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Author(s)	Kametani, Yukio; Nagatomo, Yasuyuki
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CONSTRUCTION OF c_2-SELF-DUAL BUNDLES ON A QUATERNIONIC PROJECTIVE SPACE

Yukio KAMETANI and Yasuyuki NAGATOMO

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1. Introduction

In the A.D.H.M.-construction of anti-self-dual bundles on S^4, the Penrose twistor transformation $\pi_Z : \boldsymbol{P}^3 \to S^4$ plays an important role [2],[6]. This fibration is easily obtaind when we identify S^4 with $\boldsymbol{HP}^1$ and $\boldsymbol{C}^2$ with $\boldsymbol{H}$. An anti-self-dual vector bundle F with unitary structure on S^4 is lifted to a holomorphic vector bundle $\tilde{F}$ on $\boldsymbol{P}^3$ with respect to the induced connection. $\boldsymbol{P}^3$ has a real structure σ induced by the right multiplication by the unit quaternion j which is different from the usual real strucure and preserves the fibration. Although σ on $\boldsymbol{P}^3$ has no fixed points, it does have fixed lines. These are precisely the fibres of π_Z. So the fibres of π_Z are called *real lines* and are denoted by $\boldsymbol{P}_x$ ($x \in S^4$). Then the above holomorphic vector bundle $\tilde{F}$ has the following properties.

(1) $\tilde{F}$ restricted to an arbitrary real line $\boldsymbol{P}_x$ is holomorphically trivial.

(2) The cohomology group $H^1(\boldsymbol{P}^3, \tilde{F}(-2))$ vanishes.

By the definition of $\tilde{F}$, (1) is trivial. But (2) is a deep result given by Drinfeld-Manin [6], Rawnsley [16], Douady [5] and Hitchin [8]. On the other hand, Barth-Hulek [3] have shown the following. If a holomorphic vector bundle E on $\boldsymbol{P}^3$ satisfies $E \cong E^*$(E^* is the dual bundle) and $H^1(\boldsymbol{P}^3, E(-2)) = 0$ and if E restricted to some line is holomorphically trivial, then E is constructed by some monad. Consequently, A.D.H.M-construction is completed.

We take $\boldsymbol{HP}^n$ instead of $S^4 \cong \boldsymbol{HP}^1$. Differential-geometrically, $\boldsymbol{HP}^n$ is one of the quaternionic Kähler manifolds which are $4n$-dimensional oriented Riemannian manifolds whose holonomy groups are contained in the subgroup $Sp(n) \cdot Sp(1), n \geqq 1$. Nitta [13] and Mamone Capria and Salamon [4] have developed independently higher dimensional analogues of the notion of (anti-)self-dual connections on a quaternionic Kähler manifold. Those connections are called c_1,c_2 and c_3-self-dual connections in Galicki and Poon [7] which are Yang-Mills. We use these terminology.

As for the Penrose twistor space on a half-conformally flat manifold [1], we also have a higher dimensional analogue. Salamon showed that there is a twistor space Z which has a natural complex structure on an arbitrary quaternionic Kähler manifold M [18]. If we pull back c_2-self-dual form to Z, we get (1,1) form on

Z. Therefore every c_2-self-dual bundle on M is pulled back to a holomorphic bundle on Z.

From now on, we shall confine ourselves exclusively to a quaternionic projective space $\boldsymbol{H}P^n$. The twistor fibration of $\boldsymbol{H}P^n$ is the well known one $\pi: P^{2n+1} \to \boldsymbol{H}P^n$. We have a *real structure* and *real lines* P_x ($x \in \boldsymbol{H}P^n$) in the same way as $\pi_Z: \boldsymbol{P}^3 \to S^4$. Let F be a c_2-self-dual bundle with a unitary structure on $\boldsymbol{H}P^n$. We denote by $\tilde{F}$ the pull-back bundle of F. As in the 4-dimensional case, we can easily check that the above condition (1) still holds. Moreover, the second author showed a vanishing theorem such as (2) under a general situation. Namely, if M is a compact quaternionic Kähler manifold with positive scalar curvature and Z is the twistor space of M, then $H^1(Z,\tilde{F}(-2))=0$ [12]. But, for a higher-dimensional case, further information about cohomology groups is needed for an analogue of A.D.H.M.-construction.

