_n; \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_m)$, with $\mathbf{x}$ denoting a payoff vector and $\mathfrak{K}$ a coalition structure. There cannot be more coalitions than players, i.e., $m \leqslant n$, and additionally

$$x(\mathcal{C}_j) \equiv \sum_{i \in \mathcal{C}_j} x_i = v(\mathcal{C}_j), \text{ for all } j = 1, 2, \cdots, m. \tag{2.2}$$

The meaning of Eq. (2.2) is that each proposed or formed coalition will expend neither more nor less than its value to its members.

The usage and efficiency of the characteristic function representation of a game and the $\mathcal{PC}$ representation of its outcomes are presented through the following examples taken from [29].

Example 2.1.3 (**Odd Man Out**). *Three players bargain in pairs to form a deal. The deal is an agreement on how to divide money provided by the experimenter. The amount of money that the experimenter provides depends on which pair concludes the deal. If players A and B combine, excluding C, then they divide \$4.00. If players A and C team up to* *exclude B, then they get \$5.00. And if B and C coalesce, they split \$ 6.00. Its characteristic function is defined as follows.*

$$v\{A\} = v\{B\} = v\{C\} = v\{N\} = 0;$$
$$v\{AB\} = 4;\ v\{AC\} = 5;\ v\{BC\} = 6$$

If players A and B decide to divide their joint payoff equally, the PC of the game will be

$$(2.00, 2.00, 0; AB, C).$$

If players B and C form a coalition where player C gets two-thirds of $v\{BC\}$, leads to the PC

$$(0, 2.00, 4.00; A, BC)$$

Whereas if all the players in the game are unable to reach agreement, the PC will be

(0, 0, 0; A, B, C).

Example 2.1.4 (**A Slightly Larger Market**). *Suppose a society in which there are five sandlmakers. Symmetric players A and B make only left sandals, while symmetric players C, D, and E make only right sandals. In one working period, a left sandalmaker can manufacture 17 sandals, while in the same time, a right sandalmaker can produce 10 sandals. Any single sandal is worth nothing, but a pair sells for 20. At the end of the working period, all scrap leather and unused sandals are worthless. A coalition is a binding agreement among sandalmakers to pool their output and divide their profits.*

A single player cannot earn any value, that is, $v\{i\} = 0$, for all $i \in \{A, B, C, D, E\}$. In the same way, companies making same-footed are not alowed to merge in order to get lucrative results; thus, $v\{AB\} = \{CD\} = \{CE\} = \{DE\} = \{CDE\} = 0$. Whenever a member of each subdivision makes ties, the smaller output of the right sandalmaker determines that $v\{dr\}$= 200, $d \in \{A, B\}$, $r \in \{C, D, E\}$. The value for a 4-person coalition of two of each type of sandalmaker corresponds the double value of the 2-person coalition. A 3-person coalition of one left and two right sandalmakers is constrained by the output of the left sandals, therefore $v\{drr\} = 340$. Summing a left sandalmaker to an (dr) coalition or a right sandalmaker to an (drr) coalition sums no value. Thus, the grand coalition can produce 30 pairs of sandals for 600.

The summary of all aforementioned can be observed through the characteristic function as follows.

$$\left\{\begin{array}{c} v\{A\} = v\{B\} = v\{C\} = v\{D\} = v\{E\} = 0; \\ v\{AB\} = v\{CD\} = v\{CE\} = v\{DE\} = v\{CDE\} = 0; \\ v\{AC\} = \{BC\} = \{AD\} = \{BD\} = \{AE\} = v\{BE\} = 200; \\ v\{ACD\} = v\{ACE\} = v\{ADE\} = v\{BCD\} = v\{BCE\} = v\{BDE\} = 340; \\ v\{ABC\} = v\{ABD\} = v\{ABE\} = 200; \\ v\{ACDE\} = v\{BCDE\} = 340; \\ v\{ABCD\} = v\{ABCE\} = v\{ABDE\} = 400; \\ v\{N\} = 600. \end{array}\right\} \tag{2.3}$$

Suppose now that players A and D make a deal, opposite to B and E whom decide to play independently, this could have the following $\mathcal{PC}$

$$(100, 50, 0, 100, 150; AD, BE, C);$$

moreover, finding other $\mathcal{PC}s$ is also possible as long as players keep making agreements.

2.2 Shapley Value in Transferable Utility (TU) Games

Classes of TU games

Definition 2.2.1. (Superadditive game). *A game $G = (N, v)$ is superadditive if for all S, $T \subset N$, $S \cap T = \varnothing$, holds*

$$v(S \cup T) \geqslant V(S) + V(T). \tag{2.4}$$

Superadditivity makes sense when coalitions can always work without interfering with one another; hence, the value of two coalitions will be less than the sum of their individual values. This property implies that the value of the entire set of players, i.e., the "grand coalition" is no less than the sum of the value of any nonoverlapping set of coalitions. In other words, the grand coalition has the highest payoff among all coalitional structures.

When coalitions can never affect one another, either positively or negatively, then we have additive (or *inessential*) games.

Definition 2.2.2 (Additive game). *A game $G = (N, v)$ is additive (orinessential) if for all S, $T \subset N$, $S \cap T = \varnothing$, holds*

$$v(S \cup T) = v(S) + v(T). \tag{2.5}$$

Definition 2.2.3. (Constant-sum game). *A game $G = (N, v)$ is constant sum, if for all $S \subset N$,*

$$v(S) + v(N \backslash S) = v(N). \tag{2.6}$$

Every additive game is necessarily constant sum, but not vice versa. In noncooperative game theory, the most commonly studied constant-sum games are *zero-sum games.*

Definition 2.2.4. (Convex game) *A game $G = (N, v)$ is called convex if for all S, $T \subset N$,*

$$v(S \cup T) \geqslant v(S) + v(T) - v(S \cap T). \tag{2.7}$$

Convexity is a stronger condition than superadditivity. Whereas convex games may therefore appear to be a very specialized class of cooperative games, these games are actually not so rare in practice [50, 11].

Definition 2.2.5. (Simple game). *A game $G = (N, v)$ is said to be simple if for all $S \subset N$, $v(S) \in (0, 1)$.*

Simple games represent a class of cooperative games with restrictions on the values that payoffs are allowed to take. This class of games are useful for modeling situations. **??**. Often is added the requirement that if a coalition wins, then all larger sets are also winning coalitions (i.e., if $v(S) = 1$, then for all $T \supset S$, $v(T) = 1$).

When simple games are also constant sum, they are called *proper simple games.* In this case, if S is a winning coalition, then $N \backslash S$ is a losing coalition.

As regard to the relationship between the different classes aforementioned in this subsection, a graphical representation taken from [11] is given in Fig. 2.2. $X \supset Y$ means that class X is a superclass of class Y. Dividing payoff to the grand coalition among the players constitutes the main question in cooperative game theory. The centralization on the grand coalition is due to the fact that most widely studied games are superadditive. Since the grand coalition is the coalition achieving the highest payoff over all the coalitions, it is expected that this coalition will appear for there may not be other alternative for players, but to form the grand coalition.

On the difficult to decide how this coalition should divide its payoffs a variety of solution concepts have been proposed. In this research, our focus is mainly the Shapley value. Therefore, as the chapter points out, our description will be made having this solution concept in mind.

A basic definition related to payoff division must be given before to proceed.

Definition 2.2.6 (Imputation). *An imputation for the n−person game v is a vector*

$$\mathbf{x} = (x_1, \ldots, x_n) \tag{2.8}$$

satisfying

(a) $\sum_{i \in N} x_i = v(N)$,

(b) $x_i \geqslant v = \{i\}$ *for all* $i \in N$.

Under an imputation, each player must be guaranteed a payoff of at least the amount that he could achieve by forming a singleton coalition. As a solution concept, the core suffers from three main drawbacks:

(i) The core can be empty;

(ii) The core can be quite large, hence selecting a suitable core allocation can be difficult, and

(iii) In many scenarios, the allocations that lie in the core can be unfair to one or more players.

These drawbacks motivated the search for a solution concept, which can associate to every coalitional game (N, v), a unique payoff vector known as the value of the game (a value quite different from the value of a coalition). Hence, a solution concept known as the Shapley value was proposed in 1953 by Lloyd Stowell Shapley [50, 63, 75]. His concept was originally defined for transferable utility games (TU games); however, extensions to nontransferable utility games (NTU games) exist [50, 29, 11, 13, 40, 42].

The Shapley value is based on a set of axioms, which are described in the next lines after the following two definitions.

Definition 2.2.7 (Carrier). *Let c be an n-person game. A coalition R such that, for any coalition T,* $c(T) = c(T \cap R)$ *is called a carrier for game c. Any player i out of the carrier is a dummy player, that is, cannot contribute to any coalition.*

Definition 2.2.8 (Permutation). *Consider* τ *a permutation of the set N, and c an n-person game. Then, game u implies* τc *such that, for any* $Q = \{r_1, r_2, \cdots, r_q\}$*. Then, changing the roles of the players in game c by the permutation* τ*, holds*

$$u(\{\tau(r_1), \tau(r_2), \cdots, \tau(r_q)\}) = c(Q), \tag{2.9}$$

that is, τc *is simply the game u such that, for any* $Q = \{r_1, r_2, \cdots, r_q\}$ *Eq. (2.9) is satisfied.*

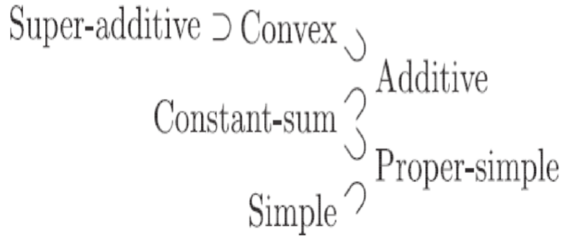

Figure 2.2: A hierarchy of coalitional game classes.

Axioms of Shapley value

Let $\varphi[c]$ an n-vector be the value of game c.

Axiom 2.2.1 (Efficiency). *Players distribute among themselves the resources available to the grand coalition, that is, if T is any carrier, then*

$$\sum_T \varphi_i[c] = c(T) \tag{2.10}$$

Axiom 2.2.2 (Symmetry). *If two players a and b in game c are symmetric, then $\varphi_a(c) = \varphi_b(c)$. This goes along with permutation, i.e., for any permutation μ, and i in N,*

$$\varphi_{\tau(i)}(\tau c) = \varphi_i(c) \tag{2.11}$$

To recall that two players a and b are said to be symmetric with respect to game c if they have the same marginal contribution to any coalition they belong to, that is, for each $T \subset N$ with $a, b \notin T$, $c(T \cup a) + c(T \cup b)$. Players are paid equal shares.

Axiom 2.2.3 (Additivity). *If u and c are any games, then*

$$\varphi_i(u + c) = \varphi_i(u) + \varphi_i(c) \tag{2.12}$$

Axiom 2.2.4 (Dummy). *If a is a dummy player, that is, $c(T \cup a) - c(T) = 0$ for every $T \subset N$, then $\varphi_a(c) = 0$.*

Theorem 2.2.1 (Shapley, 1953). *For all games there exists a unique value varphi satisfying all these axioms. This value is obtained in Eq.* (2.28).

The uniqueness [75] is sustained by the fact that n-person games have a 2^{n-1}-dimensional vector space based on a set of unanimity games. A game v_Q is called a unanimity game on the domain Q if

$$v_Q(T) = \begin{cases} 1, & \text{if } \; Q \subset T \\ 0, & \text{otherwise} \end{cases} \tag{2.13}$$

Since each player in the domain should receive an equal share of 1 and the others 0, a value that is uniquely determined on unanimity games is possible through the dummy and symmetry axioms The additivity axiom complets the uniqueness result. Furthermore, the proof of Theorem 2.2.1 follows by [50] through a sequel of lemmas and a corollary.

Lemma 2.2.1. *For any coalition T, consider ω_T a game defined as*

$$t\omega_T(R) = \begin{cases} 0, & \textit{if} \;\; T \not\subset R \\ 1, & \textit{if} \;\; T \subset R. \end{cases} \tag{2.14}$$

Follows that if by defining the number of players in T as t,

$$\varphi_i[w_T] = \begin{cases} \frac{1}{t}, & \textit{if} \;\; i \in T \\ 0, & \textit{if} \end{cases} \tag{2.15}$$

Proof. From Definition 2.2.7, it is straightforward to verify that T is a carrier for ω_T, as well as any set R containing T. From Axiom (2.2.2) holds,

$$\sum_R \varphi_i[\omega_T] = 1 \;\text{ if }\; T \subset R.$$

This implies that $\varphi_i[\omega_T] = 0$ for $i \notin T$.

If any permutation μ carries T, then $\mu\omega_T = \omega_T$. Thus, from Axiom (2.2.2) follows $\varphi_i[\omega_T] = \varphi_j[\omega_T]$ for any i, $j \in T$. Additionally, $\varphi_i[\omega_T] = \frac{1}{t}$ if $i \in T$, since there are t of those terms, and if their sum is 1. ☐

Corollary 2.2.1. *If* $e > 0$, *then*

$$\varphi_i[e\omega_t] = \begin{cases} \frac{e}{t}, & \textit{if} \quad i \in T \\ 0, & \textit{if} \quad i \notin T. \end{cases} \tag{2.16}$$

Lemma 2.2.2. *If* c *is any game, then there exist* $2^n - 1$ *real numbers* e_t *for* $T \subset N$ *such that*

$$c = \sum_{T \subset N} e_t \omega_t,$$

where, w_t *is defined in the same was as in Lemma* (2.2.1)*.*

Proof. Consider

$$e_t = \sum_{R \subset T} (-1)^{t-r} c(T) \tag{2.17}$$

with r denoting the number of players in R. The task now, is to verify that lemma (2.2.2) is satisfied by e_t. For this purpose, suppose that H is a coalition,

$$\begin{aligned} \sum_{T \subset N} e_t \omega_t(H) &= \sum_{T \subset H} e_t \\ &= \sum_{T \subset H} \left(\sum_{T \subset R} (-1)^{t-r} c(T) \right) \\ &= \sum_{T \subset H} \left(\sum_{\substack{T \subset H \\ T \supset R}} (-1)^{t-r} \right) c(R). \end{aligned} \tag{2.18}$$

For every value of t between r and h, there are $\binom{h-r}{h-t}$ set T with t elements such that $R \subset T \subset H$. Therefore, the inner parenthesis may be replaced by

$$\sum_{t=r}^{h} \binom{h-r}{h-t} (-1)^{t-r}. \tag{2.19}$$

The expression in Eq. (2.19) is the binomial expansion of $(1-1)^{h-r}$. This expression is equal to zero if $r < h$, and 1 if $r = h$. Therefore,

$$\sum_{T \subset N} e_t \omega_t(H) = c(H), \quad \text{for all} \quad H \subset N. \tag{2.20}$$

Through these two lemmas some important ideas can be summed up:

(i) By Lemma (2.2.2) any game can be written as a linear combination of games ω_t.

(ii) From Lemma (2.2.1) there exists a function ϕ, which is defined for such games.

(iii) Though some coefficients e_t are negative, by Axiom (2.2.3) assuming that u, c and $u - c$ are games, then $\phi(u - c) = \phi(u) - \phi(c)$, and consequently for all games c function *phi* is uniquely defined.

As,

$$c = \sum_{T \subset N} e_t \omega_t \tag{2.21}$$

Holds,

$$\phi_i(c) = \sum_{T \subset N} e_t \phi_i(\omega_t) = \sum_{\substack{T \subset N \\ i \in T}} e_t 1/t \tag{2.22}$$

By replacing e_t as defined in Eq. (2.17), in the previous expression, holds

$$\phi_i(c) = \sum_{\substack{T\subset N\\ i\in T}} 1/t \sum_{R\subset T} (-1)^{s-t} c(R), \tag{2.23}$$

$$\phi_i(c) = \sum_{R\subset N} \sum_{\substack{T\subset N\\ R\cup i\subset T}} (-1)^{s-t} \frac{1}{t} c(R), \tag{2.24}$$

Making

$$\psi_i(R) = \sum_{\substack{T\subset N\\ R\cup i\subset T}} (-1)^{s-t} \frac{1}{t} c(R), \tag{2.25}$$

Suppose $R = R' \cup i$ and $i \notin R'$, then we have $\psi_i(R') = -\psi_i(R)$. If these conditions are satisfied, then the terms in the right side of Eq. (2.24) turn to be the same, with exceptuation of $r = r' + 1$, and this implies a change of sign.

$$\phi_i(c) = \sum_{\substack{R\subset N\\ i\in R}} \psi_i(R) c(R) - c(R - i). \tag{2.26}$$

Now, the opposite case, i.e., $i \in R$, implies that theres exists $\binom{n-r}{t-r}$ coalitions T with t elements such that $R \subset T$.

Consequently,

$$\begin{aligned}
\psi_i &= \sum_{t=r}^{n} (-1)^{t-r} \binom{n-r}{t-r} 1/t \\
&= \sum_{t=r}^{n} (-1)^{t-r} \binom{n-r}{t-r} \int_0^1 x^{t-1} dx \\
&= \int_0^1 \sum_{t=r}^{n} (-1)^{t-r} \binom{n-r}{t-r} x^{t-1} dx \\
&= \int_0^1 x^{r-1} \sum_{s=r}^{n} (-1)^{t-r} \binom{n-r}{t-r} x^{t-1} dx \\
&= \int_0^1 x^{s-1} (1-x)^{n-r} dx
\end{aligned}$$

From Eq. (2.27), holds,

$$\psi_i(R) = \frac{(r-1)!(n-r)}{n!} \tag{2.27}$$

From Eq. (2.26) and Eq. (2.27) results,

$$\phi_i(c) = \sum_{\substack{R\subset N\\ i\in R}} \frac{(r-1)!(n-r)}{n!} [c(R) - c(R - i)]. \tag{2.28}$$

□

The expression in (2.28) represents explicitly the Shapley value, and it satisfies the aforementioned axioms. Where the numerator indicates the number of permutations of N with i being preceded by the elements of coalition R, and the total number of permutations is defined through the denominator. Since the game is superadditive, we have,

$$\phi_i(v) \geqslant v(i), \tag{2.29}$$

meaning that $\phi_i(v)$ is an imputation. Although interpreted as a measure for evaluating player's power in a cooperative game, according to Winter in [75], Shapley value is often interpreted and sometimes applied as a technique to allocating collective benefits or costs. Hence, there is a concern for solving noncooperative bargain games based on cooperative solution concepts, the so-called Nash Program [28].

Winter [75] lists Harsanyi [22] for probabily being the first to address the relationship between Shapley value and noncooperative games through his "dividend game", he makes use of the conection between the Shapley value and the decomposition of games into unanimity games. The list related to research on the relationship between noncoperative games and the Shapley value goes with studies by Gul [21], Hart and Mas-Colell [23], Perez-Castillo and Wettstein and [16, 43, 51, 44, 74].

2.3 Shapley Value in Linear Solvable Formulation and Fuzzy-Shapley Value with Constraints

A description on basic concepts related to cooperative game theory, especially those related to Shapley value, was given in the previous subsection and brief comments on noncooperative game theory were also considered. Through this section, we extend Shapley value to an linear programming by making use of techniques related to least square value in TU games [56, 57, 47], and the DEA game proposed in Nakabayashi and Kone [46] is described. Furthermore, for the sake of dealing with uncertainty in decision-making as our model is also a data driven approach there may exist uncertainty within the data. So, in order to overcome this problem we simply extend the model to fuzzy theory field in the subsequent subsection.

