Chen, Z.-Q. and Kuwae, K.
Osaka J. Math.
46 (2009), 909–930

ON DOUBLY FELLER PROPERTY

ZHEN-QING CHEN* and KAZUHIRO KUWAE†

(Received May 26, 2008)

Abstract

Let X be a Feller process that has strong Feller property. In this paper, we investigate the Feller as well as strong Feller properties of the semigroups generated by multiplicative functionals of X in open sets. Special attention is given to the Feynman-Kac and Girsanov transforms of X. Three examples of local Kato class measure that are not of Kato class are given in the last section so that Feller and strong Feller properties hold for corresponding Feynman-Kac semigroup of X in open sets.

1. Doubly Feller property of transformed semigroup

Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be a strong Markov process on a locally compact separable metric space E. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ also serves as the cemetery point for X. Recall that X is said to have *Feller property* if $P_t(C_\infty(E)) \subset C_\infty(E)$ for every $t > 0$ and $\lim_{t\to\infty} \|P_t f - f\|_\infty = 0$ for every $f \in C_\infty(E)$, where $\{P_t;\, t \geq 0\}$ defined by $P_t f(x) := \mathbf{E}_x[f(X_t)]$ is the semigroup of X. Here $C_\infty(E)$ is the space of continuous functions on E that vanishes at infinity and $\|f\|_\infty := \sup_{x\in E} |f(x)|$. The space of bounded continuous functions on E will be denoted as $C_b(E)$. The process X is said to have *strong Feller property* if $P_t(\mathcal{B}_b(E)) \subset C_b(E)$ for every $t > 0$. We say X (or its transition semigroup) has *doubly Feller property* if it has both Feller and strong Feller property. Clearly the above terminology can be formulated for any semigroup $\{T_t;\, t \geq 0\}$ acting on $\mathcal{B}_b(E)$.

Let $\{Z_t;\, t \geq 0\}$ be a positive multiplicative functional of X. It defines a semigroup

$$T_t f(x) := \mathbf{E}_x[Z_t f(X_t)] \quad \text{for} \quad t > 0 \quad \text{and} \quad f \geq 0. \tag{1.1}$$

For an open subset B of E, we also define a semigroup T_t^B by

$$T_t^B f(x) := \mathbf{E}_x[Z_t f(X_t) \colon t < \tau_B] \quad \text{for} \quad t > 0 \quad \text{and} \quad f \geq 0 \quad \text{on} \quad B, \tag{1.2}$$

2000 Mathematics Subject Classification. Primary 31C25; Secondary 60J57, 60J55, 60J25.

*The research of this author is supported in part by NSF Grant DMS-0600206.

†The research of this author is partially supported by a Grant-in-Aid for Scientific Research (C) No. 19540220 from Japan Society for the Promotion of Science.

where $\tau_B := \inf\{t > 0;\ X_t \notin B\}$ is the first exit time from B. Let $\mathcal{B}^*(E)$ be the σ-field of universally measurable subsets of E and denote by $\mathcal{B}^*_b(E)$ the family of bounded universally measurable functions on E. Note that $T_t f \in \mathcal{B}^*(E)$ when f is Borel measurable ($T_t f$ is Borel for Borel function f if Z_t is $\mathcal{F}^0_t$-measurable). An open set B ($\subset E$) is said to be *regular* if $\mathbf{P}_x(\tau_B = 0) = 1$ for any $x \in B^c = E \setminus B$. Consider the following conditions:

$$\lim_{t\to 0}\sup_{x\in K}\mathbf{E}_x[|Z_t - 1|] = 0 \quad \text{for every compact set} \quad K, \tag{1.3}$$

$$\sup_{s\in[0,t]}\sup_{x\in E}\mathbf{E}_x[Z_s] < \infty \quad \text{for some (and hence for every)} \quad t > 0, \tag{1.4}$$

and for each $t > 0$, there exists $p > 1$ (which may depend on t) such that

$$\sup_{x\in E}\mathbf{E}_x[Z_t^p] < \infty. \tag{1.5}$$

The following theorem is due to K.L. Chung [6].

Theorem 1.1 (Theorems 1, 2, 3 and Corollary in [6]). *Let B be an open subset of E and suppose that the part process X^B on B is a strong Feller process and the conditions* (1.3) *and* (1.4) *hold. Then the semigroup $\{T_t^B;\ t \geq 0\}$ defined by* (1.2) *has strong Feller property. If in addition, X has Feller property* (*that is, X is doubly Feller*)*, B is a regular set and* (1.5) *holds for every $t > 0$, then $\{T_t^B;\ t \geq 0\}$ is doubly Feller. Moreover, if in addition B is relatively compact, then $T_t^B g \in C_\infty(B)$ for every $t > 0$ and $g \in \mathcal{B}_b(B)$.*

Corollary 1.2. *Let B be an open subset of E and suppose that X^B is a strong Feller process. Assume that*

$$\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x[|Z_t - 1|] = 0. \tag{1.6}$$

Then $\{T_t^B;\ t \geq 0\}$ has strong Feller property. If further X has a Feller property, B is regular and (1.5) *holds for every $t > 0$, then $\{T_t^B;\ t \geq 0\}$ has doubly Feller property. Moreover, if in addition B is relatively compact, then $T_t^B g \in C_\infty(B)$ for every $t > 0$ and $g \in \mathcal{B}_b(B)$.*

REMARK 1.3. In the conclusion of Theorem 2 in [6], only doubly Feller property of $\{T_t;\ t \geq 0\}$ is stated. But the proof actually gives the strong Feller property of $\{T_t;\ t \geq 0\}$ under (1.3), (1.4) and the strong Feller property of $\{P_t;\ t \geq 0\}$.

We shall give another criteria for the Feller properties. First we relax the conditions

(1.3), (1.4) and (1.5) into the following: Fix an open set B.

(1.7)
$$\lim_{t\to 0}\sup_{x\in D}\mathbf{E}_x[|Z_t-1|:t<\tau_D]=0 \quad \text{for any relatively compact open set} \quad D\subset B,$$

$$\sup_{s\in[0,t]}\sup_{x\in B}\mathbf{E}_x[Z_s:s<\tau_B]<\infty \quad \text{for some (and hence for every)} \quad t>0, \tag{1.8}$$

and for each $t>0$, there exists $p>1$ (which may depend on t) such that

$$\sup_{x\in B}\mathbf{E}_x[Z_t^p:t<\tau_B]<\infty. \tag{1.9}$$

Theorem 1.4. *Let X be a doubly Feller process and B an open set in E. Suppose that* (1.7) *holds and that*

(1.10) *for each $t>0$ and compact set $K\subset B$, there exists $p>1$ such that* $\sup_{x\in K}\mathbf{E}_x[Z_t^p:t<\tau_B]<\infty.$

Then the semigroup $\{T_t^B;\, t\geq 0\}$ defined by (1.2) *has strong Feller property. Assume further that* (1.8) *and* (1.9) *hold for every $t>0$, B is regular and that* $\lim_{t\to 0}\mathbf{E}_x[|Z_t-1|:t<\tau_B]=0$ *for every $x\in B$. Then the semigroup $\{T_t^B;\, t\geq 0\}$ has Feller property.*

Proof. (i) *Strong Feller property of* $\{T_t^B;\, t\geq 0\}$: Let D be a relatively compact open subset of B. The strong Feller property of the part process X^D on D holds under the doubly Feller property of X (see Theorem 1 in Chung [6]). Applying Corollary 1.2 and (1.7) to X^D, the semigroup $\{T_t^D;\, t\geq 0\}$ defined by $T_t^D f(x):=\mathbf{E}_x[Z_t f(X_t):t<\tau_D]$, $f\in\mathcal{B}_b(D)$ has strong Feller property. Take $g\in\mathcal{B}_b(E)$. Let $\{D_n\}$ be an increasing sequence of relatively compact open sets converging to B. Then the quasi-left-continuity of X yields that $\mathbf{P}_x(\lim_{n\to\infty}\tau_{D_n}=\tau_B)=1$ for all $x\in B$. Take a compact set $K\subset B$. Then there is $n_0\geq 1$ so that $K\subset D_n$ for every $n\geq n_0$ and

$$\begin{aligned}
&\sup_{x\in K}|T_t^B g(x)-T_t^{D_n}g(x)|\\
&=\sup_{x\in K}\mathbf{E}_x[Z_t g(X_t):\tau_{D_n}\leq t<\tau_B]\\
&\leq\|g\|_{B,\infty}\left(\sup_{x\in K}\mathbf{E}_x[Z_t^p:t<\tau_B]\right)^{1/p}\left(\sup_{x\in K}\mathbf{P}_x(\tau_{D_n}\leq t<\tau_B)\right)^{1/q},
\end{aligned}$$

where $\|g\|_{B,\infty}:=\sup_{x\in B}|g(x)|$ and $q:=p/(p-1)$ is the conjugate exponent of p, which may depend on K and $t>0$.