The purpose of the present paper is to give a vanishing theorem of cohomology groups, which, together with the Beilinson's spectral sequence argument, provides a sufficient condition for the construction of some kind of bundles on $\boldsymbol{H}P^n$. A bundle F which we treat in this paper has a c_2-self-dual $Sp(r)$ connection ($r \geqq 2$) with total Chern class $c(F)=(1-x)^{-k}$ where x is the standard generator of $H^4(\boldsymbol{H}P^n, \boldsymbol{Z})$. We employ an algebraic geometrical technique and an induction on the dimension of the base space. The main idea is to intertwine the twistor spaces with even dimensional complex projective spaces. By our vanishing theorem, F is obtained by a monad of the form,

$$W^*(-1) \to \boldsymbol{H}^{r+k} \to W(1),$$

where $W=H^1(\boldsymbol{P}^{2n+1}, \tilde{F}(-1))$, as in the 4-dimensional case (see also [4]).

Finally, the second author wishes to thank Professor T. Nitta for his generous help and would like to make a grateful acknowledgement to Professors K. Ogiue and Y. Ohnita for many suggestions and kindly encouragement.

2. Preliminaries

Let M be a connected quaternionic Kähler manifold. Using the reduction theorem (see Kobayashi and Nomizu [9]), we see that the orthonormal frame bundle of the tangent bundle TM can be reduced to a principal $Sp(n)\cdot Sp(1)$-bundle P. Since the action of $Ad(g)$ ($g \in Sp(n)$) on $\mathfrak{sp}(1)$ is trivial, we take the vector bundle $E = P \times_{Ad} \mathfrak{sp}(1)$ associated with the adjoint representation. Then the vector bundle E has the following properties.

(1) E is a rank 3 subbundle of $End(TM)$.

(2) E has a local basis I,J,K such that

(i) the Riemannian metric g is hermitian for I,J,K, in the sense that

$$g_x(IX, IY) = g_x(JX, JY) = g_x(KX, KY) = g_x(X, Y),$$

for all $x \in M$ and all $X, Y \in T_xM$, and

(ii) $I^2 = J^2 = -1, \qquad IJ = -JI = K.$

(3) The connection induced by the Riemannian connection preserves E.

Conversely, the existence of such a vector bundle turns a Riemannian manifold into a quaternionic Kähler manifold. Therefore the vector bundle E is called the *quaternionic Kähler structure bundle* of M.

The vector bundle $\wedge^2 T^*M$ has the following holonomy invariant decomposition:

$$\wedge^2 T^*M = S^2\boldsymbol{H} \oplus S^2\boldsymbol{E} \oplus (S^2\boldsymbol{H} \oplus S^2\boldsymbol{E})^{\perp}, \tag{2.1}$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ are vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. In particular, $\boldsymbol{H}$ is a tautological quaternionic line bundle when the base space is a quaternionic projective space $\boldsymbol{H}P^n$. This decomposition can also be explained in terms of the Hodge $*$-operator as in the 4-dimensional case. To see this, we note that E is isomorphic to $S^2\boldsymbol{H}$ via the metric g. Explicitly, an element $A \in E_x$ is mapped to ω_A by

$$\omega_A(X, Y) = g(AX, Y) \qquad \text{for } X, Y \in T_xM.$$

Then, making use of $\omega_I, \omega_J, \omega_K$ which are locally defined 2-forms, we define a global 4-form Ω by,

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

It is known that Ω is non-degenerate and parallel on M. This form Ω is called the fundamental 4-form on M [10].

DEFINITION 2.1. ([7]) For a real number c an $\omega \in T^*M$ is called a *c-self-dual form* if

$$*\omega = c\omega \wedge \Omega^{n-1},$$

where "$*$" is the Hodge $*$-operator.

We notice that in case of $n = 1$ the above equation can be viewed as the self-dual or anti-self-dual equation on a 4-dimensional oriented Riemannian manifold.

