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MARKOV UNIQUENESS AND ESSENTIAL SELF-ADJOINTNESS OF PERTURBED ORNSTEIN-UHLENBECK OPERATORS

SHIQI SONG

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0. Introduction

Stating from a simple formula, we shall show in this paper some elementary inequalities on the Wiener space. We shall give two applications of these inequalities. The first one is a quick proof of the Markov uniqueness of the perturbations of Wiener measure. The second one is to prove the essential self-adjointness of the perturbed Ornstein-Uhlenbeck operators on Wiener space, when the perturbation satisfies some kind of Lipschitz boundedness condition.

The Markov uniqueness and essential self-adjointness problems are one of the basic questions on Dirichlet forms. There are many studies on these subjects. We mention in the references the papers of Albeverio-Kondratiev-Röckner, of Albeverio-Kusuoka, Albeverio-Röckner-Zhang, of Röckner-Zhang, of Song, of Takeda, of Wielens, etc. The present paper tries to give a simpler proof of the Markov uniqueness, and to extend the result of Wielens [11] to the Wiener space. It will be noticed that our proof of the Markov uniqueness does not use the maximality property as it did in Song [8] (cf. also Albeverio-Kusuoka-Röckner [3]), and our method to prove the essential self-adjointness is different from that used in Wielens [11].

1. Notations

In this paper $E$ denotes the space $C_0(R_+, R^d)$ and $m$ denotes the classical Wiener measure on $E$. Let $\iota$ denote the usual imbedding map from the topological dual space $E^*$ of $E$ into $E$. For any element $k \in \iota(E^*) \subset E$, we shall put $\alpha_k = \iota^{-1}(k)$. Recall that $E^*$ is a pre-Hilbert space with the inner product $\int (\alpha_k)^2(x) m(dx)$. We fixe an orthonormal basis $K$ of $E^*$. We introduce the space $FC_b(K)$ to be the family of the functions $u$ on $E$ such that there is $n \in N, f \in$
and \( k_i \in K, \ i = 1, 2, \ldots, n \), so that
\[
u(x) = f[a_{k_1}(x), \ldots, a_{k_n}(x)].
\]

For \( k \in K \), for a function \( g \in FC^\infty(K) \), \( \frac{\partial g}{\partial k} \) is defined as \( \lim_{r \to 0} \frac{1}{r}(g(x + rk) - g(x)) \).

We shall say that a function \( g \in L^2(E, m) \) is differentiable in direction \( k \in K \), if there is a function \( f \in L^2(E, m) \) such that
\[
\int \left( \frac{\partial v}{\partial k} - a_k v \right)(x) g(x) m(dx) = - \int v(x) f(x) m(dx),
\]
for any \( v \in FC^\infty(K) \). In this case we denote \( \frac{\partial g}{\partial k} = f \). Note that the two definitions of \( \frac{\partial g}{\partial k} \) coincide when \( g \in FC^\infty(K) \). Recall that the bilinear form \((u, v) \mapsto \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} dm \), defined on \( FC^\infty(K) \) is closable in \( L^2(E, m) \). We denote by \( \mathcal{E} \) its closure, which is a Dirichlet form.

In this paper we are interested in probability measures \( \mu \) on \( E \) which has the form
\[
\mu = \varphi^2 \cdot m,
\]
where \( \varphi \) is a function in \( D(\mathcal{E}) \). Let \( \Gamma \) denote the operator of carré du champs of \( \mathcal{E} \). We define
\[
Au = \sum_{k=1}^{n} \left( \frac{\partial^2 u}{\partial k^2} - a_k \frac{\partial u}{\partial k} \right) + 2 \int \Gamma(u, \log \varphi), \ u \in D(A) = FC^\infty(K),
\]
where \( \Gamma(u, \log \varphi) \) is defined as \( \frac{1}{\varphi} \Gamma(u, \varphi) \). It is easy to see that \( A \) is a symmetric operator on \( L^2(E, \mu) \). Let \( D(\mu) \) denote the family of all Dirichlet forms on \( L^2(E, \mu) \) whose generator extends \( A \). We shall say that the Markov uniqueness holds for the measure \( \mu \), if \( \#D(\mu) = 1 \). Let \( S(\mu) \) be the set of all self-adjoint operators on \( L^2(E, \mu) \) which extend \( A \). We shall say that \( A \) is essentially self-adjoint on \( FC^\infty(K) \), if \( \#S(\mu) = 1 \). Note that \( S(\mu) \supset D(\mu) \) are not empty. In fact, the pre-Dirichlet form \((u, v) \mapsto \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} d\mu \), defined for \( u, v \in FC^\infty(K) \), is closable on \( L^2(E, \mu) \) (cf. Albeverio-Röckner [4], Song [8]). If we denote by \( \mathcal{E}_\mu \) its closure, \( \mathcal{E}_\mu \in D(\mu) \).

