ON DOUBLY FELLER PROPERTY

ZHEN-QING CHEN* and KAZUHIRO KUWAE†

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Abstract

Let $X$ be a Feller process that has strong Feller property. In this paper, we investigate the Feller as well as strong Feller properties of the semigroups generated by multiplicative functionals of $X$ in open sets. Special attention is given to the Feynman-Kac and Girsanov transforms of $X$. Three examples of local Kato class measure that are not of Kato class are given in the last section so that Feller and strong Feller properties hold for corresponding Feynman-Kac semigroup of $X$ in open sets.

1. Doubly Feller property of transformed semigroup

Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, P_x, x \in E)$ be a strong Markov process on a locally compact separable metric space $E$. Let $\partial$ be a point added to $E$ so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of $E$. The point $\partial$ also serves as the cemetery point for $X$. Recall that $X$ is said to have Feller property if $P_t(C_\infty(E)) \subset C_\infty(E)$ for every $t > 0$ and $\lim_{t \to \infty} \|P_t f - f\|_\infty = 0$ for every $f \in C_\infty(E)$, where $\{P_t ; t \geq 0\}$ defined by $P_t f(x) := E_x[f(X_t)]$ is the semigroup of $X$. Here $C_\infty(E)$ is the space of continuous functions on $E$ that vanishes at infinity and $\|f\|_\infty := \sup_{x \in E} |f(x)|$. The space of bounded continuous functions on $E$ will be denoted as $C_b(E)$. The process $X$ is said to have strong Feller property if $P_t(B_b(E)) \subset C_b(E)$ for every $t > 0$. We say $X$ (or its transition semigroup) has doubly Feller property if it has both Feller and strong Feller property. Clearly the above terminology can be formulated for any semigroup $\{T_t; t \geq 0\}$ acting on $B_b(E)$.

Let $\{Z_t; t \geq 0\}$ be a positive multiplicative functional of $X$. It defines a semigroup

(1.1) $T_t f(x) := E_x[Z_t f(X_t)]$ for $t > 0$ and $f \geq 0$.

For an open subset $B$ of $E$, we also define a semigroup $T_t^B$ by

(1.2) $T_t^B f(x) := E_x[Z_t f(X_t); t < \tau_B]$ for $t > 0$ and $f \geq 0$ on $B$.

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where \( \tau_B := \inf\{t > 0; X_t \notin B\} \) is the first exit time from \( B \). Let \( B^s(E) \) be the \( \sigma \)-field of universally measurable subsets of \( E \) and denote by \( B^u_s(E) \) the family of bounded universally measurable functions on \( E \). Note that \( T_t f \in B^s(E) \) when \( f \) is Borel measurable (\( T_t f \) is Borel for Borel function \( f \) if \( Z_t \) is \( \mathcal{F}_t^0 \)-measurable). An open set \( B (\subset E) \) is said to be regular if \( P_x(\tau_B = 0) = 1 \) for any \( x \in B^c = E \setminus B \). Consider the following conditions:

\[
\text{(1.3)} \quad \lim_{t \to 0} \sup_{x \in K} E_x[|Z_t - 1|] = 0 \quad \text{for every compact set} \quad K,
\]

\[
\text{(1.4)} \quad \sup_{x \in [0,t]} \sup_{x \in E} E_x[Z_t] < \infty \quad \text{for some (and hence for every)} \quad t > 0,
\]

and for each \( t > 0 \), there exists \( p > 1 \) (which may depend on \( t \)) such that

\[
\text{(1.5)} \quad \sup_{x \in E} E_x[Z_t^p] < \infty.
\]

The following theorem is due to K.L. Chung [6].

**Theorem 1.1** (Theorems 1, 2, 3 and Corollary in [6]). Let \( B \) be an open subset of \( E \) and suppose that the part process \( X^B \) on \( B \) is a strong Feller process and the conditions (1.3) and (1.4) hold. Then the semigroup \( \{T_t^B; t \geq 0\} \) defined by (1.2) has strong Feller property. If in addition, \( X \) has Feller property (that is, \( X \) is doubly Feller), \( B \) is a regular set and (1.5) holds for every \( t > 0 \), then \( \{T_t^B; t \geq 0\} \) is doubly Feller. Moreover, if in addition \( B \) is relatively compact, then \( T_t^B g \in C_\infty(B) \) for every \( t > 0 \) and \( g \in B_b(B) \).

**Corollary 1.2.** Let \( B \) be an open subset of \( E \) and suppose that \( X^B \) is a strong Feller process. Assume that

\[
\text{(1.6)} \quad \lim_{t \to 0} \sup_{x \in E} E_x[|Z_t - 1|] = 0.
\]

Then \( \{T_t^B; t \geq 0\} \) has strong Feller property. If further \( X \) has a Feller property, \( B \) is regular and (1.5) holds for every \( t > 0 \), then \( \{T_t^B; t \geq 0\} \) has doubly Feller property. Moreover, if in addition \( B \) is relatively compact, then \( T_t^B g \in C_\infty(B) \) for every \( t > 0 \) and \( g \in B_b(B) \).

**Remark 1.3.** In the conclusion of Theorem 2 in [6], only doubly Feller property of \( \{T_t; t \geq 0\} \) is stated. But the proof actually gives the strong Feller property of \( \{T_t; t \geq 0\} \) under (1.3), (1.4) and the strong Feller property of \( \{P_t; t \geq 0\} \).

We shall give another criteria for the Feller properties. First we relax the conditions
(1.3), (1.4) and (1.5) into the following: Fix an open set $B$.

\begin{equation}
\limsup_{t \to 0} \mathbb{E}_x [Z_{t^-} - 1] : t < \tau_D = 0 \quad \text{for any relatively compact open set } D \subset B,
\end{equation}

\begin{equation}
\sup_{x \in [0, t]} \mathbb{E}_x [Z_s] : s < \tau_B < \infty \quad \text{for some (and hence for every) } t > 0,
\end{equation}

and for each $t > 0$, there exists $p > 1$ (which may depend on $t$) such that

\begin{equation}
\sup_{x \in B} \mathbb{E}_x [Z_{t^+}^p] : t < \tau_B < \infty.
\end{equation}

**Theorem 1.4.** Let $X$ be a doubly Feller process and $B$ an open set in $E$. Suppose that (1.7) holds and that

for each $t > 0$ and compact set $K \subset B$, there exists $p > 1$ such that

\begin{equation}
\sup_{x \in K} \mathbb{E}_x [Z_{t^+}^p] : t < \tau_B < \infty.
\end{equation}

Then the semigroup $\{T_t^B ; t \geq 0\}$ defined by (1.2) has strong Feller property. Assume further that (1.8) and (1.9) hold for every $t > 0$, $B$ is regular and that $\lim_{u \to 0} \mathbb{E}_x [Z_{t^-} - 1] : t < \tau_B = 0$ for every $x \in B$. Then the semigroup $\{T_t^B ; t \geq 0\}$ has Feller property.

Proof. (i) **Strong Feller property of $\{T_t^B ; t \geq 0\}$:** Let $D$ be a relatively compact open subset of $B$. The strong Feller property of the part process $X_D$ on $D$ holds under the doubly Feller property of $X$ (see Theorem 1 in Chung [6]). Applying Corollary 1.2 and (1.7) to $X_D$, the semigroup $\{T_t^D ; t \geq 0\}$ defined by $T_t^D f(x) := \mathbb{E}_x [Z_{t^-} f(X_t) : t < \tau_D]$, $f \in \mathcal{B}_b(D)$ has strong Feller property. Take $g \in \mathcal{B}_b(E)$. Let $\{D_n\}$ be an increasing sequence of relatively compact open sets converging to $B$. Then the quasi-left-continuity of $X$ yields that $P_x (\lim_{n \to \infty} \tau_{D_n} = \tau_B) = 1$ for all $x \in B$. Take a compact set $K \subset B$. Then there is $n_0 \geq 1$ so that $K \subset D_n$ for every $n \geq n_0$ and

$$
\sup_{x \in K} [T_t^B g(x) - T_t^{D_n} g(x)]
\leq \sup_{x \in K} \mathbb{E}_x [Z_{t^-} g(X_t) : \tau_{D_n} \leq t < \tau_B]
\leq \|g\|_{B, \infty} \left( \sup_{x \in K} \mathbb{E}_x [Z_{t^+}^p : t < \tau_B] \right)^{1/p} \left( \sup_{x \in K} P_x (\tau_{D_n} \leq t < \tau_B) \right)^{1/q},
$$

where $\|g\|_{B, \infty} := \sup_{x \in B} |g(x)|$ and $q := p/(p-1)$ is the conjugate exponent of $p$, which may depend on $K$ and $t > 0$.