Shapley Value in Linear Solvable Formulation

Definition 2.3.1 (Weight function). *A positive and symmetric weight function on N, is a function $m(s)$ (or simply m) such that $2^N \backslash \varnothing \rightarrow \mathbb{R}$ where $m(S) \geqslant 0$ for all $S \subseteq N$, where $m(S) > 0$ for some $S \neq N$, and such that $m(S) = m(T)$ whenever $s = t$, where s and t denote the cardinality of S and T respectively.*

Consider the following problem for each weight function m:

$$\begin{aligned} &\mathit{Minimize} \quad \sum_{S \subset R,\ S \neq R} M_{r,s} \left(v(S) - \sum_{d \in S} z_d(R,v) \right)^2 \\ &\mathit{Subject\ to} \quad \sum_{d \in R} z_d(R,v) = v(R) \end{aligned} \tag{2.30}$$

The solution for Eq. (2.30) can be found through a vector imputation $z_d^+(R,v)$ defined by the following expression:

$$z_d^+(R,v) = \frac{1}{r} \left\{ v(R) + \sum_{i \in R} (c_{di} - c_{id'}) \right\} \tag{2.31}$$

and $\Gamma(d^+,\ d'^-) = \{S \subset R |\ d \in S,\ d' \notin S\}$

$M_{r,s}$, a set of weights, is defined as:

$$M_{r,s} = \frac{1}{r-1} \binom{r-2}{s-1}^{-1} \tag{2.32}$$

This process produces a vector imputation equivalent to the Shapley value defined in Eq. (2.28) That is,

$$z_d^+(R,c) = \phi_d(R,c) \tag{2.33}$$

Consequently, $\phi_i(R,c)$ is always an imputation [56, 57].

Let ι be a sample from a dataset, and x_i and y_ι are the values of the sample. An error from data can be estimated through the following expression

$$e_\iota = f_d(\boldsymbol{A},\boldsymbol{M},\boldsymbol{v}) = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{v} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{A}\boldsymbol{z})_d \tag{2.34}$$

where $(\ )_d$ denotes the selection of the d-th row value, thus the sum of all error functions $E = \sum_i |e_i|$ using the multiple linear regression model that minimize the sum of the absolute values of the residuals. This problem can be restated as an LP problem [54]:

$$\begin{aligned} \text{Minimize} \quad & \epsilon \\ \text{Subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{v} + \boldsymbol{s}^+ - \boldsymbol{s}^- = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{A}\boldsymbol{z} \\ & \sum_{d\in K} z_d(K,v) = v(K) \\ & 0 \leqslant s^+ \leqslant \epsilon,\ 0 \leqslant s^- \leqslant \epsilon \end{aligned} \tag{2.35}$$

where $\boldsymbol{s}^+ = [s_1^+, s_2^+, s_3^+, \cdots, s_n^+]^{\mathrm{T}}$ and $\boldsymbol{s}^- = [s_1^-, s_2^-, s_3^-, \cdots, s_n^-]^{\mathrm{T}}$; $\boldsymbol{A}$ is a matrix satisfying the condition of supperaditive; $\boldsymbol{v}$ is a column matrix whose elements are the real values $v(S)$, i.e., the characteristic functions of singular players and their respective coalitions in the game; $\boldsymbol{M}$ is a diagonal matrix formed by the weights $M_{k,s}$ obtained through Eq. (2.32), and is defined as follows.

$$\boldsymbol{M} = \begin{bmatrix} M_{k,s} & 0 & \cdots & 0 \\ 0 & M_{k,s} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k,s-1} \end{bmatrix} \tag{2.36}$$

Briefly we explain how matrices and vectors of the optimization model are obtained. A more elaborated example is considered in Section 2.4.

Example 2.3.1. *Suppose a game (K,v) with three players, i.e., $K = \{1,2,\cdots,k\}$ then the matrices aforementioned have the following structures.*

$$\boldsymbol{z} = \begin{bmatrix} z_A(K,v) \\ z_B(K,v) \\ z_C(K,v) \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} v(\{A\}) \\ v(\{B\}) \\ v(\{C\}) \\ v(\{A,B\}) \\ v(\{A,C\}) \\ v(\{B,C\}) \end{bmatrix}, \tag{2.37}$$

and

$$\boldsymbol{M} = \begin{bmatrix} M_{3,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{3,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{3,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{3,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{3,2} \end{bmatrix}, \tag{2.38}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{2.39}$$

Decision making in order to be efficient and consistent in terms of results requires to consider factors such as uncertainty. With this in mind we extended the linear solvable formulation of Shapley value to fuzziness environment in order to obtain robustness solutions, which can overcome the ambiguity in the mind of DM.

Probability theory and error calculus were for centuries the only models known to deal with uncertainty. This dependence was overcome with the introduction of several models for handling incomplete numerical and linguistic information, decision maker's subjectivity, etc. Nonmeasurable elemnts such as "almost the same", or "preference", for instance, cannot be treated by the conventional set theory or probability theory [24]. However, this kind of vagueness on information can be treated nowadays by using techniques such as fuzziness proposed by Zadeh [77] who proposed the concept of fuzzy set, and has been applied to a variety of real problems. The concept by itself is a non-statistical, and can deal with situations where imprecision is evident. This technique is defined as follows.

Definition 2.3.2 (Fuzzy set). *A fuzzy set F in a universe K is a mapping from F to [0,1]. For any $k \in K$ the value $F(k)$ is called the degree of membership of k in F. K corresponds the carrier of the fuzzy set F. The degree of membership can also be represented by k instead of $F(k)$. $G(k)$ denotes the class of all fuzzy sets in K.*

Following the concept of fuzzy set the notion of fuzzy variable was also introduced [30, 78, 45]

According to Klir and Yuan [32], uncertainty can be manifested in different forms. These forms are classified in three distinct types of uncertainty within the framework of fuzzy set theory and fuzzy measure theory, while in probability theory, uncertainty is recognized only in one form. The three types are: *nonspecificity*, also known as *imprecision*, it is connected with sizes (cardinalities) of relevant sets of alternatives; *fuzzines* or *vagueness*, which results from imprecise boundaries of fuzzy sets; the last type is *strife* or *discord*, this type describes conflicts among various sets of alternatives [24].

As the progress on studies related to decision theory is evident, it is probable that uncertainty may not be limited in these three types only. Fig. 2.3 shows the three types of uncertainty.

Fuzzy-Shapley Value with Constraints

Because there may exists factors such as ambiguity while DM performs his tasks, through this subsection, we propose a fuzzy-Shapley value model.

Let $Z_d = (z_d, \zeta_d)_L$ be a fuzzy Shapley value with z_d as the center and ζ_d the width; $(\ )_L$ denotes is represented by $L(x)$ satisfying

$$\begin{array}{rcl} L(x) & = & L(-x), \\ L(0) & = & 1, \\ L(x) & is & non-increasing\ function\ in\ x \in [0, \infty). \end{array} \tag{2.40}$$

For instance, if $L(x) = \max(0, 1 - |x|)$ its fuzzy membership function is defined by $\mu_{Z_d}(y_d) = L\left(\dfrac{z_d - y_d}{\zeta_d}\right)$.

Moreover the characteristic function is also assumed to be generated by fuzzy value $V(S) = (v(S), \pi_S)_L$.

Then we can define fuzzy vector $\boldsymbol{Y} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{v}$ and $\boldsymbol{X} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{z}$ whose elements are given by the fuzzy variables $\boldsymbol{Y} = [Y_1, Y_2, Y_3, \cdots, Y_k]$ and

$$Y_d = \left(\boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v},\ |\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}\right)_L = \left(\sum_{S \subset K,\ S \neq K} b_{dS} v(S), \sum_{S \subset K,\ S \neq K} |b_{dS}| \pi_S\right)_L, \quad (d = 1, 2, \cdots, k) \tag{2.41}$$

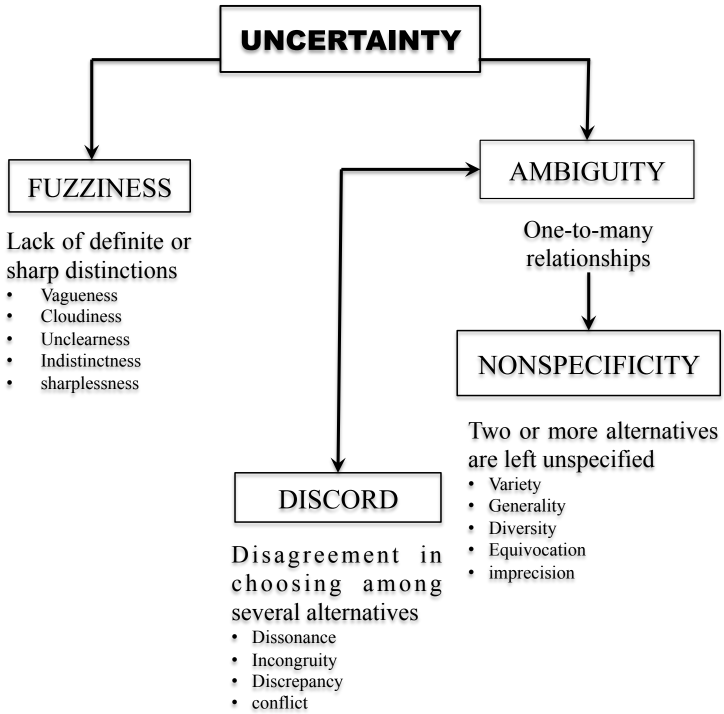

Figure 2.3: Basic types of uncertainty [32].

where $\boldsymbol{b}_d^{\mathrm{T}} \in \Re^1 \times \Re^q$ is a d-th row vector of matrix $\boldsymbol{B}^{\mathrm{T}}$ and $|\boldsymbol{b}_d^{\mathrm{T}}| = [|b_{d,1}|, |b_{d,2}|, \cdots, |b_{d,q}|]$. Vector $\boldsymbol{\pi} \in \Re^q \times \Re^1$ is the width representing ambiguity of fuzzy variables for $V(S)$.

Also $\boldsymbol{X} = [X_1, X_2, X_3, \cdots, X_k]$ is

$$X_d = \left(\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z},\ |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta}\right)_L = \left(\sum_{S\subset K,\ S\neq K} c_{dS} z_d,\ \sum_{S\subset K,\ S\neq K} |c_{dS}|\zeta_d\right)_L,\quad (d = 1, 2, \cdots, k) \tag{2.42}$$

where $\boldsymbol{c}_d^{\mathrm{T}} \in \Re^1 \times \Re^d$ is a d-th row vector of matrix $\boldsymbol{C}^{\mathrm{T}}$ and $|\boldsymbol{c}_d^{\mathrm{T}}| = [|c_{d,1}|, |c_{d,2}|, |c_{d,3}|, \cdots, |b_{d,k}|]$. The vector $\boldsymbol{\zeta} \in \Re^k \times \Re^1$ is the width representing ambiguity of fuzzy variables for z_d.

Possibility Measure for Fuzzy Shapley Value

Here we consider the following possibility of equivalent $Pos(Y_d = X_d)$ for fuzzy variables, Y_d and X_d, as follows;

$$Pos(Y_d = X_d) = \sup_{\theta\in\Re} \min\left(\mu_{Y_d}(\theta), \mu_{X_d}(\theta)\right), \tag{2.43}$$

and the condition for α level set

$$Pos\left(Y_d = X_d\right) \geqslant \alpha. \tag{2.44}$$

Such definition implies the sets of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ consisting of the fuzzy vector $\boldsymbol{X}$ under the condition as the possibility of equivalent between Y_d and X_d is larger than α.

Now that we can derive a LP problem to obtain fuzzy Shapley value by

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{d=1}^{k} |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + \beta\epsilon & (2.45)\\
\text{Subject to} \quad & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + s_d^{+} - s_d^{-} \leqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + s_d^{+} - s_d^{-} \geqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} - L^{-1}(\alpha)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \sum_{d=1}^{k} z_d(K, v) = v(K) \\
& z_d \geqslant 0,\ \zeta_d \geqslant 0,\ s_d^{+} \leqslant \epsilon,\ s_d^{-} \leqslant \epsilon \qquad (d = 1, 2, 3, \cdots, k)
\end{aligned}
$$

It is assumed that the coefficient β in model (2.45) is given by a value sufficiently large, and ϵ works significantly if $\alpha = 1$, that is, the values of the decision variables are exactly equal to Shapley value as in Eq. (2.28).

Necessity Measure for Fuzzy Shapley Value

Next, we consider the following necessity of inclusion $Nec(Y_d \supset X_d)$ for fuzzy variables, Y_d and X_d,

$$
Nec(Y_d \supset X_d) = \inf_{\theta \in \Re} \max\left(\mu_{Y_d}(\theta), 1 - \mu_{X_d}(\theta)\right), \tag{2.46}
$$

and the condition for α level set

$$
Nec\left(Y_d \supset X_d\right) \geqslant \alpha. \tag{2.47}
$$

Inequality (2.47) implies that the sets of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ have a fuzzy vector $\boldsymbol{X}$ under the condition that it is the degree of necessity. Furthermore, X_d is included by Y_d, and larger than α.

Having this condition satisfied, then we can obtain fuzzy Shapley value by solving tprogram (2.48), which corresponds the LP model in case of necessity of inclusion.

$$
\begin{aligned}
\text{Minimize} \quad & -\sum_{d=1}^{k} |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + \beta\epsilon & (2.48)\\
\text{Subject to} \quad & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} + L^{-1}(1-\alpha)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + s_d^{+} - s_d^{-} \leqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(1-\alpha)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} + s_d^{+} - s_d^{-} \geqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} - L^{-1}(\alpha)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\
& \qquad (d = 1, 2, 3, \cdots, k) \\
& \sum_{d=1}^{k} z_d(K, v) = v(K) \\
& z_d \geqslant 0,\ \zeta_d \geqslant 0,\ s_d^{+} \leqslant \epsilon,\ s_d^{-} \leqslant \epsilon \qquad (d = 1, 2, 3, \cdots, k)
\end{aligned}
$$

2.4 Numerical Example

In this section, models proposed within the chapter are tested. A comparison study with other models preceds the discussion on the results.

Table 2.2: Data of 3 shops

Shop	**Profits**	**Employee**	**Size**
A	10	9	20
B	5	16	10
C	7	20	30
Sum	22	45	60

The first example, the Market Arcade Game, corresponds to a problem taken from Cooper et al., in [15], where Shapley value was obtained by using the DEA game approach proposed by Nakabayashi and Tone in [46]. The problem is presented as follows.

A shopping mall association made the following agreement on the arcade maintenance fee: "Every shop facing the arcade street has to pay a monthly fee. The method for arriving at this fee for each shop was discussed and approved at the general meeting. Share of cost was determined based on parameters such as category of business, the size of the shop, the number of employees and so on." Suppose that only three shops A, B and C face the arcade street and they have to determine share of the arcade maintenance fee based on three criteria such as the profits, the number of employees and the size of the three shops, as exhibited in Table 2.2.

Assignment: How is share of cost determined? The values of the characteristic functions for this 3-person game are fuzzy values as mentioned previously, having $\boldsymbol{v}$ as the center and $\boldsymbol{\pi}$ as the width:

$$\boldsymbol{v} = \begin{bmatrix} v(A) \\ v(B) \\ v(C) \\ v(A,B) \\ v(A,C) \\ v(B,C) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.1667 \\ 0.3182 \\ 0.5 \\ 0.6444 \\ 0.5455 \end{bmatrix}, \qquad \boldsymbol{\pi} = \begin{bmatrix} \pi(A) \\ \pi(B) \\ \pi(C) \\ \pi(A,B) \\ \pi(A,C) \\ \pi(B,C) \end{bmatrix} = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.03 \\ 0.05 \\ 0.06 \\ 0.05 \end{bmatrix} \tag{2.49}$$

The computation of matrices $\boldsymbol{M}$, $\boldsymbol{A}^{\mathrm{T}}$, $\boldsymbol{B}^{\mathrm{T}}$ and $\boldsymbol{C}^{\mathrm{T}}$ gives

$$\boldsymbol{M} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \tag{2.50}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \tag{2.51}$$

$$\boldsymbol{B}^{\mathrm{T}} = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0.5 \end{bmatrix} \tag{2.52}$$

$$\boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} 1.5 & 0.5 & 0.5 \\ 0.5 & 1.5 & 0.5 \\ 0.5 & 0.5 & 1.5 \end{bmatrix} \tag{2.53}$$

Then, fuzzy variables $\boldsymbol{Y}$ are given by

$$\boldsymbol{Y} = \left(\begin{bmatrix} 0.67220 \\ 0.60610 \\ 0.75405 \end{bmatrix}, \begin{bmatrix} 0.065 \\ 0.055 \\ 0.070 \end{bmatrix} \right)_L \tag{2.54}$$

If $\alpha = 1$, ambiguity does not exist for ζ is null and both models have the same result as those from Shapley value in (2.28). By finding Shapley values through the possibility and necessity measures we intend to analyze the ambiguity, which usually is not considered. Ambiguity can be observed through α.

With some ambiguity, say $\alpha = 0.7$, the value for possibility condition diferes a little comparing to those from the condition of necessity of inclusion. The fuzzy Shapley value based on possibility of equality is given as follows.

$$\boldsymbol{z} = \begin{bmatrix} 0.4381 \\ 0.2047 \\ 0.3572 \end{bmatrix}, \quad \boldsymbol{\zeta} = \begin{bmatrix} 0.5475 \\ 0 \\ 0 \end{bmatrix} \tag{2.55}$$

Adequate conditions such as, a width pattern can be added in order to control the result of $\boldsymbol{\zeta}$, moreover, $\boldsymbol{C}^{\mathrm{T}}$ is not an invertible matrices. As for necessity of inclusion fuzzy Shapley values were obtained as follows.

$$\boldsymbol{z} = \begin{bmatrix} 0.3286 \\ 0.2595 \\ 0.4119 \end{bmatrix}, \quad \boldsymbol{\zeta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.56}$$

This result shows that even if there is ambiguity through the characteristic functions, necessity of inclusion does not present always ambiguity. These numerical simulations were performed on Excel solver the results obtained for possibility measure are presented in Table 2.3.

In this chapter, concepts related to CGT were presented, and briefly the main difference between CGT and NCGT was also considered. The term "cooperative" may mislead, it does not mean that there not exists cooperation among players in NCGT at all, or that they always cooperate with regard to CGT. One of the key difference is that in the former, players act strategically, while in the later they try to form coalitions. Solution concepts in cooperative games with TU and NTU with emphasis to Shapley value were considered. One of the advantages of cooperative games, unlike noncoperative games in multiperson situations, it is very practical. Real problems are easily understood into a coalitional game, which has proved more tractable than that of a noncoperative game, whether that be in normal or extensive form.

Basic concepts regarding the theory behind Shapley value was also introduced. Shapley value was proposed with the motivation of finding a mathematical concept to deal with any game, i.e., n-player games. According to [75], among all the solution concepts in cooperative game theory, the Shapley value seems to be the most cooperative, undoubtedly more than the core, and moreover the concept emerges as the outcome of a variety of noncooperative games quite different in structure and interpretation. Its main advantage is the possibility of offering fair and unique solution while the difficult on the computation due to number of combination of players in terms of coalition formation is the drawback especially if n is large.