To establish $T_t g\in C(K)$, it suffices to show

$$\lim_{n\to\infty}\sup_{x\in K}\mathbf{P}_x(\tau_{D_n}\leq t<\tau_B)=0.$$

Note that $\mathbf{P}_x(\tau_{D_n} \le t < \tau_B) = P_t^B 1(x) - P_t^{D_n} 1(x)$ is continuous in $x \in D_n$. So $\sup_{x\in K} \mathbf{P}_x(\tau_{D_n} \le t < \tau_B) = \mathbf{P}_{x_n}(\tau_{D_n} \le t < \tau_B)$ for some $x_n \in K$. There exists a subsequence (still denoted as n) and $x \in K$ such that $x_n \to x$. For $n_0 \le m < n$,

$$\mathbf{P}_{x_n}(\tau_{D_n} \le t < \tau_B) \le \mathbf{P}_{x_n}(\tau_{D_m} \le t < \tau_B).$$

By fixing $m \ge n_0$ first and taking $n \to \infty$, we have

$$\overline{\lim_{n\to\infty}} \mathbf{P}_{x_n}(\tau_{D_n} \le t < \tau_B) \le \mathbf{P}_x(\tau_{D_m} \le t < \tau_B).$$

By sending $m \to \infty$, we conclude that

$$\lim_{n\to\infty} \sup_{x\in K} \mathbf{P}_x(\tau_{D_n} \le t < \tau_B) = \lim_{n\to\infty} \mathbf{P}_{x_n}(\tau_{D_n} \le t < \tau_B) = 0.$$

This proves that $\{T_t; t \ge 0\}$ is strong Feller.

(ii) *Feller property of* $\{T_t^B; t \ge 0\}$: Since (1.9) holds for each $t > 0$ with some $p > 1$, we easily see $T_t^B f \in C_\infty(B)$ for $f \in C_\infty(B)$. Finally we prove that for each $f \in C_\infty(B)$

$$\lim_{t\to 0} \| T_t^B f - f \|_{B,\infty} = 0. \tag{1.11}$$

For $f \in C_\infty(B)$, under the conditions, we can easily see that for each $x \in B$

$$\lim_{t\to 0} T_t^B f(x) = f(x).$$

Then we can deduce (1.11) by using (1.8) Riesz-Markov-Kakutani theorem and Hahn-Banach theorem to the dual space of $C_\infty(B)$ (see Exercise (9.27) in [19]). $\square$

Corollary 1.5. *Let X be a doubly Feller process and assume* (1.7) *holds. Let B an open regular set. Suppose that*

$$\begin{aligned} &\textit{there exists } p > 1 \textit{ such that} \\ &\sup_{s\in[0,t]} \sup_{x\in B} \mathbf{E}_x[Z_s^p : s < \tau_B] < \infty \quad \textit{for some (hence every)} \quad t > 0. \end{aligned} \tag{1.12}$$

Then $\{T_t^B; t \ge 0\}$ has doubly Feller property. If in addition, B is relatively compact, then $T_t^B g \in C_\infty(B)$ for every $t > 0$ and $g \in \mathcal{B}_b(B)$.

Proof. (1.12) is stronger than (1.10). Hence the strong Feller property holds for $\{T_t^B; t \ge 0\}$ by Theorem 1.4. (1.12) also implies (1.4) and (1.5). From (1.5), we have that T_t^B maps $C_\infty(B)$ into itself. As in the proof of Theorem 1.4, we shall show

$\mathbf{E}_x[|Z_t - 1|: t < \tau_B] \to 0$ as $t \to 0$ for each $x \in B$. By condition (1.7), for a relatively compact open set $D \subset B$, it suffices to show that for each $x \in D$

$$\lim_{t\to 0} \mathbf{E}_x[|Z_t - 1|: \tau_D \le t < \tau_B] = 0.$$

We see that for each $x \in D$

$$\begin{aligned}
&\mathbf{E}_x[|Z_t - 1|: \tau_D \le t < \tau_B] \\
&= (\mathbf{E}_x[|Z_t - 1|^p: t < \tau_B])^{1/p}\mathbf{P}_x(\tau_D \le t)^{1/q} \\
&\le \mathbf{P}_x(\tau_D \le t)^{1/q} \sup_{s\in[0,T]} \sup_{y\in B}(\mathbf{E}_y[|Z_s - 1|^p: s < \tau_B])^{1/p} \to 0 \quad \text{as} \quad t \to 0.
\end{aligned}$$

The proof of $T_t^B g \in C_\infty(B)$ for $g \in \mathcal{B}_b(B)$ is similar to that of Corollary of [6]. $\square$

REMARK 1.6. (i) Without assuming the regularity of B, we can not deduce the Feller property of the semigroup $P_t^B f(x) := \mathbf{E}_x[f(X_t): t < \tau_B]$.
(ii) (1.3) (resp. (1.5)) implies (1.7) (resp. (1.10)). In view of the latter part of the proof of Corollary 1.5 and Lemma 2 in [6], (1.7) and (1.12) for an open set B together imply (1.13) below

$$\lim_{t\to 0} \sup_{x\in K} \mathbf{E}_x[|Z_t - 1|: t < \tau_B] = 0 \quad \text{for every compact set} \quad K \subset B, \tag{1.13}$$

under the Feller property of $\{P_t; t \ge 0\}$.
(iii) A sufficient condition for (1.3), (1.4), (1.5), (1.6), (1.7), (1.10) and (1.12) to hold is the following

$$\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x\left[\sup_{s\in[0,t]} (Z_s - 1)^2\right] = 0. \tag{1.14}$$

The above condition (1.14) is satisfied if Z_t is a combination of Girsanov and Feynman-Kac transform under appropriate Kato class condition (see, e.g. [4], [5], and Lemmas 2.4 and 3.2 below).
(iv) The condition (1.7) is satisfied when the multiplicative functional Z satisfies certain local Kato class condition and the condition (1.12) holds when Z is in certain extended Kato class. We will explain these points precisely in next two sections. We will show in the last section that there are examples of positive multiplicative functional Z that satisfies conditions (1.7) and (1.12) but not condition (1.6) nor condition (1.14).

2. Feynman-Kac transform

Let X be a strong Markov process on E as in the previous section and ω_∂ denote the sample path such that $\omega_\partial(t) = \partial$ for every $t \ge 0$.

DEFINITION 2.1 (AF). An $(\mathcal{F}_t)$-adapted process $A = (A_t)_{t\geq 0}$ with values in $[-\infty, \infty]$ is said to be an *additive functional* (AF in short) if there exist a *defining set* $\Xi \in \mathcal{F}_\infty$ satisfying the following conditions;
(i) $\mathbf{P}_x(\Xi) = 1$ for all $x \in E$,
(ii) $\theta_t \Xi \subset \Xi$ for all $t \geq 0$; in particular, $\omega_\partial \in \Xi$ because of $\omega_\partial = \theta_{\zeta(\omega)}(\omega)$ for all $\omega \in \Xi$,
(iii) for all $\omega \in \Xi$, $A_\cdot(\omega)$ is right continuous having left hand limits on $[0, \zeta(\omega)[$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$ and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for all $t, s \geq 0$,
(iv) for all $t \geq 0$, $A_t(\omega_\partial) = 0$; in particular, under the additivity in (iii), $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for all $t \geq \zeta(\omega)$ and $\omega \in \Xi \cap \{\zeta < \infty\}$.
An AF A is called *right-continuous with left limits* (rcll AF in brief) if $A_{\zeta(\omega)-}$ exists for each $\omega \in \Xi \cap \{\zeta < \infty\}$. An AF A is said to be *finite* (resp. *continuous additive functional* (CAF in brief)) if $|A_t(\omega)| < \infty$, $t \in [0, \infty[$ (resp. $t \mapsto A_t(\omega)$ is continuous on $[0, \infty[$) for each $\omega \in \Xi$. A $[0, \infty[$-valued AF is called a *positive additive functional* (PAF in short). Two AFs A and B are called *equivalent* if there exists a common defining set $\Xi \in \mathcal{F}_\infty$ such that $A_t(\omega) = B_t(\omega)$ for all $t \in [0, \infty[$ and $\omega \in \Xi$.

We now introduce the PAFs of Dynkin and Kato classes.

DEFINITION 2.2 (Dynkin class). A PAF A is said to be *of Dynkin class* if $\sup_{x\in E} \mathbf{E}_x[A_t] < \infty$ for some $t > 0$, or equivalently, $\sup_{x\in E} \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t}\, dA_t\right] < \infty$ for some $\alpha > 0$. By the Markov property of X, a PAF A is of Dynkin class if and only $\sup_{x\in E} \mathbf{E}_x[A_t] < \infty$ for all $t > 0$.

DEFINITION 2.3 (Kato class). A PAF A is said to be *of Kato class* (resp. *of extended Kato class* or *generalized Kato class*) if $\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x[A_t] = 0$ (resp. $\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x[A_t] < 1$); or equivalently, $\lim_{\alpha\to\infty} \sup_{x\in E} \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t}\, dA_t\right] = 0$ (resp. $\lim_{\alpha\to\infty} \sup_{x\in E} \mathbf{E}_x\left[\int_0^\infty e^{-\alpha t}\, dA_t\right] < 1$). A PAF A is said to be *of local Kato class* if for each compact set K, a PAF $\mathbf{1}_K A$ defined by $(\mathbf{1}_K A)_t = \int_0^t \mathbf{1}_K(X_{s-})\, dA_s$ is of Kato class.

