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# WEAKLY HYPERBOLIC EQUATIONS, SOBOLEV SPACES OF VARIABLE ORDER, AND PROPAGATION OF SINGULARITIES 

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## 1. Introduction

Let us recall some results about linear and semilinear wave equations. We examine the Cauchy problems

$$
\begin{equation*}
u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x) \tag{1.1}
\end{equation*}
$$

$$
\begin{align*}
& \square u=f(u), \\
& \square v=0, \tag{1.2}
\end{align*}
$$

$$
v(0, x)=u_{0}(x), \quad v_{t}(0, x)=u_{1}(x)
$$

where $\square=\partial_{t}^{2}-\triangle,(t, x) \in \mathbb{R}_{t} \times \mathbb{R}_{x}^{n}$ and $f \in C^{\infty}$ with $f(0)=0$. We suppose that the initial data $u_{0}, u_{1}$ satisfy $u_{0} \in H^{s}, u_{1} \in H^{s-1}$ for some $s>n / 2+1$. Then it is known that solutions $u, v$ exist in $C\left([0, T], H^{s}\right) \cap C^{1}\left([0, T], H^{s-1}\right)$ for some small $T>0$. Further, we assume that $u_{0}, u_{1}$ belong to $C^{\infty}$ outside some closed set of $\mathbb{R}^{n}$. If singularities starting from two different points of this set of singularities meet, nothing happens in the linear case. They ignore each other and continue on their track. However, in the semilinear case, the nonlinear interaction of singularities may generate new singularities. These are weaker than those of $v$ by at least one Sobolev order, which can be seen immediately as follows: we have $\square(u-v)=f(u) \in C\left([0, T], H^{s}\right)$; hence $u-v \in C\left([0, T], H^{s+1}\right)$.

The aim of this publication is to prove a similar result for weakly hyperbolic equations whose lower order terms satisfy sharp Levi conditions. To demonstrate the phenomena which may occur in this setting, we recall a result of [15]. Let $v=v(t, x)$ be the solution of

$$
\begin{equation*}
v_{t t}-t^{2} v_{x x}-b v_{x}=0, \quad v(0, x)=u_{0}(x), \quad v_{t}(0, x)=0, \quad x \in \mathbb{R} \tag{1.3}
\end{equation*}
$$

If $b=4 m+1$ and $m \in \mathbb{N}_{0}$, then the solution is given by

$$
\begin{equation*}
v(t, x)=\sum_{j=0}^{m} C_{j m} t^{2 j}\left(\partial_{x}^{j} u_{0}\right)\left(x+\frac{t^{2}}{2}\right) \tag{1.4}
\end{equation*}
$$

with some constants $C_{j m}$; and $C_{m m}$ does not vanish. We observe two phenomena. The first is the loss of regularity: if $u_{0} \in H^{s}$, then $v(t,.) \in H^{s-m}$. There is no classical
solution for $m>s-5 / 2$ ! The second is that singularities of the datum only propagate along one characteristic.

The loss of regularity makes the investigation of the semilinear problem

$$
\begin{equation*}
u_{t t}-t^{2} u_{x x}-b u_{x}=f(u), \quad u(0, x)=u_{0}(x), \quad u_{t}(0, x)=0 \tag{1.5}
\end{equation*}
$$

difficult, because the standard iteration procedure and fixed point principles do not work. The existence of a solution $u$ with $u(t,.) \in H^{s-M}$ for some number $M$ can be proved by a modified iteration technique, see for instance [11], [16]. See also [5] for similar considerations in the $C^{\infty}$ class. However, that method gives no information about the propagation of singularities, because only rough estimates of $M$ can be found, which are generally not sharp.

If one is interested in the propagations of singularities in Sobolev spaces, then it is of great importance to know the spaces in which the description of singularities makes sense. To clarify this point, let us consider (1.1) and (1.2) with $u_{0} \in H^{s}, u_{1} \in H^{s-1}$. It is a true statement that, for instance, $u$ and $v$ belong to $C\left([0, T], H^{s-5}\right)$, while $u-v \in C\left([0, T], H^{s-4}\right)$. However, it makes no sense to investigate singularities in that space, because the singular support is the empty set. The right statement is $u, v \in$ $C\left([0, T], H^{s}\right)$ and $u-v \in C\left([0, T], H^{s+1}\right)$, but, in general, $v \notin C\left([0, T], H^{s+1}\right)$.