Within the chapter, extension of Shapley value to optimization programs was presented. For being one of optimization approaches most used, LP models seem to be easier and faster because of the large number of softwares capable to solve a problem defined through the method. Therefore, within the chapter the conventional formulation of Shapley was, also, considered in LP model. This model was extended to fuzziness concepts, and through possibility and necessity measures we proposed models to obtain fuzzy Shapley values with fuzzy variables. The differences between these models and the classical formulation is that through them DMs can also have information regarding ambiguity, a type of uncertainty. This fact is not considered in the original case. In practical terms DM chooses the value for π in order to get a precise information while analyzing the membership function. Values are defined in terms of fuzziness, while in the previous case they are crisp values. To test the efficiency of these models, a numerical example was included in the last section. If α is equal to unity, then models become simply the classical Shapley value by returning the same results. The level of ambiguity can be observed for cases where $\alpha \neq 1$.

Table 2.3: Fuzzy Shapley value based on possibility measure with $\alpha = 1$

POS	z_1	z_2	z_3	ζ_1	ζ_2	ζ_3	a_1^+	a_2^+	a_3^+	a_1^-	a_2^-	a_3^-	ϵ	α	$L^-(\alpha)$	$L^-(1-\alpha)$
Fuzzy Shapley value	20.328083	0.261983	0.409933	0	0	0	0	0	0	0.155883	0.155883	0.155883	0.155883	1	0	
min	0	0	0	0	2.5	2.5	2.5	0	0	0	0	0	100	15.58833		
s.t.	1.5	0.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.6722	$\leqslant$	
	0.5	1.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.6061	$\leqslant$	
	0.5	0.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.75405	$\leqslant$	0.6722
	1.5	1.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.8722	$\leqslant$	0.6061
	0.5	0.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.6061	$\leqslant$	0.75405
	0.5	0.5	0.5	0	0	0	1	0	0	-1	0	0	0	0.75405	$\leqslant$	0.6722
	1	1.5	0.5	0	0	0	1	0	0	-1	0	0	0	1	$\leqslant$	0.6061
	0	0.5	0.5	0	0	0	1	0	0	-1	0	0	0	-0.15588	$\leqslant$	0.75405
	0	1	0.5	0	0	0	1	0	0	-1	0	0	0	-0.15588	$\leqslant$	1
	0	0	0	0	0	0	1	0	0	0	0	0	-1	-0.15588	$\leqslant$	0
	0	0	0	0	0	0	1	0	0	1	0	0	-1	-0.15588	$\leqslant$	0
	0	0	0	0	0	0	0	1	0	0	0	0	-1	0	$\leqslant$	0
	0	0	0	0	0	0	0	0	1	0	0	0	-1	0	$\leqslant$	0
	0	0	0	0	0	0	0	0	0	1	0	0	-1	0	$\leqslant$	0
	0	0	0	0	0	0	0	0	0	0	0	1	-1	0	$\leqslant$	0

Chapter 3

Risk Measure in Production Planning with Probability

In order to understand risk is preferable to define first its concept and what that it involves. Accordingly, risk measure can be defined as a procedure for shaping a loss distribution, for instance an investor's risk profile [61, 55]. Value-at-risk (VaR), which is a percentile of a loss distribution and conditional value-at-risk (CVaR) are among the few risk measure approaches most accepted by practitioners. There is a close correspondence between both approaches. When it is considered the same confidence level, VaR becomes a lower bound for CVaR. In terms of optimization applications CVaR is superior to VaR [61]. This chapter deals with production planning problems. Firstly, inventory management is described on the viewpoint of supplier; then, risk measures such as VaR and CVaR are introduced, followed by a treatement of multi-period production planning where Shapley value, and other proprieties of games such as group or grand rationality and individual rationality are also incorporated. Within the chapter, the relation between production planning and cooperative game theory is described, and as result three models to forecast production volume are proposed. The chapter concludes with a numerical example where the suggested models are tested.

3.1 Inventory Management

The automobile industry has been pushing forward with reduction in cost, induction of foreign capital, competition between suppliers, and so on. It becomes very important to remove waste of the production activities and to meet the demands of an individual customer and a changing market. In particular, we must satisfy the variety of customer specification in product and service, without dropping the productive efficiency in mass customization [9, 20, 52].

For example, in case of a certain car manufacturer, it is said that there is possibility of 300,000,000 ways of specification for a "customized product" in one model of a car. On manufacturing of this "customized product", leveling of production load becomes very difficult because manufacturers have to meet orders of customers based on each specification. Then the productivity decreases, finished goods in stock increase, and it becomes difficult to deal with customer specification.

There have already been some manufacturing strategies for mass customization. However, the design method of production planning and management system for it has not yet been established. Under the precondition that delivery lead time which is expected by customers is longer than production lead time which is necessary for manufacurers, Make-to-Order management system in which manufacturing starts after receiving an order has been applied for a variety of customer specifications as production planning and management system for mass customization. However, it is often the case that delivery lead time becomes shorter than production lead time. Therefore manufacturers have to start manufacturing before they receive an order from customers.

For a variety of customer specifications, Make-to-Order management system and Parts Oriented Production System (POPS) type module manufacturing system are proposed [14]. The Material Requirement Planning (MRP) [64] and Advanced Planning & Scheduling (APS) [36] are presented to plan and manage such systems. This MRP satisfies various demands from customers by promoting modularization in production. By making use of modularization, the MRP makes production lead time shorter, and it manages Make-to-Order management system on precondition that delivery lead time is longer than production lead time. When delivery lead time is shorter than production lead time, we consider the minimum stock as necessary beforehand. But in these management systems, the mechanism between substitutability of unfulfilled order and stocks on mass customization has not been discussed quantitatively.

On precondition that delivery lead time is shorter than production lead time, manufacture seat system for both prospective stocks and order stocks has been proposed in [17]. A study on the analysis of manufacture seat system coverage has also been [34]. However, it has been pointed out that further theoretical work about the decision method for proper manufacture seat should be done [66].

Mass Customization Environment

In order to implement mass customization for each item (or part) it is necessary a colaboration between manufacturers and suppliers. Three types of order information (forecast order) from manufacturer to suplier can be considered, that is, a monthly forecast order which gives the prospected order, say 3 months before; the weekly forecast order informing the prediction value for weeks before, and the delivery instruction to supplier which is given as the firm order 1-3 days before delivery due date. The increase of customer specifications, in mass customization environment, implies increase of production specifications in manufacturer. This means that the quantities of firm order reflecting customer needs have large fluctuation.

For supplier, the standard production lead time for 100-200 kinds of products (or parts) usually takes about 1 week. Therefore production lead time becomes longer than delivery lead time. Supplier must start production in advance based on production plan by using MRP, for example, according to weekly forecast order. Furthermore, the supplier must perform the production planning while at the same time avoiding unfulfilled order to the firm order which is given 1-3 days before the delivery due date.

We regard mass customization as fluctuation of order from customer to manufacturer and from manufacturer to supplier. Then it can be grasped from the supplier's point of view as the fluctuation of order quantities of target part from manufacturer to supplier. This means that the fluctuation of order quantities can be represented by standard deviation around average that forecast order gives.

Assuming that σ_0 denotes the order fluctuation of concerned model of car from customer to manufacturer and σ denotes the order fluctuation about concerned production (or parts) from manufacturer to suppliers. Then the production planning and management system implementing mass customization in suppliers side can be considered by the problem such as how to product and to stock individual parts in advance in order not to run short of supplies for σ which gives fluctuation of firm order from manufacturer. In this chapter, we assume that σ is given and demand is defined as a normal distribution with time variant average and time variant deviation of order. We here focus on the supplier's production planning system, so the effect of σ only are discussed in this model. However it is important to investigate the impact of σ_0 on the model, which should be included in the future research.

Unfulfilled Order Rate for Supplier's Production Planning and Management System

Derivation of Unfulfilled Order Rate

In mass customization environment, we formulate problem which determines proper production quantities among n periods to optimize inventory change at final production stage of supplier, corresponding to fluctuation of demand from manufacturer. In this section, we discuss about one certain part.

[Notation]

i **:** Period ($i \leqslant n$).

d_i **:** Firm order of manufacturer at period i.

x_i **:** Production quantity at period i of supplier.

S_i **:** Inventory quantity at period i of supplier.

p_i **:** Manufacturing cost per module at period i of supplier.

h_i **:** Inventory holding cost per module at period i of supplier.

r **:** Total product quantity until n period of supplier.

R **:** Set of linear production constraints of supplier.

SO_i **:** Unfulfilled order rate until i period of supplier.

β **:** Unfulfilled order rate of planned target of supplier.

It is assumed that d_i obeies an average $\bar{d}_i$ and standard deviation ω_i where d_i, d_j $(i \neq j)$ are independent each other, and $\omega_i = \sigma \cdot \bar{d}_i$, where σ is the deviation of order. Let initial inventory be S_0. R is denoted by $(x_1, x_2, \cdots, x_n) \in R$ where R denotes linear production constraints, and is convex. SO_i is probability function which is in short of delivery to the firm order at least untill i period.

The inventory quantity S_i at ith period is given by

$$S_i = S_0 + \sum_{t=1}^{i} x_t - \sum_{t=1}^{i} d_t \tag{3.1}$$

where d_i is random variable so that S_i becomes random variable. It obeys normal distribution which has the following time variant average and variance.

$$\textit{Average} \qquad m_i = S_0 + \sum_{t=1}^{i} x_t - \sum_{t=1}^{i} \bar{d}_t, \tag{3.2}$$

$$\textit{Variance} \qquad \sigma_i^2 = \sum_{t=1}^{i} \omega_t^2. \tag{3.3}$$

S_i can be replaced by

$$y_i = \frac{S_i - m_i}{\sigma_i}. \tag{3.4}$$

We consider an unfulfilled order rate SO_n which represents probability that $S_1 \geqslant 0, S_2 \geqslant 0, S_3 \geqslant 0, \cdots, S_n \geqslant 0$ is not satisfied. The probability function M_i which satisfies inventory quantity $S_i \geqslant 0$ at i period is derived by

$$M_i = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(S_i - m_i)^2}{2\sigma_i^2}} dS_i = \int_{-\frac{m_i}{\sigma_i}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i. \quad (3.5)$$

Thus

$$(i) \qquad M_i = 0.5 + \int_0^{\frac{m_i}{\sigma_i}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \qquad (m_i \geqslant 0), \quad (3.6)$$

$$(ii) \qquad M_i = 0.5 - \int_0^{-\frac{m_i}{\sigma_i}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i \qquad (m_i < 0). \quad (3.7)$$

These equations can be represented by

$$M_i = 0.5 + sgn(m_i) \int_0^{|-\frac{m_i}{\sigma_i}|} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_i^2}{2}} dy_i. \quad (3.8)$$

The upper bound of the unfulfilled order rate SO_n is given by $1 - \prod_{t=1}^{n} M_t$. Since it is difficult to obtain the true value of the unfulfilled order rate, we define the unfulfilled order rate SO_n by

$$SO_n = 1 - \prod_{t=1}^{n} M_t. \quad (3.9)$$

By considering such unfulfilled order rate, we can construct the production planning on safety side. In addition, SO_n can be rewritten by applying the integration by parts to M_t as follows;

$$SO_n = 1 - A \quad (3.10)$$

where

$$A = \prod_{t=1}^{n} \left\{ 0.5 + sgn(m_t) \frac{e^{-\frac{1}{2}\left(\frac{m_t}{\sigma_t}\right)^2}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{|-\frac{m_t}{\sigma_t}|^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)} \right\}. \quad (3.11)$$

Property of Unfulfilled Order Rate

We clarify the property of unfulfilled order rate to understand effect of some variables.

Lemma 1 SO_n is monotonous decrease function of x_i.

Lemma 2 SO_n is monotonous decrease function of S_0.

Lemma 3 SO_n is monotonous increase function of $\bar{d}_i$.

Lemma 4 SO_n is monotonous increase function of ω_i in $m_i > 0$ $(i \leqslant n)$.

Proof. We apply partial derivative SO_n with respect to x_i, S_0, $\bar{d}_i$ and ω_i. Let α be arbitrary variable.

$$\frac{\partial SO_n}{\partial \alpha} = - \frac{\partial M_1}{\partial \alpha} M_2 \cdots M_n - M_1 \frac{\partial M_2}{\partial \alpha} M_3 \cdots M_n - M_1 M_2 \cdots M_{n-1} \frac{\partial M_n}{\partial \alpha}. \quad (3.12)$$

Since $M_i > 0$ and

$$\frac{\partial}{\partial \alpha} \int_{g(\alpha)}^{\infty} f(y_i) dy_i = -f(g(\alpha)) \frac{\partial g(\alpha)}{\partial \alpha}, \quad (3.13)$$

by substituting α into x_i, S_0, $\bar{d}_i$ and ω_i, we can derive the following properties.

$$\frac{\partial M_i}{\partial \alpha} = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(-\frac{m_i}{\sigma_i}\right)^2} > 0 \qquad (\alpha \leftarrow x_i, S_0), \quad (3.14)$$

$$\frac{\partial M_i}{\partial \alpha} = -\frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{1}{2}\left(-\frac{m_i}{\sigma_i}\right)^2} < 0 \qquad (\alpha \leftarrow \bar{d}_i), \quad (3.15)$$

$$\frac{\partial M_i}{\partial \alpha} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(-\frac{m_i}{\sigma_i}\right)^2} \frac{m_i}{\sigma_i^2} \cdot \frac{\omega_i}{\sigma_i} < 0 \quad (\alpha \leftarrow \omega_i,\ m_i > 0). \quad (3.16)$$

□

(3.14) and $M_i > 0$ $(i \leqslant n)$ lead to Lemmas 1 - 4.

Production Planning for Implementing Mass Customization with Multi-item

Problem Formulation of Mass Customization with Multi-item

In this subsection, we define supplier's production planning and management system as the stochastic programming problem to find production plan minimizing manufacture and inventory cost under both unfulfilled order rate constraint and production constraint. In order to consider the multi-item case, previous notations are extended by adding information about jth part of product $(j \leqslant m)$. For example, d_i^j, x_i^j and S_i^j denote the firm order about jth part of manufacture at period i with time-variant average $\bar{d}_i^j$, the production quantity of jth part at period i and inventory quantity of jth part at period i respectively. Also p_i^j and h_i^j denote manufacturing cost per module about jth part of manufacture at period i and inventory holding cost per module about jth part of manufacture at period i, respectively. Those parameters, p_i^j and h_i^j, will be regarded as the same value about each part without losing generality and practicality in order to simplify a transformation of objective function. And S_0^j denotes initial inventory of jth part, β^j is targeted unfulfilled order rate of jth part, and Q_i denotes upper bound of total product quantity at period i.

$$\text{minimize} \quad \mathrm{E}[\sum_{j=1}^{m}\{\sum_{i=1}^{n} p_i^j x_i^j + \sum_{i=1}^{n} h_i^j S_i^j\}] \tag{3.17}$$

$$\text{s.t.} \quad S_0^j + \sum_{t=1}^{i} x_t^j - \sum_{t=1}^{i} \bar{d}_t^j \geqslant 0 \qquad (\forall i, j) \tag{3.18}$$

$$\sum_{i=1}^{n} x_i^j = r^j \qquad (\forall j) \tag{3.19}$$

$$SO_n^j \leqslant \beta^j \qquad (\forall j) \tag{3.20}$$

$$\sum_{j=1}^{m} x_i^j \leqslant Q_i \qquad (\forall i) \tag{3.21}$$

$$(x_1^j, x_2^j, \cdots, x_n^j) \in R^j \qquad (\forall j) \tag{3.22}$$

$$x_i^j \geqslant 0 \qquad (\forall i, j) \tag{3.23}$$

The evaluation function (eq. (3.17)) expresses that we must find optimal solutions $(x_1^1, \cdots, x_n^m)$ that minimize the expected value (expectation) of the sum of production cost and inventory cost. Eq. (3.18) is non negative condition about inventory quantity that the demand is not over the forecast order. Eq. (3.19) is constraint about target production quantity. Eq. (3.20) gives constraint about unfulfilled order rate, Eq. (3.21) denotes limit of production capacity at each period and Eq. (3.23) represents linear constraint about production.

The above problem is a manufacture/inventory problem where demand changes stochastically [68]. Although the (s, S) strategy [17, 68] is known as an effective tool for such a problem, it is not applicable when the problem has many conditions about production. Thus we propose practical and effective algorithm by solving linear problem as partial problems repeatedly.

3.2 Risk Measure Approaches

To understand risk and all that it is involved become easier bu defining first the concept itself. The most known definition of risk, according to Holton [26], is credited to Frank Knight [33], while was actively researching the foundations of probability.

Regarding subjective versus objective interpretations of probability, the list of researchers includes John Maynard Keynes [31], Richard von Mises [41], and Andrey Kolmogorov[35]. For the objective interpretations probability is real, and may be visualized by logic or evaluated through statistical analyses. The subjective side determines probability as human beliefs which are not intrinsic to nature. They may be stipulated to determine the uncertainty. Hume [27] is believed to be the pioneer of the philosophical roots of subjective interpretations of probability, as he states on this quote:

> Though there be no such thing as *Chance* in the world; our ignorance of the real cause of any event has the same influence on the understanding, and begets a like species of belief or opinion.

Concepts related to measurement of risk have been associated to probability theory by mathematicians since the early stage of studies on risk as Daniel Bernoulli [7] stated:

> *Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of possible cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon.*

He explains his idea through the following fundamental rule:

> *If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divided the sum of these products by the total number of possible cases, a mean utility (or moral expectation) will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question.*

Thus, according to Bernoulli, in order to get a valid measurement of the value of a risk one must consider its *utility*. However, this utility depends on the circumstances which makes it hard to be generalized.

3.2.1 Value-at-Risk (VaR)

Let X be a loss random variable with the cumulative distribution function $F_x(z) = P\{X \geqslant z\}$.

Definition 3.2.1 (VaR). *The value-at-risk of X with confidence level $\alpha \in (0,1)$ is given by*

$$VaR_{1-\alpha}(X) = min\{z\colon F_x(z) \geqslant \alpha\} \tag{3.24}$$

i.e., $VaR_{1-\alpha}(X)$ is a lower α-percentile of X and is proportional to the standard deviation if X is normally distributed, that is, $X \sim N(\mu, \sigma^2)$ then,

$$k(\alpha) = \sqrt{2}\mathrm{erf}^{-1}(2\alpha - 1) \tag{3.25}$$

and

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-x^2}\, dt. \tag{3.26}$$

$\mathrm{VaR}_\alpha(X)$ is a nonconvex and discontinuous function of the confidence level α for discrete distributions; non-subadditive and has several extrema for discrete which makes it to be difficult to control or optimize for nonnormal distributions [39].

3.2.2 Conditional Value-at-Risk (CVaR)

Definition 3.2.2 (CVaR). *$CVaR_\alpha(X)$ equals the conditional expectation of $\boldsymbol{X}$ subject to $\boldsymbol{X} \geqslant \boldsymbol{VaR}_\alpha(\boldsymbol{X})$ for random variables with continuous distribution. Formally,*

$$CVaR_\alpha(X) = \{\mathrm{E[X]}\; s.t.\mathrm{X} \geqslant \mathrm{VaR}_\alpha(X)\} \tag{3.27}$$

The CVaR of $\boldsymbol{X}$ with $\alpha \in (0,1)$ is the mean of the generalized α-tail distribution:

$$CVaR_\alpha(X) = \int_{-\infty}^{\infty} \mathrm{z} dF_x^\alpha(z) \tag{3.28}$$

where

$$F_x^\alpha(z) = \begin{cases} 0, & \text{for } z < VaR_\alpha(X) \\ \frac{F_x(z)-\alpha}{1-\alpha}, & \text{for } z \geqslant VaR_\alpha(X) \end{cases} \tag{3.29}$$

$\mathrm{CVaR}_\alpha(X)$ is not equal to an average of outcomers greater than $\mathrm{VaR}_\alpha(X)$.

