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MARKOV UNIQUENESS AND ESSENTIAL SELF-ADJOINTNESS OF PERTURBED ORNSTEIN-UHLENBECK OPERATORS

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0. Iutroduction

Stating from a simple formula, we shall show in this paper some elementary inequalities on the Wiener space. We shall give two applications of these inequalities. The first one is a quick proof of the Markov uniqueness of the perturtions of Wiener measure. The second one is to prove the essential self-adjointness of the perturbed Ornstein-Uhlenbeck operatros on Wiener space, when the perturbation satisfies some kind of Lipschitz boundedness condition.

The Markov uniqueness and essential self-adjointness problems are one of the basic questions on Dirichlet forms. There are many studies on these subjects. We mention in the references the papers of Albeverio-Kondratiev-Röckner, of Albeverio-Kusuoka, Albeverio-Röckner-Zhang, of Röckner-Zhang, of Song, of Takeda, of Wielens, etc. The present paper tries to give a simpler proof of the Markov uniqueness, and to extend the result of Wielens [11] to the Wiener space. It will be noticed that our proof of the Markov uniqueness does not use the maximality property as it did in Song [8] (cf. also Albeverio-Kusuoka-Röckner [3]), and our method to prove the essential self-adjointness is different from that used in Wielens [11].

1. Notations

In this paper E denotes the space $C_0(\boldsymbol{R}_+, \boldsymbol{R}^d)$ and m denotes the classical Wiener measure on E. Let ι denote the usual imbedding map from the topological dual space E^* of E into E. For any element $k\in\iota(E^*)\subset E$, we shall put $\alpha_k=\iota^{-1}(k)$. Recall that E^* is a pre-Hilbert space with the inner product $\int(\alpha_k)^2(x)m(dx)$. We fixe an orthonormal basis K of E^*. We introduce the space $FC_b^\infty(K)$ to be the family of the functions u on E such that there is $n\in N$, $f\in$

$C_b^\infty(\boldsymbol{R}^n)$, and $k_i \in K$, $i=1, 2, \cdots, n$, so that

$$u(x)=f[\alpha_{k_1}(x), \cdots, \alpha_{k_n}(x)].$$

For $k \in K$, for a function $g \in FC_b^\infty(K)$, $\frac{\partial g}{\partial k}$ is defined as $\lim_{r \to 0} \frac{1}{r}(g(x+rk)-g(x))$. We shall say that a function $g \in L^2(E, m)$ is differentiable in direction $k \in K$, if there is a function $f \in L^2(E, m)$ such that

$$\int \left(\frac{\partial v}{\partial k} - \alpha_k v\right)(x) g(x) m(dx) = -\int v(x) f(x) m(dx),$$

for any $v \in FC_b^\infty(K)$. In this case we denote $\frac{\partial g}{\partial k}=f$. Note that the two definition of $\frac{\partial g}{\partial k}$ coincide when $g \in FC_b^\infty(K)$. Recall that the bilinear form $(u, v) \longrightarrow \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} dm$, defined on $FC_b^\infty(K)$ is closable in $L^2(E, m)$. We denote by $\mathcal{E}$ its closure, which is a Dirichlet form.

In this paper we are interested in probability measures μ on E which has the form $\mu=\varphi^2 \cdot m$, where φ is a function in $D(\mathcal{E})$. Let Γ denote the operator of carré du champs of $\mathcal{E}$. We define

$$Au=\sum_{k \in K}\left(\frac{\partial^2 u}{\partial k^2} - \alpha_k \frac{\partial u}{\partial k}\right) + 2\Gamma(u, \log \varphi), \; u \in D(A)=FC_b^\infty(K),$$

where $\Gamma(u, \log \varphi)$ is defined as $\frac{1}{\varphi}\Gamma(u, \varphi)$. It is easy to see that A is a symmetric operator on $L^2(E, \mu)$. Let $D(\mu)$ denote the family of all Dirichlet forms on $L^2(E, \mu)$ whose generator extends A. We shall say that the *Markov uniqueness* holds for the measure μ, if $\# D(\mu)=1$. Let $S(\mu)$ be the set of all self-adjoint operators on $L^2(E, \mu)$ which extend A. We shall say that A is *essentially self-adjoint* on $FC_b^\infty(K)$, if $\# S(\mu)=1$. Note that $S(\mu) \supset D(\mu)$ are not empty. In fact, the pre-Dirichlet form $(u, v) \longrightarrow \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} d\mu$, defined for $u, v \in FC_b^\infty(K)$, is closable on $L^2(E, \mu)$ (cf. Albeverio-Röckner [4], Song [8]). If we denote by $\mathcal{E}_\mu$ its closure, $\mathcal{E}_\mu \in D(\mu)$.