We shall denote by \( R_\lambda \) (resp. by \( U_\lambda \)) the resolvent operator of \( \mathcal{E} \) (resp. of \( \mathcal{E}_\mu \)). The generator of \( \mathcal{E}_\mu \) will be denoted by \( L \). The space \( D(\mathcal{E}_\mu) \) (resp. the space \( D(L) \)) will be considered as a Hilbert space with the inner product \( \langle u, Lu \rangle_{L^2(\mu)} \).
2. Resolvent $R_\lambda$

We present some elementary properties of the resolvent operator $R_\lambda$.

**Lemma 1.** For any $k \in K$, for any bounded function $f$ we have the following formula:

$$\frac{\partial}{\partial k} R_\lambda f(x) = \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt \int a_k(y) f(e^{-t}x + \sqrt{1 - e^{-2t}} y) m(dy).$$

Proof. Note we have $\frac{\partial g}{\partial k} dm = \int a_k g dm$, for any $k \in K$, for $g \in FC^\infty(K)$. Using this relation the lemma can be easily proved when $f \in FC^\infty(K)$. For a general bounded function $f$, choose a uniformly bounded sequence of functions $f_n \in FC^\infty(K)$ such that $f_n \to f$ in $L^2(E, m)$. Let $v \in FC^\infty(K)$. We have:

$$\int v(x) \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt \int a_k(y) f(e^{-t}x + \sqrt{1 - e^{-2t}} y) m(dy) m(dx)$$

$$= \lim_{n \to \infty} \int v(x) \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt \int a_k(y) f_n(e^{-t}x + \sqrt{1 - e^{-2t}} y) m(dy) m(dx)$$

$$= \lim_{n \to \infty} \int v(x) \frac{\partial}{\partial k} R_\lambda f_n(x) m(dx)$$

$$= - \lim_{n \to \infty} \int \left( \frac{\partial v}{\partial k} - a_k v \right) R_\lambda f_n(x) m(dx)$$

$$= - \int \left( \frac{\partial v}{\partial k} - a_k v \right) R_\lambda f(x) m(dx).$$

This achieves the proof of the lemma. □

**Lemma 2.** For any bounded function $f$, we have the inequality:

$$\sup_{x \in E} \sup_{\lambda \geq 0} \lambda \Gamma(R_\lambda f, R_\lambda f)(x) \leq \langle C_\infty \rangle \|f\|_{L^2},$$

where $C_\infty = \sup_{k > 0} \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt < \infty$.

Proof. We have:

$$\lambda \Gamma(R_\lambda f, R_\lambda f)(x) = \lambda \sum_{k \in K} \left( \frac{\partial}{\partial k} R_\lambda f(x) \right)^2$$

$$\leq C_\infty \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt \sum_{k \in K} \left( \int a_k(y) f(e^{-t}x + \sqrt{1 - e^{-2t}} y) m(dy) \right)^2$$

$$\leq C_\infty \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} dt \|f(e^{-t}x + \sqrt{1 - e^{-2t}} \cdot)\|_2^2,$$

because $a_k$ forms an orthonormal system in $L^2(E, m)$,
\[ \leq C_\lambda \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} \, dt \| f \|_w^2, \]

where

\[ C_\lambda = \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t} - 1}} \, dt \]

\[ = \sqrt{\lambda} \int_0^\infty (\lambda + 2) e^{-(\lambda + 1)/2} \sqrt{1 - e^{-2t}} \, dt \]

\[ \leq \sqrt{\lambda} \int_0^\infty (\lambda + 2) e^{-(\lambda + 1)/2} \sqrt{2t} \, dt \]

\[ = \int_0^\infty \frac{\lambda + 2}{\lambda} e^{-(1 + 1/\lambda) u} \sqrt{2u} \, du \]

\[ \rightarrow \int_0^\infty e^{-u} \sqrt{2u} \, du < \infty, \text{ when } \lambda \to \infty. \]