For a PAF A, let $\mathrm{Exp}(A)_t$ be the Stieltjes exponential of A, that is, $\mathrm{Exp}(A)_t$ is the unique solution of Z_t of

$$Z_t = 1 + \int_{]0,t]} Z_{s-}\, dA_s.$$

By (A4.17) of Sharpe [19],

$$\mathrm{Exp}(A)_t = \exp(A_t^c) \prod_{0<s\leq t} (1 + \Delta A_s) = \exp\left(A_t^c + \sum_{0<s\leq t} \log(1 + \Delta A_s)\right), \tag{2.1}$$

where $\Delta A_s := A_s - A_{s-}$ and A_t^c denotes the continuous part of A_t. Since $\log(1+x) \leq x$, $x \in]0, \infty[$, we see $\mathrm{Exp}(A)_t \leq \exp A_t$. From (2.1), we have for $p \geq 1$

$$(\mathrm{Exp}(A)_t)^p = \mathrm{Exp}(A^{(p)})_t, \tag{2.2}$$

where $A_t^{(p)} := pA_t^c + \sum_{0<s\leq t}((1+\Delta A_s)^p - 1)$.

Lemma 2.4. *Let A be a PAF of Kato class and put $Z_t := \mathrm{Exp}(A)_t$. Then* (1.6) *holds for Z. Moreover*
(i) *if the AF B defined by $B_t := \sum_{0<s\leq t}(\Delta A_s)^2$ is of Dynkin class, then* (1.5) *holds for some* (*hence every*) *$t > 0$ and some $p > 1$,*
(ii) *if $|\Delta A| \leq M$ $\mathbf{P}_x$-a.s. for some $M > 0$, then* (1.5) *holds for some* (*hence every*) *$t > 0$ and any $p > 1$,*
(iii) *if B is of Kato class* (*resp. $|\Delta A| \leq M$ $\mathbf{P}_x$-a.s. for some $M > 0$*), *then for any $p \in [1, 2]$* (*resp. for any $p \in [1, \infty[$*)

$$\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x\left[\sup_{s\in[0,t]} |Z_s - 1|^p\right] = 0.$$

Proof. In view of the proof of Khas'minskii's lemma (see Lemma 2.1 (a) in [20]), we have

$$\sup_{x\in E} \mathbf{E}_x[\mathrm{Exp}(A)_t] \leq \frac{1}{1 - \sup_{x\in E} \mathbf{E}_x[A_t]}$$

for sufficiently small $t > 0$ with $\sup_{x\in E} \mathbf{E}_x[A_t] < 1$. From this,

$$\sup_{x\in E} \mathbf{E}_x[|Z_t - 1|] \leq \frac{\sup_{x\in E} \mathbf{E}_x[A_t]}{1 - \sup_{x\in E} \mathbf{E}_x[A_t]},$$

which converges to 0 as $t \to 0$. Next we prove the second statement. We first assume that B is of Dynkin class and $p \in]1, 2]$. Letting p be close to 1, we have $\sup_{x\in E} \mathbf{E}_x[pA_t + (p-1)B_t]] < 1$ for sufficiently small $t > 0$. Since $(1+x)^{p-1} - 1 \leq (p-1)x$ for $x > -1$, we get $(1+x)^p - 1 \leq (p-1)x^2 + px$ for $x > -1$. Then we have

$$\begin{aligned}\sup_{x\in E} \mathbf{E}_x[Z_t^p] &= \sup_{x\in E} \mathbf{E}_x\left[\mathrm{Exp}\left(pA^c + \sum_{0<s\leq\cdot}(1+\Delta A_s)^p - 1\right)_t\right] \\ &\leq \frac{1}{1 - \sup_{x\in E} \mathbf{E}_x[pA_t + (p-1)B_t]} < \infty\end{aligned}$$

for such $p > 1$ and $t > 0$ by way of the argument as above. Next we assume that ΔA is bounded above by $M > 0$ $\mathbf{P}_x$-a.s. Then the PAF $A^{(p)}$ is of Kato class for any $p > 1$.

Indeed, set $n := [p] + 1 \in \mathbb{N}$. Then we see

$$A_t^{(p)} \leq nA_t^c + \frac{(1+M)^n - 1}{M} A_t^d.$$

Hence

$$\begin{aligned}\sup_{x\in E} \mathbf{E}_x[Z_t^p] &= \sup_{x\in E} \mathbf{E}_x[\mathrm{Exp}(A^{(p)})_t] \\ &\leq \frac{1}{1 - \sup_{x\in E} \mathbf{E}_x[A_t^{(p)}]}.\end{aligned}$$

Finally we show the last statement. Assume that B is of Kato and $p \in [1, 2]$. Using $(x-1)^p \leq x^p - 1$ for $x \geq 1$, we have

$$\begin{aligned}\sup_{x\in E} \mathbf{E}_x\left[\sup_{s\in[0,t]} |\mathrm{Exp}(A)_s - 1|^p\right] &\leq \sup_{x\in E} \mathbf{E}_x\left[\mathrm{Exp}\left(pA^c + \sum_{0<s\leq \cdot} (1+\Delta A_s)^p - 1\right)_t - 1\right] \\ &\leq \frac{\sup_{x\in E} \mathbf{E}_x[pA_t + (p-1)B_t]}{1 - \sup_{x\in E} \mathbf{E}_x[pA_t + (p-1)B_t]} \to 0 \quad \text{as} \quad t \to 0.\end{aligned}$$

The proof for the case of ΔA being bounded is similar. □

Theorem 2.5. *Let A be a PAF of local and extended Kato class. Suppose that a PAF B defined by $B_t := \sum_{0<s\leq t} (\Delta A_s)^2$ is of Dynkin class. Put $Z_t := \mathrm{Exp}(A)_t$. Let B an open regular set. Then* (1.7) *and* (1.12) *hold for Z and B. Consequently, the semigroup $\{T_t^B;\, t \geq 0\}$ defined by* (1.2) *has doubly Feller property provided X is doubly Feller.*

Proof. First we show the condition (1.7) follows from the local Kato property of A. This is because for any compact set K,

$$\begin{aligned}\mathbf{E}_x[|\mathrm{Exp}(A)_t - 1| : t < \tau_K] &= \mathbf{E}_x[\mathrm{Exp}(A)_t - 1 : t < \tau_K] \\ &= \mathbf{E}_x[\mathrm{Exp}(\mathbf{1}_K A)_t - 1 : t < \tau_K] \\ &\leq \mathbf{E}_x[\mathrm{Exp}(\mathbf{1}_K A)_t - 1],\end{aligned}$$

which converges to 0 uniformly on E by Lemma 2.4. Here we used the fact that on $\{t < \tau_K\}$, $X_{s-} \in K$ for all $s \in]0, t]$.

For (1.12), it suffices to show that there exists $p > 1$ such that

$$pA_t^c + \sum_{0<s\leq t} ((1+\Delta A_s)^p - 1)$$

is of extended Kato. For $p \in]1, 2]$ and $x > -1$, recall $(1+x)^p - 1 \leq (p-1)x^2 + px$. Since A is of extended Kato class, $\lambda := \sup_{x\in E} \mathbf{E}_x[A_T] < 1$ for some $T > 0$. For such

$T > 0$, we set $l := \sup_{x\in E} \mathbf{E}_x\left[\sum_{0<s\le T}(\Delta A_s)^2\right] < \infty$ and take $p \in]1, 2\wedge(1+l)/(\lambda+l)[$. Then

$$\sup_{x\in E} \mathbf{E}_x\left[pA_T^c + \sum_{0<s\le T}((1+\Delta A_s)^p - 1)\right] \le (p-1)l + p\lambda < 1.$$

Hence (1.12) holds by way of Khas'minskii's lemma (see Lemma 2.1 (a) in [20]). □

3. Girsanov transform

In this section, we assume that X is an m-symmetric doubly Feller process, where m is a positive Radon measure on E with full support and that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X is regular on $L^2(E; m)$. In this case, the transition kernel $\{p_t;\, t > 0\}$ of X satisfies the absolute continuity condition with respect to m, i.e. $p_t(x, N) = 0$ if $m(N) = 0$ for each $N \in \mathcal{B}(E)$, $x \in E$ and $t > 0$. For $\alpha > 0$, there exists a α-order resolvent kernel $r_\alpha(x, y)$ which is defined for all $x, y \in E$ (see Lemma 2.14 in [8]). A Borel measure ν is said to be of *Dynkin class* (resp. *Kato class*) if $\sup_{x\in E} R_\alpha\nu(x) < \infty$ for some $\alpha > 0$ (resp. $\lim_{\alpha\to\infty}\sup_{x\in E} R_\alpha\nu(x) = 0$), and ν is in the *local Kato class* if $\mathbf{1}_K\nu$ is in the Kato class for every compact set $K \subset E$. The measure is said to be of *extended Kato class* if $\lim_{\alpha\to\infty}\sup_{x\in E} R_\alpha\nu(x) < 1$. Here $R_\alpha\nu(x) := \int_E r_\alpha(x, y)\,\nu(dy)$. Since X is a Feller process, its Lévy system (N, H) exists and is defined under $\mathbf{P}_x$ for every $x \in E$. Denote by S_1 (resp. S_{00}) the family of smooth measures in the strict sense (resp. measures of finite energy integrals with bounded potentials) (see (2.2.10) and p. 195 in [8]). Note that any Radon measure of Dynkin class always belongs to S_1 in view of Proposition 3.1 in [12].

