



Title	Markov uniqueness and essential self-adjointness of perturbed Ornstein-Uhlenbeck operators
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Citation	Osaka Journal of Mathematics. 1995, 32(3), p. 823-832
Version Type	VoR
URL	https://doi.org/10.18910/6196
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Song, S.
Osaka J. Math.
32 (1995), 823-832

MARKOV UNIQUENESS AND ESSENTIAL SELF-ADJOINTNESS OF PERTURBED ORNSTEIN-UHLENBECK OPERATORS

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(Received December 6, 1993)

0. Introduction

Starting from a simple formula, we shall show in this paper some elementary inequalities on the Wiener space. We shall give two applications of these inequalities. The first one is a quick proof of the Markov uniqueness of the perturbations of Wiener measure. The second one is to prove the essential self-adjointness of the perturbed Ornstein-Uhlenbeck operators on Wiener space, when the perturbation satisfies some kind of Lipschitz boundedness condition.

The Markov uniqueness and essential self-adjointness problems are one of the basic questions on Dirichlet forms. There are many studies on these subjects. We mention in the references the papers of Albeverio-Kondratiev-Röckner, of Albeverio-Kusuoka, Albeverio-Röckner-Zhang, of Röckner-Zhang, of Song, of Takeda, of Wielens, etc. The present paper tries to give a simpler proof of the Markov uniqueness, and to extend the result of Wielens [11] to the Wiener space. It will be noticed that our proof of the Markov uniqueness does not use the maximality property as it did in Song [8] (cf. also Albeverio-Kusuoka-Röckner [3]), and our method to prove the essential self-adjointness is different from that used in Wielens [11].

1. Notations

In this paper E denotes the space $C_0(\mathbf{R}_+, \mathbf{R}^d)$ and m denotes the classical Wiener measure on E . Let ι denote the usual imbedding map from the topological dual space E^* of E into E . For any element $k \in \iota(E^*) \subset E$, we shall put $\alpha_k = \iota^{-1}(k)$. Recall that E^* is a pre-Hilbert space with the inner product $\int (\alpha_k)^2(x) m(dx)$. We fixe an orthonormal basis K of E^* . We introduce the space $FC_b^\infty(K)$ to be the family of the functions u on E such that there is $n \in N$, $f \in$

$C_b^\infty(\mathbf{R}^n)$, and $k_i \in K$, $i=1, 2, \dots, n$, so that

$$u(x) = f[\alpha_{k_1}(x), \dots, \alpha_{k_n}(x)].$$

For $k \in K$, for a function $g \in FC_b^\infty(K)$, $\frac{\partial g}{\partial k}$ is defined as $\lim_{r \rightarrow 0} \frac{1}{r}(g(x+rk) - g(x))$.

We shall say that a function $g \in L^2(E, m)$ is differentiable in direction $k \in K$, if there is a function $f \in L^2(E, m)$ such that

$$\int \left(\frac{\partial v}{\partial k} - \alpha_k v \right)(x) g(x) m(dx) = - \int v(x) f(x) m(dx),$$

for any $v \in FC_b^\infty(K)$. In this case we denote $\frac{\partial g}{\partial k} = f$. Note that the two definition of $\frac{\partial g}{\partial k}$ coincide when $g \in FC_b^\infty(K)$. Recall that the bilinear form $(u, v) \mapsto \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} dm$, defined on $FC_b^\infty(K)$ is closable in $L^2(E, m)$. We denote by \mathcal{E} its closure, which is a Dirichlet form.

In this paper we are interested in probability measures μ on E which has the form $\mu = \varphi^2 \cdot m$, where φ is a function in $D(\mathcal{E})$. Let Γ denote the operator of carré du champs of \mathcal{E} . We define

$$Au = \sum_{k \in K} \left(\frac{\partial^2 u}{\partial k^2} - \alpha_k \frac{\partial u}{\partial k} \right) + 2\Gamma(u, \log \varphi), \quad u \in D(A) = FC_b^\infty(K),$$

