Kamae, M. and Takahashi, S.
Osaka J. Math.
33 (1996), 915–925

A SHAPE THEOREM FOR THE SPREAD OF EPIDEMICS AND FOREST FIRES IN TWO-DIMENSIONAL EUCLIDEAN SPACE

MOTOKO KAMAE and SATOSHI TAKAHASHI

(Received October 12, 1995)

1. Introduction and statement of results

J.T. Cox and R. Durrett (1988, [2]) considered a model of epidemics and forest fires in $\boldsymbol{Z}^2$ with nearest neighbor interactions,and have shown the shape theorem for the spread of epidemics. Y. Zhang (1993, [6]) has dealt with a model which had not only nearest neighbor interactions but also interactions between further hosts, and proved the shape theorem for the model.

We consider a model of epidemics or forest fires in $\boldsymbol{R}^2$. Hosts are distributed according to a Poisson point process $X_\lambda=\{X_i\in\boldsymbol{R}^2\}_{i=1}^{\infty}$ of intensity $\lambda(>0)$ in $\boldsymbol{R}^2$. Each individual X_i, $i\in\boldsymbol{N}$ can be in one of three states 1,2 or 0. In the epidemic interpretation, 1 = susceptible, 2 = infected, and 0 = immune, while for a forest fire, 1 = alive, 2 = on fire, and 0 = burnt. The state of the process is represented by a function $\eta_t(X_i)\in\{0,1,2\}$, $i\in\boldsymbol{N}$, which is the state of X_i at time t. An individual stays infected for the amount of time 1, then it recovers and becomes immune. Once an individual becomes immune, it will be never infected. An infected individual X_i emits germs after a random time T_i from its infection. Germs emitted from X_i go to all individuals in $S_i=\{x\in\boldsymbol{R}^2 \mid |x-X_i|\leq 1\}$, the disk of radius 1 with its center at X_i. (If $T_i>1$, X_i really does not emit germs.) Let T_i, $i\in\boldsymbol{N}$ be nonnegative, i.i.d. random variables with distribution function F. We have $F(0)<1$ and $F(1)>0$. If a germ goes to a susceptible individual, then the individual immediately becomes infected. If the germ goes to an infected or immune individual, the individual does not change its state at all. Initially all points of X_λ in $S_0=\{x\in\boldsymbol{R}^2 \mid |x|\leq 1\}$, the disk of radius 1 with its center at 0, are infected and all other points of X_λ are susceptible:

$$\eta_0(X_i)=\begin{cases}2, & \text{if } X_i\in S_0\\ 1, & \text{otherwise.}\end{cases}$$

We construct the probability space representing our epidemic model (or our forest fire model). Let Ω_λ be the Poisson point process of intensity λ in $\boldsymbol{R}^2$ which consists of countable points valued in $[0,\infty)$ independently according to the distribution function F. An element of Ω_λ is denoted by $\{(X_i,T_i)\in \boldsymbol{R}^2\times[0,\infty)\}_{i=1}^{\infty}$, and $\{X_i\}_{i=1}^{\infty}$ by X_λ. Let $A(V,t,k)$ be the event that exactly k points of X_λ in V have values less than t:

$$A(V,t,k)=\{\{(X_i,T_i)\}_{i=1}^{\infty} \mid \#\{i\in \boldsymbol{N}\mid X_i\in V,\ T_i\le t\}=k\},$$

where $V\in\mathscr{B}(\boldsymbol{R}^2)$ is bounded, $t\ge 0$, and $k\in \boldsymbol{N}\cup\{0\}$. Let

$$P_\lambda(A(V,t,k))=\frac{(\lambda F(t)\,|\,V|)^k}{k!}e^{-\lambda F(t)|V|},$$

$$\mathscr{B}(\Omega_\lambda)=\sigma(\{A(V,t,k)\mid V\in\mathscr{B}(\boldsymbol{R}^2)\text{ is bounded, } t\ge 0,\ k\in \boldsymbol{N}\cup\{0\}\}).$$

Clearly $(\Omega_\lambda,\mathscr{B}(\Omega_\lambda),P_\lambda)$ $(\lambda>0)$ is a probability space. From now on we deal with this probability space which represents our model.

