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COMPLEX STRUCTURES AND NON-DEGENERATE CLOSED 2-FORMS OF COMPACT REAL PARALLELIZABLE NILMANIFOLDS

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Abstract

In this paper, we consider a relation of non-degenerate closed 2-forms and complex structures on compact real parallelizable nilmanifolds.

1. Introduction

Let $\Gamma\backslash G$ be a compact complex parallelizable manifold, where G is a simply connected complex Lie group, and Γ is a lattice in G. H. C. Wang [9] has proven that if $\Gamma\backslash G$ has a Kähler structure, then G is abelian, in particular, $\Gamma\backslash G$ is biholomorphic to a complex torus. If $(\Gamma\backslash G, J)$ has a pseudo-Kählerian structure, then G is 2-step solvable ([8], [11, 12]). Dorfmeister-Guan [6] have proven that a compact homogeneous pseudo-Kähler manifold is biholomorphic to the product of a homogeneous rational manifold and a complex torus. Thus, we are interested in the case of compact non-homogeneous complex manifolds which have a non-degenerate closed 2-form. By a compact *real* parallelizable manifold we mean a compact manifold of the form $\Gamma\backslash G$, where G is a simply connected *real* Lie group, and Γ is a lattice in G, that is, a discrete co-compact subgroup of G. It is known due to Auslander [2] that a solvmanifold has a solvmanifold of the form $\Gamma\backslash G$ as a finite covering, where G is a simply connected solvable Lie group, and Γ is a discrete subgroup.

In this paper, we consider a relation of non-degenerate closed 2-forms and complex structures on compact real parallelizable nilmanifolds, and we prove following result.

Theorem 1.1. *There exists a compact real parallelizable nilmanifold $\Gamma\backslash N$ and its complex structures J_1, J_2 such that $(\Gamma\backslash N, J_1)$ has a pseudo-Kähler structure and no holomorphic symplectic structures, however $(\Gamma\backslash N, J_2)$ has a holomorphic symplectic structure and no pseudo-Kähler structures.*

2. Non-degenerate closed 2-forms and complex structures

In this section, we consider a relation of non-degenerate closed 2-forms and complex structures on compact real parallelizable manifolds.

Definition 2.1. Let (M, J) be a complex manifold. A *pseudo-Kähler structure* on (M, J) is

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a real closed non-degenerate $(1,1)$-form. Let $\dim_{\mathbb{C}} M = 2m$. Then a *holomorphic symplectic structure* is a closed holomorphic 2-form on (M, J) of maximal rank $2m$.

Note that if Ω is a holomorphic symplectic form on (M, J), then $\Omega + \bar{\Omega}$ is a symplectic form on M.

Proposition 2.2 ([5]). *Let $\Gamma\backslash N$ be a compact complex parallelizable nilmanifold. If $\Gamma\backslash N$ has a pseudo-Kähler structure, then $\Gamma\backslash N$ is a complex torus.*

Let N be a simply connected real nilpotent Lie group, and $\mathfrak{n}$ its Lie algebra. It is well known that N has a lattice if and only if $\mathfrak{n}$ has a rational structure, i.e., there exists a rational Lie subalgebra $\mathfrak{n}_{\mathbb{Q}}$ such that $\mathfrak{n} \cong \mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$.

Let us consider following nilpotent Lie groups N_1, N_2 with left-invariant complex structures:

$$N_1 = \left\{ \begin{pmatrix} 1 & w_1 & w_3 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, w_i \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & w_4 \\ 0 & 1 \end{pmatrix} \middle| \, w_4 \in \mathbb{C} \right\},$$

$$N_2 = \left\{ \begin{pmatrix} 1 & \bar{z}_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, z_i \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & \bar{z}_4 \\ 0 & 1 \end{pmatrix} \middle| \, z_4 \in \mathbb{C} \right\}.$$