Results of the following general type tell us that the strongest singularities of $u$ and $v$ coincide: we construct two function spaces $B_{2} \subset B_{1}$, where the functions of $B_{2}$ have higher smoothness than that of $B_{1}$. Then we prove that $u, v \in B_{1}$ and $u-v \in B_{2}$. These statements are sharp in the sense that examples show that $v \notin B_{2}$.

The usual iteration procedure, even in its modified version mentioned above, is not able to give us a sharp description of the smoothness of $v$ and $u$ solving (1.3), (1.5). Another way to attack (1.5) consists in the construction of a special function space. This space contains all functions $w(t, x)$ with $\vartheta(t, \xi) \hat{w}(t, \xi) \in C\left([0, T], L^{2}\right)$, where $\vartheta(0, \xi)=\mathscr{O}\left(\langle\xi\rangle^{s}\right)$ and $\vartheta(t, \xi)=\mathscr{O}\left(\langle\xi\rangle^{s-m}\right)(t \neq 0)$ for $\langle\xi\rangle \rightarrow \infty$. Utilizing this idea, in [7] it was shown that the solution $u$ of (1.5) belongs to $C\left([0, T], H^{s-m}\right)$ and that $u-v \in C\left([0, T], H^{s-m+1 / 2}\right)$. By (1.4), $v \notin C\left([0, T], H^{s-m+1 / 2}\right)$. In other words, also the strongest singularities of $u$ propagate only along one characteristic. The idea to assign a weight $\vartheta(t, \xi)$ to the hyperbolic operator and to estimate a certain norm of the product $\vartheta(t, \xi) \hat{w}(t, \xi)$ goes back to [17]. The coefficients in the Cauchy problems of [7] and [17] did not depend on $x$. Therefore it was possible to apply partial Fourier transform and study the arising ODEs, in contrast to the situation in this publication, where the coefficients may depend on $x$, too.

Consider the model problems

$$
\begin{aligned}
& u_{t t}-t^{2} u_{x x}-(4 m(x)+1) u_{x}=f(u), \quad u(0, x)=u_{0}(x), \quad u_{t}(0, x)=0, \\
& v_{t t}-t^{2} v_{x x}-(4 m(x)+1) v_{x}=0, \quad v(0, x)=u_{0}(x), \quad v_{t}(0, x)=0,
\end{aligned}
$$

where $m \in C_{b}^{\infty}(\mathbb{R})$ and $m \geq 0$. One expects a loss of $m(x)$ derivatives at the point
$x$. In Section 7, we will show that this variable loss of regularity happens indeed. As explained above, a reasonable description of singularities in spaces of Sobolev type requires to find the right spaces. Therefore we have to employ Sobolev spaces with variable order of differentiation $H^{s+m(x)}$ :

$$
H^{s+m(x)}:=\left\{w \in L^{2}(\mathbb{R}):\left\langle D_{x}\right\rangle^{s+m(x)} w(x) \in L^{2}(\mathbb{R})\right\}, \quad s \geq 0
$$

The main results of this publication (Theorems 4.1, 6.1 and 6.2) applied to these model problems yield:

Proposition 1.1. Assume that $f=f(u)$ is entire analytic with $f(0)=0$ and $u_{0} \in$ $H^{s+m(x)}$ with $s>5 / 2$. Then some $T>0$ exists with

$$
u, v \in C\left([0, T], H^{s}\right), \quad u-v \in C\left([0, T], H^{s+1 / 2}\right)
$$

Examples show that the statement about the regularity of $v$ is the best possible.

The Cauchy problems to be studied in this paper have the form

$$
\begin{array}{lll}
L u=f(t, x, u), & u(0, x)=u_{0}(x), & u_{t}(0, x)=u_{1}(x) \\
L v=0, & v(0, x)=u_{0}(x), & v_{t}(0, x)=u_{1}(x)
\end{array}
$$

where

$$
\begin{aligned}
L= & D_{t}^{2}+2 \sum_{j=1}^{n} t^{l_{*}} c_{j}(t, x) D_{t} D_{x_{j}}-\sum_{i, j=1}^{n} t^{2 l_{*}} a_{i j}(t, x) D_{x_{i}} D_{x_{j}} \\
& -\sum_{j=1}^{n} i l_{*} t^{l_{*}-1} b_{j}(t, x) D_{x_{j}}+i c_{0}(t, x) D_{t}, \quad l_{*} \in \mathbb{N}_{+}
\end{aligned}
$$