Before stating our result, we define some notions. For $\{(X_i,T_i)\}_{i=1}^{\infty}\equiv\omega\in\Omega_\lambda$, let

$$\tau_i=\begin{cases}T_i, & \text{if } T_i\le 1\\ \infty, & \text{if } T_i>1\end{cases}\qquad i\in \boldsymbol{N}.$$

τ_i is the time lag from the infection of X_i until X_i emit germs. We say that X_i is open if $\tau_i<\infty$, and closed otherwise. When X_i is open and infected, it emits germs and infects all points of X_λ in the disk S_i of radius 1 with its center at X_i. When X_i is closed, it cannot infect any points of X_λ in S_i. For $X_i,X_j\in X_\lambda$, we call $\{X_{i_1},\cdots,X_{i_K}\}$ a path from X_i to X_j if the following hold:

(i) $X_{i_1}=X_i,\quad X_{i_K}=X_j$

(ii) $|X_{i_K}-X_{i_{k+1}}|\le 1$ and $X_{i_K}\ne X_{i_{k+1}},\quad \forall k=1,\cdots,K-1$

In addition to the above, we say that $\{X_{i_1},\cdots,X_{i_K}\}$ is an open path from X_i to X_j if the following holds:

(iii) X_{i_K} is open, $\forall k=1,\cdots,K$

Let

$$C_0(\omega)=\left\{X_i\ \middle|\ \begin{array}{l}\text{There is a path from a point of } X_\lambda \text{ in}\\ S_0 \text{ to } X_i \text{ denoted by } \{X_{i_1},\cdots,X_{i_K}\},\text{ and}\\ X_{i_1},\cdots,X_{i_{K-1}} \text{ are open.}\end{array}\right\}. \tag{1.1}$$

We call $C_0(\omega)$ the cluster contaning the origin 0. In the epidemics interpretation, $C_0(\omega)$ is the set of points of X_λ that X_i will ever becomes infected if initially all points of X_λ in S_0 are infected and all other points of X_λ susceptible. Let

$$\lambda_c=\inf\{\lambda\mid P_\lambda(|C_0|=\infty)>0\}.$$

λ_c is the critical value whether the epidemic spreads infinitely or not. For $\omega \in \Omega_\lambda$, let

$$t(X_i,X_j)=\inf\{\sum_{l=1}^{L-1} \tau_{i_l} \mid \{X_{i_1},\cdots,X_{i_L}\} \text{ is a path from } X_i \text{ to } X_j.\},\quad X_i,X_j \in X_\lambda.$$

$t(X_i,X_j)=\infty$ if there is no path from X_i to X_j, and $t(X_i,X_j)=0$ if $X_i=X_j$. $t(X_i,X_j)$ is the minimum time of the infection of X_j if only X_i is infected initially. For $x,y \in \boldsymbol{R}^2$, let

$$t'(x,y)(\omega)=\inf_{\substack{S_x \ni X_i \\ S_y \ni X_j}} t(X_i,X_j),$$

where $S_x=\{x' \in \boldsymbol{R}^2 \mid |x-x'| \leq 1\}$. If there is no point of X_λ in S_x nor S_y, we let $t'(x,y)(\omega)=\infty$. $t'(x,y)$ is the minimum time for a point of X_λ in S_y to be infected if the points of X_λ in S_x are infected and others susceptible initially. Let $e_1=(1,0)$, then $\liminf_{n\to\infty}\frac{t'(0,ne_1)}{n}$ is almost surely constant ([2]), and we denote it by γ. γ is the average time for the epidemic to go the unit distance. Let

$$\zeta_t(\omega)=\{X_i \in X_\lambda \mid \eta_t(X_i)=0\},$$
$$\xi_t(\omega)=\{X_i \in X_\lambda \mid \eta_t(X_i)=2\}.$$

ζ_t is the set of points of X_λ which are immune at time t, and ξ_t is the set of points of X_λ which are infected at time t.

Here we consider in what shape it spreads out.