Let Γ_1, Γ_2 be lattices in N_1, N_2, respectively. Then,

$$W_1 = \frac{\partial}{\partial w_1}, W_2 = \frac{\partial}{\partial w_2} + w_1 \frac{\partial}{\partial w_3}, W_3 = \frac{\partial}{\partial w_3}, W_4 = \frac{\partial}{\partial w_4}$$

is a basis of left-invariant vector fields of type $(1,0)$ on N_1, and

$$\tau_1 = dw_1, \tau_2 = dw_2, \tau_3 = dw_3 - w_1 dw_2, \tau_4 = dw_4$$

is its dual basis. Thus,

$$d\tau_1 = d\tau_2 = d\tau_4 = 0, d\tau_3 = -\tau_1 \wedge \tau_2.$$

Hence, $\Gamma_1\backslash N_1$ has the following two holomorphic symplectic structures Ω_1, Ω_2, which are not cohomologous:

$$\Omega_1 = \tau_1 \wedge \tau_3 + \tau_2 \wedge \tau_4,$$
$$\Omega_2 = \tau_1 \wedge \tau_4 + \tau_2 \wedge \tau_3.$$

On the other hand,

$$Z_1 = \frac{\partial}{\partial z_1}, Z_2 = \frac{\partial}{\partial z_2} + \bar{z}_1 \frac{\partial}{\partial z_3}, Z_3 = \frac{\partial}{\partial z_3}, Z_4 = \frac{\partial}{\partial z_4}$$

is a basis of left-invariant vector fields of type $(1,0)$ on N_2, and

$$\omega_1 = dz_1, \omega_2 = dz_2, \omega_3 = dz_3 - \bar{z}_1 dz_2, \omega_4 = dz_4$$

is its dual basis. Thus,

$$d\omega_1 = d\omega_2 = d\omega_4 = 0, d\omega_3 = -\bar{\omega}_1 \wedge \omega_2.$$

Hence, $\Gamma_2\backslash N_2$ has the following pseudo-Kähler structure ω, and holomorphic symplectic structure Ω:

$$\omega = \mathrm{Re}(\omega_1 \wedge \bar{\omega}_3 + \omega_2 \wedge \bar{\omega}_4),$$
$$\Omega = \omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_3,$$

where $\mathrm{Re}(\omega_1 \wedge \bar{\omega}_3 + \omega_2 \wedge \bar{\omega}_4)$ denotes the real part of $\omega_1 \wedge \bar{\omega}_3 + \omega_2 \wedge \bar{\omega}_4$.

We can easily see that N_1 and N_2 are isomorphic as *real* Lie groups. Indeed, let $W_i = U_i + \sqrt{-1}V_i$, $Z_i = X_i + \sqrt{-1}Y_i$ ($i = 1, 2, 3$), where U_i, V_i, X_i, and Y_i are real vector fields. Then the structures of Lie algebras of N_1 and N_2 are following:

$$[U_1, U_2] = \frac{1}{2}U_3, \quad [-V_1, V_2] = \frac{1}{2}U_3, [-V_1, U_2] = -\frac{1}{2}V_3, \quad [U_1, V_2] = \frac{1}{2}V_3,$$
$$[X_1, X_2] = \frac{1}{2}X_3, \qquad [Y_1, Y_2] = \frac{1}{2}X_3, [Y_1, X_2] = -\frac{1}{2}Y_3, \qquad [Y_1, Y_2] = \frac{1}{2}Y_3,$$

and the other brackets being zero. In particular, we can assume that $\Gamma_1 = \Gamma_2$.

Remark 2.3. Let us consider the following subgroups of N_1

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & \mu_1 & \mu_3 \\ 0 & 1 & \mu_2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, \mu_i \in \mathbb{Z}[\sqrt{-1}] \right\} \times \left\{ \begin{pmatrix} 1 & \mu_4 \\ 0 & 1 \end{pmatrix} \,\middle|\, \mu_4 \in \mathbb{Z}[\sqrt{-1}] \right\}$$

$$H_1 = \left\{ \begin{pmatrix} 1 & w_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, w_1 \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & w_4 \\ 0 & 1 \end{pmatrix} \,\middle|\, w_4 \in \mathbb{C} \right\}.$$

Then $\Gamma_1 \cap H_1\backslash H_1$ is a holomorphic Lagrangian submanifold of $(\Gamma_1\backslash N_1, \Omega)$.