To establish $T_t g \in C(K)$, it suffices to show

$$
\lim_{n \to \infty} \sup_{x \in K} P_x (\tau_{D_n} \leq t < \tau_B) = 0.
$$
Note that $P_x(\tau_{D_n} \leq t < \tau_B) = P^B_1(x) - P^D_1(x)$ is continuous in $x \in D_n$. So
\[
\sup_{x \in K} P_x(\tau_{D_n} \leq t < \tau_B) = P_{x_n}(\tau_{D_n} \leq t < \tau_B)
\]
for some $x_n \in K$ and $x \in K$ such that $x_n \to x$. For $n_0 \leq m < n$,
\[
P_{x_n}(\tau_{D_n} \leq t < \tau_B) \leq P_{x_n}(\tau_{D_n} \leq t < \tau_B).
\]
By fixing $m \geq n_0$ first and taking $n \to \infty$, we have
\[
\lim_{n \to \infty} P_{x_n}(\tau_{D_n} \leq t < \tau_B) = P_{x_n}(\tau_{D_n} \leq t < \tau_B).
\]
By sending $m \to \infty$, we conclude that
\[
\lim_{n \to \infty} \sup_{x \in K} P_x(\tau_{D_n} \leq t < \tau_B) = \lim_{n \to \infty} P_{x_n}(\tau_{D_n} \leq t < \tau_B) = 0.
\]
This proves that $\{T_t; t \geq 0\}$ is strong Feller.

(ii) **Feller property of** $\{T^B_t; t \geq 0\}$: Since (1.9) holds for each $t > 0$ with some $p > 1$, we easily see $T^B_t f \in C_\infty(B)$ for $f \in C_\infty(B)$. Finally we prove that for each $f \in C_\infty(B)$
\[
\lim_{t \to 0} \|T^B_t f - f\|_{\infty} = 0.
\]
For $f \in C_\infty(B)$, under the conditions, we can easily see that for each $x \in B$
\[
\lim_{t \to 0} T^B_t f(x) = f(x).
\]
Then we can deduce (1.11) by using (1.8) Riesz-Markov-Kakutani theorem and Hahn-Banach theorem to the dual space of $C_\infty(B)$ (see Exercise (9.27) in [19]).

**Corollary 1.5.** Let $X$ be a doubly Feller process and assume (1.7) holds. Let $B$ an open regular set. Suppose that
\[
\text{there exists } p > 1 \text{ such that}
\]
\[
\sup_{x \in [0,t]} \mathbb{E}_x[Z^p_s; s < \tau_B] < \infty \quad \text{for some (hence every) } t > 0.
\]
Then $\{T^B_t; t \geq 0\}$ has doubly Feller property. If in addition, $B$ is relatively compact, then $T^B_t g \in C_\infty(B)$ for every $t > 0$ and $g \in \mathcal{B}_0(B)$.

**Proof.** (1.12) is stronger than (1.10). Hence the strong Feller property holds for $\{T^B_t; t \geq 0\}$ by Theorem 1.4. (1.12) also implies (1.4) and (1.5). From (1.5), we have that $T^B_t$ maps $C_\infty(B)$ into itself. As in the proof of Theorem 1.4, we shall show
\[ \mathbb{E}_x [\{ Z_t - 1 : t < \tau_B \} \mid t \to 0 \text{ as } t \to 0 \text{ for each } x \in B. \] By condition (1.7), for a relatively compact open set \( D \subset B \), it suffices to show that for each \( x \in D \)

\[
\lim_{t \to 0} \mathbb{E}_x [\{ Z_t - 1 : \tau_D \leq t < \tau_B \} = 0.
\]

We see that for each \( x \in D \)

\[
\mathbb{E}_x [\{ Z_t - 1 : \tau_D \leq t < \tau_B \} = (\mathbb{E}_x [\{ Z_t - 1 \} : t < \tau_B])^{1/p} \mathbb{P}_x (\tau_D \leq t)^{1/q} \leq \mathbb{P}_x (\tau_D \leq t)^{1/q} \sup_{s \in [0,T]} \sup_{y \in B} \mathbb{E}_y [\{ Z_s - 1 \} : s < \tau_B])^{1/p} \to 0 \text{ as } t \to 0.
\]

The proof of \( T^B_t g \in C_\infty (B) \) for \( g \in \mathcal{B}_b(B) \) is similar to that of Corollary of [6]. \( \square \)

**Remark 1.6.** (i) Without assuming the regularity of \( B \), we cannot deduce the Feller property of the semigroup \( P^B_t f(x) := \mathbb{E}_x [f(X_t) : t < \tau_B] \).

(ii) (1.3) (resp. (1.5)) implies (1.7) (resp. (1.10)). In view of the latter part of the proof of Corollary 1.5 and Lemma 2 in [6], (1.7) and (1.12) for an open set \( B \) together imply (1.13) below

\[
\lim_{t \to 0} \sup_{x \in K} \mathbb{E}_x [\{ Z_t - 1 : t < \tau_B \} = 0 \text{ for every compact set } K \subset B,
\]

under the Feller property of \( \{ P_t : t \geq 0 \} \).

(iii) A sufficient condition for (1.3), (1.4), (1.5), (1.6), (1.7), (1.10) and (1.12) to hold is the following

\[
\lim_{t \to 0} \sup_{x \in E} \mathbb{E}_x \left[ \sup_{s \in [0,t]} (Z_s - 1)^2 \right] = 0.
\]

The above condition (1.14) is satisfied if \( Z_t \) is a combination of Girsanov and Feynman-Kac transform under appropriate Kato class condition (see, e.g. [4], [5], and Lemmas 2.4 and 3.2 below).

(iv) The condition (1.7) is satisfied when the multiplicative functional \( Z \) satisfies certain local Kato class condition and the condition (1.12) holds when \( Z \) is in certain extended Kato class. We will explain these points precisely in next two sections. We will show in the last section that there are examples of positive multiplicative functional \( Z \) that satisfies conditions (1.7) and (1.12) but not condition (1.6) nor condition (1.14).

2. **Feynman-Kac transform**

Let \( X \) be a strong Markov process on \( E \) as in the previous section and \( \omega_0 \) denote the sample path such that \( \omega_0(t) = \partial \) for every \( t \geq 0 \).
\textbf{Definition 2.1 (AF).} An \((\mathcal{F}_t)\)-adapted process \(A = (A_t)_{t \geq 0}\) with values in \([-\infty, \infty]\) is said to be an \emph{additive functional} (AF in short) if there exist a \emph{defining set} \(\Xi \in \mathcal{F}_\infty\) satisfying the following conditions:

(i) \(\mathbf{P}_x(\Xi) = 1\) for all \(x \in E\),

(ii) \(\theta_t \Xi \subseteq \Xi\) for all \(t \geq 0\); in particular, \(\omega_0 \in \Xi\) because of \(\omega_0 = \theta_{\zeta(\omega)}(\omega)\) for all \(\omega \in \Xi\),

(iii) for all \(\omega \in \Xi\), \(A(\omega)\) is right continuous having left hand limits on \([0, \zeta(\omega)]\), \(A_0(\omega) = 0\), \(|A_t(\omega)| < \infty\) for \(t < \zeta(\omega)\) and \(\theta_t A_0(\omega) = A_t(\omega) + A_0(\theta_t \omega)\) for all \(t, s \geq 0\),

(iv) for all \(t \geq 0\), \(A_t(\omega_0) = 0\); in particular, under the additivity in (iii), \(A_t(\omega) = A_{\zeta(\omega)}(\omega)\) for all \(t \geq \zeta(\omega)\) and \(\omega \in \Xi \cap [\zeta < \infty]\).

An AF \(A\) is called \emph{right-continuous with left limits} (rcll AF in brief) if \(A_{\zeta(\omega)}(\omega)_-\) exists for each \(\omega \in \Xi \cap [\zeta < \infty]\). An AF \(A\) is said to be \emph{finite} (resp. \emph{continuous additive functional} (CAF in brief)) if \(|A_t(\omega)| < \infty\), \(t \in [0, \infty[\) (resp. \(t \mapsto A_t(\omega)\) is continuous on \([0, \infty[\)) for each \(\omega \in \Xi\). A \([0, \infty[\)-valued AF is called a \emph{positive additive functional} (PAF in short). Two AFs \(A\) and \(B\) are called \emph{equivalent} if there exists a common defining set \(\Xi \in \mathcal{F}_\infty\) such that \(A_t(\omega) = B_t(\omega)\) for all \(t \in [0, \infty[\) and \(\omega \in \Xi\).

We now introduce the PAFs of Dynkin and Kato classes.

\textbf{Definition 2.2 (Dynkin class).} A PAF \(A\) is said to be \emph{of Dynkin class} if \(\sup_{t \leq s \in E} E_s[A_t] < \infty\) for some \(s > 0\), or equivalently, \(\sup_{t \leq s \in E} E_s\left[\int_0^\infty e^{-\alpha t} dA_t\right] < \infty\) for some \(\alpha > 0\). By the Markov property of \(X\), a PAF \(A\) is of Dynkin class if and only \(\sup_{t \leq s \in E} E_s[A_t] < \infty\) for all \(t > 0\).