Theorem 1. *Assume $\gamma>0$. Let D be the disk of radius $\frac{1}{\gamma}$ with its center at the origin. If $\lambda>\lambda_c$, then for any $\varepsilon>0$, we have*

$$P_\lambda(C_0 \cap t(1-\varepsilon)D \subset \zeta_t \subset t(1+\varepsilon)D \quad \text{for all sufficiently large } t)=1,$$
$$P_\lambda(\xi_t \subset t(1+\varepsilon)D \backslash t(1-\varepsilon)D \quad \text{for all sufficiently large } t)=1.$$

2. Probability estimates of events in the model in Z_n^2

We prove Theorem 1 by approximating $\boldsymbol{R}^2$ with the lattice $\boldsymbol{Z}_n^2$:

$$\boldsymbol{Z}_n^2=\{(\frac{x}{n},\frac{y}{n}) \mid (x,y) \in \boldsymbol{Z}^2\},\quad n=2,3,\cdots$$

We construct a site percolation in $\boldsymbol{Z}_n^2$ which corresponds to that in $\boldsymbol{R}^2$. For $z=(z_1,z_2) \in \boldsymbol{Z}_n^2$, let

$$B_n(z)=[z_1-\frac{1}{2n}, z_1+\frac{1}{2n})\times[z_2-\frac{1}{2n}, z_2+\frac{1}{2n}).$$

For $z\in \boldsymbol{Z}_n^2$ and $\omega\in\Omega_\lambda$, we define T_z by

$$T_z(\omega)=\begin{cases}\inf\limits_{X_i\in B_{n(z)}} T_i, & \text{if } B_n(z)\cap X_\lambda\neq\emptyset\\ \infty, & \text{if } B_n(z)\cap X_\lambda=\emptyset\end{cases}$$

T_z, $z\in\boldsymbol{Z}_n^2$ are i.i.d. random variables. We say that a site $z\in\boldsymbol{Z}_n^2$ is open if $T_z\leq 1$, and closed otherwise. Then for each site z, z is open or closed independently, and by using the distribution function F of T_i,

$$P_\lambda(z \text{ is open})=1-e^{-\lambda F(1)n^{-2}}\equiv p_n(\lambda),$$
$$P_\lambda(z \text{ is closed})=e^{-\lambda F(1)n^{-2}}.$$

For $z_1,z_2\in\boldsymbol{Z}_n^2$, we say that z_1 and z_2 are adjacent if $|z_1-z_2|\leq 1-\frac{\sqrt{2}}{n}$. If $|z_1-z_2|\leq 1-\frac{\sqrt{2}}{n}$, then when an open point of X_λ in $B_n(z_1)$ is infected, all healthy points of X_λ in $B_n(z_2)$ are infected. By this adjacency relation, we define N_z, the neighbor of $z\in\boldsymbol{Z}_n^2$, as

$$N_z=\{z'\in\boldsymbol{Z}_n^2 \mid |z-z'|\leq 1-\frac{\sqrt{2}}{n}\}.$$

To any $z'(\neq z)\in N_z$, we give a bond oriented from z to z', and denote it by $\langle z,z'\rangle_n$. We let $M_n\equiv 1-\frac{\sqrt{2}}{n}$, and denote by $\boldsymbol{Z}_n^2(M_n)$ all oriented bonds $\langle z_1,z_2\rangle_n$ with $|z_1-z_2|\leq M_n$. For any oriented bond $\langle z_1,z_2\rangle_n\in\boldsymbol{Z}_n^2(M_n)$, we say that $\langle z_1,z_2\rangle_n$ is open if z_1 and z_2 are open, and closed otherwise. For $z_1,z_2\in\boldsymbol{Z}_n^2$, we call $\{z_{i_1},\cdots,z_{i_K}\}$ a path of $\boldsymbol{Z}_n^2(M_n)$ from z_1 to z_2 if the follwing hold:

(i) $z_{i_1}=z_1,\quad z_{i_K}=z_2$

(ii) $|z_{i_k}-z_{i_{k+1}}|\leq M_n$ and $z_{i_k}\neq z_{i_{k+1}},\quad \forall k=1,\cdots,K-1$

In addition to the above, we say that $\{z_{i_1},\cdots,z_{i_K}\}$ is an open path of $\boldsymbol{Z}_n^2(M_n)$ from z_1 to z_2 if the following holds:

(iii) $\langle z_{i_k},z_{i_{k+1}}\rangle_n$ is open, $\quad\forall k=1,\cdots,K-1$

