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## ON THE CAUCHY PROBLEM FOR KOWALEWSKI SYSTEMS

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The present paper is concerned with the global Cauchy problem for a Kowalewski system of partial differential equations of the form

$$\frac{\partial u_\mu}{\partial t} = \sum_{\nu=1}^k \left\{ \sum_{j=1}^m A_{\mu\nu j}(t, x) \frac{\partial u_\nu}{\partial x_j} + B_{\mu\nu}(t, x) u_\mu \right\} + f_\mu(t, x) \quad (0.1)$$

with the initial conditions

$$u_\mu(0, x) = \varphi_\mu(x), \quad \mu = 1, 2, \dots, k \quad (0.2)$$

where  $x=(x_1, \dots, x_m)$  is the generic point in the  $m$ -dimensional Euclidean space  $R^m$ .

In this paper we shall not impose any condition on the characteristics of this system.

In [2] S. Mizohata studied the global uniqueness of solutions of the Cauchy problem within the class of tempered distributions under the conditions that the coefficients  $A_{\mu\nu j}(t, x)$  and  $B_{\mu\nu}(t, x)$  of (0.1) are bounded continuous functions in  $(t, x)$  whose Fourier transforms with respect to  $x$  are measures with compact supports. In [1] A.G. Kostjutschenko and G.E. Shilov considered the uniqueness of solutions of the Cauchy problem for the system of type (0.1)-(0.2) within a class of functions which satisfy the inequality  $|u(x)| \leq M e^{-|x|^p}$  for some constants  $M$  and  $p$ , under the conditions that the coefficients  $A_{\mu\nu j}(t, x)$  and  $B_{\mu\nu}(t, x)$  of (0.1) are independent of  $t$  and are bounded continuous functions of  $x$  whose Fourier transforms are exponentially decreasing measures. On the other hand in [5] T. Yamanaka investigated the uniqueness of solutions of the Cauchy problem for the system (0.1) within a class of distributions with a finite growth order under the condition that the coefficients  $B_{\mu\nu}(t, x)$  are of the form  $B_{\mu\nu}(t, x) = P_{\mu\nu}(x) B'_{\mu\nu}(t, x)$  where  $P_{\mu\nu}(x)$  are any polynomials in  $x$  and  $B'_{\mu\nu}(t, x)$  are bounded continuous functions of  $(t, x)$  whose Fourier transforms with respect to  $x$  are exponentially decreasing measures, and the coefficients  $A_{\mu\nu j}(t, x)$  are the same type of functions as  $B'_{\mu\nu}(t, x)$ .

In [4] the author studied the existence and the uniqueness of global solutions of the Cauchy problem for the system (0.1). The uniqueness of solutions was proved within a class of functions which satisfy the inequality

$$|u(t, x)| \leq M \exp(ae^{b|x|}) \text{ in } [0, T] \times R^m$$

for some constants  $a$ ,  $b$  and  $M$ , under some conditions on the coefficients of (0,1) specified in the main theorems.

The purpose of this paper is to give a revised and complete proof of the results obtained in [4]. The method of the existence proof used here is essentially based on that of M. Nagumo [2].

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## 1. Assumptions and Notations

We denote by  $R^m$  and  $C^m$  the  $m$ -dimensional real and complex Euclidean space respectively, and denote by  $x=(x_1, \dots, x_m)$  and  $z=x+\sqrt{-1}y=(x_1+\sqrt{-1}y_1, \dots, x_m+\sqrt{-1}y_m)$  ( $x, y \in R^m$ ) their generic point respectively. For positive numbers  $T$  and  $\gamma$ ,  $D(T)$  and  $\mathfrak{D}_\gamma(T)$  are defined as follows:

$$D(T) = \{(t, x); 0 \leq t \leq T, x \in R^m\}$$

$$\mathfrak{D}_\gamma(T) = \{(t, x); 0 \leq t \leq T, z \in C^m, |y_j| < \gamma, j = 1, \dots, m\}.$$