We shall denote by R_λ (resp. by U_λ) the resolvent operator of $\mathcal{E}$ (resp. of $\mathcal{E}_\mu$). The generator of $\mathcal{E}_\mu$ will be denoted by L. The space $D(\mathcal{E}_\mu)$ (resp. the space $D(L)$) will be considered as a Hilbert space with the inner product $\mathcal{E}_{\mu,1}$ (resp. $\|u-Lu\|_{L^2(\mu)}$).

2. Resolvent R_λ

We present some elementary properties of the resolvent operator R_λ.

Lemma 1. *For any $k\in K$, for any bounded function f we have the following formula:*

$$\frac{\partial}{\partial k}R_\lambda f(x)=\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\int a_k(y)f(e^{-t}x+\sqrt{1-e^{-2t}}\,y)m(dy).$$

Proof. Note we have $\int\frac{\partial g}{\partial k}dm=\int a_k g dm$, for any $k\in K$, for $g\in FC_b^\infty(K)$. Using this relation the lemma can be easily proved when $f\in FC_b^\infty(K)$. For a general bounded function f, choose a uniformly bounded sequence of functions $f_n\in FC_b^\infty(K)$ such that $f_n\to f$ in $L^2(E, m)$. Let $v\in FC_b^\infty(K)$. We have:

$$\begin{aligned}
&\int v(x)\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\int a_k(y)f(e^{-t}x+\sqrt{1-e^{-2t}}\,y)m(dy)m(dx)\\
&=\lim_{n\to\infty}\int v(x)\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\int a_k(y)f_n(e^{-t}x+\sqrt{1-e^{-2t}}\,y)m(dy)m(dx)\\
&=\lim_{n\to\infty}\int v(x)\frac{\partial}{\partial k}R_\lambda f_n(x)m(dx)\\
&=-\lim_{n\to\infty}\int\left(\frac{\partial v}{\partial k}-a_k v\right)R_\lambda f_n(x)m(dx)\\
&=-\int\left(\frac{\partial v}{\partial k}-a_k v\right)R_\lambda f(x)m(dx).
\end{aligned}$$

This achieves the proof of the lemma. □

Lemma 2. *For any bounded function f, we have the inequality:*

$$\sup_x\sup_{\lambda>0}\lambda\Gamma(R_\lambda f,\ R_\lambda f)(x)\leq(C_\infty)^2\|f\|_\infty^2,$$

where $C_\infty=\sup_{\lambda>0}\sqrt{\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt<\infty.$

Proof. We have:

$$\begin{aligned}
\lambda\Gamma(R_\lambda f,\ R_\lambda f)(x)&=\lambda\sum_{k\in K}\left(\frac{\partial}{\partial k}R_\lambda f(x)\right)^2\\
&\leq C_\lambda\sqrt{\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\sum_{k\in K}\left(\int a_k(y)f(e^{-t}x+\sqrt{1-e^{-2t}}\,y)m(dy)\right)^2\\
&\leq C_\lambda\sqrt{\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\|f(e^{-t}x+\sqrt{1-e^{-2t}}\,\cdot)\|_2^2,
\end{aligned}$$

because a_k forms an orthonormal system in $L^2(E, m)$,

$$\leq C_\lambda\sqrt{\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\|f\|_\infty^2,$$

where

$$\begin{aligned}C_\lambda&=\sqrt{\lambda}\int_0^\infty e^{-\lambda t}\frac{1}{\sqrt{e^{2t}-1}}dt\\&=\sqrt{\lambda}\int_0^\infty(\lambda+2)e^{-(\lambda+2)t}\sqrt{e^{2t}-1}\,dt\\&=\sqrt{\lambda}\int_0^\infty(\lambda+2)e^{-(\lambda+1)t}\sqrt{1-e^{-2t}}\,dt\\&\leq\sqrt{\lambda}\int_0^\infty(\lambda+2)e^{-(\lambda+1)t}\sqrt{2t}\,dt\\&=\int_0^\infty\frac{\lambda+2}{\lambda}e^{-(1+1/\lambda)u}\sqrt{2u}\,du\\&\to\int_0^\infty e^{-u}\sqrt{2u}\,du<\infty,\ \text{when}\ \lambda\to\infty.\quad\square\end{aligned}$$