\textbf{Definition 2.3 (Kato class).} A PAF \(A\) is said to be \emph{of Kato class} (resp. \emph{of extended Kato class} or \emph{generalized Kato class}) if \(\lim_{t \to 0} \sup_{t \leq s \in E} E_s[A_t] = 0\) (resp. \(\lim_{t \to 0} \sup_{t \leq s \in E} E_s[A_t] < 1\)); or equivalently, \(\lim_{t \to \infty} \sup_{t \leq s \in E} E_s\left[\int_0^\infty e^{-\alpha t} dA_t\right] = 0\) (resp. \(\lim_{t \to \infty} \sup_{t \leq s \in E} E_s\left[\int_0^\infty e^{-\alpha t} dA_t\right] < 1\)). A PAF \(A\) is said to be \emph{of local Kato class} if for each compact set \(K\), a PAF \(1_K A\) defined by \((1_K A)_t = \int_0^t 1_K(X_{s-}) dA_s\) is of Kato class.

For a PAF \(A\), let \(\text{Exp}(A)_t\) be the Stieltjes exponential of \(A\), that is, \(\text{Exp}(A)_t\) is the unique solution of \(Z_t\) of

\[ Z_t = 1 + \int_{[0,t]} Z_{s-} \ dA_s. \]

By (A4.17) of Sharpe [19],

\[(2.1) \quad \text{Exp}(A) = \exp(A^c) \prod_{0<s \leq t} (1 + \Delta A_s) = \exp\left(A^c_t + \sum_{0<s \leq t} \log(1 + \Delta A_s)\right)\]
where $\Delta A_t := A_t - A_s$ and $A_t^c$ denotes the continuous part of $A_t$. Since $\log(1+x) \leq x$, $x \in [0, \infty[$, we see $\operatorname{Exp}(A)_t \leq \exp A_t$. From (2.1), we have for $p \geq 1$

\[(2.2) \quad (\operatorname{Exp}(A)_t)^p = \operatorname{Exp}(A^{(p)})_t,\]

where $A_t^{(p)} := pA_t^c + \sum_{0<s\leq t} ((1 + \Delta A_s)^p - 1)$.

**Lemma 2.4.** Let $A$ be a PAF of Kato class and put $Z_t := \operatorname{Exp}(A)_t$. Then (1.6) holds for $Z$. Moreover

(i) if the AF $B$ defined by $B_t := \sum_{0<s\leq t}(\Delta A_s)^2$ is of Dynkin class, then (1.5) holds for some (hence every) $t > 0$ and some $p > 1$,

(ii) if $|\Delta A| \leq M \mathbb{P}_x$-a.s. for some $M > 0$, then (1.5) holds for some (hence every) $t > 0$ and any $p > 1$,

(iii) if $B$ is of Kato class (resp. $|\Delta A| \leq M \mathbb{P}_x$-a.s. for some $M > 0$), then for any $p \in [1, 2]$ (resp. for any $p \in [1, \infty[$)

\[
\lim_{t \to 0} \sup_{x \in E} \mathbb{E}_x \left[ \sup_{s \in [0, t]} |Z_s - 1|^p \right] = 0.
\]

**Proof.** In view of the proof of Khas’minskii’s lemma (see Lemma 2.1 (a) in [20]), we have

\[
\sup_{x \in E} \mathbb{E}_x[\operatorname{Exp}(A)_t] \leq \frac{1}{1 - \sup_{x \in E} \mathbb{E}_x[A_t]}
\]

for sufficiently small $t > 0$ with $\sup_{x \in E} \mathbb{E}_x[A_t] < 1$. From this,

\[
\sup_{x \in E} \mathbb{E}_x[|Z_t - 1|] \leq \frac{\sup_{x \in E} \mathbb{E}_x[A_t]}{1 - \sup_{x \in E} \mathbb{E}_x[A_t]},
\]

which converges to 0 as $t \to 0$. Next we prove the second statement. We first assume that $B$ is of Dynkin class and $p \in [1, 2]$. Letting $p$ be close to 1, we have $\sup_{x \in E} \mathbb{E}_x[pA_t + (p - 1)B_t] < 1$ for sufficiently small $t > 0$. Since $(1 + x)^{p-1} - 1 \leq (p-1)x$ for $x > -1$, we get $(1 + x)^p - 1 \leq (p-1)x^2 + px$ for $x > -1$. Then we have

\[
\sup_{x \in E} \mathbb{E}_x[Z_t^p] = \sup_{x \in E} \mathbb{E}_x \left[ \operatorname{Exp} \left( pA_t^c + \sum_{0<s\leq t} (1 + \Delta A_s)^p - 1 \right) \right] 
\]

\[\leq \frac{1}{1 - \sup_{x \in E} \mathbb{E}_x[pA_t + (p - 1)B_t]} < \infty\]

for such $p > 1$ and $t > 0$ by way of the argument as above. Next we assume that $\Delta A$ is bounded above by $M > 0 \mathbb{P}_x$-a.s. Then the PAF $A^{(p)}$ is of Kato class for any $p > 1$. 

Indeed, set \( n := \lfloor p \rfloor + 1 \in \mathbb{N} \). Then we see
\[
A_t^{(p)} \leq n A_t^c + \frac{(1 + M)^n - 1}{M} A_t^d.
\]
Hence
\[
\sup_{x \in E} E_x[Z_t^p] = \sup_{x \in E} E_x[\text{Exp}(A^{(p)} t)] \\
\leq \frac{1}{1 - \sup_{x \in E} E_x[A_t^{(p)}]}.
\]
Finally we show the last statement. Assume that \( B \) is of Kato and \( p \in [1, 2] \). Using \((x - 1)^p \leq x^p - 1 \) for \( x \geq 1 \), we have
\[
\sup_{x \in E} E_x\left[ \sup_{s \in [0, t]} \left| \text{Exp}(A_s) - 1 \right|^p \right] \leq \sup_{x \in E} E_x\left[ \text{Exp} \left( p A^c + \sum_{0 < s \leq t} \left( 1 + \Delta A_s \right)^p - 1 \right) - 1 \right] \\
\leq \frac{\sup_{s \in E} E_s[\text{Exp}(p A_t + (p - 1) B_t)]}{1 - \sup_{s \in E} E_s[\text{Exp}(p A_t + (p - 1) B_t) - 1]} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\]
The proof for the case of \( \Delta A \) being bounded is similar.

**Theorem 2.5.** Let \( A \) be a PAF of local and extended Kato class. Suppose that a PAF \( B \) defined by \( B_t := \sum_{0 < s \leq t} (\Delta A_s)^2 \) is of Dynkin class. Put \( Z_t := \text{Exp}(A)_t \). Let \( B \) an open regular set. Then (1.7) and (1.12) hold for \( Z \) and \( B \). Consequently, the semigroup \( \{T_t^B; t \geq 0\} \) defined by (1.2) has doubly Feller property provided \( X \) is doubly Feller.

Proof. First we show the condition (1.7) follows from the local Kato property of \( A \). This is because for any compact set \( K \),
\[
E_x[\text{Exp}(A)_t - 1; t < \tau_K] = E_x[\text{Exp}(A)_t - 1; t < \tau_K] \\
= E_x[\text{Exp}(I_K A)_t - 1; t < \tau_K] \\
\leq E_x[\text{Exp}(I_K A)_t - 1],
\]
which converges to 0 uniformly on \( E \) by Lemma 2.4. Here we used the fact that on \( \{t < \tau_K\}, X_s \in K \) for all \( s \in [0, t] \).

For (1.12), it suffices to show that there exists \( p > 1 \) such that
\[
p A^c_t + \sum_{0 < s \leq t} \left( (1 + \Delta A_s)^p - 1 \right)
\]
is of extended Kato. For \( p \in [1, 2] \) and \( x > -1 \), recall \( (1 + x)^p - 1 \leq (p - 1)x^2 + px \). Since \( A \) is of extended Kato class, \( \lambda := \sup_{x \in E} E_x[A_T] < 1 \) for some \( T > 0 \). For such
$T > 0$, we set $l := \sup_{x \in E} E_x \left[ \sum_{0 < x \leq T} (\Delta A_x)^2 \right] < \infty$ and take $p \in \mathbb{I}, 2 \wedge (1 + l)/(\lambda + l)$. Then

$$
\sup_{x \in E} E_x \left[ pA_T^2 + \sum_{0 < x \leq T} \left( (1 + \Delta A_x)^p - 1 \right) \right] \leq (p - 1)l + p\lambda < 1.
$$

Hence (1.12) holds by way of Khas’minskii’s lemma (see Lemma 2.1 (a) in [20]).