For any non-negative integer  $h$  we denote by  $C_{(t,z)}^h[\mathfrak{D}_\gamma(T)]$  the class of all complex valued functions whose derivatives of order up to  $h$  are continuous in  $\mathfrak{D}_\gamma(T)$ . By  $A_{(z)}[\mathfrak{D}_\gamma(T)]$ , we denote the class of all complex valued functions defined in  $\mathfrak{D}_\gamma(T)$  and holomorphic with respect to  $z$  when  $t$  is fixed in  $[0, T]$ . For any positive constants  $a$  and  $b$ , the class of all continuously differentiable functions which satisfy the inequality  $|f(t, x)| \leq M \exp(ae^{b|x|})$  in  $D(T)$  for some positive constant  $M$  is denoted by  $\mathfrak{F}(a, b)$ ,  $M$  being allowed to be dependent on the individual  $f$ .

We now state the assumptions on the coefficients of (0,1) here.

### Assumptions

(I)  $A_{\mu\nu j}(t, z)$ ,  $B_{\mu\nu}(t, z)$  and  $f_\mu(t, z)$  ( $\mu, \nu=1, \dots, k; j=1, \dots, m$ ) are continuous functions in  $\mathfrak{D}_\gamma(T)$ .

(II)  $A_{\mu\nu j}(t, z)$  and  $B_{\mu\nu}(t, z)$  ( $\mu, \nu=1, \dots, k; j=1, \dots, m$ ) are holomorphic functions with respect to  $z$  in the domain:  $\{z \in C^m; -\infty < x_j < +\infty, |y_j| < \gamma, j=1, 2, \dots, m\}$  for each fixed  $t$  in  $[0, T]$ , and there exist positive constants  $A$  and

$B$  such that  $|A_{\mu\nu j}(t, z)| \leq A$ ,  $|B_{\mu\nu}(t, z)| \leq B$  in  $\mathfrak{D}_\gamma(T)$ .

(III)  $f_\mu(t, z)$  ( $\mu=1, 2, \dots, k$ ) are holomorphic functions with respect to  $z$  in  $\{z \in C^m; -\infty < x_j < +\infty, |y_j| < \gamma, j=1, \dots, m\}$  for each  $t$  in  $[0, T]$ , and  $\varphi_\mu(z)$  ( $\mu=1, \dots, m$ ) are holomorphic functions in  $\{z \in C^m; -\infty < x_j < +\infty, |y_j| < \gamma, j=1, \dots, m\}$ .

## 2. Main Theorems

In this section we shall state an existence theorem and a uniqueness theorem for the system (0,1) with the initial conditions (0,2). Proofs of these theorems will be given in section 4.

### Theorem 1. (existence of solutions)

Under the assumptions (I), (II) and (III) on the coefficients of (0,1) and on the initial conditions (0,2), there exist positive numbers  $T_1$  and  $\gamma_1$  ( $0 < T_1 \leq T$ ,  $0 < \gamma_1 < \gamma$ ) and a system of solutions  $u_\mu(t, z)$  ( $\mu=1, \dots, k$ ) of (0,1) satisfying the initial conditions (0,2), and belonging to  $C_{(t,z)}^1[\mathfrak{D}_{\gamma_1}(T_1)] \cap A_{(z)}[\mathfrak{D}_{\gamma_1}(T_1)]$ .

### Theorem 2. (uniqueness of solutions)

Suppose that the assumptions (I) and (II) are satisfied. If  $u_\mu(t, x)$  and  $v_\mu(t, x)$  ( $\mu=1, 2, \dots, k$ ) are two continuously differentiable solutions of (0,1) in  $D(T)$  satisfying the same initial conditions (0,2) and belonging to  $\mathfrak{F}(a, b)$  for some constants  $a$  and  $b$ , then  $u_\mu(t, x) \equiv v_\mu(t, x)$  ( $\mu=1, 2, \dots, k$ ) in  $D(T)$ .

## 3. Preliminary Lemmas

We begin this section with the following basic lemma.