where $\Gamma(u, \log \varphi)$ is defined as $\frac{1}{\varphi} \Gamma(u, \varphi)$. It is easy to see that A is a symmetric operator on $L^2(E, \mu)$. Let $D(\mu)$ denote the family of all Dirichlet forms on $L^2(E, \mu)$ whose generator extends A . We shall say that the *Markov uniqueness* holds for the measure μ , if $\#D(\mu)=1$. Let $S(\mu)$ be the set of all self-adjoint operators on $L^2(E, \mu)$ which extend A . We shall say that A is *essentially self-adjoint* on $FC_b^\infty(K)$, if $\#S(\mu)=1$. Note that $S(\mu) \supset D(\mu)$ are not empty. In fact, the pre-Dirichlet form $(u, v) \mapsto \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} dm$, defined for $u, v \in FC_b^\infty(K)$, is closable on $L^2(E, \mu)$ (cf. Albeverio-Röckner [4], Song [8]). If we denote by \mathcal{E}_μ its closure, $\mathcal{E}_\mu \in D(\mu)$.

We shall denote by R_λ (resp. by U_λ) the resolvent operator of \mathcal{E} (resp. of \mathcal{E}_μ). The generator of \mathcal{E}_μ will be denoted by L . The space $D(\mathcal{E}_\mu)$ (resp. the space $D(L)$) will be considered as a Hilbert space with the inner product $\mathcal{E}_{\mu,1}$ (resp. $\|u - Lu\|_{L^2(\mu)}$).

2. Resolvent R_λ

We present some elementary properties of the resolvent operator R_λ .

Lemma 1. *For any $k \in K$, for any bounded function f we have the following formula :*

$$\frac{\partial}{\partial k} R_\lambda f(x) = \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \int \alpha_k(y) f(e^{-t}x + \sqrt{1-e^{-2t}}y) m(dy).$$

Proof. Note we have $\int \frac{\partial g}{\partial k} dm = \int \alpha_k g dm$, for any $k \in K$, for $g \in FC_b^\infty(K)$.

Using this relation the lemma can be easily proved when $f \in FC_b^\infty(K)$. For a general bounded function f , choose a uniformly bounded sequence of functions $f_n \in FC_b^\infty(K)$ such that $f_n \rightarrow f$ in $L^2(E, m)$. Let $v \in FC_b^\infty(K)$. We have :

$$\begin{aligned} & \int v(x) \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \int \alpha_k(y) f(e^{-t}x + \sqrt{1-e^{-2t}}y) m(dy) m(dx) \\ &= \lim_{n \rightarrow \infty} \int v(x) \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \int \alpha_k(y) f_n(e^{-t}x + \sqrt{1-e^{-2t}}y) m(dy) m(dx) \\ &= \lim_{n \rightarrow \infty} \int v(x) \frac{\partial}{\partial k} R_\lambda f_n(x) m(dx) \\ &= -\lim_{n \rightarrow \infty} \int \left(\frac{\partial v}{\partial k} - \alpha_k v \right) R_\lambda f_n(x) m(dx) \\ &= -\int \left(\frac{\partial v}{\partial k} - \alpha_k v \right) R_\lambda f(x) m(dx). \end{aligned}$$

This achieves the proof of the lemma. \square

Lemma 2. *For any bounded function f , we have the inequality :*

$$\sup_x \sup_{\lambda > 0} \lambda \Gamma(R_\lambda f, R_\lambda f)(x) \leq (C_\infty)^2 \|f\|_\infty^2,$$

where $C_\infty = \sup_{\lambda > 0} \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt < \infty$.

Proof. We have :

$$\begin{aligned} \lambda \Gamma(R_\lambda f, R_\lambda f)(x) &= \lambda \sum_{k \in K} \left(\frac{\partial}{\partial k} R_\lambda f(x) \right)^2 \\ &\leq C_\lambda \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \sum_{k \in K} \left(\int \alpha_k(y) f(e^{-t}x + \sqrt{1-e^{-2t}}y) m(dy) \right)^2 \\ &\leq C_\lambda \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \|f(e^{-t}x + \sqrt{1-e^{-2t}} \cdot)\|_2^2, \end{aligned}$$

because α_k forms an orthonormal system in $L^2(E, m)$,

$$\leq C_\lambda \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \|f\|_\infty^2,$$