We generalize our examples above to the other Lie groups. Let us consider the following Lie algebra over $\mathbb{R}$:

$$\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{b},$$

where $\mathfrak{a}$ is abelian, and $\mathfrak{b}$ is an ideal. Take bases of Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$:

$$\mathfrak{a} = \mathrm{span}_{\mathbb{R}}\{U_1^1, \ldots, U_p^1\},$$
$$\mathfrak{b} = \mathrm{span}_{\mathbb{R}}\{V_1^1, \ldots, V_q^1\}.$$

Consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. Since $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$, ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ has the following basis:

$$\{U_1^1, \ldots, U_p^1, U_1^2, \ldots, U_p^2, V_1^1, \ldots, V_q^1, V_1^2, \ldots, V_q^2\},$$

where $U_i^2 = \sqrt{-1}U_i^1$, $V_j^2 = \sqrt{-1}V_j^1$. Let J be a complex structure on ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$JU_i^1 = U_i^2 \ (JU_i^2 = -U_i^1), \ JV_j^1 = V_j^2 \ (JV_j^2 = -V_j^1)$$

for $i = 1, \ldots, p$, $j = 1, \ldots, q$. Note that $({}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}), J)$ is a complex Lie algebra.

Let ${}_{\mathbb{R}}(G^{\mathbb{C}})$ be the simply connected real Lie group corresponding to ${}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. We have the following.

Proposition 2.4 ([7], [10]). *If $\mathfrak{g}$ has a non-degenerate closed 2-form, then $({}_{\mathbb{R}}(G^{\mathbb{C}}), J)$ has a left-invariant holomorphic symplectic form.*

We define other complex structure on $_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$ by the following:

$$\tilde{J}U_i^1 = -U_i^2\ (\tilde{J}U_i^2 = U_i^1),\ \tilde{J}V_j^1 = V_j^2\ (\tilde{J}V_j^2 = -V_j^1)$$

for $i = 1, \ldots, p, j = 1, \ldots, q$.

Proposition 2.5. *$\tilde{J}$ is integrable on $_{\mathbb{R}}(G^{\mathbb{C}})$.*

Proof. We show that the Nijenhuis tensor $N_{\tilde{J}}$ vanishes. Let $X \in \mathfrak{a}$, $Y \in \mathfrak{b}$. Note that $J \circ ad(Z) = ad(Z) \circ J$ for $Z \in {}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}})$. Then,

$$\begin{aligned} N_{\tilde{J}}(X, Y) &= [X, Y] + \tilde{J}[\tilde{J}X, Y] + \tilde{J}[X, \tilde{J}Y] - [\tilde{J}X, \tilde{J}Y] \\ &= [X, Y] + J[-JX, Y] + J[X, JY] - [-JX, JY] \\ &= 2([X, Y] + [JX, JY]) \\ &= 2([X, Y] + J^2[X, Y]) = 0. \end{aligned}$$

The other cases are similar, and hence omitted. □

Moreover, assume that $\mathfrak{b}$ is a direct sum: $\mathfrak{b}_1 + \mathfrak{b}_2$, where $\mathfrak{b}_1$ is abelian and $[\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}_2$, and that $\dim \mathfrak{a} = \dim \mathfrak{b}_2$. We take bases of the Lie subalgebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$:

$$\begin{aligned} \mathfrak{b}_1 &= \mathrm{span}_{\mathbb{R}}\{V_1^1, \ldots, V_r^1\}, \\ \mathfrak{b}_2 &= \mathrm{span}_{\mathbb{R}}\{W_1^1, \ldots, W_p^1\}. \end{aligned}$$

Let

$$\{\alpha_1^1, \ldots, \alpha_p^1, \beta_1^1, \ldots, \beta_r^1, \gamma_1^1, \ldots, \gamma_p^1, \alpha_1^2, \ldots, \alpha_p^2, \beta_1^2, \ldots, \beta_r^2, \gamma_1^2, \ldots, \gamma_p^2\}$$

be the dual basis of

$$\{U_1^1, \ldots, U_p^1, V_1^1, \ldots, V_r^1, W_1^1, \ldots, W_p^1, U_1^2, \ldots, U_p^2, V_1^2, \ldots, V_r^2, W_1^2, \ldots, W_p^2\}.$$