3.3 Multi-period Production Planning by Shapley Value

Game theory can be applied to production planning process. Comparison between the main element for both techniques are presented in Table 3.1. In our methodology we use the corresponding second column to perform the computations. The estimated demand can be represented as $\boldsymbol{d} = [d_1, d_2, \dots, d_n]$, and is normally distributed, i.e., $d \sim N(\bar{d}, \Sigma)$., with $\bar{\boldsymbol{d}}$ denoting the expected demand.

Shapley Value in Quadratic Form

In their study, [57] proved Shapley value to be a least square value (LS) with its weight function $M_{n,s}$ defined by

$$M_{n,s} = \frac{1}{n-1}\binom{n-2}{s-1}^{-1} \tag{3.30}$$

where s denotes the cardinality of any coalition S, and n the number of players in the game.

Model 1 (**Shapley value**)

$$\begin{aligned} \min \sum_{s \subset N} & \Big(v(S) - x(S)\Big)^2 m(s) \\ \text{s.t.} \sum_{i \in N} & x_i = v(N) \\ & x_i \geqslant v(i), \;\; (\forall\, i \in N) \end{aligned} \tag{3.31}$$

where, $m(s) = \mathrm{M}_{n,s}$ is the weight function, x a payoff vector and

$$x(S) = \sum_{i \in S} x_i \tag{3.32}$$

for any coalition S. As demonstrated in [56, 57], Eq. (3.31) is equivalent to Eq. (2.28) , but in QP form with individual rationality and group rationality as constraints.

Production Planning

Production managers are responsible for decision making regarding planning of production for certain period. To this end, consider the following elements [67]:

- The cumulative demand $D = d_1 + d_2 + \cdots + d_n$, where d_i, $(i = 1, 2, \cdots, n)$ expresses the demand for product at certain period i of planning. The demand $d = [d_1, d_2, \dots, d_n]$ follows the normal distribution, i.e., $d_i \sim N(\bar{d}, w_i^2)$ $\forall i$, with $\bar{d_\imath}$ denoting the expected demand. Moreover, $Cov(d_i, d_j) = 0$ $\forall i, j, i \neq j$.

- The inventory for the i^{th}-period is given by S_i, $(i = 1, 2, \cdots, n)$, and calculated through Eq. (3.33). At the beginning of the process the initial inventory S_0 is assumed to be known.

$$S_i = S_0 + \sum_{t=1}^{i} x_t^* - \sum_{t=1}^{i} \bar{d}_t \tag{3.33}$$

  where, x_i^* denotes the production level set for each period i, $(i = 1, 2, \cdots, n)$ which is computed by applying Eq. (3.41).

- $1 - \alpha$ is the confidence level, often set to 0.95 or 0.99 [61].

Relation between Production Planning and Cooperative Game Theory

As a technique used for strategic decision making, game theory can be applied to several problems including production planning. A summary on the relation between the former and latter approaches is described in Table 3.1 [67]. On the top of the table both approaches are stated; the first column describes the comon element between them, and the following two columns describe the meaning of each element in production planning and coalitional game theory, respectively. The i^{th}-period in production planning is described as the i^{th}-player in cooperative games; subset S, $(S \subseteq N)$, a coalition in game theory represents a set of periods in production planning; while the characteristic functions are denoted by $v(S)$ in game theory their correspondence in production planning is CVaR with a confidence level of $1-\alpha$; in the last row, φ_i indicates the estimated demand in production planning, hereafter we call it allocated risk, and Shapley value is its correspondence in coalitional game theory point of view.

Consider $CVaR_{D|D}(1-\alpha)$ for a cumulative demand D. Production planning problem in a multi-period context [67] consists, basically, in finding the volume of production for each period in order to satisfy $CVaR_{D|D}(1-\alpha)$ using Shapley value. The avarage value-at-risk is given by

$$CVaR_{D|D}(1-\alpha) = E[D|D > VaR_D(1-\alpha)] \tag{3.34}$$

Given an initial inventory S_0, the process starts with x_1 production volume, an average demand $\bar{d}_1$ is estimed; in the next period the stock is evaluated and it is expected to increase the average of the demand, as well as high level of production. This dynamics is performed for all n periods as shown in Figure 3.1.

Characteristic Function

The characteristic function $v(S)$ for any member of S is equal to $CVaR_{(1-\alpha)}$, and can be computed by using Eq. (3.35) defined as follows.

$$\begin{aligned} v(S) = CVaR(S)_{(1-\alpha)} &= \sum_{i\in S} d_i \\ &+ \sum_{i\in S}\sum_{j\in S} \sigma_{ij} \frac{\vartheta(z_{1-\alpha})}{1-\Phi(z_{1-\alpha})} \end{aligned} \tag{3.35}$$

where,

$$z_{1-\alpha} = \frac{VaR_D(1-\alpha) - \sum_{i\in S} d_i}{\sqrt{\sum_{i\in S}\sum_{j\in S} \sigma_{ij}}} \tag{3.36}$$

Here, ϑ is the standard normal density; Φ denotes the cumulative function and σ_{ij} the elements of the covariance matrix Σ as shown in (3.44) .

Table 3.1: Relation between production planning problem and game theory

	Production Planning	**Cooperative Game Theory**
i	Period	Player
S	Set of Periods	Coalition
Characteristic function	$\text{CVaR}_{(1-\alpha)}$	$v(S)$
φ_i	Estimated demand	Shapley value

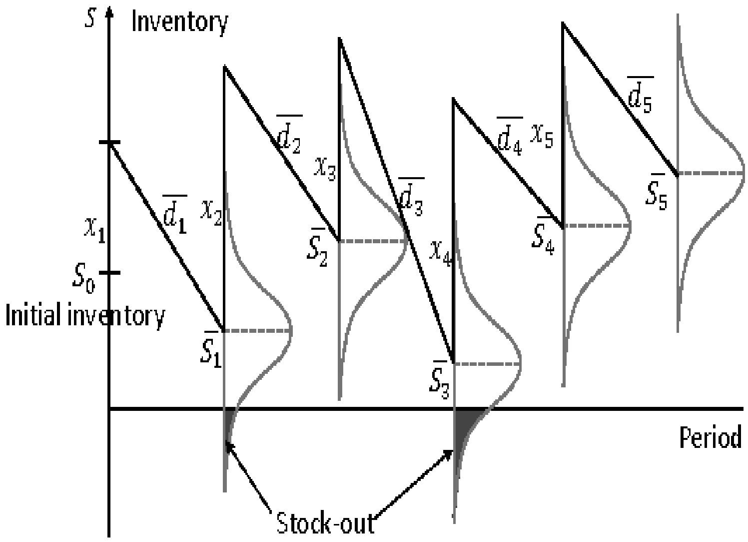

Figure 3.1: Dynamics of the production planning

This typical characteristic function is subadditive [61], that is, given $S, T \subset N$ such that $S \cap T = \varnothing$,

$$v(S \cup T) \leqslant v(S) + v(T). \tag{3.37}$$

Thus, $v(S)$ is negative, as well as Shapley value in order to satisfy superadditivity.

Production Constraints

The production manager can obtain information regarding the volume of production for each i period by analyzing the relation between the production constraints (x_i) and Shapley values (φ_i).

Formally, this volume is computed [67] by solving the following equation.

- For $i = 1$:

$$x_1^* = -\varphi_1 - S_0 \tag{3.38}$$

- For $i \geqslant 2$:

$$x_i^* = -\varphi_i + \varphi_{i-1} + \bar{d}_{i-1} \tag{3.39}$$

That is,

$$\begin{cases} x_1^* = -\varphi_1 - S_0 \\ x_2^* = -\varphi_2 + \varphi_1 + \bar{d}_1 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ x_n^* = -\varphi_n + \varphi_{n-1} + \bar{d}_{n-1} \end{cases} \tag{3.40}$$

A production volume will be accepted if $x_i^* \geqslant 0$; in case where $x_i^* \leqslant 0$, the process will be interpreted as $x_i^* = 0$, i.e., nothing to be considered.

System (3.40) can be rewritten in compact form as

$$\mathbf{X} = \mathbf{Q}\varphi + \tilde{\mathbf{d}}, \tag{3.41}$$

where,

$$\mathbf{X} = \begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \\ \vdots \\ x_n^* \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_n \end{pmatrix} \tag{3.42}$$

$$\tilde{\mathbf{d}} = \begin{pmatrix} -S_0 \\ \bar{d}_1 \\ \bar{d}_2 \\ \vdots \\ \bar{d}_n \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \tag{3.43}$$

The covariance matrix (3.44) of cumulative demand D is computed by finding first the variance ω_i and performing the arithmetic within the matrix.

$$\begin{aligned} \mathbf{\Sigma} &= \begin{bmatrix} \omega_1^2 & \omega_1^2 & \cdots & \omega_1^2 \\ \omega_1^2 & \omega_1^2 + \omega_2^2 & \cdots & \omega_1^2 + \omega_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^2 & \omega_1^2 + \omega_2^2 & \cdots & \omega_1^2 + \omega_2^2 + \cdots + \omega_n^2 \end{bmatrix} \\ &\equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix} \end{aligned} \tag{3.44}$$

with $\sigma_{ij} = \omega_1^2 + \omega_2^2 + \cdots + \omega_i^2, \quad (i < j)$.

Models

Methods to solve a typical production planning problem with uncertainty demand are several. For instance, one can employ stochastic models or safety inventory, which are probability of stock-out methods based approaches. In this research, we took a different direction by basing our analysis on the cumulative demand.

In Ueno, et al., [67] a similar problem were treated by applying Shapley value to risk management considering several periods. Through this chapter, we aim to extend that study by taking as initial state the quadratic form of Shapley value defined in Ruiz, et al,. in [57, 56]; three nonlinear models are presented. In the first model we added new constraints, production constraints, directly into the QP model in order to find optimal and feasible solutions within the production planning point of view; in the second model, a new constraint is considered to estimate the penalty of individual rationality as regard to risk in order to support policy makers. Hence, the model is also called as Constrained Model. Now, because of some weakness this model was extended the Available Model. Computations regarding the efficiency of the models proposed in the chapter can be found in the next section through a numerical example, followed by its discussion.

Now, introducing a set of productions constraints into (3.31) in order to satisfy the two aforementioned properties we proposed the following models.

Model 1.1

$$
\begin{aligned}
&\min \sum_{s\subset N} (v(S) - x(S))^2 m(s) \\
&\text{s.t. } \sum_{i\in N} \varphi_i = v(N) \\
&\qquad \varphi_i \geqslant v(i) \\
&\qquad x_i^* \geqslant 0 \\
&\qquad \varphi_i \geqslant \beta, \quad (\exists\, i \in N)
\end{aligned}
\tag{3.45}
$$

where β represents a nonnegative value interpreted by the decision-maker as upper or lower value in production volume, this helps DM to analyze the perfomance of the process in a specific period; x_i^* denotes the production volume as defined is (3.40).

Model 1.1 is simply an extension of Shapley value in (3.31). The motivation to propose Model 1.1 or Eq. (3.45) is that Model 1 or Eq. (3.31) has direct application in cooperative games, but it is inefficient to solve problems related to production planning. The diference between these two models lies, essentially, on the constraints added, that is, $x_i^* \geqslant 0$ and φ_i.

These new constraints, particularly $x_i^* \geqslant 0$, allows the decider to evaluate problems related to production planning under cooperative game theory framework. Furthermore:

- $x_i^* \geqslant 0$ informs DMs about the level of production.
- The relation between Shapley value φ_i and production volume x_i^* is described through Eq. (3.40), i.e., production volume is linearly proportional to the difference between Shapley value and the initial inventory in first period, and proportional to the difference between Shapley solution and the i^{th} for the subsequent periods.
- By adding β the decision-maker would be able to estimate if the level of production in the j^{th}-period is upper or lower comparing to other periods. For instance, suppose he desires to reach a high volume of production on the last day of the week, then he would attribute a certain value to β while forecasting the production for that particular day.

Although it satisfies individual and group rationalities, as one can observe in the last three columns in Table 3.3 (cases 2-12 and 2-13), Model 1.1 does not have solution always. Therefore, in order to solve this inefficiency we proposed Model 2 which is formally given by

Model 2(Constrained model)

$$
\begin{aligned}
&\min_{\varphi} \sum_{|S|} M(v(S) - \sum_{i\in S} \varphi_i(N,v))^2 \\
&\text{s.t. } \sum_{i\in S} \varphi_i = v(N) \\
&\qquad x_i^* \geqslant 0 \\
&\qquad \varphi_i \geqslant \beta, \quad (\exists\, i \in N)
\end{aligned}
\tag{3.46}
$$

where $v(S)$ represents the characteristic function computed through CVaR in (3.35); $\varphi = [\varphi_1, \varphi_2, \cdots, \varphi_n]^T$ are the decision variables; $M = diag(M_{n,s}, \cdots, M_{n,n-1})$ denotes a matrix of weights whose components are taken from Eq. (3.30), that is,

$$
\mathbf{M} = \begin{bmatrix} M_{n,s} & & & & \\ & M_{n,s} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & M_{n,n-1} \end{bmatrix} \tag{3.47}
$$

- This model is distint to the previous one by considering a matrix of weights in the objective function, and moroever, payoff vector $x(S)$ is replaced by the total reward allocated, i.e., $\varphi_i(N, v)$.
- Although it has a solution; however, the constrained model does not satisfy the individual rationality; in order to evaluate the distribution of production a penalty is chosen by the decision maker.

Models described above seemed to be ineffective to control the satisfaction of individual rationality for each period. For this reason, we proposed a new model by adding a weighting factor, w_i, $(i = 1, 2, \cdots, n)$, into QP (3.46).

Model 3 (**Available Model**)

$$\begin{aligned} \min_{\varphi} \ & \sum_{i} \sum_{|S|} w_i M(v(S) - \sum_{i \in S} \varphi_i(N, v))^2 \\ \text{s.t.} \ & \sum_{i \in S} \varphi_i = v(N) \\ & \varphi_i \geqslant \beta, \ (\exists\, i \in N) \\ & x_i^* \geqslant 0 \end{aligned} \tag{3.48}$$

- The objective function in model (3.48) indicates the badness regarding uncertainty.
- The penalty is defined by the difference between the allocated risk and the charcteristic functions, formally $\varphi_i - v\{\imath\}$. This is possible for inserting the new contraints $\sum_{i \in S} \varphi_i = v(N)$ and $\varphi_i \geqslant \beta$ into the model.
- The difference between model (3.48) and the constrained model (3.46) is essentially the weighting factor added into the ojective function of model (3.48).
- Model 3 has always feasible solution.
- The weighting factor, w_i, $(i = 1, 2, \cdots, n)$, is inserted to strictly satifies the group rationality, as well to control the satifaction of individual rationality from the point of view of cooperative game theory; while in terms of production planning individual rationality is not necessarily crucial for all periods, thus this condition may not always be satisfied.
- If the weighting factor is $w_i \gg 0$, $(i = 1, 2, \cdots, n)$, the objective funtion to be minimized will be comparativelly small, otherwise it makes it greater, which turns to be a contradiction as we pretend to decrease this value.
- The advantage of using factor w_i is that it allows policy makers to forecast precisely how much they would expect as production volume for each i-period, that is, using model (3.48) the decision maker is able to predict which period will have upper or lower performance as regards to production for using w_i, $(i = 1, 2, \cdots, n)$. This fact can be observed through cases 4-11 to 4-18 in Table 3.3.

Next section provides a numerical example where models described above were employed.

3.4 Numerical example

To analyze the efficiency of the three models, we performed different case studies. Numerical results are summarized through Tables 3.2 - 3.5.

Table 3.2: Characteristic function $v(S)$ for the 5-period game

Coalitions	***v(S)***	**Coalitions**	***v(S)***
$v\{\emptyset\}$	0	$v\{123\}$	-73.5853
$v\{1\}$	-17.9957	$v\{124\}$	-57.1545
$v\{2\}$	-31.3076	$v\{125\}$	-64.6151
$v\{3\}$	-37.8488	$v\{134\}$	-62.6151
$v\{4\}$	-21.9912	$v\{135\}$	-69.9870
$v\{5\}$	-29.8788	$v\{145\}$	-53.2845
$v\{12\}$	-43.8488	$v\{234\}$	-73.9870
$v\{13\}$	-49.9912	$v\{235\}$	-81.2845
$v\{14\}$	-33.8788	$v\{245\}$	-64.5186
$v\{15\}$	-41.5853	$v\{345\}$	-69.6978
$v\{23\}$	-61.8788	$v\{1234\}$	-85.2845
$v\{24\}$	-45.5853	$v\{1235\}$	-92.5186
$v\{25\}$	-53.1545	$v\{1245\}$	-75.6978
$v\{34\}$	-51.1545	$v\{1345\}$	-80.8287
$v\{35\}$	-58.6151	$v\{2345\}$	-91.9169
$v\{45\}$	-41.9870	$v\{12345\}$	-102.9670

3.4.1 A 5-Period Game

Consider a group of production managers intending to forecast their business for five periods. Thus, we have a problem of production planning for a set of five periods, i.e., $N = \{1, 2, 3, 4, 5\}$ as described in Table 3.1 periods denote player in cooperative game theory, thus it is a 5-persons or 5-periods game.

Initially, their inventory is assigned to be $S_0 = 10$, the estimated demand is given by $d = [d_1, d_2, d_3, d_4, d_5] = [10, 20, 24, 6, 12]$, respectively; the variance ω is equal to 3 with the level of significance $\alpha = 0.01$. The characteristic functions for each period and respective coalitions are shown in Table 3.2, and were obtained by solving Eq. (3.35).

From Eq. (3.30) and Eq. (3.47) holds: $M_{5,1} = M_{5,4} = 0.25$ and $M_{5,2} = M_{5,3} = 0.0833$

$$\mathbf{M} = \begin{bmatrix} 0.2500 & 0 & 0 & 0 \\ 0 & 0.0833 & 0 & 0 \\ 0 & 0 & 0.0833 & 0 \\ 0 & 0 & 0 & 0.2500 \end{bmatrix}$$

The matrix covariance-variance (3.44) is then given by

$$\mathbf{\Sigma} = \begin{pmatrix} 9 & 9 & 9 & 9 & 9 \\ 9 & 18 & 18 & 18 & 18 \\ 9 & 18 & 27 & 27 & 27 \\ 9 & 18 & 27 & 36 & 36 \\ 9 & 18 & 27 & 36 & 45 \end{pmatrix}$$

3.4.2 Results and discussion

To discuss the results from models proposed in this chapter, we start by explaning the content of each table throug columns to rows, which indicate the performance of each tested case.

Table 3.3 shows the allocated risk φ_i for all periods.

- The first column lists all case studies considered in the problem, namely Shapley value Eq. (3.31) was employed to case 2-11; cases 2-12 and 2-13 were analyzed by employing Model 1.1, that is, Eq. (3.45), while the Constrained Model Eq. (3.46) was used to solve cases 3-11 to 3-15, and finally cases 4-11 to 4-18 were computed by using Mode 3 in Eq. (3.48).