where

$$\begin{aligned} C_\lambda &= \sqrt{\lambda} \int_0^\infty e^{-\lambda t} \frac{1}{\sqrt{e^{2t}-1}} dt \\ &= \sqrt{\lambda} \int_0^\infty (\lambda+2) e^{-(\lambda+2)t} \sqrt{e^{2t}-1} dt \\ &= \sqrt{\lambda} \int_0^\infty (\lambda+2) e^{-(\lambda+1)t} \sqrt{1-e^{-2t}} dt \\ &\leq \sqrt{\lambda} \int_0^\infty (\lambda+2) e^{-(\lambda+1)t} \sqrt{2t} dt \\ &= \int_0^\infty \frac{\lambda+2}{\lambda} e^{-(1+1/\lambda)u} \sqrt{2u} du \\ &\rightarrow \int_0^\infty e^{-u} \sqrt{2u} du < \infty, \text{ when } \lambda \rightarrow \infty. \quad \square \end{aligned}$$

3. A resolvent change formula

Lemma 3. *For any $f \in FC_b^\infty(K)$, $\Gamma(R_\lambda f, \log \varphi) \in L^2(E, \mu)$, and the following formula holds :*

$$U_\lambda f = R_\lambda f + 2U_\lambda [\Gamma(R_\lambda f, \log \varphi)].$$

Proof. It is enough to remark that $R_\lambda f \in FC_b^\infty(K) \subset D(\mathcal{E}_\mu)$, and

$$(\lambda - L)R_\lambda f = (1 - A)R_\lambda f = f - 2\Gamma(R_\lambda f, \log \varphi). \quad \square$$

Lemma 4. *The formula in Lemma 3 also holds for any bounded function. Moreover, for any bounded function f , $R_\lambda f \in D(\mathcal{E}_\mu)$ and the following inequalities hold :*

$$\|\lambda R_\lambda f\|_{L^2(\mu)} \leq \|\lambda U_\lambda f\|_{L^2(\mu)} + 2 \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty,$$

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \leq \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2} + 2C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty.$$

Proof. The two inequalities are direct consequences of Lemma 3 (cf. Song [8]) if $f \in FC_b^\infty(K)$. In fact, the equality in Lemma 3 implies immediately

$$\|\lambda R_\lambda f\|_{L^2(\mu)} \leq \|\lambda U_\lambda f\|_{L^2(\mu)} + 2\|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)}$$

Since $|\Gamma(R_\lambda f, \log \varphi)|^2 \leq \Gamma(R_\lambda f, R_\lambda f) \Gamma(\varphi, \varphi) \frac{1}{\varphi^2}$, we have

$$\begin{aligned}\|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} &\leq [\sup_y \Gamma(R_\lambda f, R_\lambda f)(y)]^{1/2} \mathcal{E}(\varphi, \varphi)^{1/2} \\ &\leq \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty,\end{aligned}$$

by Lemma 2. Similarly, we have

$$\begin{aligned}\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \\ \leq \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2} + 2 \mathcal{E}_\mu(\lambda U_\lambda \Gamma(R_\lambda f, \log \varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi))^{1/2}.\end{aligned}$$

The second term can be controlled by

$$\begin{aligned}\mathcal{E}_{\mu, \lambda}(\lambda U_\lambda \Gamma(R_\lambda f, \log \varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi)) \\ = \lambda^2 \int \Gamma(R_\lambda f, \log \varphi) U_\lambda \Gamma(R_\lambda f, \log \varphi) d\mu \\ \leq \lambda^2 \|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \|U_\lambda \Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \\ \leq \lambda \|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)}^2 \\ \leq [C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty]^2.\end{aligned}$$

We therefore proved the two inequalities for $f \in FC_b^\infty(K)$.

Now, consider any bounded function f . Let $f_n \in FC_b^\infty(K)$ be a sequence of functions converging to f in $L^2(E, \mu + m)$, and uniformly bounded by $(1 + \varepsilon) \|f\|_\infty$, where ε is an arbitrary fixed positive constant. Thanks to the second inequality, we see that $\mathcal{E}_\mu(\lambda R_\lambda f_n, \lambda R_\lambda f_n)$ is uniformly bounded. Since the function $R_\lambda f_n$ converges to $R_\lambda f$ in probability with respect to μ , and is uniformly bounded, it converges also in $L^2(E, \mu)$. We have for any $\alpha > 0$:

$$\begin{aligned}\int \alpha(1 - \alpha U_\alpha) \lambda R_\lambda f(x) \lambda R_\lambda f(x) \mu(dx) \\ = \lim_{n \rightarrow \infty} \int \alpha(1 - \alpha U_\alpha) \lambda R_\lambda f_n(x) \lambda R_\lambda f_n(x) \mu(dx) \\ \leq \sup_n \mathcal{E}_\mu(\lambda R_\lambda f_n, \lambda R_\lambda f_n) < \infty.\end{aligned}$$