3. A resolvent change formula

Lemma 3. *For any $f\in FC_b^\infty(K)$, $\Gamma(R_\lambda f, \log\varphi)\in L^2(E,\mu)$, and the following formula holds*:

$$U_\lambda f=R_\lambda f+2U_\lambda[\Gamma(R_\lambda f, \log\varphi)].$$

Proof. It is enough to remark that $R_\lambda f\in FC_b^\infty(K)\subset D(\mathcal{E}_\mu)$, and

$$(\lambda-L)R_\lambda f=(1-A)R_\lambda f=f-2\Gamma(R_\lambda f, \log\varphi).\quad\square$$

Lemma 4. *The formula in* Lemma 3 *also holds for any bounded function. Moreover, for any bounded function f, $R_\lambda f\in D(\mathcal{E}_\mu)$ and the following inequalities hold*:

$$\|\lambda R_\lambda f\|_{L^2(\mu)}\leq\|\lambda U_\lambda f\|_{L^2(\mu)}+2\frac{1}{\sqrt{\lambda}}C_\infty\mathcal{E}(\varphi,\varphi)^{1/2}\|f\|_\infty,$$

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\leq\mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}+2C_\infty\mathcal{E}(\varphi,\varphi)^{1/2}\|f\|_\infty.$$

Proof. The two inequalities are direct consequences of Lemma 3 (cf. Song [8]) if $f\in FC_b^\infty(K)$. In fact, the equality in Lemma 3 implies immediately

$$\|\lambda R_\lambda f\|_{L^2(\mu)}\leq\|\lambda U_\lambda f\|_{L^2(\mu)}+2\|\Gamma(R_\lambda f, \log\varphi)\|_{L^2(\mu)}$$

Since $|\Gamma(R_\lambda f, \log\varphi)|^2\leq\Gamma(R_\lambda f, R_\lambda f)\Gamma(\varphi,\varphi)\frac{1}{\varphi^2}$, we have

$$\begin{aligned}\|\Gamma(R_\lambda f, \log\varphi)\|_{L^2(\mu)} &\leq [\sup_y \Gamma(R_\lambda f, R_\lambda f)(y)]^{1/2}\mathcal{E}(\varphi, \varphi)^{1/2}\\ &\leq \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2}\|f\|_\infty,\end{aligned}$$

by Lemma 2. Similarily, we have

$$\begin{aligned}&\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\\ &\leq \mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2} + 2\mathcal{E}_\mu(\lambda U_\lambda \Gamma(R_\lambda f, \log\varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log\varphi))^{1/2}.\end{aligned}$$

The second term can be controlled by

$$\begin{aligned}&\mathcal{E}_{\mu,\lambda}(\lambda U_\lambda \Gamma(R_\lambda f, \log\varphi), \lambda U_\lambda \Gamma(R_\lambda f, \log\varphi))\\ &= \lambda^2 \int \Gamma(R_\lambda f, \log\varphi) U_\lambda \Gamma(R_\lambda f, \log\varphi) d\mu\\ &\leq \lambda^2 \|\Gamma(R_\lambda f, \log\varphi)\|_{L^2(\mu)} \|U_\lambda \Gamma(R_\lambda f, \log\varphi)\|_{L^2(\mu)}\\ &\leq \lambda \|\Gamma(R_\lambda f, \log\varphi)\|^2_{L^2(\mu)}\\ &\leq [C_\infty \mathcal{E}(\varphi, \varphi)^{1/2}\|f\|_\infty]^2.\end{aligned}$$

We therefore proved the two inequalities for $f \in FC_b^\infty(K)$.