3. A resolvent change formula

**Lemma 3.** For any \( f \in FC_w^\infty(K) \), \( \Gamma(R, \log \varphi) \in L^2(E, \mu) \), and the following formula holds:

\[ U_\lambda f = R \lambda f + 2 U_\lambda [\Gamma(R \lambda f, \log \varphi)]. \]

**Proof.** It is enough to remark that \( R \lambda f \in FC_w^\infty(K) \subset D(\mathcal{E}_\mu) \), and

\[ (\lambda - L)R \lambda f = (1 - A)R \lambda f = f - 2\Gamma(R \lambda f, \log \varphi). \]

**Lemma 4.** The formula in Lemma 3 also holds for any bounded function. Moreover, for any bounded function \( f, R \lambda f \in D(\mathcal{E}_\mu) \) and the following inequalities hold:

\[ \| \lambda R \lambda f \|_{L^2(\mu)} \leq \| \lambda U \lambda f \|_{L^2(\mu)} + 2 \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E} (\varphi, \varphi)^{1/2} \| f \|_w, \]

\[ \mathcal{E}_\mu(\lambda R \lambda f, \lambda R \lambda f)^{1/2} \leq \mathcal{E}_\mu(\lambda U \lambda f, \lambda U \lambda f)^{1/2} + 2 C_\infty \mathcal{E} (\varphi, \varphi)^{1/2} \| f \|_w. \]

**Proof.** The two inequalities are direct consequences of Lemma 3 (cf. Song [8]) if \( f \in FC_w^\infty(K) \). In fact, the equality in Lemma 3 implies immediately

\[ \| \lambda R \lambda f \|_{L^2(\mu)} \leq \| \lambda U \lambda f \|_{L^2(\mu)} + 2 \| \Gamma(R \lambda f, \log \varphi) \|_{L^2(\mu)} \]

Since \( |\Gamma(R \lambda f, \log \varphi)|^2 \leq \Gamma(R \lambda f, R \lambda f) \Gamma(\varphi, \varphi) \frac{1}{\varphi^2} \), we have
\[ \| \Gamma(Rf, \log \varphi) \|_{L^2(\mu)} \leq [\sup_y \Gamma(Rf, Rf)(y)]^{1/2} C(\varphi, \varphi)^{1/2} \]
\[ \leq \frac{1}{\sqrt{\lambda}} C_\infty C(\varphi, \varphi)^{1/2} \| f \|_\infty, \]

by Lemma 2. Similarly, we have

\[ \mathcal{E}_\mu(\lambda Rf, Rf)^{1/2} \]
\[ \leq \mathcal{E}_\mu(\lambda U_1 f, \lambda U_1 f)^{1/2} + 2 \mathcal{E}_\mu(\lambda U_1 \Gamma(Rf, \log \varphi), \lambda U_1 \Gamma(Rf, \log \varphi))^{1/2}. \]

The second term can be controlled by

\[ \mathcal{E}_\mu(\lambda U_1 \Gamma(Rf, \log \varphi), \lambda U_1 \Gamma(Rf, \log \varphi)) \]
\[ = \lambda^2 \int \Gamma(Rf, \log \varphi) U_1 \Gamma(Rf, \log \varphi) d\mu \]
\[ \leq \lambda^2 \| \Gamma(Rf, \log \varphi) \|_{L^2(\mu)}^{1/2} \| U_1 \Gamma(Rf, \log \varphi) \|_{L^2(\mu)}^{1/2} \]
\[ \leq \lambda \| \Gamma(Rf, \log \varphi) \|_{L^2(\mu)}^{1/2} \]
\[ \leq [C_\infty C(\varphi, \varphi)^{1/2} \| f \|_\infty]^2. \]

We therefore proved the two inequalities for \( f \in FC_0^c(K) \).