**Lemma 1.** Let  $f(z_1, \dots, z_m)$  be a function which is holomorphic in the domain  $G(\delta) = \{z = x + \sqrt{-1}y \in C^m; |z_j| < \delta, j=1, 2, \dots, m\}$  and satisfies the inequality

$$|f(z_1, \dots, z_m)| \leq M\rho^{-\alpha} \tag{3.1}$$

there for some positive constants  $M$  and  $\alpha$ , where  $\rho = \delta - \text{Max}_{1 \leq j \leq m} |z_j|$ . Then the following inequality holds in  $G(\delta)$  for each  $j=1, 2, \dots, m$ ;

$$\left| \frac{\partial f}{\partial x_j}(x_1 + \sqrt{-1}y_1, \dots, x_m + \sqrt{-1}y_m) \right| \leq \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} M\rho^{-\alpha-1} \tag{3.2}$$

Proof. For arbitrary  $z^0 = (z_1^0, \dots, z_m^0) \in G(\delta)$  we take a circle  $C_j$  in the  $z_j$ -plane with center  $z_j^0$  and radius  $\frac{\rho}{1+\alpha}$  where  $\rho = \delta - \text{Max}_{1 \leq j \leq m} |z_j|$ . If  $z_j \in C_j$  ( $j=1, 2, \dots, m$ ), then  $\delta - |z_j| \geq \delta - |z_j^0| - |z_j - z_j^0| \geq \frac{\alpha}{1+\alpha} \rho$  and hence we have the inequality

$$|f(z)| \leq M \left( \rho - \frac{\alpha}{1+\alpha} \right) \rho^{-\alpha} = \frac{(1+\alpha)^\alpha}{\alpha^\alpha} M \rho^{-\alpha}.$$

Thus in view of Cauchy's integral formula we get the inequalities

$$\left| \frac{\partial f}{\partial x_j}(z^0) \right| \leq \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} M \rho^{-\alpha-1}, \quad j = 1, 2, \dots, m$$

Q.E.D.

In the proof of Theorem 1 and Theorem 2 we may suppose without loss of generality that the initial values  $\varphi_\mu$  all vanish. Then the system (0.1)–(0.2) is equivalent to the following system of integro-differential equations:

$$u_\mu(t, x) = \Phi_\mu[u(t, x)], \quad \mu = 1, 2, \dots, k \quad (3.3)$$

where for every  $\mu$

$$\begin{aligned} \Phi_\mu[u] = & \sum_{\nu=1}^k \left\{ \sum_{j=1}^m \int_0^t A_{\mu\nu j}(\tau, x) \frac{\partial u_\nu}{\partial x_j}(\tau, x) d\tau + \int_0^t B_{\mu\nu}(\tau, x) u_\nu(\tau, x) d\tau \right\} \\ & + \int_0^t f_\mu(\tau, x) d\tau. \end{aligned}$$

Therefore in order to prove Theorem 1 and Theorem 2, it is sufficient to prove the existence and the uniqueness of solutions of (3.3) respectively.

First of all we shall prove the following local existence theorem.

**Lemma 2.** *Suppose that the assumptions (I), (II) and (III) are satisfied. Then for arbitrary  $x^0 \in R^m$  there exists a solution  $u(t, z)$  of (3.3) which is continuously differentiable in  $(t, z)$  and holomorphic with respect to  $z$  in*

$$\Delta(x^0) = \{(t, z); 0 \leq t \leq T_1, |z_j - x_j^0| < R_1 - L_1 t, j = 1, 2, \dots, k\}$$

where  $0 < R_1 < \text{Min} \left\{ 1, \gamma, \left( \frac{1+\alpha}{\alpha} \right)^{1+\alpha} (1-\alpha) \frac{mA}{B} \right\}$

$$L_1 = \frac{mkA}{\kappa} \left( \frac{1+\alpha}{\alpha} \right)^{1+\alpha},$$

$$T_1 = \text{Min} \{T, R_1/L_1\}$$

with any fixed constants  $\alpha$  and  $\kappa$  satisfying  $0 < \alpha < 1$  and  $0 < \kappa < 1$ .