In [1] and [18], special linear model equations of this kind have been investigated and it was shown that the propagation of singularities depends in a sensitive way on lower order terms. Similarly to (1.3), sometimes the singularities propagate only in one direction. The special choice of $t$-exponents reflects the so-called Levi conditions which are necessary and sufficient conditions for the $C^{\infty}$ well-posedness, compare [9] and [14]. For related results on propagation of $C^{\infty}$ singularities, for example see [10].

Our assumptions are the following:

$$
\begin{align*}
& a_{i j}, c_{j}, b_{j} \in C_{b}^{\infty}\left(\left[0, T_{0}\right] \times \mathbb{R}^{n}, \mathbb{R}\right)  \tag{1.8}\\
& \sum_{i, j=1}^{n} a_{i j}(t, x) \xi_{i} \xi_{j}+\left(\sum_{j=1}^{n} c_{j}(t, x) \xi_{j}\right)^{2} \geq \alpha_{0}|\xi|^{2}  \tag{1.9}\\
& \quad \alpha_{0}>0, \quad(t, x, \xi) \in\left[0, T_{0}\right] \times \mathbb{R}^{2 n}
\end{align*}
$$

$$
\begin{align*}
& f(t, x, u)=\sum_{j=1}^{\infty} f_{j}(t, x) u^{j}, \quad f_{j} \in C_{b}^{\infty}\left(\left[0, T_{0}\right] \times \mathbb{R}^{n}\right),  \tag{1.10}\\
& \sup _{\left[0, T_{0}\right]} \sum_{j=1}^{\infty}\left\|f_{j}(t, .)\right\|_{C_{b}^{k}} u^{j}<\infty \quad \forall k \in \mathbb{N}, \quad \forall u \in \mathbb{R} . \tag{1.11}
\end{align*}
$$

The paper is organised as follows. A theory of Sobolev spaces of variable order is presented in Section 2. We also list some properties of special symbol classes which are a main ingredient in the construction of parametrix in Section 3. This fundamental solution is utilized to prove the well-posedness of the linear Cauchy problem in Section 4. The investigation of the semilinear Cauchy problem in Section 6 relies on the algebra property of our function spaces proved in Section 5. Finally, we show that our results are sharp and discuss an application to propagation of singularities in Section 7.

## 2. Function spaces and symbol classes

For convenience, we introduce the notation $\lambda(t)=t^{l_{*}}, \Lambda(t)=\int_{0}^{t} \lambda(\tau) d \tau$. If $X$ is some set from $\mathbb{R}^{K}$, then $C_{b}^{m}(X)$ denotes the space of all functions, whose derivatives up to the order $m$ are continuous and bounded functions over $X$. Let $S_{\varrho, \delta}^{m}$ and $\Psi_{\varrho, \delta}^{m}$ denote the usual spaces of symbols and pseudodifferential operators, respectively,

$$
\begin{aligned}
& S_{\varrho, \delta}^{m}:=\left\{p(x, \xi) \in C^{\infty}\left(\mathbb{R}_{x}^{n} \times \mathbb{R}_{\xi}^{n}\right): \quad \pi_{\alpha, \beta}^{m}(p)<\infty \quad \forall \alpha, \beta \in \mathbb{N}^{n}\right\}, \\
& \pi_{\alpha, \beta}^{m}(p):=\sup \left\{\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} p(x, \xi)\right|\langle\xi\rangle^{-m+\varrho|\beta|-\delta|\alpha|}:(x, \xi) \in \mathbb{R}^{2 n}\right\},
\end{aligned}
$$

where we employ the usual multi-index notation and $\langle\xi\rangle=\left(1+|\xi|^{2}\right)^{1 / 2}$. To simplify notation, we adjust the measure in the cotangent space, $d \xi=(2 \pi)^{-n} d \xi_{1} \cdots d \xi_{n}$. If $p(x, \xi)$ is some symbol, the corresponding pseudodifferential operator will be denoted by the upper case letter, $P=P\left(x, D_{x}\right)$. If $P\left(x, D_{x}\right)$ and $Q\left(x, D_{x}\right)$ are pseudodifferential operators, then $P \cdot Q=(P \cdot D)\left(x, D_{x}\right)$ is defined as the operator with the symbol $(\operatorname{sym}(P \cdot Q))(x, \xi):=p(x, \xi) q(x, \xi)$. If $p(x, \xi)$ and $q(x, \xi)$ are symbols, the symbol $(p \circ q)(x, \xi)$ is defined by the asymptotic expansion