Let $\phi\colon E_\partial \times E_\partial \to \mathbb{R}$ be a Borel function that vanishes along the diagonal. The following lemma is a slightly modified version of [2, Lemma 3.2].

Lemma 3.1. *Assume $N(|\phi| \wedge |\phi|^2)\mu_H \in S_1$. Then there exists a local martingale additive functional M of purely discontinuous type such that $M_t - M_{t-} = \phi(X_{t-}, X_t)$ for all $t \in]0, \infty[$ $\mathbf{P}_x$-a.s. Moreover, if $N(\phi^2)\mu_H \in S_1$, then such M is locally square integrable.*

Proof. Let $M^{(2)}$ be the AF defined by

$$M_t^{(2)} := \sum_{s\le t}(\mathbf{1}_{\{|\phi|>1\}}\phi)(X_{s-}, X_s) - \int_0^t N(\mathbf{1}_{\{|\phi|>1\}}\phi)(X_s)\,dH_s.$$

Then $M^{(2)}$ is a local MAF in the strict sense ($M^{(2)}$ is locally square integrable provided $N(\phi^2)\mu_H \in S_1$). For $n \ge 2$, define AF M^n by

$$M_t^n := \sum_{s\le t}(\mathbf{1}_{\{1/n<|\phi|\le 1\}}\phi)(X_{s-}, X_s) - \int_0^t N(\mathbf{1}_{\{1/n<|\phi|\le 1\}}\phi)(X_s)\,dH_s,$$

which is a locally square integrable MAF in the strict sense. For $n > m > 1$

$$[M_t^m - M^n]_t = \sum_{s \le t} (\mathbf{1}_{\{1/m < |\phi| \le 1/n\}} \phi^2)(X_{s-}, X_s),$$

and so

$$\langle M_t^m - M^n \rangle_t = \int_0^t N(\mathbf{1}_{\{1/m < |\phi| \le 1/n\}} \phi^2)(X_s)\, dH_s.$$

Therefore the limit

$$M_t^{(1)} := \lim_{n \to \infty} M_t^n$$

exists and defines a locally square integrable MAF in the strict sense of purely discontinuous type. Therefore, $M_t := M_t^{(1)} + M_t^{(2)}$ is the desired local MAF of purely discontinuous type. □

Hereafter we fix a continuous locally square integrable MAF M^c and a Borel function $\phi \colon E_\partial \times E_\partial \to \mathbb{R}$ with $\phi(x, y) > -1$ for all $x, y \in E_\partial$ and $\phi(x, x) = 0$ for $x \in E_\partial$. We use $\mu_{\langle M^c \rangle}$ to denote the Revuz measure of $\langle M^c \rangle$.

Lemma 3.2. *Suppose that* $N(\phi - \log(1 + \phi))\mu_H \in S_1$ *and assume that* $\nu := N(\phi^2)\mu_H + (1/2)\mu_{\langle M^c \rangle}$ *is a Radon measure of extended Kato class.*
(i) *There exists a locally square integrable MAF* M^d *of purely discontinuous type such that* $\Delta M_t^d = \phi(X_{t-}, X_t)$ $t \in]0, \infty[$ $\mathbf{P}_x$*-a.s.*
(ii) *There exist* $t > 0$ *and* $p > 1$ *such that*

$$\sup_{x \in E} \mathbf{E}_x \left[\sup_{s \in [0,t]} Z_s^p \right] < \infty, \tag{3.1}$$

where $Z_t := \mathrm{Exp}(M)_t$ *is the solution of Doléan-Dade equation*

$$Z_t = 1 + \int_{]0,t]} Z_{s-}\, dM_s$$

for $M_t := M_t^c + M_t^d$*. In particular,* Z_t *is a martingale.*
(iii) *If* $\log(1 + \phi)$ *is bounded and* ν *is of Kato, then* (3.1) *holds for any* $t > 0$ *and* $p \ge 1$*. Moreover, for any* $p \ge 1$*, we have*

$$\lim_{t \to 0} \sup_{x \in E} \mathbf{E}_x \left[\sup_{s \in [0,t]} |Z_s - 1|^p \right] = 0 \tag{3.2}$$

and $\tilde{Z}_t := \exp(M_t)$ *satisfies that for any* $p \geq 1$

$$\lim_{t\to 0} \sup_{x\in E} \mathbf{E}_x \left[\sup_{s\in[0,t]} |\tilde{Z}_s - 1|^p \right] = 0.$$

Proof. (i): Under the conditions, the measure $N(\phi^2)\mu_H$, hence $N(|\phi| \wedge |\phi|^2)\mu_H$, is smooth in the strict sense (see Proposition 3.1 in [12]). By Lemma 3.1, it is easy to see the existence of the locally square integrable MAF M^d of purely discontinuous type such that $M_t^d - M_{t-}^d = \phi(X_{t-}, X_t)$ for all $t \in]0, \infty[$ $\mathbf{P}_x$-a.s.

(ii): Since $N(\phi - \log(1 + \phi))\mu_H$ is a smooth measure in the strict sense, there exists a local MAF L of purely discontinuous type such that $L_t - L_{t-} = (\phi - \log(1 + \phi))(X_{t-}, X_t)$ for all $t \in]0, \infty[$ $\mathbf{P}_x$-a.s. Set $J := M^d - L$ and $U := J + M^c$. Note that

$$Z_t = \exp(U_t - C_t),$$

where C is a PCAF in the strict sense defined by $C_t := \int_0^t N(\phi - \log(1+\phi))(X_s)\, dH_s + (1/2)\langle M^c\rangle_t$. Take $p > 1$ and $q > 1$ with $pq \in]1, 2]$ and define $\phi_{pq} := (1 + \phi)^{pq} - 1$. Recalling the inequality $(1+x)^r \leq 1 + rx + (r-1)x^2$ for $x > -1$ and $r \in [1, 2]$, we see that $N(\phi_{pq} - \log(1+\phi_{pq}))\mu_H \in S_1$ under the conditions. Let $M^{(pq),d}$ be a locally square integrable MAF of purely discontinuous type with $M_t^{(pq),d} - M_{t-}^{(pq),d} = \phi_{pq}(X_{t-}, X_t)$ for all $t \in]0, \infty[$ $\mathbf{P}_x$-a.s., which can be similarly constructed by using of ϕ_{pq} instead of ϕ. Set $M_t^{(pq)} := M_t^{(pq),d} + pqM_t^c$ and $Z_t^{(pq)} := \mathrm{Exp}(M^{(pq)})_t$. Then

$$Z_t^{(pq)} = \exp(pqU_t - C_t^{(pq)}),$$

where $C_t^{(pq)} := \int_0^t N(\phi_{pq} - \log(1 + \phi_{pq}))(X_s)\, dH_s + (p^2q^2/2)\langle M^c\rangle_t$. We then see that

$$\begin{aligned}
\mathbf{E}_x[Z_t^p] &= \mathbf{E}_x[\exp(pU_t - pC_t)] \\
&= \mathbf{E}_x\left[(Z_t^{(pq)})^{1/q} \exp\left(\frac{1}{q}C_t^{(pq)} - pC_t\right)\right] \\
&\leq \mathbf{E}_x[Z_t^{(pq)}]^{1/q}\mathbf{E}_x\left[\exp\left(\frac{1}{q-1}C_t^{(pq)} - \frac{pq}{q-1}C_t\right)\right]^{(q-1)/q} \\
&\leq \mathbf{E}_x\left[\exp\left(\frac{1}{q-1}\int_0^t N((1+\phi)^{pq} - 1 - pq\phi)(X_s)\, dH_s \right.\right. \\
&\qquad \left.\left. + \frac{pq}{2(q-1)}(pq-1)\langle M^c\rangle_t\right)\right]^{(q-1)/q}.
\end{aligned} \tag{3.3}$$

Since $pq \in]1, 2]$, (3.3) is estimated by

$$\mathbf{E}_x\left[\exp\left(\frac{pq-1}{q-1}\left(\int_0^t N(\phi^2)(X_s)\, dH_s + \frac{pq}{2}\langle M^c\rangle_t\right)\right)\right]^{(q-1)/q}.$$

Put $l := \lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x\big[\int_0^t N(\phi^2)(X_s)\,dH_s + (1/2)\langle M^c\rangle_t\big] < 1$. Letting p and q be sufficiently close to 1, we have

$$\frac{pq-1}{q-1}\cdot pql < 1,$$

which shows $\sup_{x\in E}\mathbf{E}_x[Z_t^p] < \infty$ for some (hence all) $t > 0$ and some $p > 1$ in view of Khas'minskii's lemma. Let $\{T_n\}$ be an increasing sequence of stopping times such that $Z_{t\wedge T_n}$ is a martingale for each $n \in \mathbb{N}$. In the same way, we see that

$$\sup_{n\in\mathbb{N}}\sup_{x\in E}\mathbf{E}_x[Z_{t\wedge T_n}^p] < \infty$$

for such $t > 0$ and $p > 1$, which yields the martingale property of Z. (3.1) follows from Doob's inequality.