For two open paths of $\boldsymbol{Z}_n^2(M_n)$ denoted by r_1, r_2, we say that r_1 and r_2 are connected in $\boldsymbol{Z}_n^2(M_n)$ if there exist open sites $z_1\in r_1$, $z_2\in r_2$ such that $z_1=z_2$ or $|z_1-z_2|\leq M_n$. Let

$$D_z^{(n)}=\{z'\in \boldsymbol{Z}_n{}^2 \mid \text{There is an open path of } \boldsymbol{Z}_n{}^2(M_n) \text{ from } z \text{ to } z'.\}$$

be the open cluster containing $z\in \boldsymbol{Z}_n{}^2$. Let $p_c(\boldsymbol{Z}_n{}^2(M_n))$ be the critical probabilty of the site percolation with the adjacency relation as mentioned above, then from $0<p_c(\boldsymbol{Z}_n{}^2(M_n))<1$ ([3]), there exists $\lambda_c{}^{(n)}$, where $0<\lambda_c{}^{(n)}<\infty$, such that

$$p_c(\boldsymbol{Z}_n{}^2(M_n))=1-e^{-\lambda_c{}^{(n)}F(1)n^{-2}}.$$

We call $\lambda_c{}^{(n)}$ the critical value in $\boldsymbol{Z}_n{}^2$, as it satisfies

$$\lambda_c{}^{(n)}=\inf\{\lambda \mid P_\lambda(|D_0^{(n)}|=\infty)>0\}.$$

Lemma 2.1 ([7], [8]). *Let*

$$\lambda_c=\inf\{\lambda \mid P_\lambda(|C_0|=\infty)>0\},$$
$$\lambda_T=\inf\{\lambda \mid E_\lambda|C_0|=\infty\},$$

where $E_\lambda|C_0|=\int_{\Omega_\lambda}|C_0|dP_\lambda$, *then* $\lim_{n\to\infty}\lambda_c{}^{(n)}=\lambda_c=\lambda_T$ *and* $0<\lambda_c<\infty$.

Later we will use Lemma 2.1 to prove Theorem 1 in Section 3.

For $\omega\in\Omega_\lambda$, let $m(z)$, $z\in \boldsymbol{Z}_n{}^2$ be the minimum $m\in \boldsymbol{N}$ $(m>2n)$ satisfying the following:

(i) There is an open path of $\boldsymbol{Z}_n{}^2(M_n)$ from $z+[-\frac{m}{2n},\frac{m}{2n}]^2$ to ∞ in $\boldsymbol{Z}_n{}^2\backslash\{z+[-\frac{m}{2n},\frac{m}{2n}]^2\}$.

(ii) There is an open circuit of $\boldsymbol{Z}_n{}^2(M_n)$ in the annulus $\{z+[-\frac{3m}{2n},\frac{3m}{2n}]^2\}\backslash\{z+[-\frac{m}{2n},\frac{m}{2n}]^2\}$.

Corollary 2.2 ([6]). *If* $\lambda>\lambda_c{}^{(n)}$, *then there exist positive constants* $K(\lambda)$ *and* β *such that for any* $z\in \boldsymbol{Z}_n{}^2$, *we have*

$$P_\lambda(m(z)>2^l K(\lambda))\le\beta e^{-2^l\gamma},\quad \gamma=\log 2,\quad \forall l=0,1,\cdots$$

By Corollary 2.2, *we have*

$$P_\lambda(m(z)<\infty,\quad \forall z\in \boldsymbol{Z}_n{}^2)=1.$$

In Section 3 *we let* $\Omega'_\lambda=\{\omega \mid m(z)<\infty,\ \forall z\in \boldsymbol{Z}_n{}^2\}$ *and consider events in* Ω'_λ.

3. Proof of Theorem 1

In this section we prove Theorem 1 by using the probability estimates in Section 2.

From now on we assume $\lambda>\lambda_c$. We approximate our system in $\boldsymbol{R}^2$ by a lattice system in Z_n^2. By Lemma 2.1 we have that $\lambda > \lambda_c^{(n)}$ for $\lambda > \lambda_c$ with large enough n. Hereafter we consider Z_n^2 for this n.