This proves $R_\lambda f \in D(\mathcal{E}_\mu)$. It now is clear that $R_\lambda f_n$ converges to $R_\lambda f$ weakly in $D(\mathcal{E}_\mu)$. By continuity and by Banach-Saks theorem (cf. Ma-Röckner [6]), we can prove that the above two inequalities hold for $R_\lambda f$.

To prove the equality in Lemma 3 for $R_\lambda f$, we notice that $\Gamma(R_\lambda f_n, \log \varphi)$ converges to $\Gamma(R_\lambda f, \log \varphi)$ in probability with respect to μ , and

$$\|\Gamma(R_\lambda f_n, \log \varphi)\|_{L^2(\mu)} \leq (1 + \varepsilon) \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty.$$

These facts imply that $\Gamma(R_\lambda f, \log \varphi)$ is in $L^2(E, \mu)$. It now becomes clear that $\Gamma(R_\lambda f_n, \log \varphi)$ converges to $\Gamma(R_\lambda f, \log \varphi)$ in $L^1(E, \mu)$, and consequently converges weakly in $L^2(E, \mu)$. Finally, we can prove the equality in Lemma 3 by Banach-Saks theorem and by continuity. \square

REMARK. We in fact have proved

$$\|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \leq \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty.$$

for any bounded function f . \square

Corollary 5. *For any bounded function f , $R_\lambda f \in D(L)$. Moreover,*

$$\begin{aligned} \|LR_\lambda f\|_{L^2(\mu)} &\leq \|f - \lambda U_\lambda f\|_{L^2(\mu)} + 2\|\Gamma(R_\lambda f, \log \varphi) - \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \\ &\leq 2\|f\|_{L^2(\mu)} + 4\frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty. \end{aligned}$$

Proof. We note that for any $g \in L^2(E, \mu)$, $U_\lambda g \in D(L)$. Now, this lemma is a direct consequence of Lemma 3 and Lemma 4. \square

4. Markov uniqueness

Lemma 6. *Let \widehat{D} denote the closure of $FC_b^\infty(K)$ for the norm $\|u - Au\|_{L^2(\mu)}$. Let f be a bounded function. Then, for any fixed $\lambda > 0$, $R_\lambda f \in \widehat{D}$.*

REMARK. The space \widehat{D} is a closed subspace in $D(L)$, because L is an extention of A .

Proof. We regard \widehat{D} as a Hilbert space with the inner product $\|u - Lu\|_{L^2(\mu)}^2$. Let f_n be a sequence of functions in $FC_b^\infty(K)$ which tend to f in $L^2(E, \mu + m)$. We shall suppose that f_n 's are uniformly bounded by $2\|f\|_\infty$. Then, $R_\lambda f_n \in FC_b^\infty(K)$ for each $n \in N$. Furthermore, according to Corollary 5, the family of functions $R_\lambda f_n$ is a bounded family in \widehat{D} .

Now, the closed bounded balls in \widehat{D} are weakly compact, we can suppose that $R_\lambda f_n$ converges weakly to an element g in \widehat{D} . According to the Banach-Saks theorem we can even suppose that the Cesaro mean v_n of $R_\lambda f_n$ converges strongly in \widehat{D} to g . It is clear that $R_\lambda f_n$ converges to $R_\lambda f$ in probability with respect to μ . Hence, the only limit for v_n must be $R_\lambda f$. We thus have proved that $R_\lambda f = g \in \widehat{D}$. \square

Lemma 7. *Let $\alpha > 0$. Let A^* denote the adjoint operator of A . Let h be a bounded solution of the equation $(A^* - \alpha)h = 0$. Then, $h \in D(\mathcal{E}_\mu)$.*