(iii): We may assume $p > 1$. Note that the function $f_p(x) := ((1+x)^p - 1 - px)/x^2$ is bounded above over $[-1+\varepsilon, -1+\varepsilon^{-1}]$ for each $\varepsilon \in]0,1[$ and $p > 1$ and set $D_\varepsilon^{(p)} := \sup_{x\in[-1+\varepsilon,-1+\varepsilon^{-1}]} f_p(x) > 0$. Assuming that $|\log(1+\phi)|$ is bounded above by $|\log\varepsilon|$, (3.3) is estimated by

$$\mathbf{E}_x\left[\exp\left(\frac{D_\varepsilon^{(pq)}}{q-1}\int_0^t N(\phi^2)(X_s)\,dH_s + \frac{pq}{2(q-1)}(pq-1)\langle M^c\rangle_t\right)\right]^{(q-1)/q}.$$

Since ν is of Kato, we then have that $\sup_{x\in E}\mathbf{E}_x[Z_t^p] < \infty$ for some (hence all) $t > 0$ and any $p > 1$. The rest is similar as in (ii). Finally we prove the last statement. Owing to Doob's inequality, it suffices to show

$$\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x[|Z_t - 1|^p] = 0.$$

Noting that $|x-1|^p \le |x^p - 1|$ for $x > 0$ and $\mathbf{E}_x[Z_t^p] \ge (\mathbf{E}_x[Z_t])^p = 1$, we have

$$\begin{aligned}\mathbf{E}_x[|Z_t - 1|^p]^2 &\le \mathbf{E}_x[|Z_t^p - 1|]^2\\ &\le \mathbf{E}_x[|Z_t^p - 1|^2] = \mathbf{E}_x[Z_t^{2p} - 2Z_t^p + 1] \le \mathbf{E}_x[Z_t^{2p}] - 1\\ &\le \mathbf{E}_x[\exp(A_t^{(2p,q)})]^{(q-1)/q} - 1,\end{aligned}$$

where $A_t^{(2p,q)} := D_\varepsilon^{(2pq)}\int_0^t N(\phi^2)(X_s)\,dH_s + (2pq/(2(q-1)))(2pq-1)\langle M^c\rangle_t$ is a PCAF of Kato class, which shows $\sup_{x\in E}\mathbf{E}_x[|Z_t - 1|^p]^2 \le \sup_{x\in E}\mathbf{E}_x[\exp(A_t^{(2p,q)})] - 1 \to 0$ as $t \to 0$. By $|x+y|^p \le 2^{p-1}(|x|^p + |y|^p)$, $x, y \in \mathbb{R}$, we have

$$|\tilde{Z}_s - 1|^p \le 2^{p-1}(\mathrm{Exp}(\tilde{A}^{(p)})_s|Z_s - 1|^p + \mathrm{Exp}(\tilde{A}^{(p)})_s - 1),$$

where $\tilde{A}^{(p)}$ is a PAF such that its continuous part is $(p/2)\langle M^c\rangle_t$ and $\Delta\tilde{A}_t^{(p)} = \exp(p(\phi -$

$\log(1+\phi))(X_{t-}, X_t)) - 1$ for all $t \in]0,\infty[$ $\mathbf{P}_x$-a.s. Since $|\log(1+\phi)|$ is bounded by $|\log\varepsilon|$,

$$\{\log(1+\phi)\}^2 \le \left(\frac{\log\varepsilon}{\varepsilon-1}\right)^2\phi^2,$$

where we use the inequality $0 \le \log(1+x)/x \le \log\varepsilon/(\varepsilon-1)$ for $x \in [-1+\varepsilon, -1+\varepsilon^{-1}]$. Hence there exists $C_\varepsilon > 0$ such that $\phi - \log(1+\phi) \le C_\varepsilon\phi^2$. This implies that $\tilde{A}^{(p)}$ is of Kato under the conditions. Therefore we obtain the desired assertion. $\square$

Theorem 3.3. *Assume that* $\log(1+\phi)$ *is bounded on* $K \times E$ *for each compact set* K *and* $\nu := N(\phi^2)\mu_H + (1/2)\mu_{\langle M^c\rangle}$ *is a positive Radon measure of local and extended Kato class. Put* $Z_t := \mathrm{Exp}(M)_t$. *Then* (1.7) *and* (1.12) *hold for* Z *and for every regular open set* B. *In particular,* Z *is a martingale under the conditions. Consequently, the semigroup* $\{T_t^B;\, t \ge 0\}$ *defined by* (1.2) *has doubly Feller property.*

Proof. First we show that $N(\phi - \log(1+\phi))\mu_H$ is a smooth measure in the strict sense. Let K be a compact set in E. Then $\log(1+\phi_K)$ is bounded on $E_\partial \times E_\partial$, where $\phi_K(x,y) := \mathbf{1}_K(x)\phi(x,y)$. So there exists $C_K > 0$ such that $\phi_K - \log(1+\phi_K) \le C_K\phi_K^2$ on $E_\partial \times E_\partial$. This implies that $\mathbf{1}_K N(\phi - \log(1+\phi))\mu_H$ is of Kato class, because $\mathbf{1}_K N(\phi^2)\mu_H$ is a finite measure of Kato class, hence it is in S_{00} in view of the proof of Proposition 3.1 in [12]. Then we have that $N(\phi - \log(1+\phi))\mu_H$ is smooth in the strict sense. We can apply Lemma 3.2 (ii), consequently (1.12) is obtained. Next we prove (1.7). By assumption, $(\mathbf{1}_K * (\langle M^d\rangle + (1/2)\langle M^c\rangle))_t = \langle \mathbf{1}_K * M^d\rangle_t + (1/2)\langle \mathbf{1}_K * M^c\rangle_t$ is a PCAF of Kato class. Hence we can apply Lemma 3.2 (iii). Therefore

$$\begin{aligned}\lim_{t\to 0}\sup_{x\in K}\mathbf{E}_x\left[\sup_{s\in[0,t]}|Z_s - 1| : t < \tau_K\right] &= \lim_{t\to 0}\sup_{x\in K}\mathbf{E}_x\left[\sup_{s\in[0,t]}|\mathrm{Exp}(\mathbf{1}_K * M)_s - 1| : t < \tau_K\right]\\ &\le \lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x\left[\sup_{s\in[0,t]}|\mathrm{Exp}(\mathbf{1}_K * M)_s - 1|\right] = 0,\end{aligned}$$

which implies (1.7). $\square$

4. Examples

We show in each of the three examples in this section, there is a positive measure μ whose associated PAF A^μ is in the local Kato class but not in Kato class and that $Z_t := e^{A_t^\mu}$ satisfies conditions (1.7) and (1.12) but not (1.6) nor (1.14). Here A^μ is the positive continuous additive functional of X having Revuz measure μ. These examples are to illustrate Theorem 2.5. From them, the readers can easily come up examples of Girsanov transform in the same spirit but Theorem 3.3 is still applicable.

EXAMPLE 4.1 (Brownian motion). Let $X = (\Omega, X_t, \mathbf{P}_x)$ be d-dimensional Brownian motion on $\mathbb{R}^d$. A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of Kato class* if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}\frac{|\mu|(dy)}{|x-y|^{d-2}} = 0 \quad \text{when} \quad d \geq 3,$$

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}(\log|x-y|^{-1})\,|\mu|(dy) = 0 \quad \text{when} \quad d = 2,$$

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\leq 1}|\mu|(dy) < \infty \quad \text{when} \quad d = 1.$$

Here $|\mu| := \mu^+ + \mu^-$ is the total variation measure of μ. A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of local Kato class* if $\mathbf{1}_K\mu$ is of Kato class for every compact subset K of $\mathbb{R}^d$. By definition, any measure μ of local Kato class is always a signed Radon measure. Denote by $\mathbf{K}_d$ (resp. $\mathbf{K}_d^{\mathrm{loc}}$) the family of Kato class (resp. local Kato class) measures on $\mathbb{R}^d$. It is essentially proved in [1] that a positive measure μ is in Kato class $\mathbf{K}_d$ if and only if μ is a smooth measure in the strict sense and

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^t p_s(x,y)\,ds\right)\mu(dy) = \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\mathbf{E}_x[A_t^\mu] = 0,$$

where A^μ is a PCAF of X admitting no exceptional set associated to μ under Revuz correspondence. Assume now that $d \geq 2$. We will show that there is a positive measure $\mu \in \mathbf{K}_d^{\mathrm{loc}} \setminus \mathbf{K}_d$ satisfying

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^t p_s(x,y)\,ds\right)\mu(dy) < \infty \quad \text{for some (and hence for all)} \quad t > 0 \tag{4.1}$$

and that $Z_t := e^{A_t^\mu}$ satisfies conditions (1.7) and (1.12) but not (1.6) nor (1.14).