- The second column displays the constraints added, only the first case does not consider new constraints for being the one that employs the original model Eq. (3.31). These constraints are chosen by the decision-maker; all values of φi are negative, with $\phi_i = -\varphi_i$. In all cases we took the last period as the sample representation with exception to case 3-15. For instance, suppose that the decider wishes to evaluate his choices for each cases with respect to the 5^{th} period.

- From the third to the seventh column, the numbers indicate Shapley values in the first cell of each column and the allocated risk in the rest of cells for all periods, respectively. As for difference on terminologies with regard to cells, please see Table 3.1.

- The eighth column indicates the objective function to be minimized in each case. This value refers to the level of badness while implementing the Eq. (3.48).

- In the ninth column is presented the total penalty over individual rationality for each case. These values represent overall quantity on how individual rationality is affected with regard to each model when using the constraints, and is found through Table 3.4.

- The next two columns list the satisfaction or dissatisfaction of group rationality and individual rationality, respectively.

- The last column lists the state of satisfaction of constraints in each case.

- In the first row Shapley values are indicated by φ_i, they are all negatives due to the characteristic function being subadditive. Consider, for instance, case 2-11 vector φ =[-12.761, -24.616, - 30.300, -13.886, -21.405] represents Shapley values for all periods, respectively. This case corresponds to Shapley model (3.31), therefore there is not constraints added; optimal solution is found with the objective function equal to 141.928; this model satisfies both group and individual rationalities.

- Hereafter we set $\phi_i = -\varphi_i$. In the second row, case 2-12 has $\beta = 20$, then the last period is forecast to be no great than 20, that is, $\phi_5 \geqslant 20$; this case is equal to Shapley value model despite of using a constraint that is not considered in the conventional method; depending on the constraints chosen by the decision-maker this model have the same solution with those in Shapley model. As can be observed the constraint added by the decision-maker is satisfied with $\phi_i = 21.405$.

- Case 2-13, in the third row, with constraint set to $\phi_5 \geqslant 25$, model 1.1 satisfy all properties; the highest value is 25.

- From the 4^{th} to the 7^{th} rows, i.e., case 3-11 to 3-14, employing model 2 (3.46) individual rationality is not observed, but all cases have solution. The highest is the value of β the highest is the total penalty, which is 20.121 cprresponding to case 3-14.

- In the 8^{th} row, case 3-15, the penalty affects the 4^{th} and the 5^{th} periods for the decision-maker forecasts the production by setting the sum of these two period to not be less than 60. These correspond to cases where Model 2 was considered.

- From case 4-11 to 4-18, corresponding to the last eight rows, the Available Model (3.48) was used to evaluate the performance. The weighting factor is set to $w_i = 100$. For instance, in case 4-12 $w_1 = 100$, and ϕ_1 is pretty close to $v\{1\}$ in Table 3.2. Whenever w_i is applied to a certain period the total penalty is found essentially high at that period.

- In most of cases individual rationality is not satisfied, while group rationality is always satisfied. Additionally, all cases where model (3.48) was used the total penalty seems to be very small implying that the solution from this model satisfies group rationality by using production constraints, (cf. ninth column of Table 3.3). Furthermore, this solution is quite close to those from Shapley value and thus, model (3.48) is efficient to solve problems regarding production planing.

Table 3.4 highlights the concept of penalty for each period, which as given by the difference between the allocated risk and the characteristic functions.

- The meaning of this table is to evaluate if individual rationality is satisfied. This is done by quantifying the level of penalty, computed through the difference between the allocated risk φ_i in each period and the characteristic functions $v\{i\}$ for each period, i.e., $\varphi_i - v\{i\}$.

- The first column describes the cases, and all other columns show the penalties for each period, respectively.

- Note, for instance in Case 4-14, that whenever the weighting factor w_i applied to a certain period is large, the penalty over that period is essentially low.

Table 3.5 shows data related to production volume as computed through Eq. (3.40), and added as constraints, $x_i^* \geqslant 0$, in the models.

- Production volume is the same for cases 2-11 and 2-12 although diferent models were employed. High level of production is expected mostly for periods where $w_i = 100$ is considered. Because there exists uncertainty the nature of the problem requires other factors to be considered. Thus, adding constraints to the model will produce robust results in order to overcome uncertainty, and this brings up the efficiency of the proposed model. Information obtained for inserting these constraints allows managers to predict the outcomes from their policies.

- Having a balanced penalty over individual rationality in case 2-13 is expected low volume of production in period 3.

- Particular atention must be given to the 4^{th}, 5^{th}, 7^{th} and 8^{th} rows corresponding to cases 3-11, 3-12, 3-14 and 3-15 in the first period, their production volume is estimated to be null. The reason for this is that the initial inventory S_0 is too large, which means that there is not necessity for production at that particular time. This can be verified, numerically by computing x_i^* in Eq. (3.40).

- Contrary to other cases in the group case 3-13, in the 6^{th} row, is forecast to have a minimal volume of production in the first and the fourth periods, but increasing in others especially the third period.

- Likewise, cases 4-11, 4-13, 4-15 to 4-18 in the 9th, 11th, and 13th to the 16th rows are all null in period 1, due to the same reason as explained in similar situations.

Table 3.3: Risk allocated φ_i for 5 periods

Cases	**Production Constraints**	φ_1	φ_2	φ_3	φ_4	φ_5	**Objective Function**	**Total Penalty**	**Group Rationality**	**Individual Rationality**	**Constraints Satisfaction**
2-11		-12.761	-24.616	-30.300	-13.886	-21.405	141.928	0.000	○	○	X
2-12	$\phi_5 \geqslant 20$	-12.761	-24.616	-30.300	-13.886	-21.405	141.928	0.000	○	○	○
2-13	$\phi_5 \geqslant 25$	-11.862	-23.717	-29.401	-12.987	-25.000	158.080	0.000	○	○	○
3-11	$\phi_5 \geqslant 40$	-10.000	-19.338	-25.021	-8.607	-40.000	578.906	-10.121	○	X	○
3-12	$\phi_5 \geqslant 45$	-10.000	-17.671	-23.355	-6.941	-45.000	850.972	-15.121	○	X	○
3-13	$\phi_5 \geqslant 30$	-10.612	-22.467	-28.151	-11.737	-30.000	234.274	-0.121	○	X	○
3-14	$\phi_5 \geqslant 50$	-10.000	-16.005	-21.688	-5.274	-50.000	1189.705	-20.121	○	X	○
3-15	$\phi_4 + \phi_5 \geqslant 60$	-10.000	-13.642	-19.325	-26.240	-33.760	695.707	-8.130	○	X	○
4-11	$\phi_5 \geqslant 25,\ w_2 = 100$	-10.000	-30.909	-26.736	-10.322	-25.000	231.403	0.000	○	X	○
4-12	$\phi_5 \geqslant 25,\ w_1 = 100$	-17.682	-21.777	-27.461	-11.047	-25.000	205.684	0.000	○	X	○
4-13	$\phi_5 \geqslant 30,\ w_2 = 100$	-10.000	-30.814	-24.283	-7.869	-30.000	340.254	-0.121	○	X	○
4-14	$\phi_5 \geqslant 30,\ w_1 = 100$	-17.618	-20.132	-25.815	-9.401	-30.000	303.251	-0.121	○	X	○
4-15	$\phi_5 \geqslant 25,\ w_3 = 100$	-10.000	-17.844	-37.062	-13.062	-25.000	270.083	0.000	○	X	○
4-16	$\phi_5 \geqslant 30,\ w_3 = 100$	-10.000	-13.494	-36.736	-12.736	-30.000	420.508	-0.121	○	X	○
4-17	$\phi_5 \geqslant 25,\ w_4 = 100$	-10.000	-20.386	-26.069	-21.512	-25.000	262.115	0.000	○	X	○
4-18	$\phi_5 \geqslant 30,\ w_4 = 100$	-10.000	-17.933	-23.617	-21.417	-30.000	377.634	-0.121	○	X	○

○ Satisfied

X Unsatisfied

Table 3.4: Penalty over each period

No.	Period 1	Period 2	Period 3	Period 4	Period 5
2-11	5.235	6.691	7.549	8.106	8.474
2-12	5.235	6.691	7.549	8.106	8.474
2-13	6.134	7.590	8.448	9.004	4.879
3-11	7.996	11.969	12.827	13.384	-10.121
3-12	7.996	13.636	14.494	15.051	-15.121
3-13	7.384	8.840	9.698	10.254	-0.121
3-14	7.996	15.303	16.161	16.717	-20.121
3-15	7.996	17.666	18.524	-4.249	-3.881
4-11	7.996	0.398	11.113	11.669	4.879
4-12	0.314	9.530	10.388	10.945	4.879
4-13	7.996	0.493	13.565	14.122	-0.121
4-14	0.377	11.176	12.033	12.590	-0.121
4-15	7.996	13.464	0.787	8.930	4.879
4-16	7.996	17.813	1.112	9.255	-0.121
4-17	7.996	10.922	11.780	0.479	4.879
4-18	7.996	13.375	14.232	0.574	-0.121

In the followig lines we treat the problem of production planning by using the opmization model Section 2.3 from the previous chapter.

Consider a production planning for five periods, that is, $N = \{1, 2, 3, 4, 5\}$. Suppose initial inventory is evaluated to be $S_0 = 10$, the estimated demand given by $d = [10, 20, 24, 6, 12]$ for each period, respectively, with the variance $\omega = 3$ and level of significance $\alpha = 0.01$. Table 3.6 presents the characteristic functions $v(\mathcal{H})$ for singular period (or player) and respective coalitions obtained from solving Eq. (3.34).

One can easily observe that Shapley's efficiency (Group rationality) property is fulfilled, that is, the overall sum of the values gives the same value for the grand coalition $v(\mathcal{H})$ as in Table 3.6.

Table Table 3.7 shows Shapley values for the five periods. The first row contains the solution set using Eq. (2.28) from Section 2.2, and in the second row displayed the solution from Program Section 2.3. Through this solution one can obtain information regarding risk in each period. As expected both approaches present the same results, i.e., the Shapley values for each period or player π_1, π_2, π_3, π_4 and π_5 are respectively, 17.58, 15.13, 32.47, 26.38 and 39.73.

Other slack variables of the problem are: $\boldsymbol{s}_5^- = \boldsymbol{s}_6^+ = \boldsymbol{s}_6^- = 0$ (for all cases).

In Table 3.8, we imposed some new constraints into problem Section 2.3 in order to analyze the distribution of the risk among the periods. The problem was analyzed through different cases as defined below.

- Case 1: $\pi_5 \geqslant 3\pi_1$
- Case 2: $\pi_5 \geqslant 48$
- Case 3: $\pi_1 \leqslant 10$
- Case 4: $\pi_5 \geqslant 3\pi_2$
- Case 5: $\pi_5 \leqslant 35$
- Case 6: $\pi_2 \leqslant 6$
- Case 7: $\pi_3 \leqslant 20$

Table 3.5: Production volume for each periods

Production volume					
Cases	**Period 1**	**Period 2**	**Period 3**	**Period 4**	**Period 5**
2-11	2.761	21.855	25.684	7.586	13.519
2-12	2.761	21.855	25.684	7.586	13.519
2-13	1.862	21.856	25.683	7.586	18.013
3-11	0.000	19.338	25.683	7.586	37.393
3-12	0.000	21.856	25.683	7.586	44.059
3-13	0.612	21.856	25.683	7.586	24.263
3-14	0.000	16.005	25.683	7.586	50.726
3-15	0.000	21.856	25.683	30.915	13.519
4-11	0.000	30.909	15.827	7.586	20.678
4-12	7.682	14.095	25.683	7.586	19.953
4-13	0.000	30.814	13.469	7.586	28.131
4-14	7.682	12.513	25.683	7.586	26.599
4-15	0.000	17.844	39.218	0.000	17.938
4-16	0.000	13.494	43.242	0.000	23.264
4-17	0.000	20.386	25.683	19.443	9.488
4-18	0.000	17.933	25.683	21.800	14.583

- Case 8: $\pi_4 \leqslant 12$
- Case 9: $\pi_5 \leqslant 24$

Now, we desire to estimate how much is the difference of the cost or risk for each period as compared to the main case presented in Tables 3.7 and 3.8. The result of this process is displayed in Table 3.9.

Subtracting the results from Table 3.8 we quantified the difference on the estimated demand for each period, this arithmetic is presented in Table 3.9. Moreover, the planning process, in this table, is explained as follows: one can predict a higher demand in the last period while the estimated demand for other periods is expected to decrease considerably since the corresponding Shapley value for the fifth period increases n times the first period (**case 1**); if the estimated demand is increased, for instance, 4 times in the last period, we should expect a decreasing of the demand in the first and second periods and as well as a significant fall off in the third period (**case 2**); in **case 3**, allocating the lowest demand, 10, to the first period will have a positive impact on the last two periods in terms of demand, i.e. 2.46 and 6.36, respectively. Hence, in general, positive or negative impacts regarding demands and risks depend on the constraints the model is subject to, for this reason we consider our model to be a Shapley constraint approach.

In this chapter, production planning problems under framework of cooperative game theory with CVaR were analyzed.

CVaR was defined as the characteristic function for singular periods, and coalitions of periods. The relation between production planning and cooperative game theory was also described.

Opposite to other methods in production planning field in this chapter we considered uncertainty on the cumulative demand. Three models based on Shapley value in QP Eq. (3.31) form were proposed in Section 3.3 to solve a multi-period production planning. The first model, Model 1.1 Eq. (3.45), is based, essentially, on Shapley value with the slight difference residing on the new constraints (production constraints) inserted to the optimization program; both models satisfies the grand and individual rationalities and have optimal solution. The second model, the Constrained Model Eq. (3.46), is based

Table 3.6: Coalitions' characteristic functions

Coalitions	$v(\mathcal{H})$= - CVaR$_{\mathcal{H}}(1-\alpha)$	Coalitions	$v(\mathcal{H})$= - CVaR$_{\mathcal{H}}(1-\alpha)$
$v\{\emptyset\}$	0	$v\{123\}$	83.91695542
$v\{1\}$	17.99564266	$v\{124\}$	62.51854666
$v\{2\}$	31.30754629	$v\{125\}$	73.98257064
$v\{3\}$	37.84885933	$v\{134\}$	73.92263887
$v\{4\}$	21.99128532	$v\{135\}$	80.85219835
$v\{5\}$	29.87880051	$v\{145\}$	65.50288835
$v\{12\}$	47.87880051	$v\{234\}$	88.34575512
$v\{13\}$	53.58524469	$v\{235\}$	92.64063772
$v\{14\}$	37.15448205	$v\{245\}$	79.54657798
$v\{15\}$	44.61509258	$v\{345\}$	87.23018516
$v\{23\}$	67.98692798	$v\{1234\}$	103.7939385
$v\{24\}$	51.28444217	$v\{1235\}$	110.5178543
$v\{25\}$	58.51854666	$v\{1245\}$	94.62220772
$v\{34\}$	58.82869959	$v\{1345\}$	101.9327724
$v\{35\}$	65.91695542	$v\{2345\}$	116.229087
$v\{45\}$	50.96687924	$v\{12345\}$	131.297273

essentially on a set of constraints inserted by the DM. This model has feasible solution in most of the cases, and satisfy the condition of grand rationality the refore can be used in production planning; finally, the Avaliable Solution Eq. (3.48) insert a weighted factor controled by DM; the concept of penalty was introduced, and through tables the numerical results for different simulations were displayed.

This chapter constitues the practical contribution of this research for production managers can make use of the models proposed in their work to forecast production volume. The difference between the sugested models beyond connecting to the field of cooperative games they allow the deecision maker to get information regarding risk through the penalty factor, however the manager can control by choosing the right weight factor, and consequently be ready to forecast production efficiently.

Table 3.7: Shapley values for the 5 periods

Methods	**Periods**					
	$\boldsymbol{\pi}_1$	$\boldsymbol{\pi}_2$	$\boldsymbol{\pi}_3$	$\boldsymbol{\pi}_4$	$\boldsymbol{\pi}_5$	$\boldsymbol{v\{\Omega\}}$
Conventional Shapley value	17.58	15.13	32.47	26.38	39.73	131.30
LP model	17.58	15.13	32.47	26.38	39.73	131.30

Table 3.8: Constrained Shapley value indicating risk distribution among the 5 periods

Cases	$\boldsymbol{\epsilon}'$	$\boldsymbol{\pi}_1$	$\boldsymbol{\pi}_2$	$\boldsymbol{\pi}_3$	$\boldsymbol{\pi}_4$	$\boldsymbol{\pi}_5$	$\boldsymbol{s}_1^+$	$\boldsymbol{s}_1^-$	$\boldsymbol{s}_2^+$	$\boldsymbol{s}_2^-$	$\boldsymbol{s}_3^+$	$\boldsymbol{s}_3^-$	$\boldsymbol{s}_4^+$	$\boldsymbol{s}_4^-$	$\boldsymbol{s}_5^+$
Main	0.614	17.58	15.13	32.47	26.38	39.73	0.614	0	0.614	0	0.614	0	0.614	0	0.614
Case 1	2.948	14.02	14.51	31.98	28.71	42.07	0	2.948	0	0	0.121	0	0	2.948	2.948
Case 2	8.877	16.97	14.51	26.05	25.76	48.00	0	0	0	0	0	5.807	0	0	8.877
Case 3	6.970	10.00	14.51	31.86	28.83	46.09	0	6.970	0	0	0	0	0	3.068	6.970
Case 4	1.11	18.077	13.41	32.72	26.86	40.22	1.11	0	0	0	0	0	0	0	0
Case 5	3.13	16.97	14.44	35.00	25.76	39.12	0	0	0	0	0	0	0	0	0
Case 6	8.51	16.97	6.00	31.86	28.83	47.63	0	0	0	0	0	0	0	0	0
Case 7	11.86	16.97	14.51	20.00	37.62	42.19	0	0	0	0	0	0	0	0	0
Case 8	13.76	16.97	14.51	34.93	12.00	52.89	0	0	0	0	0	0	0	0	0
Case 9	15.12	16.97	14.51	34.93	40.88	24.00	0	0	0	0	0	0	0	0	0

Table 3.9: Differences between the estimated risk in the 5 periods

Cases	$\epsilon^{'}$	**1**	**2**	**3**	**4**	**5**	s_1^+	s_1^-	s_2^+	s_3^+	s_3^-	s_4^+	s_5^+
Case 1 - Main	2.33	-3.56	-3.07	-0.49	2.33	2.33	-0.61	2.95	-0.61	-0.49	0	2.33	2.33
Case 2 - Main	8.26	-0.61	-0.61	-6.42	-0.61	8.27	-0.61	0	-0.61	-0.61	5.81	-0.61	8.26
Case 3 - Main	6.36	-7.58	-0.61	-0.61	2.46	6.36	-0.61	6.97	-0.61	-0.61	0	2.45	6.36
Case 4 - Main	0.49	-0.49	-4.18	0.24	0.49	0.49	0.50	0	-0.61	-0.61	0	-0.61	-0.61
Case 5 - Main	2.53	-0.61	-3.14	2.53	-0.61	-0.61	- 0.61	0	-0.61	-0.61	0	-0.61	-0.61
Case 6 - Main	7.90	-0.61	-11.58	-0.61	2.46	7.90	-0.61	0	-0.61	-0.61	0	-0.61	-0.61
Case 7 - Main	11.24	-0.61	-3.07	-12.47	11.25	2.46	-0.61	0	-0.61	-0.61	0	-0.61	-0.61
Case 8 - Main	13.15	-0.61	-3.07	-2.46	-14.38	13.15	-0.61	0	-0.61	-0.61	0	-0.61	-0.61
Case 9 - Main	14.50	-6.61	-3.07	-2.46	14.51	-15.73	-0.61	0	-0.61	2.45	0	14.50	-0.61

Chapter 4

Robust Measure in Regional Strategy with Ambiguity

In this chapter, we treat uncertainty via fuzzy concepts by using Shapley value. Though the notation may appear to be the same, however here we deal with other types of models applied to different situations. We intend to extend fuzzy Shapley in cases where there exists a set of α-cuts which is a distint case comparing to the what we treated in the referred chapter. Through the chapter, we propose minimax models for both possibility and necessity measures. In the last part of the chapter we apply the models to real-situation problems by combining fuzzy concepts and cooperative game theory to the field of water resource management.