3. Girsanov transform

In this section, we assume that $X$ is an $m$-symmetric doubly Feller process, where $m$ is a positive Radon measure on $E$ with full support and that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $X$ is regular on $L^2(E; m)$. In this case, the transition kernel $\{p_t; t > 0\}$ of $X$ satisfies the absolute continuity condition with respect to $m$, i.e. $p_t(x, N) = 0$ if $m(N) = 0$ for each $N \in \mathcal{B}(E), x \in E$ and $t > 0$. For $\alpha > 0$, there exists an $\alpha$-order resolvent kernel $r_\alpha(x, y)$ which is defined for all $x, y \in E$ (see Lemma 2.14 in [8]). A Borel measure $\nu$ is said to be of Dynkin class (resp. Kato class) if $\sup_{x \in E} R_\alpha \nu(x) < \infty$ for some $\alpha > 0$ (resp. $\lim_{\alpha \to \infty} \sup_{x \in E} R_\alpha \nu(x) = 0$), and $\nu$ is in the local Kato class if $1_\nu$ is in the Kato class for every compact set $K \subset E$. The measure is said to be of extended Kato class if $\lim_{\alpha \to \infty} \sup_{x \in E} R_\alpha \nu(x) < 1$. Here $R_\alpha \nu(x) := \int_E r_\alpha(x, y) \nu(dy)$. Since $X$ is a Feller process, its Lévy system $(N, H)$ exists and is defined under $\mathbb{P}_x$ for every $x \in E$. Denote by $S_1$ (resp. $S_{00}$) the family of smooth measures in the strict sense (resp. measures of finite energy integrals with bounded potentials) (see (2.2.10) and p. 195 in [8]). Note that any Radon measure of Dynkin class always belongs to $S_1$ in view of Proposition 3.1 in [12].

Let $\phi: E_0 \times E_\beta \to \mathbb{R}$ be a Borel function that vanishes along the diagonal. The following lemma is a slightly modified version of [2, Lemma 3.2].

**Lemma 3.1.** Assume $N(|\phi| \wedge |\phi|^2)\mu_H \in S_1$. Then there exists a local martingale additive functional $M$ of purely discontinuous type such that $M_t - M_{t-} = \phi(X_{t-}, X_t)$ for all $t \in ]0, \infty[ \mathbb{P}_x$-a.s. Moreover, if $N(\phi^2) \mu_H \in S_1$, then such $M$ is locally square integrable.

**Proof.** Let $M^{(2)}$ be the AF defined by

$$
M^{(2)}_t := \sum_{s \leq t} (1_{|\phi| > 1}\phi)(X_{s-}, X_s) - \int_0^t N(1_{|\phi| > 1}\phi)(X_s) dH_s.
$$

Then $M^{(2)}$ is a local MAF in the strict sense ($M^{(2)}$ is locally square integrable provided $N(\phi^2) \mu_H \in S_1$). For $n \geq 2$, define AF $M^n$ by

$$
M^n_t := \sum_{s \leq t} (1_{1/n < |\phi| \leq 1}\phi)(X_{s-}, X_s) - \int_0^t N(1_{1/n < |\phi| \leq 1}\phi)(X_s) dH_s,
$$

where $n \in \mathbb{N}$. For $n \to \infty$, $M^n$ converges to $M^{(2)}$ in $L^2$. Therefore $M^n \in S_1$ for $n \geq 2$. Note that $M^n$ converges to $M^{(2)}$ under $\mathbb{P}_x$ for every $x \in E$.
which is a locally square integrable MAF in the strict sense. For \( n > m > 1 \)

\[
[M^n_t - M^m_t]_t = \sum_{s \leq t} (1[|1/m - |\phi|_1/|n|] \phi^2)(X_{s-}, X_s),
\]

and so

\[
(M^n_t - M^m_t)_t = \int_0^t N(1[|1/m - |\phi|_1/|n|] \phi^2)(X_s) \, dH_s.
\]

Therefore the limit

\[
M^{(1)}_t := \lim_{n \to \infty} M^n_t
\]

exists and defines a locally square integrable MAF in the strict sense of purely discontinuous type. Therefore, \( M_t := M^{(1)}_t + M^{(2)}_t \) is the desired local MAF of purely discontinuous type.

Hereafter we fix a continuous locally square integrable MAF \( M^c \) and a Borel function \( \phi : E_\delta \times E_\delta \rightarrow \mathbb{R} \) with \( \phi(x, y) > -1 \) for all \( x, y \in E_\delta \) and \( \phi(x, x) = 0 \) for \( x \in E_\delta \). We use \( \mu_{(M^c)} \) to denote the Revuz measure of \( (M^c) \).

**Lemma 3.2.** Suppose that \( N(\phi - \log(1 + \phi))\mu_H \in S_1 \) and assume that \( \nu := N(\phi^2)\mu_H + (1/2)\mu_{(M^c)} \) is a Radon measure of extended Kato class.

(i) There exists a locally square integrable MAF \( M^d \) of purely discontinuous type such that \( \Delta M^d_t = \phi(X_t-, X_t) \) \( t \in ]0, \infty[ \) \( \mathbb{P}_\mathbb{x} \)-a.s.

(ii) There exist \( t > 0 \) and \( p > 1 \) such that

\[
\sup_{x \in E} \left[ \sup_{s \in [0,t]} Z^p_s \right] < \infty,
\]

where \( Z_t := \exp(M)_t \) is the solution of Doléan-Dade equation

\[
Z_t = 1 + \int_{[0,t]} Z_{s-} \, dM_s
\]

for \( M_t := M^c_t + M^d_t \). In particular, \( Z_t \) is a martingale.

(iii) If \( \log(1 + \phi) \) is bounded and \( \nu \) is of Kato, then (3.1) holds for any \( t > 0 \) and \( p \geq 1 \). Moreover, for any \( p \geq 1 \), we have

\[
\lim_{t \to 0} \sup_{x \in E} \left[ \sup_{s \in [0,t]} |Z_s - 1|^p \right] = 0
\]
and $\tilde{Z}_t := \exp(M_t)$ satisfies that for any $p \geq 1$

$$\lim_{t \to 0} \sup_{s \in E} \left[ \sup_{s \in [0,t]} |\tilde{Z}_s - 1|^p \right] = 0.$$ 

Proof. (i): Under the conditions, the measure $N(\phi^2)\mu_H$, hence $N(|\phi| + |\phi|^2)\mu_H$, is smooth in the strict sense (see Proposition 3.1 in [12]). By Lemma 3.1, it is easy to see the existence of the locally square integrable MAF $M^d$ of purely discontinuous type such that $M^d = \phi(X_{t-}, X_t)$ for all $t \in [0, \infty[ \ P$-a.s.

(ii): Since $N(\phi - \log(1 + \phi))\mu_H$ is a smooth measure in the strict sense, there exists a local MAF $L$ of purely discontinuous type such that $L_t - L_{t-} = (\phi - \log(1 + \phi))(X_{t-}, X_t)$ for all $t \in [0, \infty[ \ P$-a.s. Set $J := M^d - L$ and $U := J + M^c$. Note that

$$Z_t = \exp(U_t - C_t),$$

where $C$ is a PCAF in the strict sense defined by $C_t := \int_0^t \phi - \log(1 + \phi))((X_s) dH_s + (1/2)\langle M^c \rangle_t)$. Take $p > 1$ and $q > 1$ with $pq \in [1, 2]$ and define $\phi_{pq} := (1 + \phi)^{pq} - 1$. Recalling the inequality $(1 + x)^q \leq 1 + rx + (r - 1)x^2$ for $x > -1$ and $r \in [1, 2]$, we see that $N(\phi_{pq} - \log(1 + \phi_{pq}))\mu_H \in S_1$ under the conditions. Let $M^{(pq), d}$ be a locally square integrable MAF of purely discontinuous type with $M^{(pq), d} = \phi_{pq}(X_{t-}, X_t)$ for all $t \in [0, \infty[ \ P$-a.s., which can be similarly constructed by using of $\phi_{pq}$ instead of $\phi$. Set $M^{(pq), c} := M^{(pq), d} + pq M^c$ and $Z^{(pq), c} := \exp(M^{(pq), c})$. Then

$$Z^{(pq), c}_t = \exp(pq U_t - C^{(pq), c}_t),$$

where $C^{(pq), c}_t := \int_0^t N(\phi_{pq} - \log(1 + \phi_{pq}))((X_s) dH_s + (p^2 q^2 / 2)\langle M^c \rangle_t)$. We then see that

$$E_x[Z^{(pq), c}_t] = E_x[\exp(pU_t - pC_t)]$$

$$= E_x \left[ (Z^{(pq), c}_t)^{1/q} \exp \left( \frac{1}{q} C^{(pq), c}_t - pC_t \right) \right]$$

$$\leq E_x \left[ \exp \left( \frac{1}{q - 1} \int_0^t N((1 + \phi)^{pq} - 1 - pq\phi)(X_s) dH_s \right) \right]$$

$$\leq E_x \left[ \exp \left( \frac{1}{q - 1} \int_0^t N((1 + \phi)(X_s) dH_s + \frac{pq}{2(q - 1)}\langle M^c \rangle_t) \right) \right]^{(q - 1)/q}.$$ 