Now, consider any bounded function \( f \). Let \( f_n \in FC_0^c(K) \) be a sequence of functions converging to \( f \) in \( L^2(E, \mu + m) \), and uniformly bounded by \((1 + \epsilon)\| f \|_\infty\), where \( \epsilon \) is an arbitrary fixed positive constant. Thanks to the second inequality, we see that \( \mathcal{E}_\mu(\lambda Rf_n, \lambda Rf_n) \) is uniformly bounded. Since the function \( Rf_n \) converges to \( Rf \) in probability with respect to \( \mu \), and is uniformly bounded, it converges also in \( L^2(E, \mu) \). We have for any \( a > 0 \):

\[ \int a(1 - a U_0) \lambda Rf(x) \lambda Rf(x) \mu(dx) \]
\[ = \lim_{n \to \infty} \int a(1 - a U_0) \lambda Rf_n(x) \lambda Rf_n(x) \mu(dx) \]
\[ \leq \sup_n \mathcal{E}_\mu(\lambda Rf_n, \lambda Rf_n) < \infty. \]

This proves \( Rf \in D(\mathcal{E}_\mu) \). It now is clear that \( Rf_n \) converges to \( Rf \) weakly in \( D(\mathcal{E}_\mu) \). By continuity and by Banach-Saks theorem (cf. Ma-Röckner [6]), we can prove that the above two inequalities hold for \( Rf \).

To prove the equality in Lemma 3 for \( Rf \), we notice that \( \Gamma(Rf_n, \log \varphi) \) converges to \( \Gamma(Rf, \log \varphi) \) in probability with respect to \( \mu \), and

\[ \| \Gamma(Rf_n, \log \varphi) \|_{L^2(\mu)} \leq (1 + \epsilon) \frac{1}{\sqrt{\lambda}} C_\infty C(\varphi, \varphi)^{1/2} \| f \|_\infty. \]

These facts imply that \( \Gamma(Rf, \log \varphi) \) is in \( L^2(E, \mu) \). It now becomes clear that \( \Gamma(Rf_n, \log \varphi) \) converges to \( \Gamma(Rf, \log \varphi) \) in \( L^1(E, \mu) \), and consequently converges weakly in \( L^2(E, \mu) \). Finally, we can prove the equality in Lemma 3 by Banach-Saks theorem and by continuity. \( \square \)
Remark. We in fact have proved
\[ \| \Gamma(R_\lambda f, \log \varphi) \|_{L^2(\mu)} \leq \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \| f \|_\infty. \]
for any bounded function \( f \).

**Corollary 5.** For any bounded function \( f \), \( R_\lambda f \in D(L) \). Moreover,
\[
\| LR_\lambda f \|_{L^2(\mu)} \leq \| f - \lambda U_\lambda f \|_{L^2(\mu)} + 2 \| \Gamma(R_\lambda f, \log \varphi) - \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi) \|_{L^2(\mu)}
\]
\[
\leq 2 \| f \|_{L^2(\mu)} + \frac{4}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \| f \|_\infty.
\]

Proof. We note that for any \( g \in L^2(E, \mu) \), \( U_\lambda g \in D(L) \). Now, this lemma is a direct consequence of Lemma 3 and Lemma 4.

4. Markov uniqueness

**Lemma 6.** Let \( \tilde{D} \) denote the closure of \( FC_\infty^c(K) \) for the norm \( \| u - Au \|_{L^2(\mu)} \). Let \( f \) be a bounded function. Then, for any fixed \( \lambda > 0 \), \( R_\lambda f \in \tilde{D} \).

Remark. The space \( \tilde{D} \) is a closed subspace in \( D(L) \), because \( L \) is an extension of \( A \).

Proof. We regard \( \tilde{D} \) as a Hilbert space with the inner product \( \| u - L u \|_{L^2(\mu)}^2 \). Let \( f_n \) be a sequence of functions in \( FC_\infty^c(K) \) which tend to \( f \) in \( L^2(E, \mu + \lambda) \). We shall suppose that \( f_n \)'s are uniformly bounded by \( 2 \| f \|_\infty \). Then, \( R_\lambda f_n \in FC_\infty^c(K) \) for each \( n \in \mathbb{N} \). Furthermore, according to Corollary 5, the family of functions \( R_\lambda f_n \) is a bounded family in \( \tilde{D} \).

Now, the closed bounded balls in \( \tilde{D} \) are weakly compact, we can suppose that \( R_\lambda f_n \) converges weakly to an element \( g \) in \( \tilde{D} \). According to the Banach-Saks theorem we can even suppose that the Cesaro mean \( v_n \) of \( R_\lambda f_n \) converges strongly in \( \tilde{D} \) to \( g \). It is clear that \( R_\lambda f_n \) converges to \( R_\lambda f \) in probability with respect to \( \mu \). Hence, the only limit for \( v_n \) must be \( R_\lambda f \). We thus have proved that \( R_\lambda f = g \in \tilde{D} \).