Put

$$\lambda_i = \alpha_i^1 + \sqrt{-1}(-\alpha_i^2),\ \mu_j = \beta_j^1 + \sqrt{-1}\beta_j^2,\ \nu_k = \gamma_k^1 + \sqrt{-1}\gamma_k^2$$

for $i, k = 1, \ldots, p$, $j = 1, \ldots, r$ (note that $\tilde{J}U_i^1 = -U_i^2$). Then, we can take a basis of left-invariant $(1, 0)$-forms of $({}_{\mathbb{R}}(G^{\mathbb{C}}), \tilde{J})$ as follows:

$$\{\lambda_i, \mu_j, \nu_k\}_{i,k=1,\ldots,p, j=1,\ldots,r}.$$

Proposition 2.6. *If $\mathfrak{a} + \mathfrak{b}_2$ has a non-degenerate 2-form $\omega_{\mathfrak{a}\mathfrak{b}_2} = \sum_{k,h} P_{kh}\alpha_k^1 \wedge \gamma_h^1 \in \mathfrak{a}^* \wedge \mathfrak{b}_2^*$ which is closed on $\mathfrak{g}$. Then, $({}_{\mathbb{R}}(G^{\mathbb{C}}), \tilde{J})$ has a left-invariant pseudo-Kähler structure.*

Proof. Let

$$\tau = \sum_{k,h} P_{kh}(\bar{\lambda}_k \wedge \nu_h + \lambda_k \wedge \bar{\nu}_h) = 2\sum_{k,h} P_{kh}(\alpha_k^1 \wedge \gamma_h^1 - \alpha_k^2 \wedge \gamma_h^2).$$

Since $d\omega_{\mathfrak{a}\mathfrak{b}_2} = 0$, and $\tau(X, \tilde{J}Y) = 0$ for $X, Y \in \mathfrak{g}$, we see

$$d\tau(X, Y, Z) = d\tau(\tilde{J}X, \tilde{J}Y, \tilde{J}Z) = d\tau(\tilde{J}X, Y, Z) = 0$$

for $X, Y, Z \in \mathfrak{g}$. Let $X, Y \in \mathfrak{b}, Z \in \mathfrak{g}$. Since $\tau(JX, JY) = -\tau(X, Y)$,

$$\begin{aligned} d\tau(\tilde{J}X, \tilde{J}Y, Z) &= -\tau([\tilde{J}X, \tilde{J}Y], Z) + \tau([\tilde{J}X, Z], \tilde{J}Y) - \tau([\tilde{J}Y, Z], \tilde{J}X) \\ &= \tau([X, Y], Z) - \tau([X, Z], Y) + \tau([Y, Z], X) \\ &= -2d\omega_{\mathfrak{ab}_2}(X, Y, Z) = 0. \end{aligned}$$

Similarly, we see

$$d\tau(\tilde{J}X, \tilde{J}Y, Z) = 2d\omega_{\mathfrak{ab}_2}(X, Y, Z) = 0 \qquad \text{for } X \in \mathfrak{a}, Y \in \mathfrak{b}, Z \in \mathfrak{g}$$
$$d\tau(\tilde{J}X, \tilde{J}Y, Z) = -2d\omega_{\mathfrak{ab}_2}(X, Y, Z) = 0 \qquad \text{for } X, Y \in \mathfrak{a}, Z \in \mathfrak{g}.$$