Theorem 2.2. ([7]) *An $\omega \in \wedge^2 T^*M$ is a non-zero c-self-dual form if and only if*

(1) $\omega \in S^2\boldsymbol{H}$ *and* $c = c_1 = \dfrac{6n}{(2n+1)!}$,

(2) $\omega \in S^2E$ *and* $c=c_2=\dfrac{-1}{(2n-1)!}$,

(3) $\omega \in (S^2H \oplus S^2E)^{\perp}$ *and* $c=c_3=\dfrac{3}{(2n-1)!}$.

We shall treat metric connections on a complex vector bundle F equipped with a hermitian metric h.

DEFINITION 2.3. ([7]) A connection ∇ on F is called *c-self-dual* if its curvature 2-form R^{∇} is a End F-valued c-self-dual form.

REMARK. As we pointed out in the introduction, Nitta and Mamone Capria-Salamon have found these connections independently ([4],[13]). But they used different terminology. A'_2, B_2 and A''_2-connections used in Nitta's paper correspond to c_1,c_2 and c_3-self-dual connections, respectively. On the other hand, by Mamone Capria-Salamon, a c_2-self-dual connection is called a *self-dual* connection.

Theorem 2.4. ([4],[7],[13]) *Every c-self-dual connection is a Yang-Mills connection.*

REMARK. Moreover, if M is compact, then for $i=1$ or 2 an arbitrary c_i-self-dual connection minimizes the Yang-Mills functional [4],[7]. It is known that we have essentially unique non-flat c_1-self-dual connection over a simply connected quaternionic Kähler manifold whose dimension is greater than or equals to 8 [11].

In this paper, we are concerned with c_2-self-dual connections. Therefore, we give a property of c_2-self-dual forms. The decomposition (2.1) is based on the adjoint representation of $Sp(n)\cdot Sp(1)$. When the decomposition $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$ is regarded as one of Lie subalgebras of $\mathfrak{so}(4n)$, the subspace on which the adjoint action of $Sp(1)$ is trivial is just $\mathfrak{sp}(n)$. This observation implies the following.

Lemma 2.5. ([4; Proposition 1]). *An* $\omega \in \wedge^2 T^*M$ *is a* c_2*-self-dual form if and only if*

$$\omega_x(IX,IY)=\omega_x(JX,JY)=\omega_x(KX,KY),$$
$$\text{for all } x \in M \text{ and all } X, Y \in T_xM,$$

where I,J,K is a local basis which satisfies the condition (2) *of the quaternionic Kähler structure bundle E.*

Next, we illustrate the relation between the twistor fibration $\pi: \boldsymbol{P}^{2n+1} \to \boldsymbol{H}P^n$ and c_2-self-dual bundles briefly. We identify the complex numbers $\boldsymbol{C}$ with the subfield of the quaternions $\boldsymbol{H}$ generated by 1 and i. Similarly $\boldsymbol{H}^{n+1}$ gets identified with $\boldsymbol{C}^{2n+2}$ by writing $q_p = z_{0p} + jz_{1p}$, $0 \leqq p \leqq n$. A point of $\boldsymbol{H}P^n$ is defined by homogeneous coordinates $q_0, q_1, \cdots, q_n$, up to *right* multiplication by a non-zero quaternion. In terms of homogeneous coordinates, this fibration is given by

$$\pi: [z_{00}, z_{10}, \cdots, z_{0n}, z_{1n}] \to [q_0, q_1, \cdots, q_n]. \tag{2.2}$$

So we get a fibre bundle $\pi: \boldsymbol{P}^{2n+1} \to \boldsymbol{H}P^n$ with fibre $\boldsymbol{P}^1$. Then we obtain

$$\pi^*(S^2 E_{\boldsymbol{H}P_n}) \subset \wedge^{1,1} T^* \boldsymbol{P}^{2n+1}.$$

(see, for example, [4].) Consequently, we have the following theorem.