*Proof.* In the first place we note that

$$g_\mu(t, z) \in C_{(t, z)}^1[\mathfrak{D}_\gamma(T)] \cap A_{(z)}[\mathfrak{D}_\gamma(T)]$$

implies

$$\Phi_\mu[g(t, z)] \in C_{(t, z)}^1[\mathfrak{D}_\gamma(T)] \cap A_{(z)}[\mathfrak{D}_\gamma(T)].$$

Next consider the sequence of functions  $u_\mu^{(n)}(t, z)$  defined inductively as follows:

$$\begin{aligned} u_\mu^{(0)}(t, z) &\equiv 0 \\ u_\mu^{(n+1)}(t, z) &= \Phi_\mu[u^{(n)}(t, z)], \quad n = 0, 1, \dots \end{aligned} \quad (3.4)$$

Then from  $u_\mu^{(0)}(t, z) \in C_{(t, z)}^1[\mathfrak{D}_\gamma(T)] \cap A_{(z)}[\mathfrak{D}_\gamma(T)]$ , it follows that  $u_\mu^{(n+1)}(t, z) \in C_{(t, z)}^1[\mathfrak{D}_\gamma(T)] \cap A_{(z)}[\mathfrak{D}_\gamma(T)]$  for all positive integers  $n$ .

$$\text{Let} \quad \Psi_\mu[u] = \Phi_\mu[u] - \int_0^t f_\mu(\tau, z) d\tau.$$

Then  $u_\mu^{(h+1)} - u_\mu^{(h)} = \Psi_\mu[u^{(h)} - u^{(h-1)}]$ ,  $h = 1, 2, \dots$ . To demonstrate the convergence of the sequence  $\{u_\mu^{(n)}(t, z)\}$  we consider the series:

$$\begin{aligned} u_\mu^{(n+1)}(t, z) &= \sum_{h=1}^n \{u_\mu^{(h+1)}(t, z) - u_\mu^{(h)}(t, z)\} + u_\mu^{(1)}(t, z) \\ &= \sum_{h=1}^n \Psi_\mu[u^{(h)} - u^{(h-1)}] + u_\mu^{(1)}(t, z). \end{aligned}$$

It is obvious that for given  $\alpha$  with  $0 < \alpha < 1$ , there exists a positive constant  $M$  such that

$$|u_\mu^{(1)} - u_\mu^{(0)}| \leq \int_0^t |f_\mu(\tau, z)| d\tau \leq M \rho^{-\alpha} \quad \text{in } \Delta(x^0) \quad (3.5)$$

where  $\rho = (R_1 - L_1 t - \text{Max}_{1 \leq j \leq m} \{|z_j - x_j^0|\})$ , and hence we get

$$\int_0^t |u_\mu^{(1)} - u_\mu^{(0)}| d\tau \leq \frac{M}{(1-\alpha)L_1} R_1^{1-\alpha} \quad \text{in } \Delta(x^0).$$

From Lemma 1 and (3.5) we obtain the inequality

$$\left| \int_0^t \frac{\partial(u_\nu^{(1)} - u_\nu^{(0)})}{\partial x_j} d\tau \right| \leq \left( \frac{1+\alpha}{\alpha} \right)^{1+\alpha} \frac{M}{L_1} (\rho^{-\alpha} - R_1^{-\alpha}).$$

Hence we easily get the following inequality:

$$|u_\mu^{(2)} - u_\mu^{(1)}| \leq mkA \left( \frac{1+\alpha}{\alpha} \right)^{1+\alpha} \frac{M}{L_1} (\rho^{-\alpha} - R_1^{-\alpha}) + \frac{kBM}{(1-\alpha)L_1} R_1^{1-\alpha}.$$

The assumptions on the constants  $L_1$  and  $R_1$  lead to the estimates

$$|u_\mu^{(2)} - u_\mu^{(1)}| \leq \kappa M \rho^{-\alpha} \quad (\mu = 1, \dots, k) \text{ in } \Delta(x^0).$$

Therefore we obtain inductively for all positive integers  $n$

$$|u_\mu^{(n+1)} - u_\mu^{(n)}| \leq \kappa^n M \rho^{-\alpha} \quad (\mu = 1, \dots, k) \text{ in } \Delta(x^0). \quad (3.6)$$

This proves that the sequence  $\{u_\mu^{(n)}(t, z)\}$  converges uniformly to a function  $u_\mu(t, z)$  on any closed subdomain of  $\Delta(x^0)$ , and therefore  $u_\mu(t, z)$  belongs to  $C_{(t, z)}^1[\Delta(x^0)] \cap A_{(z)}[\Delta(x^0)]$  and  $u_\mu(t, z) = \Phi_\mu[u(t, z)]$  in  $\Delta(x^0)$ . Q.E.D.