Since $pq \in [1, 2]$, (3.3) is estimated by

$$E_x \left[ \exp \left( \frac{pq - 1}{q - 1} \left( \int_0^t N(\phi^2)(X_s) dH_s + \frac{pq}{2} \langle M^c \rangle_t \right) \right) \right]^{(q - 1)/q}.$$
Put \( l := \lim_{t \to 0} \sup_{x \in \mathbb{E}} \mathbf{E}_x \left[ \int_0^t N(\phi^2)(X_s) \, dH_s + (1/2)(M^c)_t \right] < 1 \). Letting \( p \) and \( q \) be sufficiently close to 1, we have

\[
\frac{pq - 1}{q - 1} \cdot q^l < 1,
\]

which shows \( \sup_{x \in \mathbb{E}} \mathbf{E}_x [Z^p_t] < \infty \) for some (hence all) \( t > 0 \) and some \( p > 1 \) in view of Khas’minskii’s lemma. Let \( \{T_n\} \) be an increasing sequence of stopping times such that \( Z_{t \wedge T_n} \) is a martingale for each \( n \in \mathbb{N} \). In the same way, we see that

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{E}} \mathbf{E}_x [Z^p_{t \wedge T_n}] < \infty
\]

for such \( t > 0 \) and \( p > 1 \), which yields the martingale property of \( Z \). (3.1) follows from Doob’s inequality.

(iii): We may assume \( p > 1 \). Note that the function \( f_p(x) := ((1+x)^p - 1 - px)/x^2 \) is bounded above over \([-1+\epsilon, -1+e^{-1}] \) for each \( \epsilon \in ]0, 1[ \) and \( p > 1 \) and set \( D^p := \sup_{x \in [-1+\epsilon, -1+e^{-1}]} f_p(x) > 0 \). Assuming that \( |\log(1+\phi)| \) is bounded above by \( |\log \epsilon| \), (3.3) is estimated by

\[
\mathbf{E}_x \left[ \exp \left( \frac{D^p}{q - 1} \int_0^t N(\phi^2)(X_s) \, dH_s + \frac{pq}{2(q - 1)} (pq - 1)(M^c)_t \right) \right]^{(q-1)/q}.
\]

Since \( \nu \) is of Kato, we then have that \( \sup_{x \in \mathbb{E}} \mathbf{E}_x [Z^p_t] < \infty \) for some (hence all) \( t > 0 \) and any \( p > 1 \). The rest is similar as in (ii). Finally we prove the last statement. Owing to Doob’s inequality, it suffices to show

\[
\lim_{t \to 0} \sup_{x \in \mathbb{E}} \mathbf{E}_x [\left| Z_t - 1 \right|^p] = 0.
\]

Noting that \( |x - 1|^p \leq |x^p - 1| \) for \( x > 0 \) and \( \mathbf{E}_x [Z^p_t] \geq (\mathbf{E}_x [Z_t])^p = 1 \), we have

\[
\mathbf{E}_x [\left| Z_t - 1 \right|^p]^2 \leq \mathbf{E}_x [\left| Z^p_t - 1 \right|^2]
\]

\[
\leq \mathbf{E}_x [\left| Z^p_t - 1 \right|^2] = \mathbf{E}_x [Z^2_t - 2Z^p_t + 1] \leq \mathbf{E}_x [Z^2_t - 1]
\]

\[
\leq \mathbf{E}_x [\exp(A_t^{(2,p)})]^{(q-1)/q} - 1,
\]

where \( A_t^{(2,p,q)} := D^p \int_0^t N(\phi^2)(X_s) \, dH_s + (2pq/(2(q - 1)))(2pq - 1)(M^c)_t \) is a PCAF of Kato class, which shows \( \sup_{x \in \mathbb{E}} \mathbf{E}_x [\left| Z - 1 \right|^p]^2 \leq \sup_{x \in \mathbb{E}} \mathbf{E}_x [\exp(A_t^{(2,p,q)})] - 1 \to 0 \) as \( t \to 0 \). By \( |x + y|^p \leq 2^{p-1}(|x|^p + |y|^p), x, y \in \mathbb{R} \), we have

\[
|\tilde{Z}_t - 1|^p \leq 2^{p-1}(\exp(A_t^{(p)}) - 1)^p + \exp(A_t^{(p)}) - 1,
\]

where \( A_t^{(p)} \) is a PAF such that its continuous part is \( (p/2)(M^c)_t \) and \( \Delta \tilde{A}_t^{(p)} = \exp(p(\phi -
log(1 + φ)(X_{t-}, X_t)) - 1 for all \( t \in [0, \infty[ \) \( \mathbf{P}_x \)-a.s. Since \( |\log(1 + φ)| \) is bounded by \(|\log ε|\),

\[
|\log(1 + φ)|^2 \leq \left( \frac{\log ε}{ε - 1} \right)^2 φ^2,
\]

where we use the inequality \( 0 \leq \log(1 + x)/x \leq \log ε/(ε - 1) \) for \( x \in [-1 + ε, -1 + ε^{-1}] \). Hence there exists \( C_ε > 0 \) such that \( φ - \log(1 + φ) \leq C_ε φ^2 \). This implies that \( \dot{A}^{(p)} \) is of Kato under the conditions. Therefore we obtain the desired assertion.

**Theorem 3.3.** Assume that \( \log(1 + φ) \) is bounded on \( K \times E \) for each compact set \( K \) and \( \nu := N(φ^2)μ_H + (1/2)μ_{(M)} \) is a positive Radon measure of local and extended Kato class. Put \( Z_t := \text{Exp}(M)_t \). Then (1.7) and (1.12) hold for \( Z \) and for every regular open set \( B \). In particular, \( Z \) is a martingale under the conditions. Consequently, the semigroup \( \{T_t^B; t \geq 0\} \) defined by (1.2) has doubly Feller property.

Proof. First we show that \( N(φ - \log(1 + φ))μ_H \) is a smooth measure in the strict sense. Let \( K \) be a compact set in \( E \). Then \( \log(1 + φ_K) \) is bounded on \( E_β × E_β \), where \( φ_K(x, y) := 1_K(x)φ(x, y) \). So there exists \( C_K > 0 \) such that \( φ_K - \log(1 + φ_K) \leq C_K φ_k^2 \) on \( E_β × E_β \). This implies that \( 1_K N(φ - \log(1 + φ))μ_H \) is of Kato class, because \( 1_K N(φ^2)μ_H \) is a finite measure of Kato class, hence it is in \( S_{00} \) in view of the proof of Proposition 3.1 in [12]. Then we have that \( N(φ - \log(1 + φ))μ_H \) is smooth in the strict sense. We can apply Lemma 3.2 (ii), consequently (1.12) is obtained. Next we prove (1.7). By assumption, \( (1_K * ((M^d) + (1/2)(M^c))_t = (1_K * M^d)_t + (1/2)(1_K * M^c)_t \) is a PCAF of Kato class. Hence we can apply Lemma 3.2 (iii). Therefore

\[
\lim sup_{t \to 0} \mathbf{E}_x \left[ \sup_{s \in [0, t]} |Z_s - 1|: t < τ_K \right] = \lim sup_{t \to 0} \mathbf{E}_x \left[ \sup_{s \in [0, t]} |\text{Exp}(1_K * M)_s - 1|: t < τ_K \right] \leq \lim sup_{t \to 0} \mathbf{E}_x \left[ \sup_{s \in [0, t]} |\text{Exp}(1_K * M)_s - 1| \right] = 0,
\]

which implies (1.7).