4.1 Regional Strategy

Nations, regions, states, and cities all require clear economic strategies that engage all stakeholders, boost innovation and ultimately improve productivity. A collaborative strategy, which is especially critical in times of austerity or economic distress, requires setting priorities and moving beyond long lists of discrete recommendations [53].

In order to preserve the economical development and security of their citizens group of countries are assembled through regional organizations. For instance the North Atlantic Treaty Organization (NATO), the African Union (AU), etc., are some of regional organizations around the world All these organizations work for the safety of specific or global interest depending on the level of the organization.

To promote regional economic integration through integrated water resources management (IWRM) in the southern part of Africa [60], there exist the regional cooperation in water resources management document which orients the participant countries the laws to adopt in terms of using and sharing river basin water. Most countries within Southern African Development Community (SADC) depend, for social and economic development, upon water that flows from and/or out of their political boundaries. The integrated development and management of water resources at a national and regional level therefore provides an opportunity to contribute to the achievement of the SADC strategic priorities of regional integration and poverty reduction. This must consider the issues of balance, equity, sustainability and mutual benefit between Member States, taking its lead from the SADC Treaty, the Regional Indicative Strategic Development Plan (RISDP) and The Southern African Vision for Water, Life and the Environment in the 21st Century.

SADC has made significant progress in promoting its core objective of regional integration through the adoption and implementation of various protocols and policies in areas such as trade, transport, energy and health. The water sector is contributing to regional economic integration through the integrated management of shared watercourses, with the associated building of confidence and trust between Member States. Recognition of the imperative for joint water resources multi-purpose development projects has contributed to the realisation that economic integration depends upon shared management of resources. Uneven economic development and widespread poverty between Member States presents a challenge to adopt regional approaches to economic integration, development and the creation of economies of scale, through the integrated management of water resources and sharing of capacity. Integrated management of shared watercourses may be hindered in promoting or accomplishing economic integration at a regional level, unless the following institutional and structural challenges within and between Member States are addressed and overcome:

- Sometimes historic considerations of sovereignty by Member States tend to limit integration both for the development and management of water resources and more broadly for economic integration. However, it is recognised that good progress has been and is still being made by Member States to cooperate in order to achieve this over-arching objective.

- Poor governance and inconsistent policy and legal (enabling) environments of the water and related sectors within and between Member States are potential barriers to integration. It is however recognised that significant progress has already been made in this area.

- Inadequate institutional capacity at the regional, shared watercourse and national level restricts the ability to promote and support integration

4.2 Minimax Fuzzy-Shapley Value Model

4.2.1 Shapley Value Defined by Linear Problem

In Chapter 1 we described a linear solvable formulation based on least square value in transferable utility game. The solution of weighted least squared error, if it satisfies the condition of the constraints it is given by the following expression that denotes an inner product.

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{A}\boldsymbol{z} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{v} \tag{4.1}$$

where, for example of case $K = \{\varphi_1, \varphi_2, \varphi_3, \cdots, \varphi_k\}$ and $q = \sum_{j=1}^{k-1} {}_kC_j$ then

$$\boldsymbol{z} = \begin{bmatrix} z_{\varphi_1}(K,v) \\ z_{\varphi_2}(K,v) \\ z_{\varphi_3}(K,v) \\ \vdots \\ z_{\varphi_k}(K,v) \end{bmatrix} \in \Re^k \times \Re^1, \qquad \boldsymbol{v} = \begin{bmatrix} v(\varphi_1) \\ v(\varphi_2) \\ \vdots \\ v(\varphi_k) \\ v(\varphi_1,\varphi_2) \\ v(\varphi_1,\varphi_3) \\ \vdots \\ v(\varphi_1,\varphi_k) \\ \vdots \\ v(\varphi_{k-1},\varphi_k) \\ \vdots \\ v(\varphi_1,\varphi_2,\cdots,\varphi_{k-1}) \\ \vdots \\ v(\varphi_2,\varphi_3,\cdots,\varphi_k) \end{bmatrix} \in \Re^q \times \Re^1, \tag{4.2}$$

and

$$\boldsymbol{M} = \begin{bmatrix} M_{k,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{k,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{k,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{k,2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{k,k-1} \end{bmatrix} \in \Re^q \times \Re^q, \tag{4.3}$$

$$\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & 0 & \vdots & \cdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \cdots & 1 \end{bmatrix} \in \Re^k \times \Re^q \tag{4.4}$$

Taking a sample d from a dataset, $\boldsymbol{v}$ and $\boldsymbol{z}$ as the value of referred sample the error obtained is the expression in

$$e_d = f_d(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{v}) = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{v} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{z})_d \tag{4.5}$$

where $\boldsymbol{B}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{M} \in \Re^k \times \Re^q$, $\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} \in \Re^k \times \Re^k$ are matrices with crisp values and $(\ )_d$ denotes selection of the d-th row value, thus the sum of all error functions $\epsilon = \sum_d |e_d|$ using the multiple linear regression model that minimize the sum of the absolute values of the residuals.

The linear solvable formulation to obtain Shapley value [3] is defined by

$$\begin{array}{lll} \min\limits_{\boldsymbol{z}} & \sum\limits_{d}^{k} |e_d| & \quad (4.6) \\ \text{Subject to} & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} \leqslant |e_d|, & (d = 1, 2, 3, \cdots, k) \\ & -\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} + \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} \leqslant |e_d|, & (d = 1, 2, 3, \cdots, k) \\ & \sum\limits_{d=1}^{k} z_d(K, v) = v(K) & \\ & z_d \geqslant 0 & (d = 1, 2, 3, \cdots, k) \end{array}$$

We define matrix $\boldsymbol{B}'^{\mathrm{T}}$, vector $\boldsymbol{v}'$ and matrix $\boldsymbol{C}'^{\mathrm{T}}$ as follows:

$$\boldsymbol{B}'^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{B}^{\mathrm{T}} & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1 \end{bmatrix} \in \Re^{k+1} \times \Re^{q+1}, \qquad \boldsymbol{v}' = \begin{bmatrix} \boldsymbol{v} \\ v(K) \end{bmatrix} \in \Re^{q+1} \times \Re^{1} \tag{4.7}$$

$$\boldsymbol{C}'^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{C}^{\mathrm{T}} \\ \boldsymbol{1}^{\mathrm{T}} \end{bmatrix} \in \Re^{k+1} \times \Re^{k} \tag{4.8}$$

with $\boldsymbol{0}^{\mathrm{T}} = [0, 0, \cdots, 0] \in \Re^1 \times \Re^q$ and $\boldsymbol{1}^{\mathrm{T}} = [1, 1, \cdots, 1] \in \Re^1 \times \Re^{k+1}$.

If we impose e_{k+1} to be 0, then Section 4.2.1 becomes strictly equivalent to the following formulation, which is a linear solvable formulation to obtain Shapley value:

$$\begin{array}{lll} \min\limits_{\boldsymbol{z}} & \sum\limits_{d}^{k+1} |e_d| & \quad (4.9) \\ \text{Subject to} & \boldsymbol{c}_d'^{\mathrm{T}}\boldsymbol{z} - \boldsymbol{b}_d'^{\mathrm{T}}\boldsymbol{v} \leqslant |e_d|, & (d = 1, 2, 3, \cdots, k, k+1) \\ & -\boldsymbol{c}_d'^{\mathrm{T}}\boldsymbol{z} + \boldsymbol{b}_d'^{\mathrm{T}}\boldsymbol{v} \leqslant |e_d|, & (d = 1, 2, 3, \cdots, k, k+1) \\ & z_d \geqslant 0 & (d = 1, 2, 3, \cdots, k) \end{array}$$

where $\boldsymbol{b}_d'^{\mathrm{T}} \in \Re^1 \times \Re^q$ is a d-th row vector of matrix $\boldsymbol{B}'^{\mathrm{T}}$ and $\boldsymbol{c}_d'^{\mathrm{T}} \in \Re^1 \times \Re^d$ is a d-th row vector of matrix $\boldsymbol{C}'^{\mathrm{T}}$.

Consider $Z_i = (z_i, \zeta_i)_L$ a fuzzy Shapley value with z_d and ζ_d its center and width, respectively. Moreover, $(\ )_L$ defines the type of triangle, i.e., $L - L$ type, the fuzzy membership function $\mu_{Z_i}(y_i)$ is represented by $L(x)$ and satisfies

$$\begin{array}{lll} L(x) & = & L(-x), \\ L(0) & = & 1, \\ L(x) & \text{is} & \text{a non-increasing function in } x \in [0, \infty). \end{array} \tag{4.10}$$

4.2.2 Possibility Measure for Robust Fuzzy Shapley Value

Here we consider the following possibility of equivalence $Pos(Y_d = X_d)$ for fuzzy variables, Y_d and X_d, as follows;

$$Pos(Y_d = X_d) = \sup_{\theta \in \Re} \min\left(\mu_{Y_d}(\theta), \mu_{X_d}(\theta)\right), \tag{4.11}$$

and the condition for α_i level set

$$Pos\left(Y_d = X_d\right) \geqslant \alpha_i. \tag{4.12}$$

Such definition implies that the set of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ consist of elements from the fuzzy vector $\boldsymbol{X}$ under the condition that the possibility of equivalence between Y_d and X_d is larger than α_i.

Once satisfied these conditions we can propose a minimax problem to obtain robust fuzzy Shapley value under existence of ambiguity. Here we assume that ambiguities are generated from any element of $\alpha = [\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_I]$, then we have

$$\begin{aligned} \min_{\boldsymbol{z},\,\boldsymbol{\zeta}} \ \max_{\alpha} \quad & \sum_{d=1}^{k+1} |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \\ \text{Subject to} \quad & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (d = 1, 2, 3, \cdots, k, k+1) \\ & -\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant -\boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (d = 1, 2, 3, \cdots, k, k+1) \\ & z_d \geqslant 0,\ \zeta_d \geqslant 0 \qquad (d = 1, 2, 3, \cdots, k) \end{aligned} \tag{4.13}$$

In the minimax model for robustness, the condition to $d = k+1$ is related to the grand coalition rationality $Pos\left(\sum_{d=1}^{k} z_d(K, v) = v(K)\right) \geqslant \alpha_i$, so $\boldsymbol{c}_{k+1}^{\mathrm{T}} = [1, 1, 1, \cdots, 1] \in \Re^1 \times \Re^d$, $\boldsymbol{b}_{k+1}^{\mathrm{T}}\boldsymbol{v}$ is given by $v(K)$ and $|\boldsymbol{b}_{k+1}^{\mathrm{T}}|\boldsymbol{\pi}$ is given by π_K, respectively.

The above the minimax problem can be transformed to an LP problem as follows.

$$\begin{aligned} \min_{\boldsymbol{z},\,\boldsymbol{\zeta}} \quad & \epsilon \\ \text{Subject to} \quad & \sum_{d=1}^{k+1} |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant \epsilon \\ & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (i = 1, 2, 3, \cdots, I;\ d = 1, 2, 3, \cdots, k, k+1) \\ & -\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} - L^{-1}(\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant -\boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (i = 1, 2, 3, \cdots, I;\ d = 1, 2, 3, \cdots, k, k+1) \\ & z_d \geqslant 0,\ \zeta_d \geqslant 0 \qquad (d = 1, 2, 3, \cdots, k) \end{aligned} \tag{4.14}$$

4.2.3 Necessity Measure for Robust Fuzzy Shapley Value

Next, we consider the necessity measure $Nec(Y_d \supset X_d)$ for fuzzy variables, Y_d and X_d,

$$Nec(Y_d \supset X_d) = \inf_{\theta \in \Re} \max\left(\mu_{Y_d}(\theta), 1 - \mu_{X_d}(\theta)\right), \tag{4.15}$$

and the condition for α_i level set

$$Nec\left(Y_d \supset X_d\right) \geqslant \alpha_i. \tag{4.16}$$

This definition implies that the set of vectors $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ whose elements are fuzzy vectors $\boldsymbol{X}$. The condition for the degree of necessity is that X_d included in Y_d is larger than α_i.

Thus, we can derive an LP problem to obtain robust fuzzy Shapley values by

$$\begin{aligned} \max_{\boldsymbol{z},\,\boldsymbol{\zeta}} \quad & \epsilon \\ \text{Subject to} \quad & \sum_{d=1}^{k+1} |\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \geqslant \epsilon \\ & -\boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} + L^{-1}(1-\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant -\boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (i = 1, 2, 3, \cdots, I;\ d = 1, 2, 3, \cdots, k, k+1) \\ & \boldsymbol{c}_d^{\mathrm{T}}\boldsymbol{z} + L^{-1}(1-\alpha_i)|\boldsymbol{c}_d^{\mathrm{T}}|\boldsymbol{\zeta} \leqslant \boldsymbol{b}_d^{\mathrm{T}}\boldsymbol{v} + L^{-1}(\alpha_i)|\boldsymbol{b}_d^{\mathrm{T}}|\boldsymbol{\pi}, \\ & \qquad (i = 1, 2, 3, \cdots, I;\ d = 1, 2, 3, \cdots, k, k+1) \\ & z_d \geqslant 0,\ \zeta_d \geqslant 0 \qquad (d = 1, 2, 3, \cdots, k) \end{aligned} \tag{4.17}$$

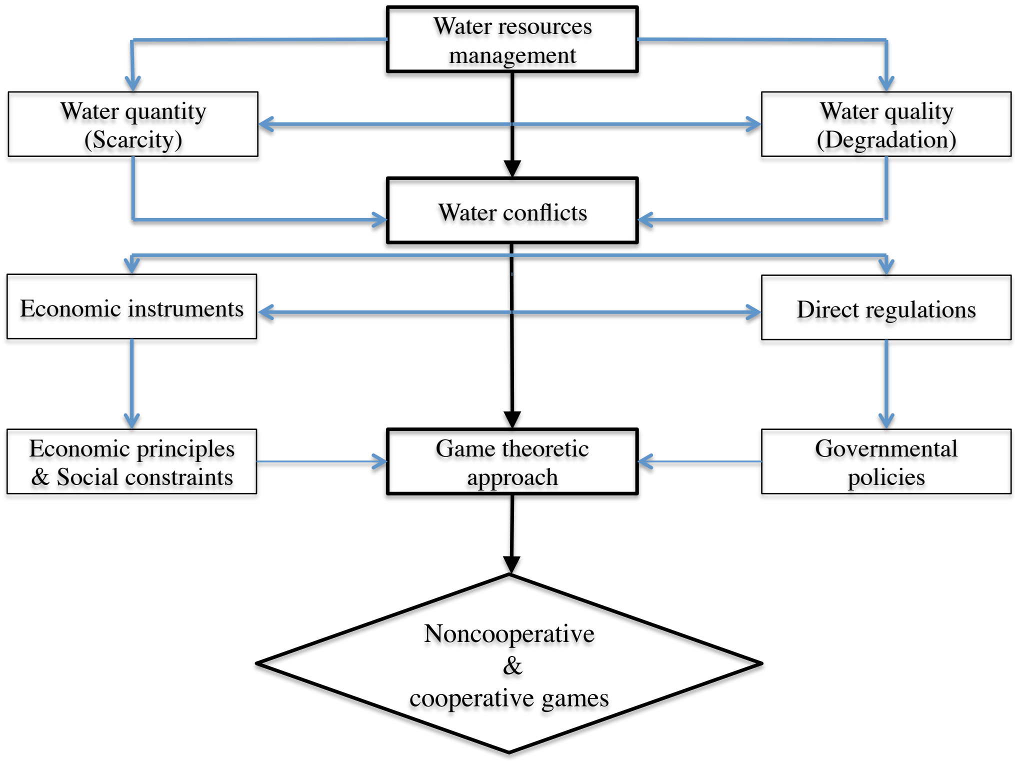

Figure 4.1: Instruments to solve water conflicts. (Wei, 2008)

The similarity between models introduced in this chapter with those in Section 2.2 of Chapter 2 is that in all of them we deal with fuzzy concepts. However, although having this particularity these models are distinct as follows.

In Chapter 2, fuzzy Shapley values for necessity and possibility measures are obtained using one single value for α. However, the robust fuzzy Shapley models here, evaluate a set of α in just one time, in this way the DM is able to answer about ambiguity in a certain bound of random values for α. For instance, the numerical results of the decision variables in Table 4.3 are fuzzy Shapley values, and respective ambiguity with α=0.7, while in Tables 4.4 - 4.6 these variables represent the robust fuzzy Shapley values with different values of α. The reasoning of the DM while using the robust models is in general cases for he is able to get an idea about a specific bound for random values. Initially, these are crisp values, but in order to understand the ambiguity the DM chooses the values of the width for each fuzzy variable.

In practical terms DMs are responsible to analyze the distribution of the width while employing the models, i.e., DM choose values for π in order to obtain the width of the fuzzy variable.

4.3 Water Resource Management and Game Theory

Water has an economic value, this idea is commonly accepted among water resource community [76], which means that economical benefits can arise from better usage of water. The term value in this context refer to a particular economic value for a specific location and point in time , such as a household with private connection using water for domestic purposes, or a farmer abstracting water for irrigation. The economic user value of water is the amount of money a user will be willing to give up to obtain more water and it will be determined both by the use to which this water will be put and the amount of money the user has, [69, 70, 72, 73]

The main task of water resource management (WRM) is essentially to support the coordinated water use in order to maximize the objectives welfare (economic, social and environmental) using a certain class of principles (equity, efficiency and sustainability).

Allocating water is essential to the management of water resources. due to geographically and temporarlly unevebly distributed precipitation [5]. Conflicts often arise when different water users compete for limited water supply. Hence, the need to establish approapriate water allocation methodologies is extremely important, and have been pointed out by management institutions and all entities involve in the process. Since the allocation of water resources often engages multiple parties with conflicting interests as described in Fang et al., [18, 19].

To achieve equitable and efficient water allocation requires the cooperation of all stakeholders in sharing water resources. Cooperative game theory can be utilized to study the fair allocation of common pool resources by Owen [50], and has been applied to the following types of problems in water resources management: cost allocation of water resources development projects, including joint waste water treatment and disposal facilities (Giglio and Wrightington, 1972; Dinar and Howitt, 1997), and water supply development projects (Young et al., 1982; Driessen and Tijs, 1985; Dufournaud and Harrington, 1990, 1991; Dinar et al., 1992; Lejano and Davos, 1995; Lippai and Heaney, 2000); (2) equitable allocation of waste loads to a common receiving medium (Kilgour et al., 1988; Okada and Mikami, 1992); and (3) allocation of water rights (Tisdell and Harrison, 1992). There are only a limited number of models employing cooperative game theory in water allocation, and these models have none or else simple hydrological constraints. Tisdell and Harrison (1992) use a number of different cooperative games to model the efficient and socially equitable reallocation of water among six representative farms in Queensland, Australia. Rogers (1969) uses linear programming to compute the optimum benefits of six strategies of India and East Pakistan (acting singly or in cooperation) in the international Ganges-Brahmaputra river basin, and then analyzes the strategies by a nonzero-sum game for the two countries. Incorporating Nepal into his analysis, Rogers (1993a, b) outlines the applicability of cooperative game theory and Pareto frontier analyses to water resources allocation problems. Okada and Sakakibara (1997) also applied a hierarchical cooperative game model to analyze cost/benefit allocation in a basin-wide reservoir redevelopment as part of water resources reallocation.