From the above proof of Lemma 2, we see obviously the following corollary.

**Corollary 1.** *Let the functions  $u_\mu(t, z)$  ( $\mu=1, \dots, k$ ) be a solution obtained in Lemma 2, then we have the following inequalities:*

$$|u_\mu(t, z)| \leq \frac{M}{1-\kappa} \rho^{-\alpha} \quad \text{in } \Delta(x^0), \quad \mu = 1, \dots, k,$$

where  $M = \sup_{\substack{(t, z) \in \Delta(x^0) \\ 1 \leq \mu \leq k}} \{\rho^\alpha T_1 |f_\mu(t, z)|\}$ .

From now on we shall denote by  $u_\mu(t, z, x^0)$  the solution of (3.3) in  $\Delta(x^0)$  constructed in the proof of Lemma 2.

#### 4. Proof of Theorems

Proof of Theorem 1. We shall show that by the analytic continuation with respect to  $z$  of the local solutions whose existence was established in Lemma 2, we get a global solution of (3.3). For this purpose it suffices to prove that for arbitrary  $z \in \Delta(x^0) \cap \Delta(x^1)$  ( $x^0, x^1 \in R^m$ ),  $u_\mu(t, z, x^0)$  is equals to  $u_\mu(t, z, x^1)$  ( $\mu=1, \dots, k$ ).

Letting  $v_\mu(t, z) = u_\mu(t, z, x^0) - u_\mu(t, z, x^1)$ , we have  $v_\mu(0, z) \equiv 0$ ,  
 $v_\mu(t, z) = \Psi_\mu[v(t, z)]$  in  $\Delta(x^0) \cap \Delta(x^1)$ .

If  $R$  is a such positive number that

$$\Delta' = \left\{ (t, z); 0 \leq t \leq T_2, \left| z_j - \frac{x_j^0 + x_j^1}{2} \right| < \tilde{R} - L_1 t, j = 1, \dots, k \right\} \subset \Delta(x^0) \cap \Delta(x^1),$$

then we get the following inequalities as in the proof of Lemam 2:

$$|v_\mu(t, z)| = |\Psi_\mu[v(t, z)]| \leq \kappa \tilde{M} \tilde{\rho}^{-\alpha} \quad \text{in } \Delta', \quad \mu = 1, \dots, k$$

where  $\tilde{\rho} = \left( \tilde{R} - L_1 t - \text{Max}_{1 \leq j \leq m} \left\{ \left| z_j - \frac{x_j^0 + x_j^1}{2} \right| \right\} \right)$ ,

$$\tilde{M} = \sup_{\substack{(t, z) \in \Delta' \\ 1 \leq \mu \leq k}} \{\tilde{\rho}^\alpha |v_\mu(t, z)|\}.$$

From these inequalities it follows that  $\tilde{M} \tilde{\rho}^{-\alpha} \leq \kappa \tilde{M} \tilde{\rho}^{-\alpha}$  for given  $\kappa$  such that  $0 < \kappa < 1$ . This shows that  $\tilde{M} = 0$  and hence  $v_\mu(t, z) \equiv 0$  ( $\mu=1, \dots, k$ ) in  $\Delta'$ . Thus in view of the analyticity of  $v_\mu$  with respect to  $z$  we get  $v_\mu(t, z) \equiv 0$  in  $\Delta(x^0) \cap \Delta(x^1)$  ( $\mu=1, \dots, k$ ), obtaining a global solution  $u_\mu(t, z)$  of (3.3) in  $\mathfrak{D}_{\gamma_1}(T_1)$ .