DEFINITION 3.1. For $\omega \in \Omega_\lambda$, $z \in Z_n^2$, let

$$\Delta_1(z) = z + [-\frac{m(z)-1}{2n}, \frac{m(z)-1}{2n})^2,$$

$$\Delta_2(z) = z + [-\frac{m(z)+1}{2n}, \frac{m(z)+1}{2n})^2,$$

$$\Delta_3(z) = z + [-\frac{3m(z)+1}{2n}, \frac{3m(z)+1}{2n})^2.$$

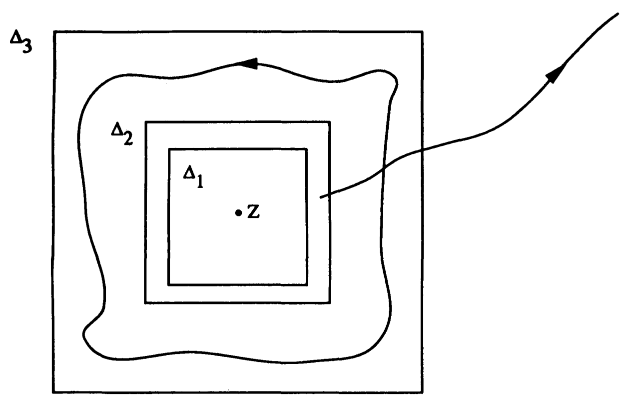

Fig. 1. Illustration of Δ_1, Δ_2 and Δ_3.

By the definition of $m(z)$, $z \in Z_n^2$ in Section 2, we have the following:

(i) There is an open path from $\Delta_2(z)$ to ∞ in $R^2 \backslash \Delta_1(z)$.

(ii) There is an open circuit in the annulus $\Delta_3(z) \backslash \Delta_2(z)$.

DEFINITION 3.2. For $\omega \in \Omega_\lambda$, $z_1, z_2 \in Z_n^2$, by using $t(X_i, X_j)$ let

$$\hat{t}(z_1, z_2)(\omega) = \inf_{\substack{\Delta_2(z_1) \ni X_i \\ \Delta_2(z_2) \ni X_j}} t(X_i, X_j).$$

$\hat{t}(z_1, z_2)$ is the minimum time of the infection of a point of X_λ in $\Delta_2(z_2)$ if the points of X_λ in $\Delta_2(z_1)$ are infected and the other susceptible initially. We extend the domain of $\hat{t}$ to all R^2 by letting $\hat{t}(x,y) = \hat{t}(\pi_n(x), \pi_n(y))$ for $x, y \in R^2$, where $\pi_n(x)$ is the element of Z_n^2 such that $x \in B_n(\pi_n(x))$. If $\{z_1, \cdots, z_k\}$ is a path of $Z_n^2(M_n)$

from $z_1=\pi_n(x)$ to $z_2=\pi_n(y)$, then there is an open path from $\Delta_2(\pi_n(x))$ to $\Delta_2(\pi_n(y))$ contained in $\bigcup_{i=1}^{k}\Delta_3(z_i)$, so $\hat{t}$ is finite in Ω'_λ.

Definition 3.3. For $\omega\in\Omega'_\lambda$, $z\in \boldsymbol{Z}_n{}^2$, $u(z)$ is defined as

$$u(z)=\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

By the following lemma we have that $u(z)$ is the upper bound of $t(X_i,X_j)$ for $X_i,X_j\in\Delta_3(z)\cap X_\lambda$ with $t(X_i,X_j)<\infty$.

Lemma 3.4. *For $\omega\in\Omega_\lambda$, $z\in \boldsymbol{Z}_n{}^2$, we have*

$$\sup_{\substack{\Delta_3(z)\cap X_\lambda\ni X_i,X_j\\ t(X_i,X_j)<\infty}} t(X_i,X_j)\le u(z)(\omega).$$

Proof. For $X_i,X_j\in\Delta_3(z)\cap X_\lambda$, assume that there is an open path from X_i to X_j in $\Delta_3(z)$. Now $t(X_i,X_j)<\infty$. Let $r=\{X_{i_1},\cdots,X_{i_K}\}$ be an open path from X_i to X_j in $\Delta_3(z)$ with the smallest K. Then $|X_{i_k}-X_{i_{k+1}}|\le 1$ for $k=1,\cdots,K-1$ and $|X_{i_k}-X_{i_{k'}}|>1$ for $k,k'=1,\cdots,K$ with $|k-k'|>1$. Hence we have

$$S'_{i_{2k+1}}\cap S'_{i_{2k'+1}}=\emptyset,\qquad \forall k,k'\ (k\neq k'),$$

where $S'_{i_{2k+1}}=\{x\in \boldsymbol{R}^2\mid |x-X_{i_{2k+1}}|\le\frac{1}{2}\}$. We can put at most $\frac{4(3m(z)+n+1)^2}{\pi n^2}$ disks of radii $\frac{1}{2}$ nonintersecting each other in $z+[-\frac{3m(z)+1}{2n}-\frac{1}{2},\frac{3m(z)+1}{2n}+\frac{1}{2}]^2$. From this we obtain

$$K\le\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

Therefore since X_i stays infected for time interval 1,

$$t(X_i,X_j)\le\frac{8(3m(z)+n+1)^2}{\pi n^2}.$$

By the definitions and the lemma above, if $t(X_i,X_j)<\infty$, then

$$\hat{t}(X_i,X_j)\le t(X_i,X_j)\le \hat{t}(X_i,X_j)+u(\pi_n(X_i))+u(\pi_n(X_j)). \tag{3.1}$$