Proof. Note that by the preceding lemma, $\int h(L - \alpha)R_\lambda f d\mu = 0$ for any bounded function f . Let $g_\lambda = \lambda U_\lambda h$. We have :

$$\begin{aligned}
0 &= \int h(L - \alpha) \lambda R_\lambda g_\lambda d\mu \\
&= \int h L(\lambda U_\lambda g_\lambda - 2\lambda U_\lambda [\Gamma(R_\lambda g_\lambda, \log \varphi)]) d\mu - \alpha \int h \lambda R_\lambda g_\lambda d\mu \\
&= \int g_\lambda L g_\lambda d\mu - 2 \int h L(\lambda U_\lambda [\Gamma(R_\lambda g_\lambda, \log \varphi)]) d\mu - \alpha \int h \lambda R_\lambda g_\lambda d\mu \\
&= -\mathcal{E}_\mu(g_\lambda, g_\lambda) - 2 \int h \lambda (\lambda U_\lambda [\Gamma(R_\lambda g_\lambda, \log \varphi)] - \Gamma(R_\lambda g_\lambda, \log \varphi)) d\mu \\
&\quad - \alpha \int h \lambda R_\lambda g_\lambda d\mu \\
&= -\mathcal{E}_\mu(g_\lambda, g_\lambda) - 2 \int g_\lambda \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu + 2 \int h \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu \\
&\quad - \alpha \int h \lambda R_\lambda g_\lambda d\mu.
\end{aligned}$$

From this equality we obtain :

$$\begin{aligned}
\mathcal{E}_\mu(g_\lambda, g_\lambda) &= -2 \int g_\lambda \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu + 2 \int h \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu \\
&\quad - \alpha \int h \lambda R_\lambda g_\lambda d\mu \\
&\leq 2(\|g_\lambda\|_\infty + \|h\|_\infty) \int |\Gamma(\lambda R_\lambda g_\lambda, \log \varphi)| d\mu + \alpha \|h\|_\infty^2 \\
&\leq 4\|h\|_\infty \mathcal{E}_\mu(\lambda R_\lambda g_\lambda, \lambda R_\lambda g_\lambda)^{1/2} \mathcal{E}(\varphi, \varphi)^{1/2} + \alpha \|h\|_\infty^2.
\end{aligned}$$

By Lemma 4 we have :

$$\mathcal{E}_\mu(\lambda R_\lambda g_\lambda, \lambda R_\lambda g_\lambda)^{1/2} \leq \mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2} + 2C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|g_\lambda\|_\infty.$$

Putting $C = \|h\|_\infty + (1 + C_\infty) \|h\|_\infty \mathcal{E}(\varphi, \varphi)^{1/2}$, we obtain :

$$\mathcal{E}_\mu(g_\lambda, g_\lambda) \leq 4C(\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2} + 2C) + \alpha C^2$$

or equivalently,

$$(\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2} - 2C)^2 \leq (12 + \alpha)C^2.$$

Finally, $\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2} \leq (6 + \sqrt{\alpha})C$. By this uniform boundedness, by the fact that $h = \lim_{\lambda \rightarrow \infty} \lambda U_\lambda h$ in $L^2(E, \mu)$, we conclude that $h \in D(\mathcal{E}_\mu)$. \square

Lemma 8. *The function h is the same as that in the preceding lemma. Then $\mathcal{E}_{\mu, \alpha}(h, h) = 0$.*

Proof. Let $\alpha > 0$. By the definition of h , for any $v \in FC_b^\infty(K)$,

$$\mathcal{E}_{\mu, \alpha}(h, v) = - \int h(A - \alpha)v d\mu = 0.$$

But $FC_b^\infty(K)$ is dense in $(D(\mathcal{E}_\mu), \mathcal{E}_{\mu, \alpha})$, we therefore conclude $\mathcal{E}_{\mu, \alpha}(h, h) = 0$. \square

Theorem 9. *The measure μ has Markov uniqueness.*

Proof. Let $\mathcal{E}' \in D(\mu)$. Let V_λ be its resolvent operator. We can easily see that $D(\mathcal{E}_\mu) \subset D(\mathcal{E}')$, and, for any bounded function f , $V_\alpha f - U_\alpha f \in \text{Ker}(A^* - \alpha)$ for any $\alpha > 0$. By Lemma 8, $V_\alpha f = U_\alpha f$. This implies $\mathcal{E}' = \mathcal{E}_\mu$. \square

5. Essential self-adjointness

In this section we suppose in addition that the density function φ of μ is such that $\text{ess.sup } \Gamma(\log \varphi, \log \varphi) \leq M^2$, where M is a constant.