By Corollary 1 and Theorem 1 we can show without difficulty:

**Corollary 2.** *If  $|f_\mu(t, z)| \leq M \exp(-ae^{b|x^1|})$  ( $\mu=1, \dots, k$ ) in  $\mathfrak{D}_\gamma(T)$  for some positive constants  $a, b$  and  $M$ , then for any given positive number  $a' (< a)$*

there exist positive numbers  $M'$  and  $T_1$  such that the solution  $u_\mu(t, x)$  ( $\mu=1, \dots, k$ ) of (3.3) satisfies the inequality

$$|u_\mu(t, x)| \leq M' \exp(-a'e^{b|x|}) \quad \text{in } D(T_1).$$

**Lemma 3.** For arbitrarily given positive number  $\varepsilon$  and positive constants  $a$  and  $b$ , there exist positive numbers  $a'$ ,  $b'$  and  $\gamma$  such that the inequality

$$\exp\{-(a+\varepsilon)e^{b|x|}\} \geq \exp\left\{-a' \sum_{\nu=1}^m \cosh(b'z_\nu)\right\}$$

holds in  $\mathfrak{D}_\gamma(T)$ .

$$\begin{aligned} \text{Proof. } |\exp\{-a' \cosh(b'z_\nu)\}| &= \exp\{-a' \Re \cosh(b'z_\nu)\} \\ &\leq \exp\left\{-\frac{a' \cos(b'y_\nu)}{2} e^{b'|x_\nu|}\right\}. \end{aligned}$$

For fixed  $\theta$  satisfying  $0 < \theta < \frac{2}{\pi}$ ,  $\cos(b'y_\nu) \geq \cos \theta$  when  $|y_\nu| \leq \theta/b'$ .

$$\begin{aligned} \text{Thus } |\exp\{-a' \sum_{\nu=1}^m \cosh(b'z_\nu)\}| &\leq \exp\left\{-\frac{a' \cos \theta}{2} \sum_{\nu=1}^m e^{b'|x_\nu|}\right\} \\ &\leq \exp\left\{-\frac{a' \cos \theta}{2} e^{\frac{b'}{m}|x|}\right\}, \end{aligned}$$

and setting  $b' = \sqrt{m}b$  and  $a' = \frac{2(a+\varepsilon)}{\cos \theta}$  we have

$$\exp\{-(a+\varepsilon)e^{b|x|}\} \geq \exp\left\{-a' \sum_{\nu=1}^m \cosh(b'z_\nu)\right\} \quad \text{in } \mathfrak{D}_\gamma(T)$$

when  $|y_\nu| \leq \gamma = \frac{\theta}{\sqrt{m}b}$ ,  $\nu=1, 2, \dots, m$ .

Q.E.D.

Proof of Theorem 2. Set

$$L_\mu[u] = \frac{\partial u_\mu}{\partial t} - \sum_{\nu=1}^k \left\{ \sum_{j=1}^m A_{\mu\nu j}(t, x) \frac{\partial u_\mu}{\partial x_j} + B_{\mu\nu}(t, x) u_\nu \right\}$$

and for every  $\sigma$  ( $\sigma=1, 2, \dots, k$ )

$$\begin{aligned} \tilde{L}_\mu^\sigma[u] &= -\frac{\partial u_\mu}{\partial t} + \sum_{\nu=1}^k \left\{ \sum_{j=1}^m \frac{\partial}{\partial x_j} [A_{\mu\nu j}(t, x_\nu) u_\mu] - B_{\mu\nu}(t, x) u_\nu \right\} \\ &\quad - e^{-\sqrt{-1}x\xi} \exp\left\{-a' \sum_{\nu=1}^m \cosh(b'x_\nu)\right\} \cdot \delta_{\sigma\mu} \end{aligned}$$

where  $\xi$  is an arbitrary real vector and  $a'$ ,  $b'$ ,  $\gamma$  are positive constants such that the conclusion of Lemma 3 holds when  $\varepsilon > 0$  was given in advance, and  $\delta_{\mu\sigma}$  is the Kronecker's delta.



The system of equations  $\tilde{L}_\mu^\sigma[u]=0$  ( $\mu=1, \dots, k$ ) is of similar form to the system of equations considered in Theorem 1, and considering  $t$  in the negative direction in Theorem 1, we can conclude that there exist a positive number  $T_0$  ( $\leq T_1$ ) such that for any  $T \in [0, T_0]$  there is a system of solutions  $w_\mu(t, x)$  of  $\tilde{L}_\mu^\sigma[u]=0$  ( $\mu=1, \dots, k$ ) in  $D(T)$  satisfying the initial condition  $w_\mu(T, x)=0$ . Moreover in view of the Corollary 2, we get the following inequalities:

$$|w_\mu(t, x)| \leq M' \exp \left\{ - \left( a + \frac{\varepsilon}{2} \right) e^{b|x_1|} \right\} \quad \text{in } D(T) \quad (4.1)$$

( $\mu=1, \dots, k$ ) for some positive constant  $M$  depending on  $\varepsilon$ , if we choose the constant  $a$  in that Corollary appropriately.