**4. Examples**

We show in each of the three examples in this section, there is a positive measure \( \mu \) whose associated PAF \( A^μ \) is in the local Kato class but not in Kato class and that \( Z_t := e^{t A^μ} \) satisfies conditions (1.7) and (1.12) but not (1.6) nor (1.14). Here \( A^μ \) is the positive continuous additive functional of \( X \) having Revuz measure \( μ \). These examples are to illustrate Theorem 2.5. From them, the readers can easily come up examples of Girsanov transform in the same spirit but Theorem 3.3 is still applicable.
**Example 4.1 (Brownian motion).** Let \( X = (\Omega, \mathcal{F}, \mathbb{P}_x) \) be \( d \)-dimensional Brownian motion on \( \mathbb{R}^d \). A signed Borel measure \( \mu \) on \( \mathbb{R}^d \) is said to be of Kato class if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \frac{1}{r^{d-2}} \left( \int_{|x-y| < r} |\mu(dy)| \right) = 0 \quad \text{when} \quad d \geq 3,
\]

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \frac{1}{r^{d-2}} \left( \int_{|x-y| < r} \log|y| \right) |\mu(dy)| = 0 \quad \text{when} \quad d = 2,
\]

\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} |\mu(dy)| < \infty \quad \text{when} \quad d = 1.
\]

Here \( |\mu| := \mu^+ + \mu^- \) is the total variation measure of \( \mu \). A signed Borel measure \( \mu \) on \( \mathbb{R}^d \) is said to be of local Kato class if \( 1_K \mu \) is of Kato class for every compact subset \( K \) of \( \mathbb{R}^d \). By definition, any measure \( \mu \) of local Kato class is always a signed Radon measure. Denote by \( \mathcal{K}_d \) (resp. \( \mathcal{K}^{\text{loc}}_d \)) the family of Kato class (resp. local Kato class) measures on \( \mathbb{R}^d \). It is essentially proved in [1] that a positive measure \( \mu \) is in Kato class \( \mathcal{K}_d \) if and only if \( \mu \) is a smooth measure in the strict sense and

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_0^t p_s(x, y) ds \right) \mu(dy) = \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} E_x[A^\mu_t] = 0,
\]

where \( A^\mu \) is a PCAF of \( X \) admitting no exceptional set associated to \( \mu \) under Revuz correspondence. Assume now that \( d \geq 2 \). We will show that there is a positive measure \( \mu \in \mathcal{K}^{\text{loc}}_d \setminus \mathcal{K}_d \) satisfying

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_0^t p_s(x, y) ds \right) \mu(dy) < \infty \quad \text{for some (and hence for all)} \quad t > 0
\]

and that \( Z_t := e^{A^\mu_t} \) satisfies conditions (1.7) and (1.12) but not (1.6) nor (1.14).

For any Borel measure \( \mu \), we set \( N\mu(x) := \int_{\mathbb{R}^d} \mu(dy)/|y-x|^{d-2} \) the Newtonian potential of \( \mu \) if \( d \geq 3 \), and \( L\mu(x) := \int_{\mathbb{R}^d} \log|y-x|^{-1} 1_{|y-x| < 1} \mu(dy) \) the modified logarithmic potential of \( \mu \) if \( d = 2 \). For \( r > 0 \) and \( z \in \mathbb{R}^d \), let \( \sigma_{\partial B_r(z)} \) be the surface measure on the sphere \( \partial B_r(z) \) with center \( z \) and radius \( r \).

Let \( \{z_n\}_{n=1}^\infty \) be a sequence in \( \mathbb{R}^d \) such that \( |z_{n+1}| = 2|z_n|, \ n \in \mathbb{N}, |z_1| > 2 \). We define \( \mu_n := g(r_n)\sigma_{\partial B_n(z_n)} \) and \( \mu := \sum_{n=1}^\infty \mu_n \), where

\[
r_n := 8^{-n} \quad \text{and} \quad g(t) := t^{-1} \quad \text{when} \quad d \geq 3,
\]

and

\[
r_n := 8^{-n^2} \quad \text{and} \quad g(t) := t^{-1}/\log t^{-1} \quad \text{when} \quad d = 2.
\]

It is known that for \( r > 0 \) and \( z \in \mathbb{R}^d \), \( N\sigma_{\partial B_r(z)}(x) = r \min\{1, (r/|x-z|)^{d-2}\} \) when \( d \geq 3 \) and \( L\sigma_{\partial B_r(z)}(x) = r \min\{\log r^{-1}, \log|y-z|^{-1}\} \) when \( d = 2 \). It follows that when \( d \geq 3 \), \( N\mu_n \) is bounded above by \( N\mu_n(z_n) = 1 \) (resp. by \( 1/2^n(d-2) \)) on \( \mathbb{R}^d \) (resp. on \( B_{2^n r_n}(z_n) \)).
Similarly, when $d = 2$, $L \mu_n$ is bounded above by $L \mu_n(z_n) = 1$ (resp. by $1/n^2$) on $\mathbb{R}^d$ (resp. on $B_{1/8}(z_n)$). Therefore we have that for $d \geq 3$

$$
\sup_{x \in \mathbb{R}^d} N \mu(x) = \sup_{x \in \mathbb{R}^d} \sum_{n=1}^{\infty} N \mu_n(x) 
\leq \sup_{x \in \mathbb{R}^d} \left( \sum_{n=1}^{\infty} 1_{B^{2n}(z_n)}(x)N \mu_n(x) + \sum_{n=1}^{\infty} 1_{B^{2n}(z_n)^c}(x)N \mu_n(x) \right) 
\leq \left( 1 + \sum_{n=1}^{\infty} \frac{1}{2^n(d-2)} \right) < \infty,
$$

which implies (4.1). Moreover, for all $r > r_n$

$$
\int_{|x - y| < r} \frac{\mu(dy)}{|z_n - y|^{d-2}} \geq N \mu_n(z_n) = 1,
$$

which implies $\mu \notin K_d$. It is easy to see $\mu \in K_d^{\text{loc}}$ from $\mu_n \in K_d$. Similarly, we have the same conclusion for the case of $d = 2$.

For such $\mu \in K_d^{\text{loc}} \setminus K_d$, let $\tilde{\mu} := \mu / (\|U_1 \mu\|_{\infty} + \varepsilon)$ and $A$ the PCAF of $X$ having Revuz measure $\tilde{\mu}$, where $\varepsilon > 0$ and $U_1 \tilde{\mu}(x) := \mathbf{E}_x \left[ \int_0^\infty e^{-t} dA_t \right]$. Then $A$ is of local Kato class and of extended Kato class. Hence the multiplicative functional $Z_t := \exp(A_t)$ satisfies conditions (1.7) and (1.12) in view of Theorem 2.5. However $Z$ does not satisfy condition (1.6), not to mention condition (1.14). This is because by Jensen’s inequality, as the positive measure $\mu$ is not in $K_d$,

$$
\limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} \mathbf{E}_x [Z_t - 1] = \limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} \mathbf{E}_x [Z_t - 1] \geq \limsup_{t \to 0} \left( \sup_{x \in \mathbb{R}^d} \exp(\mathbf{E}_x [A_t]) - 1 \right) > 0.
$$

**EXAMPLE 4.2 (Relativistic stable processes).** We fix $\alpha \in [0, 2]$ and $m \geq 0$. Let $X = (\Omega, \mathcal{F}, \mathbf{P}_x)_{x \in \mathbb{R}^d}$ be a Lévy process on $\mathbb{R}^d$ with

$$
\mathbf{E}_0 [e^{\sqrt{-1} \langle \xi, X_t \rangle}] = e^{-t(\|\xi\|^2 + m^{2\alpha}/\alpha^2 - m)}.
$$

If $m > 0$, it is called the relativistic $\alpha$-stable process with mass (see [15]). In particular, if $\alpha = 1$ and $m > 0$, it is called the relativistic free Hamiltonian process (see [11]). When $m = 0$, $X$ is nothing but the usual symmetric $\alpha$-stable process. It is known (see, e.g., [15]) that the transition density function $p_t(x, y)$ of $X$ is given by

$$
p_t(x, y) = e^{mt} \int_0^\infty \left( \frac{1}{4\pi s} \right)^{d/2} e^{-|x-y|^2/(4s)} e^{-sm^{2\alpha}/\alpha^2} \theta_{\alpha/2}(t, s) ds,
$$

where $\theta_{\delta}(t, s)$, $\delta$, $t$, $s > 0$, is the transition density function of the subordinator whose
Laplace transform is given by
\[ \int_{0}^{\infty} e^{-\lambda t} \theta_{\delta}(t, s) \, ds = e^{-\lambda \delta}. \]

A signed Borel measure \( \mu \) on \( \mathbb{R}^d \) is said to be of Kato class if
\[
\limsup_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{|\mu(dy)|}{|x-y|^{d-\alpha}} = 0 \quad \text{when} \quad d > \alpha,
\]
\[
\limsup_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|^{-1})|\mu(dy)| = 0 \quad \text{when} \quad d = \alpha,
\]
\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} |\mu(dy)| < \infty \quad \text{when} \quad d = 1 < \alpha.
\]

A Borel measure \( \mu \) on \( \mathbb{R}^d \) is said to be of local Kato class if \( 1_K \mu \) is of Kato class for every compact subset \( K \) of \( \mathbb{R}^d \). By definition, any measure \( \mu \) of local Kato class is always a signed Radon measure. Denote by \( K_{d,\alpha} \) (resp. \( K_{d,\alpha}^{\text{loc}} \)) the family of Kato class (resp. local Kato class) measures on \( \mathbb{R}^d \).