Theorem 2.6. ([4],[13]) *An arbitrary c_2-self-dual bundle F on $\boldsymbol{H}P^n$ is pulled back to a holomorphic bundle $\tilde{F}$ on $\boldsymbol{P}^{2n+1}$.*

Finally, we remark on the notations used in this paper. $\mathcal{O}(1)$ denotes the hyperplane bundle on $\boldsymbol{P}^m$. $\mathcal{O}(p)$, $p \in \boldsymbol{Z}$ is used as usual. $\tilde{F}(p)$ means $\tilde{F} \otimes \mathcal{O}(p)$. We do not distinguish between bundles and locally free sheaves.

We are in a position to state two main theorems.

Main Theorem 1. *Let F be an arbitrary c_2-self-dual bundle on $\boldsymbol{H}P^n$ with a unitary structure. Then, the following vanishing holds;*

$$H^1(\boldsymbol{P}^{2n+1}, \tilde{F}(p)) = 0 \qquad \textit{for } p \leqq -2,$$

where $\tilde{F}$ is the pull-back bundle of F to $\boldsymbol{P}^{2n+1}$.

The above theorem is easily obtained by using the vanishing theorem $H^1(Z, \tilde{F}(-2)) = 0$ which holds in a general context [12] and the standard exact sequence on $\boldsymbol{P}^{2n+1}$, but we reprove it in another way. Employing the same argument, we get a vanishing theorem for some special c_2-self-dual bundle, which gives us enough condition to construct such a bundle as the cohomology of a monad via the theorem of Beilinson.

Main Theorem 2. *Let F be an arbitrary c_2-self-dual bundle on $\boldsymbol{H}P^n$ with $Sp(r)$-structure and with total Chern class $c(F) = (1-x)^{-k}$, where $r \geqq 2$, $k \geqq 1$ and x is the standard generator of $H^4(\boldsymbol{H}P^n, \boldsymbol{Z})$. Then, we have*

$$H^i(\boldsymbol{P}^{2n+1},\tilde{F}(p))=0 \qquad \textit{for } 2\leqq i\leqq n \textit{ and } p\in\boldsymbol{Z},$$

where $\tilde{F}$ *is the pull-back bundle of* F *to* $\boldsymbol{P}^{2n+1}$.

REMARK 1. Applying Serre duality to Main Theorem 2, from a symplectic structure of F, we know

$$H^i(\boldsymbol{P}^{2n+1},\tilde{F}(p))=0 \qquad \text{for } 2\leqq i\leqq 2n-1 \text{ and } p\in\boldsymbol{Z},$$

REMARK 2. Main Theorem 2 is actually proved under a weaker assumption that

$$c_2(F)=k, \qquad \text{and} \qquad c_4(F)=\frac{1}{2}k(k+1).$$

Combined with Beilinson's spectral sequence (see [15 p.240]), our vanishing theorems yields the next classification.

Corollary. *If the hypothesis of* Main Theorem 2 *is satisfied, then* $\tilde{F}$ *is constructed by a monad of the following form;*

$$W(-1)\to V\to W^*(1),$$

where $W^*=H^1(\boldsymbol{P}^{2n+1},\tilde{F}(-1))$ *and* $V=H^1(\boldsymbol{P}^{2n+1},\tilde{F}\otimes\Omega^1)$ (*see* [4], *or* [16]).

However, the existence of such a vector bundle is another question.

REMARK. When $n=1$ or the base space is $\boldsymbol{H}P^1\cong S^4$, anti-self-dual forms are called c_2-self-dual forms in this paper.

3. Proof of Main Theorem 1

We begin with a lemma.