**Lemma 7.** Let \( \alpha > 0 \). Let \( A^* \) denote the adjoint operator of \( A \). Let \( h \) be a bounded solution of the equation \( (A^* - \alpha)h = 0 \). Then, \( h \in D(\mathcal{E}_\mu) \).

Proof. Note that by the preceding lemma, \( \int h(L - \alpha) R_\lambda f d\mu = 0 \) for any bounded function \( f \). Let \( g_\lambda = \lambda U_\lambda h \). We have:
0 = \int h(L - \alpha)\lambda R \mu \, d\mu

= \int hL(\lambda U, g, 2\lambda U, [\Gamma(R, g), \log \varphi]) \, d\mu - \alpha \int h \lambda R \mu \, d\mu

= \int g_{*}L \mu \, d\mu - 2 \int hL(\lambda U, [\Gamma(R, g), \log \varphi]) \, d\mu - \alpha \int h \lambda R \mu \, d\mu

= - \mathcal{E}_{\mu}(g_{*}, g_{*}) - 2 \int h \lambda \mu \lambda U_{*}[\Gamma(R, g), \log \varphi] - \Gamma(R, g, \log \varphi) \, d\mu

= - \mathcal{E}_{\mu}(g_{*}, g_{*}) - 2 \int h \lambda \mu \lambda R g, \log \varphi) \, d\mu + 2 \int h \lambda \mu \lambda R g, \log \varphi) \, d\mu

= - \mathcal{E}_{\mu}(g_{*}, g_{*}) + 2 \int h \lambda \mu \lambda R g, \log \varphi) \, d\mu.

From this equality we obtain:

\mathcal{E}_{\mu}(g_{*}, g_{*}) = - 2 \int g_{*} \lambda \mu \lambda R g, \log \varphi) \, d\mu + 2 \int h \lambda \mu \lambda R g, \log \varphi) \, d\mu

\leq 2(\|g_{*}\|_{\infty} + \|h\|_{\infty}) \int |\Gamma(\lambda R, g, \log \varphi)| \, d\mu + \alpha \|h\|_{\infty}^{2}

\leq 4 \|h\|_{\infty} \mathcal{E}_{\mu}(\lambda \mu \lambda R g, \lambda \mu \lambda R g) + \alpha \|h\|_{\infty}^{2}.

By Lemma 4 we have:

\mathcal{E}_{\mu}(\lambda \mu \lambda R g, \lambda \mu \lambda R g) \leq \mathcal{E}_{\mu}(g_{*}, g_{*}) + 2 C_{\infty} \mathcal{E}(\varphi, \varphi)^{1/2} \|g_{*}\|_{\infty}.

Putting \( C = \|h\|_{\infty} + (1 + C_{\infty}) \|h\|_{\infty} \mathcal{E}(\varphi, \varphi)^{1/2} \), we obtain:

\mathcal{E}_{\mu}(g_{*}, g_{*}) \leq 4 \mathcal{E}(\mathcal{E}(g_{*}, g_{*}) + 2 C) + a C^{2}

or equivalently,

(\mathcal{E}(g_{*}, g_{*}) - 2 C)^{2} \leq (1 + a) C^{2}.

Finally, \( \mathcal{E}(g_{*}, g_{*}) \leq (6 + \sqrt{2}) C \). By this uniform boundedness, by the fact that \( h = \lim_{\lambda \to \infty} \lambda U_{h} \) in \( L^{2}(E, \mu) \), we conclude that \( h \in D(\mathcal{E}_{\mu}) \).

Lemma 8. The function \( h \) is the same as that in the preceding lemma. Then \( \mathcal{E}_{\mu, \alpha}(h, h) = 0 \).

Proof. Let \( \alpha > 0 \). By the definition of \( h \), for any \( v \in FC_{\varphi}(K) \),

\[ \mathcal{E}_{\mu, \alpha}(h, v) = - \int h(A - \alpha) v \, d\mu = 0. \]

But \( FC_{\varphi}(K) \) is dense in \( D(\mathcal{E}_{\mu}), \mathcal{E}_{\mu, \alpha} \), we therefore conclude \( \mathcal{E}_{\mu, \alpha}(h, h) = 0 \).
Theorem 9. The measure $\mu$ has Markov uniqueness.