For any Borel measure μ, we set $N\mu(x) := \int_{\mathbb{R}^d}\mu(dy)/|x-y|^{d-2}$ the Newtonian potential of μ if $d \geq 3$, and $L\mu(x) := \int_{\mathbb{R}^d}\log|x-y|^{-1}\mathbf{1}_{\{|x-y|<1\}}\,\mu(dy)$ the modified logarithmic potential of μ if $d = 2$. For $r > 0$ and $z \in \mathbb{R}^d$, let $\sigma_{\partial B_r(z)}$ be the surface measure on the sphere $\partial B_r(z)$ with center z and radius r.

Let $\{z_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ such that $|z_{n+1}| = 2|z_n|$, $n \in \mathbb{N}$, $|z_1| > 2$. We define $\mu_n := g(r_n)\sigma_{\partial B_{r_n}(z_n)}$ and $\mu := \sum_{n=1}^\infty \mu_n$, where

$$r_n := 8^{-n} \quad \text{and} \quad g(t) := t^{-1} \quad \text{when} \quad d \geq 3,$$

and

$$r_n := 8^{-n^2} \quad \text{and} \quad g(t) := t^{-1}/\log t^{-1} \quad \text{when} \quad d = 2.$$

It is known that for $r > 0$ and $z \in \mathbb{R}^d$ $N\sigma_{\partial B_r(z)}(x) = r\min\{1, (r/|x-z|)^{d-2}\}$ when $d \geq 3$ and $L\sigma_{\partial B_r(z)}(x) = r\min\{\log r^{-1}, \log|x-z|^{-1}\}$ when $d = 2$. It follows that when $d \geq 3$, $N\mu_n$ is bounded above by $N\mu_n(z_n) = 1$ (resp. by $1/2^{n(d-2)}$) on $\mathbb{R}^d$ (resp. on $B_{2^n r_n}(z_n)^c$).

Similarly, when $d = 2$, $L\mu_n$ is bounded above by $L\mu_n(z_n) = 1$ (resp. by $1/n^2$) on $\mathbb{R}^d$ (resp. on $B_{1/8}(z_n)^c$). Therefore we have that for $d \geq 3$

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} N\mu(x) &= \sup_{x\in\mathbb{R}^d} \sum_{n=1}^{\infty} N\mu_n(x)\\ &\leq \sup_{x\in\mathbb{R}^d}\left(\sum_{n=1}^{\infty} \mathbf{1}_{B_{2^n r_n}(z_n)}(x) N\mu_n(x) + \sum_{n=1}^{\infty} \mathbf{1}_{B_{2^n r_n}(z_n)^c}(x) N\mu_n(x)\right)\\ &\leq \left(1 + \sum_{n=1}^{\infty} \frac{1}{2^{n(d-2)}}\right) < \infty,\end{aligned}$$

which implies (4.1). Moreover, for all $r > r_n$

$$\int_{|z_n - y| < r} \frac{\mu(dy)}{|z_n - y|^{d-2}} \geq N\mu_n(z_n) = 1,$$

which implies $\mu \notin \mathbf{K}_d$. It is easy to see $\mu \in \mathbf{K}_d^{\mathrm{loc}}$ from $\mu_n \in \mathbf{K}_d$. Similarly, we have the same conclusion for the case of $d = 2$.

For such $\mu \in \mathbf{K}_d^{\mathrm{loc}} \setminus \mathbf{K}_d$, let $\tilde{\mu} := \mu/(\|U_1\mu\|_\infty + \varepsilon)$ and A the PCAF of X having Revuz measure $\tilde{\mu}$, where $\varepsilon > 0$ and $U_1\tilde{\mu}(x) := \mathbf{E}_x\left[\int_0^\infty e^{-t} dA_t\right]$. Then A is of local Kato class and of extended Kato class. Hence the multiplicative functional $Z_t := \exp(A_t)$ satisfies conditions (1.7) and (1.12) in view of Theorem 2.5. However Z does not satisfy condition (1.6), not to mention condition (1.14). This is because by Jensen's inequality, as the positive measure μ is not in $\mathbf{K}_d$,

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbf{E}_x[|Z_t - 1|] = \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbf{E}_x[Z_t - 1] \geq \lim_{t\to 0}\left(\sup_{x\in\mathbb{R}^d} \exp(\mathbf{E}_x[A_t]) - 1\right) > 0.$$

EXAMPLE 4.2 (Relativistic stable processes). We fix $\alpha \in]0, 2]$ and $m \geq 0$. Let $X = (\Omega, X_t, \mathbf{P}_x)_{x\in\mathbb{R}^d}$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0[e^{\sqrt{-1}\langle\xi, X_t\rangle}] = e^{-t\{(|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m\}}.$$

If $m > 0$, it is called the *relativistic α-stable process with mass* (see [15]). In particular, if $\alpha = 1$ and $m > 0$, it is called the *relativistic free Hamiltonian process* (see [11]). When $m = 0$, X is nothing but the usual *symmetric α-stable process*. It is known (see, e.g., [15]) that the transition density function $p_t(x, y)$ of X is given by

$$p_t(x, y) = e^{mt} \int_0^\infty \left(\frac{1}{4\pi s}\right)^{d/2} e^{-|x-y|^2/(4s)} e^{-sm^{2/\alpha}} \theta_{\alpha/2}(t, s)\, ds,$$

where $\theta_\delta(t, s)$, $\delta, t, s > 0$, is the transition density function of the *subordinator* whose

Laplace transform is given by

$$\int_0^\infty e^{-\lambda s}\theta_\delta(t,s)\,ds = e^{-t\lambda^\delta}.$$

A signed Borel measure μ on $\mathbb{R}^d$ is said to be *of Kato class* if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}\frac{|\mu|(dy)}{|x-y|^{d-\alpha}} = 0 \quad \text{when} \quad d>\alpha,$$

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|<r}(\log|x-y|^{-1})|\mu|(dy) = 0 \quad \text{when} \quad d=\alpha,$$

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le 1}|\mu|(dy) < \infty \quad \text{when} \quad d=1<\alpha.$$

A Borel measure μ on $\mathbb{R}^d$ is said to be *of local Kato class* if $\mathbf{1}_K\mu$ is of Kato class for every compact subset K of $\mathbb{R}^d$. By definition, any measure μ of local Kato class is always a signed Radon measure. Denote by $\mathbf{K}_{d,\alpha}$ (resp. $\mathbf{K}_{d,\alpha}^{\mathrm{loc}}$) the family of Kato class (resp. local Kato class) measures on $\mathbb{R}^d$.

It is proved in [12] (see [21] for the case $d>\alpha$ with $m=0$, or $d\ge 2$, $\alpha=1$ with $m>0$) that a positive measure μ is in $\mathbf{K}_{d,\alpha}$ if and only if μ is a smooth measure in the strict sense and

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^t p_s(x,y)\,ds\right)\mu(dy) = \lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\mathbf{E}_x[A_t^\mu] = 0,$$

where A^μ is a PCAF of X admitting no exceptional set associated to μ under Revuz correspondence.

A measurable function f on $\mathbb{R}^d$ is said to be *of Kato class* (resp. *of local Kato class*) if $|f(x)|\,dx\in\mathbf{K}_{d,\alpha}$ (resp. $\in\mathbf{K}_{d,\alpha}^{\mathrm{loc}}$) and write $f\in\mathbf{K}_{d,\alpha}$ (resp. $f\in\mathbf{K}_{d,\alpha}^{\mathrm{loc}}$) for simplicity.

For any Borel measure μ, we set $R^{(\alpha)}\mu(x) := \int_{\mathbb{R}^d}\mu(dy)/(|x-y|^{d-\alpha})$ the Riesz potential of μ if $d>\alpha$, and $L\mu(x) := \int_{\mathbb{R}^d}\log|x-y|^{-1}\mathbf{1}_{\{|x-y|<1\}}\mu(dy)$ the modified logarithmic potential of μ if $d=\alpha$. If $\mu(dx)=g(x)\,dx$ for some non-negative function g, we write $R^{(\alpha)}g(x)$ (resp. $Lg(x)$) instead of $R^{(\alpha)}\mu(x)$ (resp. $L\mu(x)$).

Let $\{z_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^d$ such that $|z_{n+1}|=2|z_n|$, $n\in\mathbb{N}$, $|z_1|>2$. We define $f_n(x) := g(r_n)\mathbf{1}_{B_{r_n}(z_n)}(x)$ and $f(x) := \sum_{n=1}^\infty f_n(x)$, where

$$r_n := 8^{-n} \quad \text{and} \quad g(t) := t^{-\alpha} \quad \text{when} \quad d>\alpha$$

and

$$r_n := 8^{-n^2} \quad \text{and} \quad g(t) := \frac{t^{-\alpha}}{(1/d)+\log t^{-1}} \quad \text{when} \quad d=\alpha.$$

By utilizing a simple rearrangement inequality (see Theorem 3.4 in [14]), we see that for $d > \alpha$, $R^{(\alpha)} f_n$ is bounded above by $R^{(\alpha)} f_n(z_n) = (d/\alpha)\omega_d$ (resp. by $(1/(2^n-1)^{d-\alpha})(d/\alpha)\omega_d$) on $\mathbb{R}^d$ (resp. on $B_{2^n r_n}(z_n)^c$), where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Similarly, for $d = \alpha$, $L f_n$ is bounded above by $L f_n(z_n) = \omega_d$ (resp. by ω_d/n^2) on $\mathbb{R}^d$ (resp. on $B_{r_n+(1/8)}(z_n)^c$).