For $\omega\in\Omega'_\lambda$, $X_i\in X_\lambda$, let

$$t(0,X_i)=\inf_{S_0\ni X_j} t(X_j,X_i).$$

$t(0,X_i)$ is the minimum time of the infection of X_i if the points of X_λ in $S_0=\{x\in \boldsymbol{R}^2 \mid |x|\leq 1\}$ are infected and others susceptible initially. In the same way as $\tilde{t}(X_i,X_j)$, if $t(0,X_i)<\infty$, then

$$\tilde{t}(0,X_i)\leq t(0,X_i)\leq \tilde{t}(0,X_i)+u(\pi_n(0))+u(\pi_n(X_i)). \tag{3.2}$$

For $t'(0,x)(\omega)=\inf_{\substack{S_0\ni X_i\\ S_x\ni X_j}} t(X_i,X_j)$, $x\in \boldsymbol{R}^2$, if $t'(0,x)<\infty$, then by (3.1)

$$\tilde{t}(0,x)\leq t'(0,x)\leq \tilde{t}(0,x)+u(\pi_n(0))+u(\pi_n(x)). \tag{3.3}$$

For $\omega\in\Omega_\lambda$, $z_1,z_2\in \boldsymbol{Z}_n{}^2$, let

$$\bar{t}(z_1,z_2)(\omega)=\tilde{t}(z_1,z_2)(\omega)+u(z_2)(\omega).$$

Then for $\omega\in\Omega_\lambda$, $z_1,z_2,z_3\in \boldsymbol{Z}_n{}^2$,

$$\bar{t}(z_1,z_2)(\omega)\leq \bar{t}(z_1,z_3)(\omega)+\bar{t}(z_3,z_2)(\omega). \tag{3.4}$$

By Corollary 2.2, for $z\in \boldsymbol{Z}_n{}^2$, $k=1,2,\cdots$, we have

$$\begin{aligned} E_\lambda(u(z)^k)&=\sum_{l=0}^{\infty}\int_{2^lK(\lambda)<m(z)\leq 2^{l+1}K(\lambda)} u(z)^k dP_\lambda \\ &\leq \sum_{l=0}^{\infty}\{\frac{8(3\cdot 2^{l+1}K(\lambda)+n+1)^2}{\pi n^2}\}^k P_\lambda(m(z)>2^lK(\lambda)) \\ &\leq (\frac{8}{\pi n^2})^k\beta\sum_{l=0}^{\infty}(3\cdot 2^{l+1}K(\lambda)+n+1)^{2k}e^{-2^l\gamma}<\infty, \end{aligned} \tag{3.5}$$

and

$$E_\lambda(u(z)^k)=E_\lambda(u(0)^k) \tag{3.6}$$

Lemma 3.5 ([2]). *If $\lambda>\lambda_c^{(n)}$, then there exists a constant $\mu(z)$ for $z\in \boldsymbol{Z}_n{}^2$, so that as $m\to\infty$ $(m\in \boldsymbol{N})$,*

$$\frac{\tilde{t}(0,mz)}{m}\to\mu(z)\quad a.s..$$

By (3.3) and Lemma 3.5, we have that $\gamma=\liminf_{m\to\infty}\frac{t'(0,me_1)}{m}$ is almost surely constant.

We let $g(z)=E_\lambda(\bar{t}(0,z))$ for $z\in \boldsymbol{Z}_n{}^2$ and extend the domain of g to all of $\boldsymbol{R}^2$ by making it linear on triangles of the form $z, z+(\frac{1}{n},0)$, $z+(0,\frac{1}{n})$ and $z+(\frac{1}{n},\frac{1}{n})$, $z+(\frac{1}{n},0)$, and $z+(0,\frac{1}{n})$. Then we have the following lemma.

Lemma 3.6 ([1]). *There is a function* φ *on* $\boldsymbol{R}^2$ *such that* $\frac{g(mx)}{m}$ *converges uniformly to* φ *on compact subsets on* $\boldsymbol{R}^2$, *and* $\varphi(z)=\mu(z)$ *for* $z\in \boldsymbol{Z}_n{}^2$.