Lemma 3.1. *Let* $\boldsymbol{H}P^{n-1}$ *be an arbitrary quaternionic hyperplane of* $\boldsymbol{H}P^n$. *Then the restriction of any* c_2*-self-dual form on* $\boldsymbol{H}P^n$ *to* $\boldsymbol{H}P^{n-1}$ *is also a* c_2*-self-dual form on* $\boldsymbol{H}P^{n-1}$. *Moreover, the twistor space* $\boldsymbol{P}^{2n-1}$ *of* $\boldsymbol{H}P^{n-1}$ *is imbedded into the twistor space* $\boldsymbol{P}^{2n-1}$ *of* $\boldsymbol{H}P^{n-1}$ *is imbedded into the twistor space* $\boldsymbol{P}^{2n+1}$ *of* $\boldsymbol{H}P^n$ *in a natural way making the following diagram commutative.*

$$\begin{array}{ccc} \boldsymbol{P}^{2n-1} & \to & \boldsymbol{P}^{2n+1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \boldsymbol{H}P^{n-1} & \to & \boldsymbol{H}P^n \end{array}$$

Proof. Since $\boldsymbol{HP}^{n-1}$ is a quaternionic Kähler submanifold of $\boldsymbol{HP}^n$, the quaternionic Kähler structure bundle $E_{\boldsymbol{HP}^{n-1}}$ is obtained as the restriction of $E_{\boldsymbol{HP}^n}$. The first part of this lemma follows from lemma 2.5.

The latter part is verified by restricting the fibering (2.2) to $\boldsymbol{HP}^{n-1}$. □

Proof of Main Theorem 1. The proof is given by induction on the dimension of the base space. For $n=1$, the assertion has already been proved by Drinfeld-Manin, Hitchin and Rawnsley [6], [8], [17].

Suppose that the assertion is correct for $m-1$. We choose a quaternionic hyperplane $\boldsymbol{HP}^{m-1} \subset \boldsymbol{HP}^m$ and fix a sequence $\boldsymbol{P}^{2m-1} \subset \boldsymbol{P}^{2m} \subset \boldsymbol{P}^{2m+1}$, where $\boldsymbol{P}^{2m-1}$ and $\boldsymbol{P}^{2m+1}$ are the corresponding twistor spaces, respectively. (See Lemma 3.1.) If F is a c_2-self-dual bundle on $\boldsymbol{HP}^m$ with a unitary structure, the pull-back bundle $\tilde{F}$ is a holomorohic vector bundle on $\boldsymbol{P}^{2m+1}$ with a hermitian structure.

We restrict $\tilde{F}$ to $\boldsymbol{P}^{2m-1}$ and $\boldsymbol{P}^{2m}$. Note that $\tilde{F}|_{\boldsymbol{P}^{2m-1}}$ is the pull-back bundle of c_2-self-dual bundle $F|_{\boldsymbol{P}^{m-1}}$ from Lemma 3.1. We shall denote the restricted bundle by the same symbol $\tilde{F}$, when no confusion can arise. Then, by the induction hypothesis, we have

$$H^1(\boldsymbol{P}^{2m-1}, \tilde{F}(p))=0, \qquad \textit{for } p \leqq -2. \tag{3.1}$$

Since $\tilde{F}$ has trivial splitting type, for $p \leqq -1$, $\tilde{F}(p)$ can have no non-zero section over $\boldsymbol{P}^{2m-1}$, $\boldsymbol{P}^{2m}$ and $\boldsymbol{P}^{2m+1}$. Namely, we get

$$H^0(\boldsymbol{P}^{2m-1}, \tilde{F}(p))=H^0(\boldsymbol{P}^{2m}, \tilde{F}(p))=H^0(\boldsymbol{P}^{2m+1}, \tilde{F}(p))=0, \quad \textit{for } p \leqq -1. \tag{3.2}$$

Now consider the standard exact sequence of sheaves

$$0 \to \mathcal{O}(-1)|_{\boldsymbol{P}^{2m}} \to \mathcal{O}|_{\boldsymbol{P}^{2m}} \to \mathcal{O}|_{\boldsymbol{P}^{2m-1}} \to 0.$$

We tensor this sequence with $\tilde{F}(p)$ and take the associated long exact sequence of cohomology groups

$$\begin{aligned} \cdots \to H^0(\boldsymbol{P}^{2m-1}, \tilde{F}(p)) &\to H^1(\boldsymbol{P}^{2m}, \tilde{F}(p-1)) \\ &\to H^1(\boldsymbol{P}^{2m}, \tilde{F}(p)) \to H^1(\boldsymbol{P}^{2m-1}, \tilde{F}(p)) \to \cdots. \end{aligned}$$