Proof. Let $\mathcal{E}' \in D(\mu)$. Let $V_1$ be its resolvent operator. We can easily see that $D(\mathcal{E}_\mu) \subset D(\mathcal{E}')$, and, for any bounded function $f$, $V_\alpha f - U_\alpha f \in \text{Ker}(A^* - a)$ for any $a > 0$. By Lemma 8, $V_\alpha f = U_\alpha f$. This implies $\mathcal{E}' = \mathcal{E}_\mu$. □

5. Essential self-adjointness

In this section we suppose in addition that the density function $\varphi$ of $\mu$ is such that $\text{ess.sup} \Gamma(\log \varphi, \log \varphi) \leq M^2$, where $M$ is a constant.

Lemma 10. For $f \in L^2(E, \mu)$, $\lambda$ big enough, we have the inequalities:

- $\|\lambda R_\lambda f\|_{L^2(\mu)} \leq \left(1 - \frac{2M}{\lambda}\right)^{-1} \|f\|_{L^2(\mu)}$,
- $\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_{\lambda} f)^{1/2} \leq \left(1 - \frac{2M}{\sqrt{\lambda}}\right)^{-1} \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}$,
- $\|\Gamma(\lambda R_\lambda f, \log \varphi)\|_{L^2(\mu)} \leq M \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}$

Proof. In fact, it is enough to prove the lemma for $f \in FC_\infty(K)$. The general case can be proved by continuity. We only prove the second inequality. Using Lemma 4 we obtain the following formulae:

- $\mathcal{E}_\mu(\lambda U_\lambda \Gamma(R_\lambda f, \log \varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi))$
- $= \lambda \int \Gamma(R_\lambda f, \log \varphi) \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi) \varphi^2 dm$
- $\leq \lambda \int \Gamma(R_\lambda f, \log \varphi)^2 \varphi^2 dm$
- $\leq \lambda \int \Gamma(R_\lambda f, R_\lambda f) \Gamma(\log \varphi, \log \varphi) \varphi^2 dm$
- $\leq \frac{M^2}{\lambda} \int \Gamma(\lambda R_\lambda f, \lambda R_\lambda f) \varphi^2 dm$
- $= \frac{M^2}{\lambda} \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)$.

So,

- $\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \leq \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2} + 2 \frac{M}{\sqrt{\lambda}} \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}$,

or equivalently for $\lambda$ big enough,

- $\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \leq \left(1 - \frac{2M}{\sqrt{\lambda}}\right)^{-1} \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}$. □
**Lemma 11.** Let $a > 0$, and let $h \in L^2(E, \mu)$ such that $(A^* - a)h = 0$. Then, $h \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu,a}(h, h) = 0$.

Proof. Let $g_\lambda = \lambda U_\lambda h$. By exactly the same calculus as in the proof of Lemma 7, we have

$$\mathcal{E}_{\mu}(g_\lambda, g_\lambda) = -2 \int g_\lambda \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu + 2 \int h \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu - a \int h \lambda R_\lambda g_\lambda d\mu.$$

So, according to Lemma 10, we have

$$\mathcal{E}_{\mu}(g_\lambda, g_\lambda) \leq 4M \left(1 - 2\frac{M}{\lambda^2}\right)^{-1} \mathcal{E}_{\mu}(\lambda U_\lambda g_\lambda, \lambda U_\lambda g_\lambda)^{1/2} \|h\|_{L^2(\mu)}^2 + a \left(1 - 2\frac{M}{\lambda^2}\right)^{-1} \lambda^2 \|g_\lambda\|_{L^2(\mu)}^2$$

for $\lambda$ big enough. There exists then a constant $C = C(\alpha, M)$ such that

$$\mathcal{E}_{\mu}(g_\lambda, g_\lambda) \leq C \|h\|_{L^2(\mu)}^2$$

for $\lambda$ big enough. From this fact we deduce $h \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu,a}(h, h) = 0$. □

**Theorem 12.** The operator $A$ is essentially self-adjoint on $FC_b(K)$.

Proof. It is enough to notice that any solution in $L^2(E, \mu)$ of the equation $(A^* - a)f = 0$, $a > 0$, will be a null function by Lemma 12. □

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**References**


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