DEFINITION 3.7. For $\omega\in\Omega_\lambda$, $z\in\boldsymbol{Z}_n{}^2$, let $C(z)=\{X_{i_1},\cdots,X_{i_L}\}$ $(X_{i_1}=X_{i_L})$ be an open circuit surrounding z in $\Delta_3(z)$ with the smallest L and let $\Delta(z)$ be the interior and the boundary of the polygon whose vertices are $X_{i_1},\cdots,X_{i_L}$ in $C(z)$, and whose sides $(X_{i_l},X_{i_{l+1}})$ for $l=1,\cdots,L-1$. We define $v(z)$ as

$$v(z)=\frac{|\Delta(z)|}{\pi(\frac{1}{2})^2}+|C(z)|+1,$$

where $|\Delta(z)|$ is the area of $\Delta(z)$, and $|C(z)|$ is the number of elements in $C(z)$.

If there is a path from X_i to X_j in $\Delta(z)$ for $X_i,X_j\in\Delta(z)\cap X_\lambda$, then $t(X_i,X_j)\le v(z)$. Proof of this fact is same as Lemma 3.4. Now to prove the following lemma, we use $c(z)$ instead of the min-circuit in [2], and use the fact above.

For $\omega\in\Omega_\lambda$, let $\hat{A}_t(\omega)=\{x\in\boldsymbol{R}^2\mid \hat{t}(0,x)\le t\}$. Let $D=\{x\in\boldsymbol{R}^2\mid \varphi(x)\le 1\}$. For any $x\in\boldsymbol{R}^2$, any $k\in\boldsymbol{Q}$ $(k=\frac{q}{p},\ p,q\in\boldsymbol{N})$, from the definition of φ, we obtain

$$\frac{\varphi(kx)}{k}=\lim_{m\to\infty}\frac{g(mp\cdot\frac{q}{p}x)}{mp\cdot\frac{q}{p}}=\varphi(x).$$

Hence for any $\alpha\in\boldsymbol{R}$,

$$\frac{\varphi(\alpha x)}{\alpha}=\varphi(x).$$

Since $\gamma=\liminf_{m\to\infty}\frac{t'(0,me_1)}{m}=\varphi(e_1)$ from (3.3) and $\varphi(x)=\varphi(e_1)$ for any $x\in\boldsymbol{R}^2$ such that $|x|=1$, $\varphi(y)=|y|\gamma$ for any $y\in\boldsymbol{R}^2$. Therefore $D=\{x\in\boldsymbol{R}^2\mid |x|\le\frac{1}{\gamma}\}$.

Lemma 3.8 ([1]). *Assume* $\gamma>0$. *If* $\lambda>\lambda_c$, *then for any* $\varepsilon>0$, *we have*

$$P_\lambda((1-\varepsilon)D\subset t^{-1}\hat{A}_t\subset(1+\varepsilon)D \textit{ for all sufficiently large } t)=1.$$

Proof of Theorem 1 ([2]). Taking $k=4$ in (3.5) and (3.6), we have $E_\lambda(u(z)^2)=E_\lambda(u(0)^2)<\infty$ for any $z\in\boldsymbol{Z}_n{}^2$, then for any $\varepsilon''>0$,

$$P_\lambda(u(z)>\varepsilon''|z| \text{ i.o.})=0.$$

Hence for any $\varepsilon''>0$,

$$P_\lambda(\exists a>0 \text{ s.t. } u(z)\le\max\{a,\varepsilon''|z|\} \quad\text{for}\quad \forall z\in\boldsymbol{Z}_n{}^2)=1. \tag{3.7}$$

Taking any $\varepsilon>0$, by Lemma 3.8 we have that almost surely for all large t, if $(1-\varepsilon)tD\ni x$, then $\hat{t}(0,x)(\omega)\leq t$. Here let $t'=(1-\frac{\varepsilon}{2})t$. We know that if $(1-\varepsilon)t'D\ni x$, then $\hat{t}(0,x)(\omega)\leq(1-\frac{\varepsilon}{2})t$. Since $(1-\varepsilon)(1-\frac{\varepsilon}{2})=1-(\frac{3}{2}\varepsilon-\frac{\varepsilon^2}{2})$, with $\varepsilon'=\frac{3}{2}\varepsilon-\frac{\varepsilon^2}{2}(>0)$, we have that if $(1-\varepsilon')tD\ni x$, then $\hat{t}(0,x)(\omega)\leq(1-\frac{\varepsilon}{2})t$. If $X_i\in(1-\varepsilon')tD\cap C_0$, then (3.2) leads to

$$t(0,X_i)+1\leq\hat{t}(0,X_i)+u(0)(\omega)+u(\pi_n(X_i))(\omega)+1$$
$$\leq(1-\frac{\varepsilon}{2})t+u(0)(\omega)+u(\pi_n(X_i))(\omega)+1.$$