This, together with (3.1) and (3.2), implies that, for $p \leqq -2$,

$$H^1(\boldsymbol{P}^{2m}, \tilde{F}(p-1)) \cong H^1(\boldsymbol{P}^{2m}, \tilde{F}(p)). \tag{3.3}$$

On the other hand, applying Serre duality and the Theorem B (see, for example, [15 p.10]), we know that

$$H^1(\boldsymbol{P}^{2m}, \tilde{F}(p))=0 \qquad \text{if } |p|>p_0,$$

where p_0 is a sufficiently large integer. This, combined (3.3), yields

$$H^1(\boldsymbol{P}^{2m},\tilde{F}(p))=0 \qquad \text{for } p\leqq -2. \tag{3.4}$$

Next, we consider the standard exact sequence on $\boldsymbol{P}^{2m+1}$

$$0\to\mathcal{O}(-1)\to\mathcal{O}\to\mathcal{O}|_{\boldsymbol{P}^{2m}}\to 0.$$

Tensoring this with $\tilde{F}(p)$, we take the associated long exact sequence of cohomology groups

$$\cdots\to H^0(\boldsymbol{P}^{2m},\tilde{F}(p))\to H^1(\boldsymbol{P}^{2m+1},\tilde{F}(p-1))$$
$$\to H^1(\boldsymbol{P}^{2m+1},\tilde{F}(p))\to H^1(\boldsymbol{P}^{2m},\tilde{F}(p))\to\cdots.$$

Making use of of (3.2) and (3.4), we get

$$H^1(\boldsymbol{P}^{2m+1},\tilde{F}(p-1))\cong H^1(\boldsymbol{P}^{2m+1},\tilde{F}(p)) \qquad \text{for } p\leqq -2. \tag{3.5}$$

Again, it turns out from Serre duality, the Theorem B and (3.5) that

$$H^1(\boldsymbol{P}^{2m+1},\tilde{F}(p))=0 \qquad \text{for } p\leqq -2.$$

Main Theorem 1 is thereby proved. □

4. Proof of Main Theorem 2

We apply the argument which is used in the proof of Main Theorem 1. To do so, we consider the sequence $\boldsymbol{P}^3\subset\boldsymbol{P}^4\subset\boldsymbol{P}^5$. Of course, $\boldsymbol{P}^3$ corresponds to the twistor space of some $\boldsymbol{HP}^1\subset\boldsymbol{HP}^2$. The following fact has been already known in the preceding section.

$$H^0(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \text{for } p\leqq -1,$$
$$H^1(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \text{for } p\leqq -2.$$

Since $\tilde{F}$ has a symplectic structure, Serre duality implies

$$H^3(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \text{for } p\geqq -3,$$
$$H^4(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \text{for } p\geqq -4.$$

Making use of the Riemann-Roch Theorem on $\boldsymbol{P}^4$, we get

$$\dim H^2(\boldsymbol{P}^4,\tilde{F}(-2))=\frac{1}{12}\{k(k+1)-k(k+1)\}=0,$$

where the hypothesis $c_2(\tilde{F})=k$, $c_4(\tilde{F})=\frac{1}{2}k(k+1)$ is used.

Tensoring the standard exact sequence on $\boldsymbol{P}^4$ with $\tilde{F}(p)$ and taking the associated long exact sequence of cohomology groups, we know by the vanishing theorem $H^1(\boldsymbol{P}^3,\tilde{F}(p))=0$, $p\leqq -2$ that a homomorphism

$$H^2(\boldsymbol{P}^4,\tilde{F}(p-1))\to H^2(\boldsymbol{P}^4,\tilde{F}(p))$$

is injective for each $p\leqq -2$. Hence, $H^2(\boldsymbol{P}^4,\tilde{F}(-2))=0$ yields

$$H^2(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \text{for } p\leqq -2.$$

Again, Serre duality gives

$$H^2(\boldsymbol{P}^4,\tilde{F}(p))=0, \qquad \textit{for all } p\in \boldsymbol{Z}.$$