It is proved in [12] (see [21] for the case \( d > \alpha \) with \( m = 0 \), or \( d \geq 2, \alpha = 1 \) with \( m = 0 \)) that a positive measure \( \mu \) is in \( K_{d,\alpha} \) if and only if \( \mu \) is a smooth measure in the strict sense and
\[
\limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} \left( \int_{0}^{t} p_s(x, y) \, ds \right) \mu(dy) = \limsup_{t \to 0} \sup_{x \in \mathbb{R}^d} E_x[A_{t}^\mu] = 0,
\]
where \( A^\mu \) is a PCAF of \( X \) admitting no exceptional set associated to \( \mu \) under Revuz correspondence.

A measurable function \( f \) on \( \mathbb{R}^d \) is said to be of Kato class (resp. of local Kato class) if \( |f(x)| \, dx \in K_{d,\alpha} \) (resp. \( K_{d,\alpha}^{\text{loc}} \)) and write \( f \in K_{d,\alpha} \) (resp. \( f \in K_{d,\alpha}^{\text{loc}} \)) for simplicity.

For any Borel measure \( \mu \), we set \( R(\alpha) \mu(x) := \int_{\mathbb{R}^d} \mu(dy) / (|x-y|^{d-\alpha}) \) the Riesz potential of \( \mu \) if \( d > \alpha \), and \( L \mu(x) := \int_{\mathbb{R}^d} \log |x-y|^{-1} 1_{|x-y|<1} \mu(dy) \) the modified logarithmic potential of \( \mu \) if \( d = \alpha \). If \( \mu(dx) = g(x) \, dx \) for some non-negative function \( g \), we write \( R(\alpha) g(x) \) (resp. \( L g(x) \)) instead of \( R(\alpha) \mu(x) \) (resp. \( L \mu(x) \)).

Let \( \{z_n\}_{n=1}^{\infty} \) be a sequence in \( \mathbb{R}^d \) such that \(|z_{n+1}| = 2|z_n|\), \( n \in \mathbb{N} \), \(|z_1| > 2 \). We define \( f_n(x) := g(r_n) 1_{B_{r_n}(z_n)}(x) \) and \( f(x) := \sum_{n=1}^{\infty} f_n(x) \), where
\[
r_n := 8^{-n} \quad \text{and} \quad g(t) := t^{-\alpha} \quad \text{when} \quad d > \alpha
\]
and
\[
r_n := 8^{-n^2} \quad \text{and} \quad g(t) := \frac{t^{-\alpha}}{(1/d) + \log t^{-1}} \quad \text{when} \quad d = \alpha.
\]
By utilizing a simple rearrangement inequality (see Theorem 3.4 in [14]), we see that for \( d > \alpha \), \( R^{(\alpha)} f_n \) is bounded above by \( R^{(\alpha)} f_n(z_n) = (d/\alpha)\omega_d \) (resp. by \((1/(2^n - 1)^{d-\alpha})\omega_d\)) on \( \mathbb{R}^d \) (resp. on \( B_{2r_n}(z_n) \)), where \( \omega_d \) is the volume of the unit ball in \( \mathbb{R}^d \). Similarly, for \( d = \alpha \), \( Lf_n \) is bounded above by \( Lf_n(z_n) = \omega_d \) (resp. by \( \omega_d/n^2 \)) on \( \mathbb{R}^d \) (resp. on \( B_{r_n + (1/8)}(z_n) \)).

Thus for \( d > \alpha \)

\[
\sup_{x \in \mathbb{R}^d} R^{(\alpha)} f(x) = \sup_{x \in \mathbb{R}^d} \sum_{n=1}^{\infty} R^{(\alpha)} f_n(x) \\
\leq \sup_{x \in \mathbb{R}^d} \left( \sum_{n=1}^{\infty} 1_{B_{2r_n}(z_n)}(x) R^{(\alpha)} f_n(x) + \sum_{n=1}^{\infty} 1_{B_{2r_n}(z_n)}(x) R^{(\alpha)} f_n(x) \right) \\
\leq \frac{d}{\alpha} \omega_d \left( 1 + \sum_{n=1}^{\infty} \frac{1}{(2^n - 1)^{d-\alpha}} \right) < \infty,
\]

which implies \((4.1)\) by Lemma 4.3 in [13] with the upper estimate of \( p_t(x, y) \) discussed in Example 2.4 in [3] or Example 5.1 in [13]. Moreover, for all \( r > r_n \)

\[
\int_{|z_n - y| < r} \frac{f(y)}{|z_n - y|^{d-\alpha}} dy \geq \frac{d}{\alpha} \omega_d,
\]

which implies \( f \notin \mathbb{K}_{d, \alpha} \). It is easy to see \( f \in \mathbb{K}_{d, \alpha}^{\text{loc}} \) since \( f_n \in \mathbb{K}_{d, \alpha} \) for every \( n \geq 1 \). Similarly, we have the same conclusion for the case \( d = \alpha \).

For such \( f \in \mathbb{K}_{d, \alpha}^{\text{loc}} \setminus \mathbb{K}_{d, \alpha} \), we set \( \tilde{f} := f / (\| R_1 f \|_{\infty} + \varepsilon) \), where \( \varepsilon > 0 \) and \( R_1 f(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-t} f(X_t) dt \right] \). Then \( A \) is of local Kato class and of extended Kato class. Hence the multiplicative functional \( Z_t := \exp \left( \int_0^t \tilde{f}(X_s) ds \right) \) satisfies conditions \((1.7)\) and \((1.12)\). By the same reasoning as that at the end of Example 4.1, \( Z \) does not satisfy condition \((1.6)\), not to mention condition \((1.14)\).

Hereafter we shall focus on the case \( m = 0, \alpha \in ]0, 2[ \), that is, \( X \) is a symmetric \( \alpha \)-stable process and \((\mathcal{E}, \mathcal{F})\) the corresponding Dirichlet form on \( L^2(\mathbb{R}^d)\). \((\mathcal{E}, \mathcal{F})\) is given by

\[
\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d); \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy < \infty \right\},
\]

\[
\mathcal{E}(u, v) = \frac{A(d, -\alpha)}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy, \ u, v \in \mathcal{F},
\]

where

\[
A(d, \beta) := \frac{|\beta| \Gamma(\beta/2)}{2^{1+\beta} \pi^{d/2} \Gamma(1 + \beta/2)}, \ \beta \in ]-\infty, d[.
\]

\( X \) has a Lévy system \((N, H)\), where \( N(x, dy) := A(d, -\alpha)|x - y|^{-(d+\alpha)} dy \) and \( H_t = t \). We set \( \phi(x, y) := \tilde{f}(x)^{1/2} 1_{|x - y| \leq 1} \). Then for each compact set \( K \), \( \log(1 + \phi) \leq \)
\(\tilde{f}^{1/2}\) is bounded on \(K \times \mathbb{R}^d\) and the function \(N(\phi^2)\) is of local and extended Kato class. For locally square integrable MAF \(M\) with \(\Delta M_t = \phi(X_{t-}, X_t)\) for all \(t \in [0, \infty[\) \(P_x\)-a.s., the multiplicative functional \(Z\) defined by \(Z_t := \text{Exp}(M_t) = \exp(M_t) \prod_{0 \leq s \leq t} (1 + \phi(X_{s-}, X_s)) e^{-\phi(X_{s-}, X_s)}\) satisfies (1.7) and (1.12) in view of Theorem 3.3.

**Example 4.3** (Riemannian manifolds with lower Ricci curvature bounds). Let \((M, g)\) be a \(d\)-dimensional smooth complete but not compact Riemannian manifold with \(\text{Ric}_M \geq (d - 1)\kappa\) for some \(\kappa \in \mathbb{R}\). Since \(M\) is non-compact, \(\kappa \leq 0\) in view of Myers theorem (see [16, Theorem IV.3.1 (3)]).

Let \(m\) be the volume measure induced from the Riemannian metric \(g\) and set \(V(x, r) := m(B_r(x))\). Since \(\text{Ric}_M \geq (d - 1)\kappa\), the Bishop inequality \(V(x, r) \leq V_\kappa(r)\) and the Bishop-Gromov inequality \(V(x, R)/V_\kappa(R) \leq V(x, r)/V_\kappa(r), 0 < r < R\) hold (see [16, §IV.3]). Here \(V_\kappa(r)\) is the volume of the ball with radius \(r\) in the canonical manifold with constant sectional curvature \(\kappa\), which can be computed explicitly as follows.

\[
V_\kappa(r) := d\omega_d \int_0^r S_\kappa(s)^{d-1} ds,
\]

where

\[
S_\kappa(s) = \begin{cases} s, & \text{if } \kappa = 0, \\ \sinh s \sqrt{-\kappa}, & \text{if } \kappa < 0, \end{cases}
\]

where \(\omega_d\) is the volume of the unit ball in \(\mathbb{R}^d\). Consequently, we have the volume doubling condition \(\sup_{x \in M} V(x, 2r)/V(x, r) < \infty\) and \(\int_1^\infty s \, ds / \log V(x, s) = \infty\) which implies the stochastic completeness of the Brownian motion \(X = (\Omega, X_t, P_x)\) on \((M, g)\).