With $d=\frac{2}{\gamma}$ (the diameter of D), $|X_i|\leq(1-\varepsilon')td$. By (3.7), we obtain that

$$u(0)(\omega)\leq a,$$
$$u(\pi_n(X_i))(\omega)\leq\max\{a,\varepsilon''|\pi_n(X_i)|\},$$

and from the definition of π_n,

$$|\pi_n(X_i)|\leq|X_i|+\frac{1}{\sqrt{2}}.$$

Hence with taking t large enough, if necessary, we have

$$u(0)(\omega)+u(\pi_n(X_i))(\omega)+1\leq 3\varepsilon''(1-\varepsilon')td.$$

Taking $\varepsilon''=\frac{\varepsilon}{9d}$,

$$u(0)(\omega)+u(\pi_n(X_i))(\omega)+1\leq\frac{\varepsilon}{3}(1-\varepsilon')t.$$

Therefore we get

$$t(0,X_i)+1\leq(1-\frac{\varepsilon}{2})t+\frac{\varepsilon}{3}(1-\varepsilon')t$$
$$=(1-\frac{\varepsilon}{2}+\frac{\varepsilon}{3}-\frac{\varepsilon}{3}\varepsilon')t$$
$$\leq(1-\frac{\varepsilon}{6})t.$$

Because $t(0,X_i)$ is the time of the infection of X_i, and 1 is the time lag from the infection of X_i until X_i is immune, we have $X_i \in \zeta_t(\omega)$. Hence for $\forall \varepsilon > 0$,

$$P_\lambda((1-\varepsilon')tD \cap C_0 \subset \zeta_t \quad \text{for all sufficiently large } t) = 1.$$

On the other hand, if $X_i \in \xi_t(\omega)$ or $X_i \in \zeta_t(\omega)$, then $t(0,X_i) \leq t$ and so $\hat{t}(0,X_i) \leq t$. Therefore for $\forall \varepsilon > 0$, Lemma 3.8 leads to $X_i \in (1+\varepsilon)tD$. Hence

$$P_\lambda(\xi_t \subset (1+\varepsilon)tD \quad \text{for all sufficiently large } t) = 1,$$

$$P_\lambda(\zeta_t \subset (1+\varepsilon)tD \quad \text{for all sufficiently large } t) = 1.$$

We have completed the proof of Theorem 1.

References

[1] J. T. Cox and R. Durrett: *Some limit theorems for percolation processes with necessary and sufficient conditions*, Ann. Probab. **9** (1981), 593–603.
[2] J. T. Cox and R. Durrett: *Limit theorems for the spread of epidemics and forest fires*, Stochastic Process Appl. **30** (1988), 171–191.
[3] G. Grimmett: Percolation, Springer-Verlag, New York, 1988.
[4] H. Kesten: Percolation Theory for Mathematicians, Birkhäuser, Boston, 1982.
[5] J. F. C. Kingman: *Subadditive ergodic theory*, Ann. Probab. **6** (1973), 833–909.
[6] Y. Zhang: *A shape theorem for epidemics and forest fires with finite range interactions*, Ann. Probab. **21** (1993), 1755–1781.
[7] S. A. Zuev and A. F. Sidorenko: *Continuous models of percolation theory I*, Theoretical and Mathematical Physics **62** (1985), 76–86.
[8] S. A. Zuev and A. F. Sidorenko: *Continuous models of percolation II*, Theoretical and Mathematical Physics **62** (1985), 253–262.

Motoko Kamae
Department of Mathematics
Osaka University
1–1 Machikaneyamachou Toyonaka,
Osaka 560, Japan
Present Address: IBM Japan, Ltd.

Satoshi Takahashi
Department of Mathematics
Osaka University
1–16 Machikaneyamachou Toyonaka,
Osaka 560, Japan