Now, consider any bounded function f. Let $f_n \in FC_b^\infty(K)$ be a sequence of functions converging to f in $L^2(E, \mu+m)$, and uniformly bounded by $(1+\varepsilon)\|f\|_\infty$, where ε is an arbitrary fixed positive constant. Thanks to the second inequality, we see that $\mathcal{E}_\mu(\lambda R_\lambda f_n, \lambda R_\lambda f_n)$ is uniformly bounded. Since the function $R_\lambda f_n$ converges to $R_\lambda f$ in probability with respect to μ, and is uniformly bounded, it converges also in $L^2(E, \mu)$. We have for any $\alpha > 0$:

$$\begin{aligned}&\int \alpha(1-\alpha U_\alpha)\lambda R_\lambda f(x) \lambda R_\lambda f(x) \mu(dx)\\ &= \lim_{n\to\infty} \int \alpha(1-\alpha U_\alpha)\lambda R_\lambda f_n(x) \lambda R_\lambda f_n(x) \mu(dx)\\ &\leq \sup_n \mathcal{E}_\mu(\lambda R_\lambda f_n, \lambda R_\lambda f_n) < \infty.\end{aligned}$$

This proves $R_\lambda f \in D(\mathcal{E}_\mu)$. It now is clear that $R_\lambda f_n$ converges to $R_\lambda f$ weakly in $D(\mathcal{E}_\mu)$. By continuity and by Banach-Saks theorem (cf. Ma-Röckner [6]), we can prove that the above two inequalties hold for $R_\lambda f$.

To prove the equality in Lemma 3 for $R_\lambda f$, we notice that $\Gamma(R_\lambda f_n, \log\varphi)$ converges to $\Gamma(R_\lambda f, \log\varphi)$ in probability with respect to μ, and

$$\|\Gamma(R_\lambda f_n, \log\varphi)\|_{L^2(\mu)} \leq (1+\varepsilon)\frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2}\|f\|_\infty.$$

These facts imply that $\Gamma(R_\lambda f, \log\varphi)$ is in $L^2(E, \mu)$. It now becomes clear that $\Gamma(R_\lambda f_n, \log\varphi)$ converges to $\Gamma(R_\lambda f, \log\varphi)$ in $L^1(E, \mu)$, and consequently converges weakly in $L^2(E, \mu)$. Finally, we can prove the equality in Lemma 3 by Banach-Saks theorem and by continuity. □

REMARK. We in fact have proved

$$\|\Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \leq \frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty.$$

for any bounded function f. $\square$

Corollary 5. *For any bounded function f, $R_\lambda f \in D(L)$. Moreover,*

$$\begin{aligned}\|LR_\lambda f\|_{L^2(\mu)} &\leq \|f - \lambda U_\lambda f\|_{L^2(\mu)} + 2\|\Gamma(R_\lambda f, \log \varphi) - \lambda U_\lambda \Gamma(R_\lambda f, \log \varphi)\|_{L^2(\mu)} \\ &\leq 2\|f\|_{L^2(\mu)} + 4\frac{1}{\sqrt{\lambda}} C_\infty \mathcal{E}(\varphi, \varphi)^{1/2} \|f\|_\infty.\end{aligned}$$

Proof. We note that for any $g \in L^2(E, \mu)$, $U_\lambda g \in D(L)$. Now, this lemma is a direct consequence of Lemma 3 and Lemma 4. $\square$

4. Markov uniqueness

Lemma 6. *Let $\hat{D}$ denote the closure of $FC_b^\infty(K)$ for the norm $\|u - Au\|_{L^2(\mu)}$. Let f be a bounded function. Then, for any fixed $\lambda > 0$, $R_\lambda f \in \hat{D}$.*

REMARK. The space $\hat{D}$ is a closed subspace in $D(L)$, because L is an extention of A.

Proof. We regard $\hat{D}$ as a Hilbert space with the inner product $\|u - Lu\|^2_{L^2(\mu)}$. Let f_n be a sequence of functions in $FC_b^\infty(K)$ which tend to f in $L^2(E, \mu + m)$. We shall suppose that f_n's are uniformly bounded by $2\|f\|_\infty$. Then, $R_\lambda f_n \in FC_b^\infty(K)$ for each $n \in N$. Furthermore, according to Corollary 5, the family of functions $R_\lambda f_n$ is a bounded family in $\hat{D}$.

Now, the closed bounded balls in $\hat{D}$ are weakly compact, we can suppose that $R_\lambda f_n$ converges weakly to an element g in $\hat{D}$. According to the Banach-Saks theorem we can even suppose that the Cesaro mean v_n of $R_\lambda f_n$ converges strongly in $\hat{D}$ to g. It is clear that $R_\lambda f_n$ converges to $R_\lambda f$ in probability with respect to μ. Hence, the only limit for v_n must be $R_\lambda f$. We thus have proved that $R_\lambda f = g \in \hat{D}$. $\square$

Lemma 7. *Let $\alpha > 0$. Let A^* denote the adjoint operator of A. Let h be a bounded solution of the equation $(A^* - \alpha)h = 0$. Then, $h \in D(\mathcal{E}_\mu)$.*