The objectives and principles of water resources management are shown in Table 4.1. Different different intruments (economic instruments, direct regulation and game theory techniques) have been considered to solve possible conflicts as can be vizualized in Fig. 4.4.

Allocating resources within the river basin includes several elements, such as population around the river, water resource managers, etc, for the network includes several nodes. Fig. 4.2 shows an example of a river basin network withs its the main elements considered while

Table 4.1: Objectives and principles of WRM [65]

Objective	**Principle**	**Outcome**
Society	Equity	Provide for essential needs
Economics	Efficiency	Maximize economic value of water use
Environment	Sustainability	Maintain environmental quality

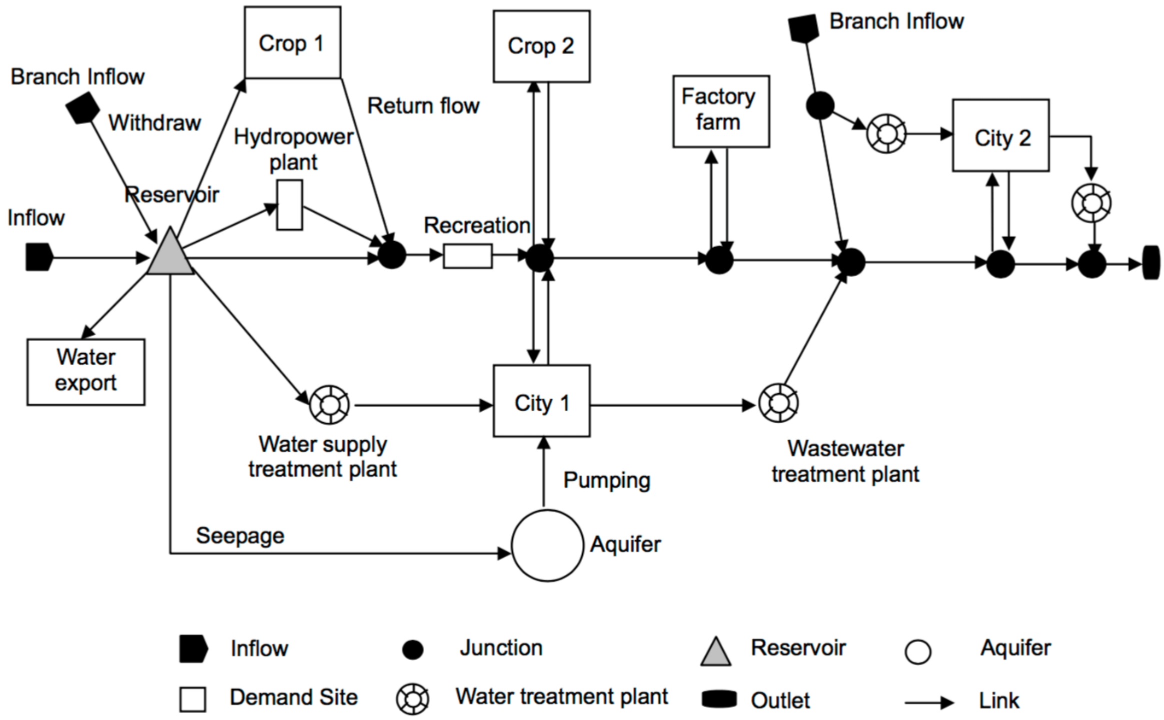

Figure 4.2: An example of a river basin network. (Wang et al., 2003)

evaluating.

Management of international water usually involves several riparians with contradictory interests, aims and strategies, which often result in water conflicts, such is the case of the 10 Nile basin coriparians, for instance. Because of their scale or multipurpose nature, water resource projects have impacts that extend across multiple political or geographic jurisdictions, that is, typically they can be qualified as multi-criteria decision making problems, which obliges decision makers (DMs) to not only focus on costs or benefits, but to extend their methods to other aspects as well [65, 38, 37].

4.4 Numerical Example

In this section we intend to apply the minimax model proposed above to water resource management. We study two cases related to river basins located in Africa. Data from Table 4.2 is only used for testing the efficiency of the models, i.e., academic purposes, for they might differ with the real data. Table 4.3 displays fuzzy Shaple values related to the three countries of the Okavango river basin, the idea is to interprete how can those countries share the water without counsing conflicts in the region for a mutual aggreement. In Tables 4.4 - 4.5 the same set of players (countries) is evaluated in terms of robustness. Through these tables we compare the distribution of ambiguity for differente values for α. Finally, in Table 4.6 the analysis is extended to other group of riparian countries, that is, those related to Nile basin. The table displays the results for robustness cases.

4.4.1 Equitable sharing of international water: the Okavango River Basin case study

The Okavango river is located in the Southern part of Africa, and runs through Angola, Namibia and Botswana as shown in Fig. 4.3. The management of the water river is coordinated by the Permanent Okavango River Basin Water Commission (OKACOM),

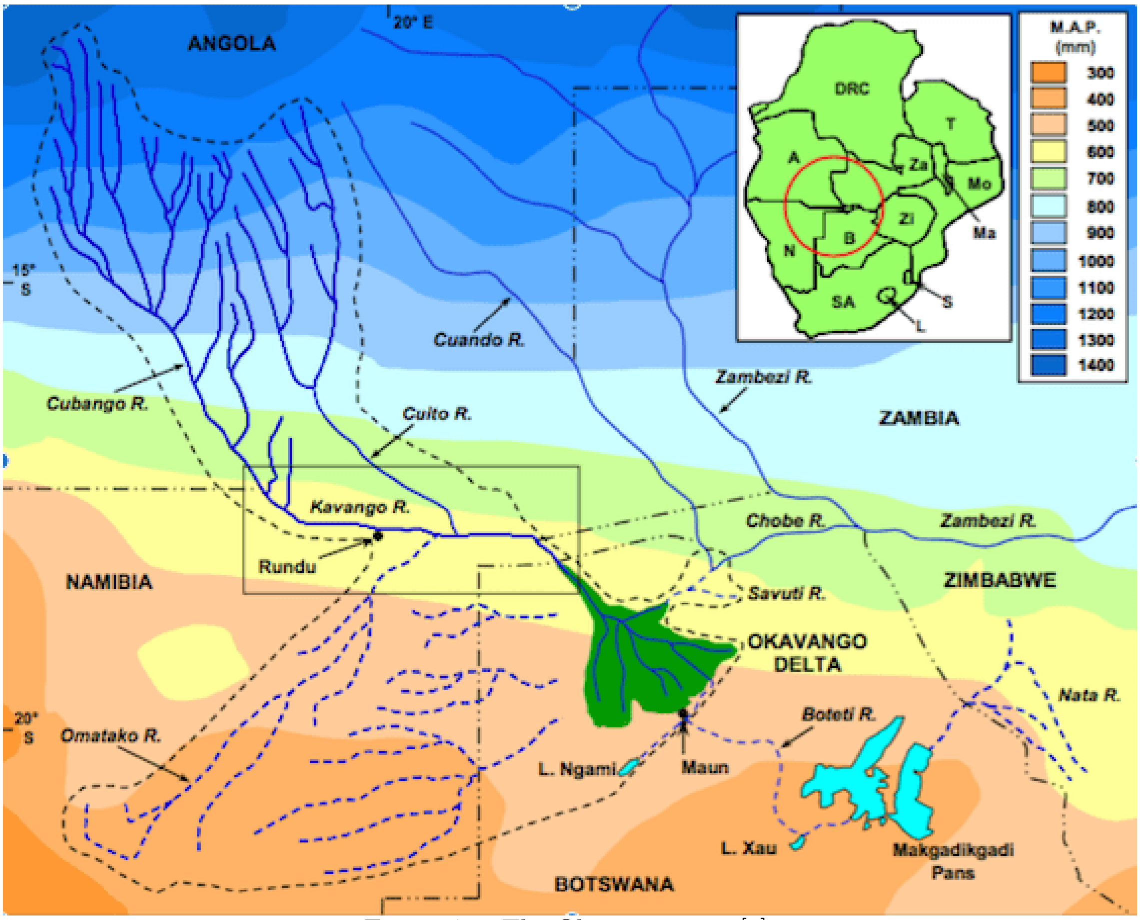

Figure 4.3: The Okavango river [6].

which was created in 1994 through the Windhoek Treaty signed by the riparians states of Angola, Botswana and Namibia with a mandate to serve as the technical adviser body to the contracting Parties on matters relating to the conservation, development and utilization of water resources of common interest in the Okavango basin system. The main task of the Comission is to investigate the pre-requisites and set-up conditions to [6]:

- Determine the long term safe yield of water available from the river
- Estimate reasonable water demand scenarios from consumers
- Prepare criteria for conservation, equitable allocation and sustainable utilization of water
- Undertake investigations related to water infraestructure
- Formulate recommended pollutation prevention measures
- Develop ,easures for the aleviation of short-term difficulties, such as temporary draughts and floods
- Generate visible impacts on poverty alleviation for teh riparian communities, emanting from applied basin resources management options.

Considering the three riparian countries as players of the game, we have a 3-Person game, with N = {Angola, Botswana, Namibia} .

Coalitions:

Table 4.2: Estimated water use in the Cubango-Okavango River Basin (in 000 m^3)

	Angola	Botswana	Namibia	Basin
Population	505 000	219 090	157 690	881 780
Irrigation	34825.4	620.0	43100.0	78545.4
Livestock sector	13163.8	4900.0	14500.0	32563.8
Use in settlements	3935.2	6850.0	8220.0	3935.2
Mining	0.0		0.0	0.0
Tourism	0.2	280.0	2530.0	2810.2
Other (e.g. aquaculture)	0.1			0.1
Est. total water use	51924.5	12650.0	68350.0	132924.5
Est. river water use	47825.4	3994.0	38270.0	90089.4

(i) Individual players:

- Angola = v{ANG};
- Namibia =v {NAM};
- Botswana =v {BOT}

(ii) Two players:

- v{ANG, NAM}, v{ANG, BOT}, v{NAM, BOT}

(iii) Three players (grand coalition):

- v{N}

- Characteristic function:

From Table 4.2 we computed the characteristic functions for all singular players and respectice coalitions. The ambiguity π related to fuzzy variable **Y** is chosen by the DM by analyzing the membership of the fuzzy fuction from the viewpoint of its center and width.

$$\boldsymbol{v} = \begin{bmatrix} v(ANG) \\ v(BOT) \\ v(NAM) \\ v(ANG,BOT) \\ v(ANG,NAM) \\ v(ANG,NAM) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.078 \\ 0.091 \\ 0.294 \\ 0.193 \\ 0.169 \end{bmatrix}, \qquad \boldsymbol{\pi} = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.03 \\ 0.05 \\ 0.06 \\ 0.05 \end{bmatrix}$$

Matrices M and A^T are given as follows

$$\boldsymbol{M} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad \boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

- Fuzzy vector **Y**:
  - Center = (0.2435, 0.065, 0.2705)
  - Width =(0.055, 0.2265, 0.07)

Table 4.3: Simplex tableau of fuzzy Shapley model for possibility measure with $\alpha = 0.7$ (The Okavango case).

POS	z_1	z_1	z_2	ζ_3	ζ_2	ζ_3	α	$L^{-1}(\alpha)$	$L^{-1}(1-\alpha)$
Fuzzy Shapley value	**0**	**0.153375**	**0.846625**	**0**	**0**	**2.4425**	0.7	0.3	
Minimize	0	0	0	2.5	2.5	2.5	6.10625		
subject to	1.5	0.5	0.5	-0.45	-0.15	-0.15	0.133625	$\leqslant$	0.263
	0.5	0.5	0.5	-0.15	-0.45	-0.15	0.287	$\leqslant$	0.287
	0.5	0.5	0.5	-0.15	-0.15	-0.15	0.2475	$\leqslant$	0.2475
	-1.5	-0.5	-0.5	-0.45	-0.15	-0.45	-0.86638	$\leqslant$	-0.263
	-0.5	-1.5	-0.5	-0.15	-0.45	-0.15	-1.01975	$\leqslant$	-0.287
	-0.5	-0.5	-1.5	-0.15	-0.15	-0.45	-2.44575	$\leqslant$	-0.2475
Grand coalition	1	1	1	0	0	0	1	=	1

Table 4.4: Simplex tableau of robust fuzzy Shapley model for possibility measure with $\alpha_1 = 0.7$ and $\alpha_2 = 0.8$ (The Okavango case).

POS	z_1	z_1	z_2	ζ_3	ζ_2	ζ_3	α	$L^{-1}(\alpha)$	$L^{-1}(1-\alpha)$
Fuzzy Shapley value	**0.1035**	**0.1275**	**0.759**	**0**	**0**	**2.27**	0.8	0.2	
Minimize	0	0	0	2.5	2.5	2.5	5.675		
subject to	1.5	0.5	0.5	-0.45	-0.15	-0.15	0.263	$\leqslant$	0.263
	0.5	0.5	0.5	-0.15	-0.45	-0.15	0.287	$\leqslant$	0.287
	0.5	0.5	0.5	-0.15	-0.15	-0.15	0.2475	$\leqslant$	0.2475
	-1.5	-0.5	-0.5	-0.45	-0.15	-0.45	-0.944	$\leqslant$	-0.263
	-0.5	-1.5	-0.5	-0.15	-0.45	-0.15	-0.968	$\leqslant$	-0.287
	-0.5	-0.5	-1.5	-0.15	-0.15	-0.45	-2.2905	$\leqslant$	-0.2475
Grand coalition	1	1	1	0	0	0	1	=	1
	1.5	0.5	0.5	-1.41	-0.47	-0.47	-0.4834	$\leqslant$	0.3046
	0.5	0.5	0.5	-0.47	-1.41	-0.47	-0.4394	$\leqslant$	0.3222
	0.5	0.5	0.5	-0.47	-0.47	-1.41	-1.9317	$\leqslant$	0.2923
	-1.5	-0.5	-0.5	-1.41	-0.47	-0.47	-1.6704	$\leqslant$	-0.3046
	-0.5	-1.5	-0.5	-0.47	-1.41	-0.47	-1.6944	$\leqslant$	-0.3222
	-0.5	-0.5	-1.5	-0.47	-0.47	-1.41	-4.4697	$\leqslant$	-0.2823

Table 4.5: Simplex tableau for Robust Fuzzy Shapley value based on possibility measure with $\alpha_1 = 0.5$; $\alpha_2 = 0.2$; $\alpha_3 = 0.01$ and $\alpha_4 = 0$ (The Okavango case).

POS	z_1	z_1	z_2	ζ_3	ζ_2	ζ_3	α	$L^{-1}(\alpha)$	$L^{-1}(1-\alpha)$
Fuzzy Shapley value	**0.1089**	**0.7967**	**0.0944**	**0**	**1.3316**	**0**	0.5	0.5	
Minimize	0	0	0	2.5	2.5	2.5	3.329		
subject to	1.5	0.5	0.5	- 0.75	-0.25	-0.25	0.276	$\leqslant$	0.276
	0.5	1.5	0.5	- 0.25	-0.75	-0.25	0.298	$\leqslant$	0.298
For $\alpha = 0.5$	0.5	0.5	1.5	- 0.25	-0.25	-0.75	0.2615	$\leqslant$	0.2615
	-1.5	-0.5	-0.5	- 0.75	-0.25	-0.25	-0.9418	$\leqslant$	-0.276
	- 0.5	-1.5	-0.5	- 0.25	-075	-0.25	-2.2954	$\leqslant$	-0.298
	-0.5	- 0.5	-1.5	- 0.25	-0.25	-0.75	-0.9273	$\leqslant$	-0.2615
	1.5	0.5	0.5	-1.2	-0.4	-0.4	-0.07626	$\leqslant$	-0.2955
	0.5	1.5	0.5	-0.4	-1.2	-0.4	- 0.30122	$\leqslant$	0.3145
For $\alpha = 0.2$	0.5	0.5	1.5	-0.4	-0.4	-1.2	0.06176	$\leqslant$	0.2825
	-1.5	-0.5	-0.5	-1.2	-0.4	-0.4	-1.14154	$\leqslant$	-0.2955
	- 0.5	-1.5	-0.5	-0.4	-1.2	-0.4	- 2.89462	$\leqslant$	-0.3145
	-0.5	- 0.5	-1.5	-0.4	-0.4	-1.2	-1.2704	$\leqslant$	-0.2825
	1.5	0.5	0.5	-1.485	-0.495	-0.495	-0.050242	$\leqslant$	0.30785
	0.5	1.5	0.5	-0.495	-1.485	-0.495	- 0680726	$\leqslant$	0.32495
For $\alpha = 0.01$	0.5	0.5	1.5	-0.495	-0.495	-1.485	-0.064742	$\leqslant$	0.2958
	-1.5	-0.5	-0.5	-1.485	-0.495	-0.495	-1.268042	$\leqslant$	-0.30785
	- 0.5	-1.5	-0.5	-0.495	-1.485	-0.495	-3.274126	$\leqslant$	-0.32495
	-0.5	- 0.5	-1.5	-0.495	-0.495	-1.485	-1.253542	$\leqslant$	-0.2958
	1.5	0.5	0.5	-1.5	-0.5	-0.5	-0.0569	$\leqslant$	0.3085
	0.5	1.5	0.5	-0.5	-1.5	-0.5	-0.7007	$\leqslant$	-0.3255
For $\alpha = 0$	0.5	0.5	1.5	-0.5	-0.5	-1.5	-0.0714	$\leqslant$	0.2965
	-1.5	-0.5	-0.5	-1.5	-0.5	-0.5	-1.2747	$\leqslant$	-0.3085
	- 0.5	-1.5	-0.5	-0.5	-1.5	-0.5	-3.2914	$\leqslant$	-0.3255
	-0.5	- 0.5	-1.5	-0.5	-0.5	-1.5	-1.2602	$\leqslant$	-0.2965
Grand coalition	1	1	1	0	0	0	1	=	1

Fuzzy Shapley values for the Okavango River Basin game with $\alpha = 0.7$

- For possibility measure:
  - Angola = 0;
  - Botswana = 0.153375;
  - Namibia = 0.846625

Table 4.3 refers to models described in Chapter 2. The second row shows the fuzzy Shapley values for the evaluated countries (0, 0.153375, 0.846625), and their values regarding ambiguity, (0, 0, 2.4425), respectively. The result and the tenth row of the table show that both individual and group rationality are satisfied, and Namibia has the highest value for ambiguity, which is 2.4425.

Now, if a DM desires to understand what could happen if he considers different values for α at the same time. In those situations, models proposed in this chapter, i.e., Eq. (4.13), Eq. (4.14) and Eq. (4.17) are useful.

In Table 4.4, we applied Eq. (4.14) for $\alpha_1 = 0.7$ and $\alpha_1 = 0.8$. The robust fuzzy Shapley values resulted are shown in the second row of the table, that is, 0.1035, 0.1275 and 0.759, respectively. As for the information regarding ambiguity: 0, 0, 2.27, respectively. Likewise, Namibia still have the highest value for ambiguity, 2.27, while the other countries are null. We performed similar computations for comparison purposes as described next.