Thus,

$$\omega = \tau + \sqrt{-1} \sum_{j=1}^{r} \mu_j \wedge \bar{\mu}_j$$

is a pseudo-Kähler structure on $(_{\mathbb{R}}(G^{\mathbb{C}}), \tilde{J})$. □

EXAMPLE 2.7. Let $H(1, n)$ be the nilpotent Lie group defined by

$$H(1, n) = \left\{ \begin{pmatrix} I_n & \mathbf{x} & \mathbf{z} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \mathbf{x}, \mathbf{z} \in \mathbb{R}^n, y \in \mathbb{R} \right\},$$

where I_n denotes the identity $n \times n$ matrix. $H(1, n)$ is called a *generalized Heisenberg group*. Let

$$N = H(1, n) \times \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \middle| \ w \in \mathbb{R} \right\}.$$

Then,

$$X_i = \frac{\partial}{\partial x_i},\ Y = \frac{\partial}{\partial y} - \sum_{i=1}^{n} x_i \frac{\partial}{\partial z_i}, Z_i = \frac{\partial}{\partial z_i}, W = \frac{\partial}{\partial w}$$

is a basis of left-invariant holomorphic vector fields on N, and

$$\alpha_i = dx_i, \beta = dy, \gamma_k = dz_k - x_k dy, \delta = dw \ (i, k = 1, \ldots, n)$$

is its dual basis. Then,

$$\sum_{i=1}^{n} \alpha_i \wedge \gamma_i + \beta \wedge \delta$$

is a symplectic structure. Let

$$\mathfrak{a} = \text{span}\{X_1, \ldots, X_n\}, \mathfrak{b}_1 = \text{span}\{Y, Z, W\}, \mathfrak{b}_2 = \{Z_1, \ldots, Z_n\}.$$

Then, $[\mathfrak{a}, \mathfrak{b}_1 + \mathfrak{b}_2] \subset \mathfrak{b}_2$, and

$$\sum_{i=1}^{n} \alpha_i \wedge \gamma_i \in \mathfrak{a}^* \wedge \mathfrak{b}_2^*$$

is non-degenerate 2-form on $\mathfrak{a} + \mathfrak{b}_2$ which is closed on the Lie algebra $\mathfrak{n}$.

Thus we see that complex structures J, $\tilde{J}$ of $_{\mathbb{R}}(N^{\mathbb{C}})$ have the properties that the complex manifold $(_{\mathbb{R}}(N^{\mathbb{C}}), J)$ admits a left-invariant holomorphic symplectic structure, and the complex manifold $(_{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ admits a left-invariant pseudo-Kähler structure. Indeed, let

$$N_2 = (_{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J}) = \left\{ \begin{pmatrix} I_n & \bar{\mathbf{v}} & \mathbf{w} \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \mathbf{v}, \mathbf{w} \in \mathbb{C}^n, u \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & \bar{v} \\ 0 & 1 \end{pmatrix} \middle| \ v \in \mathbb{C} \right\}.$$

Then,

$$\zeta_i = dv_i, \lambda_i = dw_i - \bar{v}_i du \ (i = 1, \dots, n), \mu = du, \nu = dv$$

is a basis of left-invariant $(1,0)$-forms on N_2. Thus,

$$d\zeta_i = d\mu = d\nu = 0, \ d\lambda_i = -\bar{\zeta}_i \wedge \mu \ (i = 1, \dots, n).$$

Hence,

$$\omega = \mathrm{Re}(\sum_{i=1}^{n} \zeta_i \wedge \bar{\lambda}_i + \mu \wedge \bar{\nu})$$

is a pseudo-Kähler structure.

By a straightforward computation, we see that $(_{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ has no left-invariant holomorphic symplectic structures for $n \geq 2$ and thus $(\Gamma \backslash _{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ has no holomorphic symplectic structures. Indeed, if $(\Gamma \backslash _{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ has a holomorphic symplectic structure, then $(\Gamma \backslash _{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ has an invariant holomorphic symplectic structure by results of Console-Fino [3], and Angella [1].