Proof. Note that by the preceding lemma, $\int h(L-\alpha)R_\lambda f d\mu = 0$ for any bounded function f. Let $g_\lambda = \lambda U_\lambda h$. We have :

$$
\begin{aligned}
0&=\int h(L-\alpha)\lambda R_\lambda g_\lambda d\mu\\
&=\int hL(\lambda U_\lambda g_\lambda-2\lambda U_\lambda[\Gamma(R_\lambda g_\lambda, \log\varphi)])d\mu-\alpha\int h\lambda R_\lambda g_\lambda d\mu\\
&=\int g_\lambda Lg_\lambda d\mu-2\int hL(\lambda U_\lambda[\Gamma(R_\lambda g_\lambda, \log\varphi)])d\mu-\alpha\int h\lambda R_\lambda g_\lambda d\mu\\
&=-\mathcal{E}_\mu(g_\lambda, g_\lambda)-2\int h\lambda(\lambda U_\lambda[\Gamma(R_\lambda g_\lambda, \log\varphi)]-\Gamma(R_\lambda g_\lambda, \log\varphi))d\mu\\
&\quad-\alpha\int h\lambda R_\lambda g_\lambda d\mu\\
&=-\mathcal{E}_\mu(g_\lambda, g_\lambda)-2\int g_\lambda\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu+2\int h\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu\\
&\quad-\alpha\int h\lambda R_\lambda g_\lambda d\mu.
\end{aligned}
$$

From this equality we obtain:

$$
\begin{aligned}
\mathcal{E}_\mu(g_\lambda, g_\lambda)&=-2\int g_\lambda\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu+2\int h\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu\\
&\quad-\alpha\int h\lambda R_\lambda g_\lambda d\mu\\
&\le 2(\|g_\lambda\|_\infty+\|h\|_\infty)\int|\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)|d\mu+\alpha\|h\|_\infty^2\\
&\le 4\|h\|_\infty\mathcal{E}_\mu(\lambda R_\lambda g_\lambda, \lambda R_\lambda g_\lambda)^{1/2}\mathcal{E}(\varphi, \varphi)^{1/2}+\alpha\|h\|_\infty^2.
\end{aligned}
$$

By Lemma 4 we have:

$$\mathcal{E}_\mu(\lambda R_\lambda g_\lambda, \lambda R_\lambda g_\lambda)^{1/2}\le\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2}+2C_\infty\mathcal{E}(\varphi, \varphi)^{1/2}\|g_\lambda\|_\infty.$$

Putting $C=\|h\|_\infty+(1+C_\infty)\|h\|_\infty\mathcal{E}(\varphi, \varphi)^{1/2}$, we obtain:

$$\mathcal{E}_\mu(g_\lambda, g_\lambda)\le 4C(\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2}+2C)+\alpha C^2$$

or equivalently,

$$(\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2}-2C)^2\le(12+\alpha)C^2.$$

Finally, $\mathcal{E}_\mu(g_\lambda, g_\lambda)^{1/2}\le(6+\sqrt{\alpha})C$. By this uniform boundedness, by the fact that $h=\lim_{\lambda\to\infty}\lambda U_\lambda h$ in $L^2(E, \mu)$, we conclude that $h\in D(\mathcal{E}_\mu)$. □

Lemma 8. *The function h is the same as that in the preceding lemma. Then* $\mathcal{E}_{\mu,\alpha}(h, h)=0$.

Proof. Let $\alpha>0$. By the definition of h, for any $v\in FC_b^\infty(K)$,

$$\mathcal{E}_{\mu,\alpha}(h, v)=-\int h(A-\alpha)vd\mu=0.$$

But $FC_b^\infty(K)$ is dense in $(D(\mathcal{E}_\mu), \mathcal{E}_{\mu,\alpha})$, we therefore conclude $\mathcal{E}_{\mu,\alpha}(h, h)=0$. □

Theorem 9. *The measure μ has Markov uniqueness.*

Proof. Let $\mathcal{E}'\in D(\mu)$. Let V_λ be its resolvent operator. We can easily see that $D(\mathcal{E}_\mu)\subset D(\mathcal{E}')$, and, for any bounded function f, $V_\alpha f-U_\alpha f\in\mathrm{Ker}(A^*-\alpha)$ for any $\alpha>0$. By Lemma 8, $V_\alpha f=U_\alpha f$. This implies $\mathcal{E}'=\mathcal{E}_\mu$. □

5. Essential self-adjointness

In this section we suppose in addition that the density function φ of μ is such that ess.sup $\Gamma(\log\varphi, \log\varphi)\leq M^2$, where M is a constant.