Next, the standard exact sequence on $\boldsymbol{P}^5$ gives rise to the associated long exact sequence

$$\cdots\to H^1(\boldsymbol{P}^4,\tilde{F}(p))\to H^2(\boldsymbol{P}^5,\tilde{F}(p-1))$$
$$\to H^2(\boldsymbol{P}^5,\tilde{F}(p))\to H^2(\boldsymbol{P}^4,\tilde{F}(p))\to\cdots.$$

Our vanishing theorem yields

$$H^2(\boldsymbol{P}^5,\tilde{F}(p-1))\cong H^2(\boldsymbol{P}^5,\tilde{F}(p)), \qquad \text{for } p\leqq -2.$$

From Serre duality and the Theorem B, it follows that

$$H^2(\boldsymbol{P}^5,\tilde{F}(p))=0, \qquad \text{for } p\leqq -2.$$

Moreover, the above long exact sequence implies that a homomorphism

$$H^2(\boldsymbol{P}^5,\tilde{F}(p-1))\to H^2(\boldsymbol{P}^5,\tilde{F}(p))$$

is surjective for each $p\in \boldsymbol{Z}$. So, by induction, it turns out that

$$H^2(\boldsymbol{P}^5,\tilde{F}(p))=0, \qquad \text{for all } p\in \boldsymbol{Z}.$$

From Serre duality, we have

$$H^3(\boldsymbol{P}^5,\tilde{F}(p))=0, \qquad \text{for all } p\in \boldsymbol{Z}.$$

In case of $n\geqq 3$, the proof is made along almost the same line. Assume that

$$H^i(\boldsymbol{P}^{2m-1},\tilde{F}(p))=0 \qquad \text{for } 2\leqq i\leqq m-1 \text{ and } p\in \boldsymbol{Z}.$$

The long exact sequence associated to the standard exact sequence on $\boldsymbol{P}^{2m}$ gives

$$\cdots\to H^1(\boldsymbol{P}^{2m-1},\tilde{F}(p))\to H^2(\boldsymbol{P}^{2m},\tilde{F}(p-1))$$
$$\to H^2(\boldsymbol{P}^{2m},\tilde{F}(p))\to H^2(\boldsymbol{P}^{2m-1},\tilde{F}(p))\to\cdots.$$

Main Theorem 1 and the hypothesis of the induction implies that

$$H^2(\boldsymbol{P}^{2m},\tilde{F}(p-1))\cong H^2(\boldsymbol{P}^{2m},\tilde{F}(p)), \qquad \text{for } p\leqq -2,$$

and the map

$$H^2(\boldsymbol{P}^{2m},\tilde{F}(p-1))\rightarrow H^2(\boldsymbol{P}^{2m},\tilde{F}(p)),$$

is surjective for each $p\geqq -1$. Consequently, we use Serre duality and Theorem B again to get

$$H^2(\boldsymbol{P}^{2m},\tilde{F}(p))=0, \qquad \text{for all } p\in \boldsymbol{Z}.$$

With respect to $i=3,\cdots,m$, the proof is more simple. By now familiar argument, we have

$$H^i(\boldsymbol{P}^{2m},\tilde{F}(p-1))\cong H^i(\boldsymbol{P}^{2m},\tilde{F}(p)), \qquad \text{for all } p\in \boldsymbol{Z}.$$

Hence, it follows that

$$H^i(\boldsymbol{P}^{2m},\tilde{F}(p))=0, \qquad \text{for all } p\in \boldsymbol{Z}.$$

Applying the same argument to the standard exact sequence on $\boldsymbol{P}^{2m+1}$, we obtain Main Theorem 2. □

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Yukio Kametani
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192–03, Japan

Current address
Department of Mathematics
Faculty of Science and Technology
Saga University
Saga, 840 Japan

Yasuyuki Nagatomo
Department of Mathematics
Tokyo Metropolitan University
Hachioji, Tokyo 192–03, Japan

Current address
Department of Mathematics
Sophia University
Kioicho, Chiyoda-ku, Tokyo 102 Japan