Thus for $d > \alpha$

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} R^{(\alpha)} f(x) &= \sup_{x\in\mathbb{R}^d} \sum_{n=1}^{\infty} R^{(\alpha)} f_n(x)\\ &\le \sup_{x\in\mathbb{R}^d}\left(\sum_{n=1}^{\infty} \mathbf{1}_{B_{2^n r_n}(z_n)}(x) R^{(\alpha)} f_n(x) + \sum_{n=1}^{\infty} \mathbf{1}_{B_{2^n r_n}(z_n)^c}(x) R^{(\alpha)} f_n(x)\right)\\ &\le \frac{d}{\alpha}\omega_d\left(1 + \sum_{n=1}^{\infty}\frac{1}{(2^n-1)^{d-\alpha}}\right) < \infty,\end{aligned}$$

which implies (4.1) by Lemma 4.3 in [13] with the upper estimate of $p_t(x, y)$ discussed in Example 2.4 in [3] or Example 5.1 in [13]. Moreover, for all $r > r_n$

$$\int_{|z_n-y|<r} \frac{f(y)}{|z_n-y|^{d-\alpha}}\,dy \ge R^{(\alpha)} f_n(z_n) = \frac{d}{\alpha}\omega_d,$$

which implies $f \notin \mathbf{K}_{d,\alpha}$. It is easy to see $f \in \mathbf{K}^{\mathrm{loc}}_{d,\alpha}$ since $f_n \in \mathbf{K}_{d,\alpha}$ for every $n \ge 1$. Similarly, we have the same conclusion for the case $d = \alpha$.

For such $f \in \mathbf{K}^{\mathrm{loc}}_{d,\alpha} \setminus \mathbf{K}_{d,\alpha}$, we set $\tilde{f} := f/(\|R_1 f\|_\infty + \varepsilon)$, where $\varepsilon > 0$ and $R_1 f(x) := \mathbf{E}_x\left[\int_0^\infty e^{-t} f(X_t)\,dt\right]$. Then A is of local Kato class and of extended Kato class. Hence the multiplicative functional $Z_t := \exp\left(\int_0^t \tilde{f}(X_s)\,ds\right)$ satisfies conditions (1.7) and (1.12). By the same reasoning as that at the end of Example 4.1, Z does not satisfy condition (1.6), not to mention condition (1.14).

Hereafter we shall focus on the case $m = 0$, $\alpha \in]0, 2[$, that is, X is a symmetric α-stable process and $(\mathcal{E}, \mathcal{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d)$. $(\mathcal{E}, \mathcal{F})$ is given by

$$\begin{aligned}\mathcal{F} &= \left\{u \in L^2(\mathbb{R}^d);\ \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy < \infty\right\},\\ \mathcal{E}(u, v) &= \frac{\mathcal{A}(d, -\alpha)}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}}\,dx\,dy, \quad u, v \in \mathcal{F},\end{aligned}$$

where

$$\mathcal{A}(d, \beta) := \frac{|\beta|\Gamma((d-\beta)/2)}{2^{1+\beta}\pi^{d/2}\Gamma(1+\beta/2)}, \quad \beta \in]-\infty, d[.$$

X has a Lévy system (N, H), where $N(x, dy) := \mathcal{A}(d, -\alpha)|x-y|^{-(d+\alpha)}\,dy$ and $H_t = t$. We set $\phi(x, y) := \tilde{f}(x)^{1/2}\mathbf{1}_{\{|x-y|\ge 1\}}$. Then for each compact set K, $\log(1+\phi) \le$

$\tilde{f}^{1/2}$ is bounded on $K \times \mathbb{R}^d$ and the function $N(\phi^2)$ is of local and extended Kato class. For locally square integrable MAF M with $\Delta M_t = \phi(X_{t-}, X_t)$ for all $t \in]0, \infty[$ $\mathbf{P}_x$-a.s., the multiplicative functional Z defined by $Z_t := \mathrm{Exp}(M)_t = \exp(M_t) \prod_{0<s\le t}(1 + \phi(X_{s-}, X_s))e^{-\phi(X_{s-}, X_s)}$ satisfies (1.7) and (1.12) in view of Theorem 3.3.

EXAMPLE 4.3 (Riemannian manifolds with lower Ricci curvature bounds). Let (M, g) be a d-dimensional smooth complete but not compact Riemannian manifold with $\mathrm{Ric}_M \ge (d-1)\kappa$ for some $\kappa \in \mathbb{R}$. Since M is non-compact, $\kappa \le 0$ in view of Myers theorem (see [16, Theorem IV.3.1 (3)]).

Let m be the volume measure induced from the Riemannian metric g and set $V(x, r) := m(B_r(x))$. Since $\mathrm{Ric}_M \ge (d-1)\kappa$, the Bishop inequality $V(x, r) \le V_\kappa(r)$ and the Bishop-Gromov inequality $V(x, R)/V_\kappa(R) \le V(x, r)/V_\kappa(r)$, $0 < r < R$ hold (see [16, §IV.3]). Here $V_\kappa(r)$ is the volume of the ball with radius r in the canonical manifold with constant sectional curvature κ, which can be computed explicitly as follows.

$$V_\kappa(r) := d\omega_d \int_0^r S_\kappa(s)^{d-1}\, ds,$$

where

$$S_\kappa(s) = \begin{cases} s, & \text{if } \kappa = 0, \\ \dfrac{\sinh t\sqrt{-\kappa}}{\sqrt{-\kappa}}, & \text{if } \kappa < 0, \end{cases}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Consequently, we have the volume doubling condition $\sup_{x\in M} V(x, 2r)/V(x, r) < \infty$ and $\int_1^\infty s\, ds/\log V(x, s) = \infty$ which implies the stochastic completeness of the Brownian motion $X = (\Omega, X_t, \mathbf{P}_x)$ on (M, g).

We also have the scale invariant weak Poincaré inequality (depending on κ if $\kappa < 0$) (see Saloff-Coste [17] or Theorem 5.6.5 in [18]), which implies the weak form of the weak Poincaré inequality (see Theorem 5.5.1 (i) in [18]). Then the heat kernel $p_t(x, y)$ of Brownian motion over (M, g) satisfies the following Li-Yau type estimate (see Theorems 5.5.1 and 5.5.3 in [18], cf. Theorems 6.1 and 6.2 in [9]): for each $T > 0$ there exist $C_i = C_i(T) > 0$, $i = 1, 2, 3, 4$ such that for $(t, x, y) \in]0, T[\times M \times M$

$$\frac{C_3 e^{-C_2\, d(x,y)^2/t}}{V(y, \sqrt{t})} \le p_t(x, y) \le \frac{C_4 e^{-C_1\, d(x,y)^2/t}}{V(y, \sqrt{t})}.$$

Further we assume that the injectivity radius of X (write inj_M) is positive, that is, $\mathrm{inj}_M := \inf_{x\in M} d(x, C_x) > 0$, where C_x is the cut-locus of x. Then we have the following (see the proof of Lemma 5 in [10] and Proposition 14 in [7]. Though the framework of [7] is restricted to compact Riemannian manifolds, the argument in [10] remains valid): There exists $C_d \in]0, \infty[$ such that for any $r \in]0, \mathrm{inj}_M/2[$ and $x \in M$,

$$V(x, r) \ge C_d r^d.$$

Hence we have that there exist $C_1, C_2, C_3, C_4 > 0$ such that for any $t \in]0, (\mathrm{inj}_M/2)^2[$, $x, y \in M$

$$\frac{C_3 e^{-C_2\, d(x,y)^2/t}}{t^{d/2}} \le p_t(x, y) \le \frac{C_4 e^{-C_1\, d(x,y)^2/t}}{t^{d/2}}.$$

A signed Borel measure μ is said to be of Kato class (write $\mu \in \mathbf{K}_d$) if and only if

$$\lim_{r\to 0} \sup_{x\in M} \int_{d(x,y)<r} \frac{|\mu|(dy)}{d(x, y)^{d-2}} = 0 \quad \text{when} \quad d \ge 3,$$

$$\lim_{r\to 0} \sup_{x\in M} \int_{d(x,y)<r} (\log d(x, y)^{-1})\, |\mu|(dy) = 0 \quad \text{when} \quad d = 2,$$

$$\sup_{x\in M} \int_{d(x,y)\le 1} |\mu|(dy) < \infty \quad \text{when} \quad d = 1.$$

The family of measures of local Kato class is similarly defined and will be denoted as $\mathbf{K}_d^{\mathrm{loc}}$. A function f on M is said to be *of Kato class* (write $f \in \mathbf{K}_d$ in short) if the measure $|f(x)|\, m(dx)$ is of the Kato class. Similarly, we write $f \in \mathbf{K}_d^{\mathrm{loc}}$ if the measure $|f(x)|\, m(dx)$ is so. By [13], under the above estimate, we know that $f \in \mathbf{K}_d$ if and only if $\lim_{t\to 0} \sup_{x\in M} \mathbf{E}_x\left[\int_0^t |f(X_s)|\, ds\right] = 0$. For any Borel measure μ, we set $R\mu(x) := \int_M \mu(dy)/d(x, y)^{d-2}$ the Newtonian potential of μ if $d \ge 3$, and $L\mu(x) := \int_M \log d(x, y)^{-1} \mathbf{1}_{\{d(x,y)<1\}}\, \mu(dy)$ the modified logarithmic potential of μ if $d = 2$. If $\mu(dx) = g(x)\, dx$ for some non-negative function g, we write $Rg(x)$ (resp. $Lg(x)$) instead of $R\mu(x)$ (resp. $L\mu(x)$).