Lemma 10. *For $f\in L^2(E, \mu)$, λ big enough, we have the inequalities:*

$$\|\lambda R_\lambda f\|_{L^2(\mu)}\leq\left(1-2\frac{M}{\lambda}\right)^{-1}\|f\|_{L^2(\mu)},$$

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\leq\left(1-2\frac{M}{\sqrt{\lambda}}\right)^{-1}\mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2},$$

$$\begin{aligned}\|\Gamma(\lambda R_\lambda f, \log\varphi)\|_{L^2(\mu)}&\leq M\,\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\\&\leq\frac{\lambda M}{\sqrt{\lambda}-2M}\|f\|_{L^2(\mu)}\end{aligned}$$

Proof. In fact, it is enough to prove the lemma for $f\in FC_b^\infty(K)$. The general case can be proved by continuity. We only prove the second inequality. Using Lemma 4 we obtain the following formulae:

$$\begin{aligned}&\mathcal{E}_{\mu,\lambda}(\lambda U_\lambda\Gamma(R_\lambda f, \log\varphi), \lambda U_\lambda\Gamma(R_\lambda f, \log\varphi))\\&=\lambda\int\Gamma(R_\lambda f, \log\varphi)\lambda U_\lambda\Gamma(R_\lambda f, \log\varphi)\varphi^2dm\\&\leq\lambda\int\Gamma(R_\lambda f, \log\varphi)^2\varphi^2dm\\&\leq\lambda\int\Gamma(R_\lambda f, R_\lambda f)\Gamma(\log\varphi, \log\varphi)\varphi^2dm\\&\leq\frac{M^2}{\lambda}\int\Gamma(\lambda R_\lambda f, \lambda R_\lambda f)\varphi^2dm\\&=\frac{M^2}{\lambda}\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f).\end{aligned}$$

So,

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\leq\mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}+2\frac{M}{\sqrt{\lambda}}\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2},$$

or equivalently for λ big enough,

$$\mathcal{E}_\mu(\lambda R_\lambda f, \lambda R_\lambda f)^{1/2}\leq\left(1-2\frac{M}{\sqrt{\lambda}}\right)^{-1}\mathcal{E}_\mu(\lambda U_\lambda f, \lambda U_\lambda f)^{1/2}.\ \square$$

Lemma 11. *Let $\alpha>0$, and let $h\in L^2(E, \mu)$ such that $(A^*-\alpha)h=0$. Then, $h\in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu,\alpha}(h, h)=0$.*

Proof. let $g_\lambda=\lambda U_\lambda h$. By exactly the same calculus as in the proof of Lemma 7, we have

$$\mathcal{E}_\mu(g_\lambda, g_\lambda)=-2\int g_\lambda \Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu+2\int h\Gamma(\lambda R_\lambda g_\lambda, \log\varphi)d\mu \\ -\alpha\int h\lambda R_\lambda g_\lambda d\mu.$$

So, according to Lemma 10, we have

$$\mathcal{E}_\mu(g_\lambda, g_\lambda)\leq 4M\left(1-2\frac{M}{\sqrt{\lambda}}\right)^{-1}\mathcal{E}_\mu(\lambda U_\lambda g_\lambda, \lambda U_\lambda g_\lambda)^{1/2}\|h\|_{L^2(\mu)} \\ +\alpha\left(1-2\frac{M}{\lambda}\right)^{-1}\|h\|^2_{L^2(\mu)}.$$

for λ big enough. There exists then a constant $C=C(\alpha, M)$ such that

$$\mathcal{E}_\mu(g_\lambda, g_\lambda)\leq C\|h\|^2_{L^2(\mu)}$$

for λ big enough. From this fact we deduce $h\in D(\mathcal{E}_\mu)$ and $\mathcal{E}_{\mu,\alpha}(h, h)=0$. □

Theorem 12. *The operator A is essentially self-adjoint on $FC_b^\infty(K)$.*

Proof. It is enough to notice that any solution in $L^2(E, \mu)$ of the equation $(A^*-\alpha)f=0$, $\alpha>0$, will be a null function by Lemma 12. □

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