$$
(p \circ q)(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!}\left(D_{\xi}^{\alpha} p(x, \xi)\right)\left(\partial_{x}^{\alpha} q(x, \xi)\right), \quad D_{\xi}:=-i \partial_{\xi}:=-i \frac{\partial}{\partial \xi} .
$$

This symbol is unique modulo $S^{-\infty}$. Let us derive some auxiliary results, citing ideas from [8], Chapter 22.

Definition 2.1. Let $\mathcal{K}_{\varrho, \delta}$ be the set of symbols

$$
\mathcal{K}_{\varrho, \delta}=\left\{a \in \bigcup_{m \in \mathbb{R}} S_{\varrho, \delta}^{m}:\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)\right| \leq C_{\alpha \beta} a(x, \xi)\langle\xi\rangle^{-\varrho|\beta|+\delta|\alpha|} \forall \xi, \alpha, \beta\right\}
$$

Proposition 2.2. Let $a, a_{0}, a_{1} \in \mathcal{K}_{\varrho, \delta}$. Then $a_{0} \cdot a_{1}, a^{r} \in \mathcal{K}_{\varrho, \delta}$ for all $r \in \mathbb{R}$.
Proof. The statement about $a_{0} \cdot a_{1}$ follows from the Leibniz formula. Concerning $a^{r}$, we employ Faà di Bruno's formula

$$
\begin{aligned}
& \partial_{x}^{\alpha} \partial_{\xi}^{\beta} G(a(x, \xi))=\sum_{\nu=1}^{|\alpha|+|\beta|} G^{(\nu)}(a(x, \xi)) \sum_{(\alpha, \beta, \nu)} \prod_{\mu=1}^{\nu}\left(\partial_{x}^{\alpha^{\mu}} \partial_{\xi}^{\beta^{\mu}} a(x, \xi) C_{\alpha \beta \nu \mu}\right), \\
& \sum_{(\alpha, \beta, \nu)}:=\sum_{\substack{\left(\alpha^{1}, \beta^{1}\right)+\cdots+\left(\alpha^{\nu}, \beta^{\nu}\right)=(\alpha, \beta),\left|\alpha^{\mu}\right|+\left|\beta^{3}\right|>0}},
\end{aligned}
$$

with $G(s)=s^{r}$ and the proof is complete.
2.1. Sobolev spaces of variable order The space $C\left([0, T], H^{s}\right)$ consists of all functions $w(t, x)$ with $\langle\xi\rangle^{s} \hat{w}(t, \xi) \in C\left([0, T], L^{2}\left(\mathbb{R}_{\xi}^{n}\right)\right)$. We generalize this space, replacing $\langle D\rangle^{s}$ by some pseudodifferential operator $\Theta_{M}\left(t, x, D_{x}\right)$, whose symbol $\vartheta_{M}(t, x, \xi)$ may have different growth rates with respect to $\langle\xi\rangle$, depending on $t$ and $x$. In the following, adapted symbol classes, embeddings in the usual Sobolev spaces (and vice versa), and mapping properties of operators from $\Psi_{\varrho, \delta}^{m}$ are described.

Let $\vartheta \in C\left([0, T], S_{1-\varepsilon, \varepsilon}^{K_{+}}\right), 0<\varepsilon<1 / 2$, be a symbol such that

$$
\begin{align*}
& \vartheta(t, x, \xi) \geq C_{1}\langle\xi\rangle^{K_{-}}, \quad C_{1}>0, \quad(t, x, \xi) \in[0, T] \times \mathbb{R}^{2 n},  \tag{2.1}\\
& \vartheta(t, ., .) \in \mathcal{K}_{1-\varepsilon, \varepsilon}, \text { uniformly in } t . \tag{2.2}
\end{align*}
$$

We choose some $M \in \mathbb{R}$ and define $\vartheta_{M}(t, x, \xi):=\vartheta(t, x, \xi)\langle\xi\rangle^{M}$. Let us describe the symbol $\vartheta$ more closely.