We utilize the following estimate under $\mathrm{Ric}_M \ge (d - 1)\kappa$:

Lemma 4.4. *For non-negative measurable function f on* $[0, \infty[$,

$$\int_{B_r(p)} f(d(p, x))\, m(dx) \le (1 + \theta_{d,\kappa}(r)) \int_{\{y\in\mathbb{R}^d;\ |y|<r\}} f(|z|)\, dz \tag{4.2}$$

holds for all $p \in M$, where $\theta_{d,\kappa}(r)$ is a function independent of p, but depending on d, κ such that $\lim_{r\to 0} \theta_{d,\kappa}(r) = 0$.

Proof. By Lemma II.5.4 (1) in [16], we have for any non-negative measurable function g on M,

$$\int_M g(x)\, m(dx) = \int_{]0,\infty[\times\mathbb{S}^{d-1}} g(\exp_p tu)\overline{\theta}(t, u)\, dt\, \sigma(du), \tag{4.3}$$

where $\overline{\theta}(t, u) := \theta(t, u)$ if $t < t(u)$ and $:= 0$ otherwise, and $\theta(t, u) := t^{d-1} \times \sqrt{\det(g_{ij}(\exp_p tu))_{ij}}$. Here $\exp_p \colon T_pM \to M$ is the exponential map, σ is the surface measure on $\mathbb{S}^{d-1}$ and $t(u)$ is the distance from p to the cut-point of p along γ_u (see

Chapter III, §4 in [16]). Set $g(x) := \mathbf{1}_{[0,r[}(d(p,x))f(d(p,x))$. Then

$$\int_{B_r(p)} f(d(p,x))\,m(dx) = \int_{]0,r[\times\mathbb{S}^{d-1}} f(t)\overline{\theta}(t,u)\,dt\,\sigma(du). \tag{4.4}$$

Since $\mathrm{Ric}_M \geq (d-1)\kappa$, it holds that $\overline{\theta}(t,u) \leq S_\kappa(t)^{d-1}$ for all $t > 0$ and $u \in \mathbb{S}^{d-1}$ (see Chapter IV Theorem 3.1 (2) (b) in [16]). Hence the right hand of (4.4) is estimated above by

$$\sup_{|s|\leq r}\left(\frac{S_\kappa(s)}{s}\right)^{d-1}\left(\int_0^r f(t)t^{d-1}\,dt\right)d\omega_d,$$

which implies (4.2). □

Fix a point $o \in M$. Let $\{z_n\}_{n=1}^\infty$ be a sequence in M such that $d(z_{n+1},o) = 2d(z_n,o)$, $n \in \mathbb{N}$, $d(z_1,o) > 2$. We define $f_n(x) := g(r_n)\mathbf{1}_{B_{r_n}(z_n)}(x)$ and $f(x) := \sum_{n=1}^\infty f_n(x)$, where

$$r_n := 8^{-n} \quad \text{and} \quad g(t) := t^{-2} \quad \text{when} \quad d \geq 3$$

and

$$r_n := 8^{-n^2} \quad \text{and} \quad g(t) := \frac{t^{-2}}{(1/2) + \log t^{-1}} \quad \text{when} \quad d = 2.$$

Rf_n is bounded above by $\int_{d(x,y)<3r_n} m(dy)/d(x,y)^{d-2}$ on $B_{2r_n}(z_n)$, and bounded above by $m(B_{r_n}(z_n))/r_n^{d-2}$ on $B_{2r_n}(z_n)^c$. Let $C := \sup_{n\in\mathbb{N}}(1+\theta_{d,\kappa}(3r_n))$. From Lemma 4.4, we see that for $d \geq 3$, Rf_n is bounded above by $9C\omega_d d$ (resp. by $(C3^d/(2^n-1)^{d-2})\omega_d d$) on M (resp. on $B_{2^n r_n}(z_n)^c$). Similarly, for $d = 2$, Lf_n is bounded above by $9C\omega_d d$ (resp. by $C3^d\omega_d/n^2$) on M (resp. on $B_{r_n+(1/8)}(z_n)^c$). Moreover, $Rf_n(z_n) \geq C_d$ and $Lf_n(z_n) \geq C_d/2$ for $r_n < \mathrm{inj}_M/2$.

As in the previous examples, we see $f \in \mathbf{K}_d^{\mathrm{loc}} \setminus \mathbf{K}_d$ and (4.1) holds. So the multiplicative functional $Z_t := \exp\left(\int_0^t \tilde{f}(X_s)\,ds\right)$ defined by $\tilde{f} := f/(\|R_1 f\|_\infty + \varepsilon)$ satisfies (1.7) and (1.12). By the same reasoning as that at the end of Example 4.1, Z does not satisfy condition (1.6), not to mention condition (1.14).

References

[1] M. Aizenman and B. Simon: *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–273.

[2] Z.-Q. Chen, P.J. Fitzsimmons, K. Kuwae and T.-S. Zhang: *Stochastic calculus for symmetric Markov processes*, Ann. Probab. **36** (2008), 931–970.

[3] Z.-Q. Chen and T. Kumagai: *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields **140** (2008), 277–317.
[4] Z.-Q. Chen and T.-S. Zhang: *Girsanov and Feynman-Kac type transformations for symmetric Markov processes*, Ann. Inst. H. Poincaré Probab. Statist. **38** (2002), 475–505.
[5] Z.-Q. Chen and Z. Zhao: *Diffusion processes and second order elliptic operators with singular coefficients for lower order terms*, Math. Ann. **302** (1995), 323–357.
[6] K.L. Chung: *Doubly-Feller process with multiplicative functional*; in Seminar on Stochastic Processes, 1985 (Gainesville, Fla., 1985), Progr. Probab. Statist. **12**, Birkhäuser, Boston, Boston, MA., 1986, 63–78.
[7] C.B. Croke: *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 419–435.
[8] M. Fukushima, Y. Ōshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes, de Gruyter, Berlin, 1994.
[9] A. Grigor'yan: *Heat kernels on weighted manifolds and applications*; in The Ubiquitous Heat Kernel, Contemp. Math. **398**, Amer. Math. Soc., Providence, RI., 2006, 93–191
[10] E. Hebey: *Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius*, Amer. J. Math. **118** (1996), 291–300.
[11] I.W. Herbst and A.D. Sloan: *Perturbation of translation invariant positivity preserving semi-groups on* $L^2(\mathbf{R}^N)$, Trans. Amer. Math. Soc. **236** (1978), 325–360.
[12] K. Kuwae and M. Takahashi: *Kato class functions of Markov processes under ultracontractivity*; in Potential Theory in Matsue, Adv. Stud. Pure Math. **44**, Math. Soc. Japan, Tokyo., 2006, 193–202.
[13] K. Kuwae and M. Takahashi: *Kato class measures of symmetric Markov processes under heat kernel estimates*, J. Funct. Anal. **250** (2007), 86–113.
[14] E.H. Lieb and M. Loss: Analysis, second edition, Graduate Studies in Mathematics **14**, Amer. Math. Soc., Providence, RI, 2001.
[15] M.Ł. Ryznar: *Estimates of Green function for relativistic α-stable process*, Potential Anal. **17** (2002), 1–23.
[16] T. Sakai: Riemannian Geometry, translated from the 1992 Japanese original by the author, Translations of Mathematical Monographs **149**, Amer. Math. Soc., Providence, RI, 1996.
[17] L. Saloff-Coste: *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), 417–450.
[18] L. Saloff-Coste: Aspects of Sobolev-Type Inequalities, London Mathematical Society Lecture Note Series **289**, Cambridge Univ. Press, Cambridge, 2002.
[19] M. Sharpe: General Theory of Markov Processes, Pure and Applied Mathematics **133**, Academic Press, Boston, MA, 1988.
[20] J. Ying: *Dirichlet forms perturbated by additive functionals of extended Kato class*, Osaka J. Math. **34** (1997), 933–952.
[21] Z. Zhao: *A probabilistic principle and generalized Schrödinger perturbation*, J. Funct. Anal. **101** (1991), 162–176.

Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle, WA 98195
USA
e-mail: zchen@math.washington.edu

Kazuhiro Kuwae
Department of Mathematics and Engineering
Graduate School of Science and Technology
Kumamoto University
Kumamoto 860–8555
Japan
e-mail: kuwae@gpo.kumamoto-u.ac.jp