In Table 4.5 we applied the robust fuzzy Shapley model for possibility measure to evaluate the three countries in cases where different values for α are considered. This comparison study allows us also to have an idea on the sensintive of the results. In the present case, we have α_1= 0.5, α_2= 0.2, α_3= 0.01 and $\alpha_4 = 0$; the models proposed on Chapter 2 can only evaluate one value for α each time, while the robust model could give us a general information on the ambiguity by considering different α-cut in one single step. The second row of the table present the robust fuzzy Shapley value obtained for each country, i.e., 0.1089, 0.7967 and 0.0944, respectively. As for the ambiguity represented by vector ζ= [0,1.3316,0]. Constraints regarding individual rationality are all satisfied, and in the last row group rationality is also satisified. Ambiguity is found on the second country, Botswana, that is, 1.3316; this directs DMs to check on their policies. Similar analysis is performed in Table 4.6, but for four countries with Ethiopia having the highest robust value, and consequently highest ambiguity recommending DMs to reconsider their policies.

4.4.2 Conflict resolution in international river basins: a case study of the Nile Basin

Usually, the so called Nile Economic Optimization Model (NEOM) [76] can be used to define the characteristic functions, since it is capable to estimate benefits and costs of each riparian country in terms of coalitions formation. Therefore, it is a tool to consider for conflict resolution by water resource managers. The NEOM is defined in Eq. (4.18) as folows.

$$v(S) = Maximize \sum_{c} \left\{ \sum_{i,c} P_w^{i,c} \sum_{t} Q_t^{i,c} + \sum_{i,c} P_e^{i,c} \sum_{i,c} KWH_t^{i,c} \right\} \tag{4.18}$$

The constraints of the model are:

1. Continuity constraints for reservoir nodes,

$$\begin{aligned} S_{t+1}^i &= S_t^i + I_t^i + (1 - EV_y^{j-i})R_t^i - (e_t^i - r_t^i) \\ &\times \left[^i + b^i (\frac{S_t^i + S_{t+1}^i}{2}) \right] - Q_t^{i,c} - R_t^i \text{for t=1, 2,} \cdots, 12. \end{aligned} \tag{4.19}$$

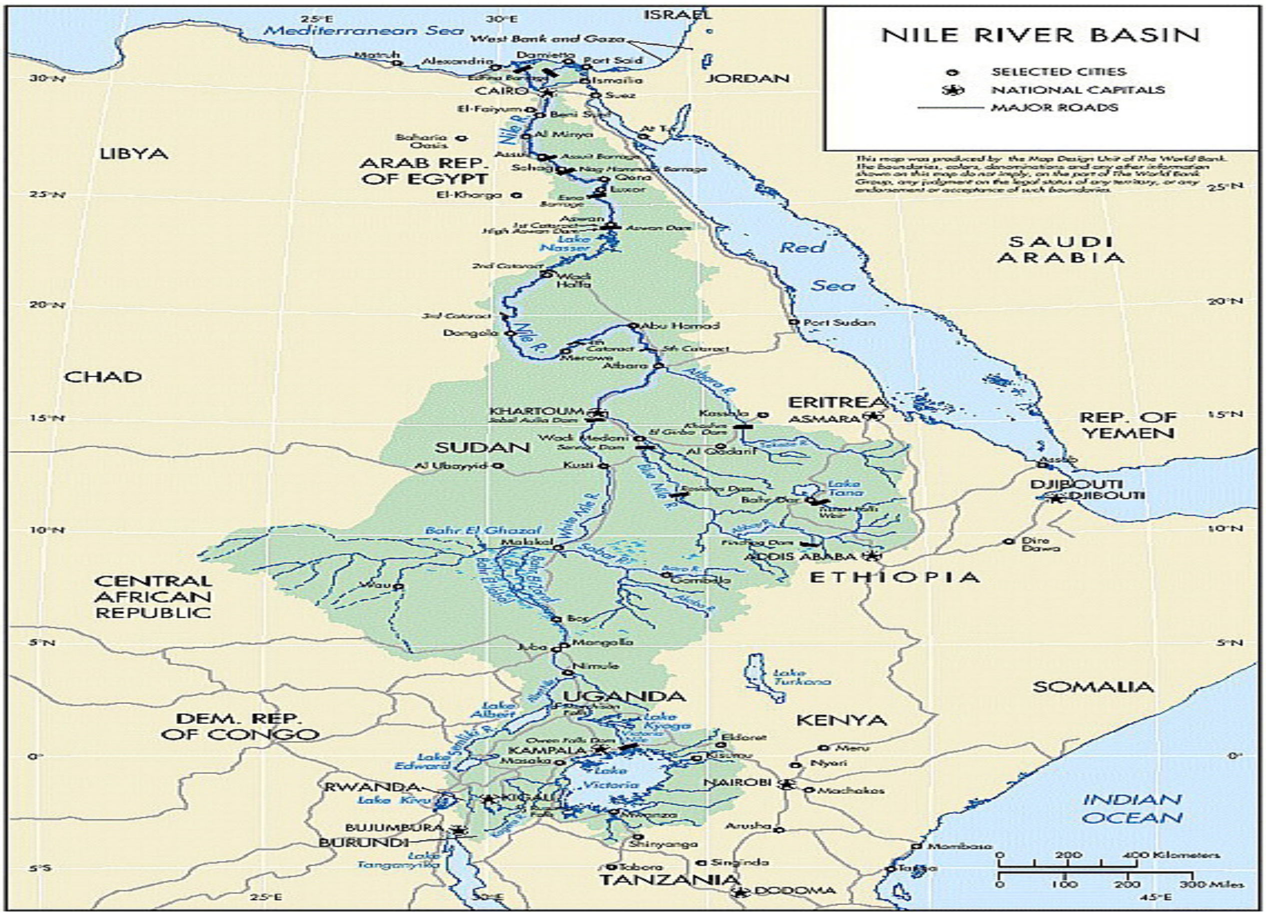

Figure 4.4: Nile Basin (Wu and Whittington, 2006).

2. Continuity constraints for intermediate nodes,

$$(1 - EV_t^{j-i})R_t^j + I_t^i = R_r^i + Q_t^{i,c} \tag{4.20}$$

for $t = 1, 2, \cdots, 12$ (j indicates nodes immediate before i and be more than one node).

3. Storage capacity constraints for reservoir nodes,

$$S_{Min}^i \leqslant S_r^i \leqslant S_{Max}^i \tag{4.21}$$

4. Irrigation water withdrawal pattern,

$$Q_t^{i,c} = Q^{i,c}\delta_t^i \tag{4.22}$$

$t = 1, 2, \cdots, 12.$

5. Hydropower generation equations,

$$KWH_t^{i,c} = \eta I_t^i f(S_t^i, S_{t+1}^i)\epsilon \tag{4.23}$$

for $t = 1, 2, \cdots, 12$.

6. Hydropower generation capacity constraints,

$$KWH_t^{i,c} =\leqslant CAP^{i,c} \tag{4.24}$$

for $t = 1, 2, \cdots, 12$.

7. Nonnegativity constraints,

$$S_t^i, R_t^i, Q_t^i, KWH_t^{i,c} \geqslant 0 \tag{4.25}$$

for all decision variables and for $t = 1, 2, \cdots, 12$.

- S_t^i is reservoir storsage for reservoir $i$in moth t; I_t^i is the inflow to site i in mnth t;
- I_t^i indicates the release (or outflow)from site i in month t;
- e_t^i is the evaporation rate at site i in month t; r_t^i is the addition to flow at site i in month t due to rainfall;
- a^i and b^i are the constant and the slope of the area storage relation of the reservoir, respectively;
- S_{Mini} and S_{Maxi} are the minimum and maximum storage for any reservoir at site i;
- $Q^{i,c}$ is the irrigation withdrawal for irrigation site i in October;
- δ_t^i corresponds the coefficients of irrigation withdrawal for site i in month t in relation to irrigation withdrawal for site i in October;
- η is unit conversion constant;
- $f(S_t^i, St + 1^i)$ is function determining average productive head;
- ϵ is hydropower efficiency;
- $CAP^{i,c}$ is the maximum hydropower that can be generated at site i in month t.

- Coalitions and respective characteristic functions
  - v{Egypt} = 1804;
  - v{Sudan}=1029
  - v{Ethiopia} = 600
  - v{Equatorial States} =1233
  - v{Egypt, Sudan}= 3107
  - v{Egypt, Ethiopia}=3759
  - v{Egypt, Equatorial States}=3731
  - v{Ethiopia, Sudan}=3131
  - v{Ethiopia, Equatorial States}=1833
  - v{Equatorial States, Sudan}=2990
  - v{Egypt, Sudan, Ethiopia}=5684
  - v{Egypt, Sudan, Equatorial States}=5509
  - v{Sudan, Ethiopia, Equatorial States}=4642
  - v{N}=9112

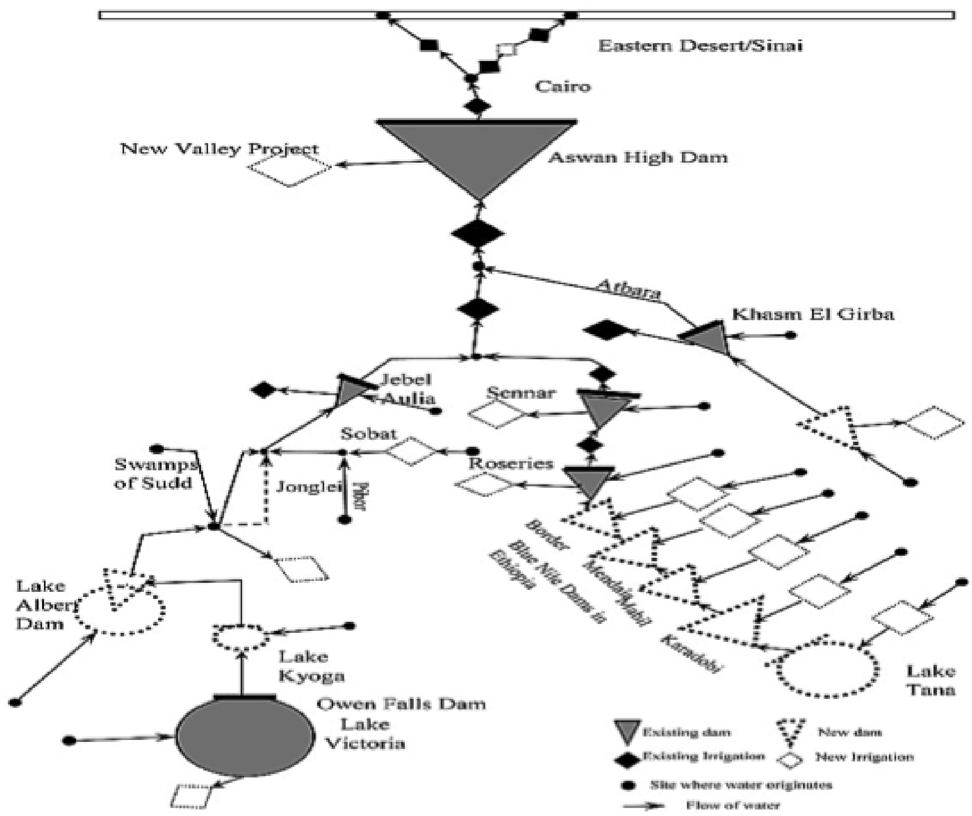

Figure 4.5: Nile Basin as represented in the Nile Economic Optimization Model (Wu and Whittington, 2006).

In this chapter, we first described concepts related to regional strategy. Afterwards, robust fuzzy Shapley model for possibility and necessity measures were introduced. The objective functions of the optimization model are designed in minimax form, i.e., the objective is to minimize the maximum of α-cut. Since minimax models can be tranformed into LP models, the correspondent LP model for possibility measure was also introduced within the chapter.

The robust model can evalute several values for α at once, and this is one of the differences between the models in this chapter and those in Chapter 2. Numerical examples were also considered by employin the model to water resoiurce management field. Basically, this chapter and Chapter 2 represent the theoretical side of this research.

Table 4.6: Simplex tableau of fuzzy Shapley model for possibility measure with $\alpha_1 = 0.4$ and $\alpha_2 = 0.7$ Nile case

POS	z_1	z_1	z_2	z_4	ζ_1	ζ_2	ζ_3	ζ_4	α	$L^{-1}(\alpha)$	$L^{-1}(1-\alpha)$
Fuzzy Shapley value	**0**	**0**	**1**	**0**	**2599.307**	**2583.267**	**962.553**	**1884.767**	0.4	0.6	
Minimize	0	0	0	0	2.3333	2.3333	2.3333	2.3333	18736.15243		
subject to	1.16667	0.3889	0.3889	0.388	-0.35	-0.116667	-0.11667	- 0.11667	-1542.93712	$\leqslant$	1889.2556
	0.3889	1.16667	0.3889	0.3889	- 0.11667	- 0.35	- 0.116667	- 0.11667	-1539.1944	$\leqslant$	1539.9722
	0.3889	0.3889	1.16667	0.3889	- 0.11667	- 0.11667	- 0.35	-0.116667	-1160.25	$\leqslant$	1162.58333
	0.3889	0.38889	0.38889	1.16667	-0.11667	- 0.11667	- 0.11667	-0.35	-1376.211111	$\leqslant$	1376.9889
	-1.16667	- 0.38889	- 0.38889	- 0.3889	- 0.35	- 0.11667	- 0.11667	-0.3	-1889.2556	$\leqslant$	-1889.2556
	- 0.3889	-1.1667	-0.3889	-0.3889	-0.116667	-0.35	-0.116667	-0.11667	-1539.9722	$\leqslant$	-1539.9722
	-0.389	-0.3889	-1.166667	-0.389	-0.116667	-0.1167	-0.35	-0.11667	-1162.5833	$\leqslant$	-1162.5834
	-0.38889	-0.3889	- 0.3889	-1.1667	-0.1167	- 0.11667	- 0.116667	- 0.35	-1376.9889	$\leqslant$	-1376.9889
Grand coalition	4446	4446	4446	4446	0	0	0	0	4446	=	4446
	1.1677	0.3889	0.3889	0.3889	-0.4667	-0.1556	-0.1556	-0.15556	-2057.3792	$\leqslant$	1161.4333
	0.3889	1.16667	0.3889	0.38889	-0.15556	-0.46667	-0.1556	-0.1556	-2052.389	$\leqslant$	945.6222
	0.3889	0.38889	1.16667	0.38889	-0.15556	-0.15556	-0.4667	-0.15556	-1547.3899	$\leqslant$	719.18899
	0.3889	0.38899	0.3899	1.16667	-0.1556	-0.15556	-0.1556	-0.46667	-1835.0778	$\leqslant$	850.15556
	-1.16667	-0.3889	-0.38889	-0.3889	-0.4667	-0.15556	-0.1556	-0.15556	-2058.156925	$\leqslant$	-1161.433
	-0.3889	-1.16667	-0.38889	-0.3889	-0.15556	-0.4667	-0.1556	-0.15556	-2053.166667	$\leqslant$	-945.6222
	-0.3889	-0.38899	-1.16667	-0.38899	-0.1556	-0.1555	-0.4667	-0.15556	-1549.7222	$\leqslant$	-719.1889
	-0.3889	-0.38899	-0.3889	-1.1667	-0.1556	-0.15556	-0.15556	-0.4667	-1835.856	$\leqslant$	-850.1556

Chapter 5

Conclusion

We are all decision makers, since we all make decisions constantly, consciously or unconcious. This research is related to consensus decision making, a dynamic process where deciders have to seek for an agreement in order to succeed in their projects. This equilibrium is extremely important for it is a bridge to achieving efficient and consistent results. If consensus is considered, then the group is able to work efficiently avoiding selfishness, or the typical Egoist's Dilemma whose model was proposed by Nakabayashi and Tone [46]. Thus, consensus decision making can be defined as a cooperative process where participants achieve their results having the benefit of the group in mind. Although the concept is used mostly by polititians or nonviolent activists, research in the field of decision theory includes this type of decision. In the process, the main point is not the individual, but the group. Sometimes, the final decision is not the best in terms of personal preference, however participants achieve an agreement by consensus as to the best performance of the group. Hence this decision is the one which satisfies the group.

Making decision includes several factors, as to mention just one uncertainty. When uncertainty is not evaluated, the adopted policies may have unexpected and negative effectes. This thesis is divided in five chapters; in the Introduction we presented aspected related to consensus decison making as well as, a background of some approaches used to treat with uncertainty. In Chapter 2, we described basic concepts related to game theory. Since it is not a general study on game theory, we focus on cooperative games, specifically on Shapley value which is one of the most used solution concepts in cooperative game theory; as referred in the chapter, there exists some applications of the value in noncooperative games. Theoretical foundations of the value were introduced. In Section 2.3, we introduced, first a linear programming model to find Shapley value, and then with this model as basis we extended again the Shapley value to fuzziness framework. The motivation behind the proposed models is that, usually, uncertainty has not been considered when one employs the solution concepts of cooperative games in most of cases. This means that in most of those cases there exists no information related to the level of ambiguity of the decision made. In the last part of the chapter we presented a numerical example to test the efficiency of the proposed models.

In Chapter 3, concepts of coalitional games were combined to risk management techniques to solve a multi-period production planning problem. The relation between production planning and coalitional games was established. Periods were defined as players in the game, etc.

Assuming that the cumulative demand is uncertain, we interpreted the problem by proposing three models to support production managers on forecasting the volume of production. The first model is based on the quadratic form of Shapley value [56, 57], defined as a least square value. The difference between the original model and ours is exactly the production constraints added into the optimization program; through the second and third models we introduced the concept of penalty and a weight factor controlled by the decision maker. The last part of the chapter discusses different case studies through which we performed numerical simulations.

Chapter 4 extends models proposed in Chapter 2 to robustness context. Basically, the difference between those models can be described as follows. In the former chapter, we proposed the fuzzy Shapley value model for possibility measure, and as well as necessity measure. These models solve the problem with one α-cut only. However in this chapter we deal with a situation where the decision maker may wish to evaluate several values of α in the same time in order to get a general information as regards to the distribution of ambiguity if diferent values are analysed in one time. Therefore, the objective function of the robust fuzzy Shapley model is defined in form of a minimax problem for the decision maker desires to minimize the maximum of those random values in order that he may be able to forecast his decisions based on the numerical information he already has by applying the model.

5.1 Research Contribution and Future Direction

Game theory has several applications in real-world situations, and as an approach for strategical decisions the need of information on uncertainty is extremely important. Absence of uncertainty analysis in decision making problems may result in inefficient and inconsistent decisions. Hence, the contribution of this work can be summarized in two parts, namely:

- First, we proposed theoretical concepts to solve decision making problems under uncertainty framework. Chapters 2 and 4 of this thesis correspond to the theoretical part of the study. By proposing models for possibility measure and necessity measure, we combine fuzzy techniques with cooperative game theory. Accordingly, the theoretical models presented in this research can support decision makers in their work.

- On the practical viewpoint, that is, the second contribution of this work is related to production planning problems. Chapter 3 addresses in practical terms how a production manager can forecast production volume at certain period.

- The necessity measure has result has presented infeasibility in some cases. Thus, an extension on the theory regarding this approach would be an interesting research to consider.

- Another direction from the viewpoint of uncertainty would be a comparison study of the proposed model with other approaches within the reasearch on uncertainty in decision making.

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