We also have the scale invariant weak Poincaré inequality (depending on \(\kappa\) if \(\kappa < 0\)) (see Saloff-Coste [17] or Theorem 5.6.5 in [18]), which implies the weak form of the weak Poincaré inequality (see Theorem 5.5.1 (i) in [18]). Then the heat kernel \(p_t(x, y)\) of Brownian motion over \((M, g)\) satisfies the following Li-Yau type estimate (see Theorems 5.5.1 and 5.5.3 in [18], cf. Theorems 6.1 and 6.2 in [9]): for each \(T > 0\) there exist \(C_i = C_i(T) > 0, i = 1, 2, 3, 4\) such that for \((t, x, y) \in ]0, T[ \times M \times M\)

\[
\frac{C_3 e^{-C_2 d(x,y)^2/t}}{V(y, \sqrt{t})} \leq p_t(x, y) \leq \frac{C_4 e^{-C_1 d(x,y)^2/t}}{V(y, \sqrt{t})}.
\]

Further we assume that the injectivity radius of \(X\) (write \(\text{inj}_M\)) is positive, that is, \(\text{inj}_M := \inf_{x \in M} d(x, C_x) > 0\), where \(C_x\) is the cut-locus of \(x\). Then we have the following (see the proof of Lemma 5 in [10] and Proposition 14 in [7]. Though the framework of [7] is restricted to compact Riemannian manifolds, the argument in [10] remains valid): There exists \(C_d \in ]0, \infty[\) such that for any \(r \in ]0, \text{inj}_M/2[\) and \(x \in M\),

\[
V(x, r) \geq C_d r^d.
\]
Hence we have that there exist $C_1, C_2, C_3, C_4 > 0$ such that for any $t \in [0, (\text{inj}_M/2)^2]$, $x, y \in M$

$$
\frac{C_3 e^{-C_2 d(x, y)^2/t}}{t^{d/2}} \leq \rho_t(x, y) \leq \frac{C_4 e^{-C_1 d(x, y)^2/t}}{t^{d/2}}.
$$

A signed Borel measure $\mu$ is said to be of Kato class (write $\mu \in K_d$) if and only if

$$
\limsup_{r \to 0} \sup_{x \in M} \int_{d(x, y) < r} \frac{|\mu|(dy)}{d(x, y)^{d-2}} = 0 \quad \text{when} \quad d \geq 3,
$$

$$
\limsup_{r \to 0} \sup_{x \in M} \int_{d(x, y) < r} \left(\log d(x, y)^{-1}\right) |\mu|(dy) = 0 \quad \text{when} \quad d = 2,
$$

$$
\sup_{x \in M} \int_{d(x, y) \leq 1} |\mu|(dy) < \infty \quad \text{when} \quad d = 1.
$$

The family of measures of local Kato class is similarly defined and will be denoted as $K_d^{loc}$. A function $f$ on $M$ is said to be of Kato class (write $f \in K_d$ in short) if the measure $|f(x)| m(dx)$ is of the Kato class. Similarly, we write $f \in K_d^{loc}$ if the measure $|f(x)| m(dx)$ is so. By [13], under the above estimate, we know that $f \in K_d$ if and only if $\lim_{r \to 0} \sup_{x \in M} E_x \left[ \int_0^1 |f(X_s)| \, ds \right] = 0$. For any Borel measure $\mu$, we set $R_\mu(x) := \int_M \mu(dy)/d(x, y)^{d-2}$ the Newtonian potential of $\mu$ if $d \geq 3$, and $L_\mu(x) := \int_M \log d(x, y)^{-1} 1_{d(x, y) < 1} \mu(dy)$ the modified logarithmic potential of $\mu$ if $d = 2$. If $\mu(dx) = g(x) \, dx$ for some non-negative function $g$, we write $R\mu(x)$ (resp. $L\mu(x)$) instead of $R_\mu(x)$ (resp. $L_\mu(x)$).

We utilize the following estimate under $\text{Ric}_M \geq (d - 1)\kappa$:

**Lemma 4.4.** For non-negative measurable function $f$ on $[0, \infty[$,

$$
\int_{B_r(p)} f(d(p, x)) \, m(dx) \leq (1 + \theta_{d, \kappa}(r)) \int_{|y| \leq \kappa : |y| < r} f(|z|) \, dz
$$

holds for all $p \in M$, where $\theta_{d, \kappa}(r)$ is a function independent of $p$, but depending on $d, \kappa$ such that $\lim_{r \to 0} \theta_{d, \kappa}(r) = 0$.

**Proof.** By Lemma II.5.4 (1) in [16], we have for any non-negative measurable function $g$ on $M$,

$$
\int_M g(x) \, m(dx) = \int_{[0, \infty[ \times \mathbb{S}^{d-1}} g(\exp_p tu) \tilde{\theta}(t, u) \, dt \, \sigma(du),
$$

where $\tilde{\theta}(t, u) := \theta(t, u)$ if $t < t(u)$ and $:= 0$ otherwise, and $\theta(t, u) := t^{d-1} \times \sqrt{\det(g_{ij}(\exp_p tu))_{ij}}$. Here $\exp_p : T_p M \to M$ is the exponential map, $\sigma$ is the surface measure on $\mathbb{S}^{d-1}$ and $t(u)$ is the distance from $p$ to the cut-point of $p$ along $\gamma_u$ (see
Since $\text{Ric}_M \geq (d - 1)\kappa$, it holds that $\bar{\theta}(t, u) \leq S_n(t)^{d-1}$ for all $t > 0$ and $u \in S^{d-1}$ (see Chapter IV Theorem 3.1 (2) (b) in [16]). Hence the right hand of (4.4) is estimated above by

$$\sup_{|t| \leq r} \left( \frac{S_n(s)}{s} \right)^{d-1} \left( \int_0^r f(t)^{d-1} \, dt \right) \, d\omega_d,$$

which implies (4.2).

Fix a point $o \in M$. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in $M$ such that $d(z_{n+1}, o) = 2d(z_n, o)$, $n \in \mathbb{N}$, $d(z_1, o) > 2$. We define $f_n(x) := g(r_n)1_{B_{r_n}(z_n)}(x)$ and $f(x) := \sum_{n=1}^{\infty} f_n(x)$, where

$$r_n := 8^{-n} \text{ and } g(t) := t^{-2} \text{ when } d \geq 3$$

and

$$r_n := 8^{-n^2} \text{ and } g(t) := \frac{t^{-2}}{(1/2) + \log t^{-1}} \text{ when } d = 2.$$

$Rf_n$ is bounded above by $\int_{d(x,y) < 3r_n} m(dy)/d(x, y)^{d-2}$ on $B_{2r_n}(z_n)$, and bounded above by $m(B_{r_n}(z_n))/r_n^{d-2}$ on $B_{2r_n}(z_n)^c$. Let $C := \sup_{n \in \mathbb{N}} (1 + \theta_{d, \kappa}(3r_n))$. From Lemma 4.4, we see that for $d \geq 3$, $Rf_n$ is bounded above by $9C\omega_d d$ (resp. by $(C^d/2^{n-1})\omega_d d$) on $M$ (resp. on $B_{2r_n}(z_n)^c$). Similarly, for $d = 2$, $Lf_n$ is bounded above by $9C\omega_d d$ (resp. by $C^d\omega_d d/2$) on $M$ (resp. on $B_{r_n+1/3}(z_n)^c$). Moreover, $Rf_n(z_n) \geq C_d$ and $Lf_n(z_n) \geq C_d/2$ for $r_n < \text{inj}_M/2$.

As in the previous examples, we see $f \in K^\text{loc}_d \setminus K_d$ and (4.1) holds. So the multiplicative functional $Z_t := \exp(\int_0^t \tilde{f}(X_s) \, ds) \text{ defined by } \tilde{f} := f/\|R_1 f\|_{\infty} + \varepsilon)$ satisfies (1.7) and (1.12). By the same reasoning as that at the end of Example 4.1, $Z$ does not satisfy condition (1.6), not to mention condition (1.14).

References


Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle, WA 98195
USA
e-mail: zchen@math.washington.edu

Kazuhiro Kuwae
Department of Mathematics and Engineering
Graduate School of Science and Technology
Kumamoto University
Kumamoto 860–8555
Japan
e-mail: kuwae@gpo.kumamoto-u.ac.jp