Next, considering other complex structure J on $_{\mathbb{R}}(N^{\mathbb{C}})$,

$$N_1 = (_{\mathbb{R}}(N^{\mathbb{C}}), J) = \left\{ \begin{pmatrix} I_n & \mathbf{z} & \mathbf{w} \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \mathbf{z}, \mathbf{w} \in \mathbb{C}^n, u \in \mathbb{C} \right\} \times \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \middle| \ v \in \mathbb{C} \right\}.$$

Then,

$$\zeta_i = dz_i, \lambda_i = dw_i - z_i du \ (i = 1, \dots, n), \mu = du, \nu = dv$$

is a basis of left-invariant holomorphic 1-forms on N_1. Thus,

$$d\zeta_i = d\mu = d\nu = 0, d\lambda_i = -\zeta_i \wedge \mu \ (i = 1, \dots, n).$$

Hence,

$$\Omega = \sum_{i=1}^{n} \zeta_i \wedge \lambda_i + \mu \wedge \nu$$

is a left-invariant holomorphic symplectic structure. By Proposition 2.2, we see that N_1 has no left-invariant pseudo-Kähker structures. We also see that $\mathfrak{h}((\Gamma \backslash _{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})) = n + 2$, and $\mathfrak{h}((\Gamma \backslash _{\mathbb{R}}(N^{\mathbb{C}}), J)) = 2n + 2$, where $\mathfrak{h}(M)$ is the Lie algebra of holomorphic vector fields on a complex manifold M.

Hence, we have the following main result.

Theorem 2.8. *There exists a compact real parallelizable nilmanifold* $\Gamma \backslash N$ *and its complex structures* J_1, J_2 *such that* $(\Gamma \backslash N, J_1)$ *has a pseudo-Kähler structure and no holomorphic symplectic structures, however* $(\Gamma \backslash N, J_2)$ *has a holomorphic symplectic structure and no pseudo-Kähler structures.*

REMARK 2.9. There exist other examples which satisfy the properties in Theorem 2.8. For example, let $H(q, p)$ be the nilpotent Lie group defined by

$$H(q,p) = \left\{ \begin{pmatrix} I_p & A & C \\ 0 & I_q & B \\ 0 & 0 & I_q \end{pmatrix} \middle| \; A, C \text{ are } p \times q \text{ matrices}, B \text{ is a diagonal matrix} \right\}.$$

We denote by $\mathfrak{h}(q, p)$ the Lie algebra of $H(q, p)$. Put $N = H(q_1, p_1) \times H(q_2, p_2)$. Then, N is a generalization of the example above (the example above is $N = H(1, n) \times H(1, 0)$). Then

$$\mathfrak{h}(q_1, p_1) \times \mathfrak{h}(q_2, p_2) = \mathrm{span}\{X_{ij}, Y_j, Z_{ij}, X'_{st}, Y'_t, Z'_{st}\}_{i=1,\ldots,p_1, j=1,\ldots,q_1, s=1,\ldots,p_2, t=1,\ldots,q_2}$$

with nontrivial equations

$$[X_{ij}, Y_j] = Z_{ij}, \ [X'_{st}, Y'_t] = Z'_{st}$$

for $i = 1, \ldots, p_1$, $j = 1, \ldots, q_1$, $s = 1, \ldots, p_2$, $t = 1, \ldots, q_2$.

Consider the Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$ defined by

$$\mathfrak{a} = \mathrm{span}\{X_{ij}, X'_{st}\}_{i=1,\ldots,p_1, j=1,\ldots,q_1, s=1,\ldots,p_2, t=1,\ldots,q_2},$$
$$\mathfrak{b} = \mathrm{span}\{Y_j, Y'_t, Z_{ij}, Z'_{st}\}_{i=1,\ldots,p_1, j=1,\ldots,q_1, s=1,\ldots,p_2, t=1,\ldots,q_2}.$$

Assume that $q_1 = q_2$. Then N has a left-invariant symplectic structure which satisfies the condition in Proposition 2.6 (cf.[4]). Thus, we see that $({}_{\mathbb{R}}(N^{\mathbb{C}}), J)$ admits a left-invariant holomorphic symplectic structure and no left-invariant pseudo-Kähler structures, and $({}_{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ admits a left-invariant pseudo-Kähler structure. By a straightforward computation, we see that $({}_{\mathbb{R}}(N^{\mathbb{C}}), \tilde{J})$ has no left-invariant holomorphic symplectic structures for $p_1 \geq 2$.

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