Lemma 10. *For $f \in L^2(E, \mu)$, λ big enough, we have the inequalities :*

$$\begin{aligned} \|\lambda R_\lambda f\|_{L^2(\mu)} &\leq \left(1 - 2\frac{M}{\lambda}\right)^{-1} \|f\|_{L^2(\mu)}, \\ \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} &\leq \left(1 - 2\frac{M}{\sqrt{\lambda}}\right)^{-1} \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}, \\ \|\Gamma(\lambda R_\lambda f, \log \varphi)\|_{L^2(\mu)} &\leq M \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \\ &\leq \frac{\lambda M}{\sqrt{\lambda} - 2M} \|f\|_{L^2(\mu)} \end{aligned}$$

Proof. In fact, it is enough to prove the lemma for $f \in FC_b^\infty(K)$. The general case can be proved by continuity. We only prove the second inequality. Using Lemma 4 we obtain the following formulae :

$$\begin{aligned} &\mathcal{E}_{\mu, \lambda}(\lambda U_\lambda \Gamma(R_\lambda f, \log \varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi)) \\ &= \lambda \int \Gamma(R_\lambda f, \log \varphi) \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi) \varphi^2 dm \\ &\leq \lambda \int \Gamma(R_\lambda f, \log \varphi)^2 \varphi^2 dm \\ &\leq \lambda \int \Gamma(R_\lambda f, R_\lambda f) \Gamma(\log \varphi, \log \varphi) \varphi^2 dm \\ &\leq \frac{M^2}{\lambda} \int \Gamma(\lambda R_\lambda f, \lambda R_\lambda f) \varphi^2 dm \\ &= \frac{M^2}{\lambda} \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f). \end{aligned}$$

So,

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \leq \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2} + 2\frac{M}{\sqrt{\lambda}} \mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2},$$

or equivalently for λ big enough,

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2} \leq \left(1 - 2\frac{M}{\sqrt{\lambda}}\right)^{-1} \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}. \quad \square$$

Lemma 11. *Let $\alpha > 0$, and let $h \in L^2(E, \mu)$ such that $(A^* - \alpha)h = 0$. Then, $h \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu, \alpha}(h, h) = 0$.*

Proof. let $g_\lambda = \lambda U_\lambda h$. By exactly the same calculus as in the proof of Lemma 7, we have

$$\begin{aligned} \mathcal{E}_\mu(g_\lambda, g_\lambda) &= -2 \int g_\lambda \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu + 2 \int h \Gamma(\lambda R_\lambda g_\lambda, \log \varphi) d\mu \\ &\quad - \alpha \int h \lambda R_\lambda g_\lambda d\mu. \end{aligned}$$

So, according to Lemma 10, we have

$$\begin{aligned} \mathcal{E}_\mu(g_\lambda, g_\lambda) &\leq 4M \left(1 - 2 \frac{M}{\sqrt{\lambda}}\right)^{-1} \mathcal{E}_\mu(\lambda U_\lambda g_\lambda, \lambda U_\lambda g_\lambda)^{1/2} \|h\|_{L^2(\mu)} \\ &\quad + \alpha \left(1 - 2 \frac{M}{\lambda}\right)^{-1} \|h\|_{L^2(\mu)}^2. \end{aligned}$$

for λ big enough. There exists then a constant $C = C(\alpha, M)$ such that

$$\mathcal{E}_\mu(g_\lambda, g_\lambda) \leq C \|h\|_{L^2(\mu)}^2$$

for λ big enough. From this fact we deduce $h \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu, \alpha}(h, h) = 0$. \square

Theorem 12. *The operator A is essentially self-adjoint on $FC_b^\infty(K)$.*

Proof. It is enough to notice that any solution in $L^2(E, \mu)$ of the equation $(A^* - \alpha)f = 0$, $\alpha > 0$, will be a null function by Lemma 12. \square

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