Suppose that  $u(t, x)$  and  $v(t, x)$  are two solutions of the system (0.1) with the initial conditions (0.2). Suppose also that  $u(t, x)$  and  $v(t, x)$  belong to  $\mathfrak{F}(a, b)$  for some positive numbers  $a$  and  $b$ . Then the function  $u-v$  satisfies

$$\begin{aligned} L_\mu[u-v] &= 0, \\ [u_\mu - v_\mu](0, x) &= 0 \quad (\mu = 1, \dots, k) \end{aligned}$$

and the inequalities

$$|[u_\mu(t, x) - v_\mu(t, x)]| \leq K \cdot \exp(ae^{b|x_1|}) \quad (4.2)$$

( $\mu=1, \dots, k$ ) in  $D(T)$  for some positive constant  $K$  and for any  $T \in [0, T_0]$ .

Since

$$\sum_{\mu=1}^k \iint_{D(T)} \{w_\mu L_\mu[u-v] - (u_\mu - v_\mu) \tilde{L}_\mu^\sigma[w]\} dx dt = 0$$

we have

$$\int_0^t dt \int_{R^m} e^{-\nu^{-1}x\xi} [(u_\sigma - v_\sigma) \exp \{-a' \sum_{\nu=1}^m \cosh(b'|x_\nu|)\}] dx = 0$$

for any  $\xi \in R^m$  and any  $t \in [0, T_0]$ . Thus for any  $\xi \in R^m$  and any  $t \in [0, T_0]$ , we obtain

$$\int_{R^m} e^{\nu^{-1}x\xi} [(u_\sigma(t, x) - v_\sigma(t, x)) \exp \{-a' \sum_{\nu=1}^m \cosh(b'|x_\nu|)\}] dx = 0 \quad (4.3)$$

Since  $|(u_\sigma - v_\sigma) \exp \{-a' \sum_{\nu=1}^m \cosh(b'|x_\nu|)\}| \leq \exp\{-\varepsilon e^{b|x_1|}\}$ ,

(4.3) means that the Fourier Transform of the integrable continuous function  $(u_\sigma - v_\sigma) \exp \{-a' \sum_{\nu=1}^m \cosh(b'|x_\nu|)\}$  vanishes identically in  $R^m$  for each  $t \in [0, T_0]$ . And since  $\exp \{-a' \sum_{\nu=1}^m \cosh(b'|x_\nu|)\} \neq 0$  in  $R^m$ , we get

$$u_\sigma(t, x) - v_\sigma(t, x) \equiv 0 \quad \text{in } D(T_0).$$

As  $\sigma$  is arbitrary,  $u_\sigma(t, x) \equiv v_\sigma(t, x)$  in  $D(T_0)$  for every  $\sigma$  ( $\sigma=1, \dots, k$ ).

Now suppose that there exists a  $T' \in [0, T]$  such that  $u_\mu(T', x) - v_\mu(T', x) \not\equiv 0$  in  $R^m$  for some  $\mu$ , and let  $T_2$  be the infimum of such  $T_1$ . Then  $u_\mu(t, x) \equiv v_\mu(t, x)$  in  $D(T_2)$ . Taking  $T'_2$  and  $T_3$  such that  $T_3 - T_2 \leq T_0$  and  $T'_2 < T_2 < T_3$ , and repeating the above argument in the interval  $[T'_2, T_3]$ , we get  $u_\mu(t, x) = v_\mu(t, x)$  for  $(t, x) \in \{D(T_3) - D(T'_2)\} = \{(t, x); T'_2 < t \leq T_3, x \in R^m\}$ . This contradicts the definition of  $T_2$ , and hence we get the conclusion

$$u_\mu(t, x) \equiv v_\mu(t, x) \quad \text{in } D(T)$$

for every  $\mu$  ( $\mu=1, \dots, k$ ).

Q.E.D.

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