Proposition 2.3. If $\vartheta \in C\left([0, T], S_{1-\varepsilon, \varepsilon}^{K_{+}}\right)$satisfies (2.1), (2.2), then

$$
\begin{equation*}
\frac{D_{x}^{\alpha} D_{\xi}^{\beta} \vartheta(t, x, \xi)}{\vartheta(t, x, \xi)} \in S_{1-\varepsilon, \varepsilon}^{-(1-\varepsilon)|\beta|+\varepsilon|\alpha|}, \quad \alpha, \beta \in \mathbb{N}^{n}, \quad 0<\varepsilon<\frac{1}{2} \tag{2.3}
\end{equation*}
$$

Proof. The assertion follows from the Leibniz rule and Proposition 2.2.
Proposition 2.4 ([8], Theorem 22.1.3). For each $\vartheta$ as above, there are symbols $\vartheta^{\sharp} \in L^{\infty}\left([0, T], S_{1-\varepsilon, \varepsilon}^{-K_{-}}\right)$and $r_{\infty} \in L^{\infty}\left([0, T], S^{-\infty}\right)$ with $\Theta^{\sharp} \Theta=I+R_{\infty}$.

The usual symbol calculus and $\vartheta(t, .,.) \in \mathcal{K}_{1-\varepsilon, \varepsilon}$ lead to the following result.

Lemma 2.5. For each $a, \tilde{b} \in L^{\infty}\left([0, T], S_{1-\varepsilon, \varepsilon}^{m}\right)$ there are symbols $\tilde{a}, b \in$ $L^{\infty}\left([0, T], S_{1-\varepsilon, \varepsilon}^{m}\right)$ with the property that

$$
a \vartheta-\tilde{a} \circ \vartheta, \quad b \vartheta-\tilde{b} \circ \vartheta \in L^{\infty}\left([0, T], S^{-\infty}\right) .
$$

Similar statements hold, when $\vartheta$ is the factor on the left.

The class of these symbols $\vartheta_{M}$ is a special case of the fairly general symbol classes $S_{\Phi, \phi}^{\mu}$ of Beals and Fefferman ([2], [3], [4]) which consist of symbols $a$ satisfying

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)\right| \leq C_{\alpha \beta} \exp (\mu(x, \xi)) \phi(x, \xi)^{-|\alpha|} \Phi(x, \xi)^{-|\beta|}
$$

with

$$
\begin{aligned}
& \phi \leq C, \quad \Phi \phi \geq c \\
& c \leq \Phi(x, \xi) \Phi(y, \eta)^{-1} \leq C \text { and } c \leq \phi(x, \xi) \phi(y, \eta)^{-1} \leq C \\
& \quad \text { if }|x-y| \leq c \phi(x, \xi) \text { and }|\xi-\eta| \leq c \Phi(x, \xi) \\
& R(x, 0) \leq C\langle x\rangle^{C}, \text { where } R=\Phi \phi^{-1} \\
& c \leq R(x, \xi) R(y, \eta)^{-1} \leq C \\
& \quad \text { if }|\xi-\eta| \leq C R(x, \xi)^{\delta+1 / 2}, \quad|x-y| \leq C R(x, \xi)^{\delta} R(y, \eta)^{-1 / 2}, \quad \delta>0 \\
& |\mu(x, \xi)-\mu(y, \eta)| \leq C \text { if }|x-y| \leq C \phi(x, \xi), \quad|\xi-\eta| \leq \Phi(x, \xi) \\
& c(\Phi \phi)^{-m} \leq e^{\mu} \Phi^{-K} \phi^{-k} \leq C(\Phi \phi)^{m} \text { for some } K, k, m
\end{aligned}
$$

In our case, $\exp \mu=\vartheta_{M}, \Phi=\langle\xi\rangle^{1-\varepsilon}, \phi=\langle\xi\rangle^{-\varepsilon}$. The results of [3] enable us to characterize Sobolev spaces of variable order, which we define now.

Definition 2.6 (Sobolev spaces of variable order). If $M+K_{-} \geq 0$, then let $B_{\vartheta, M, T}$ be the space

$$
B_{\vartheta, M, T}:=\left\{u \in C\left([0, T], L^{2}\right): \quad \Theta_{M}\left(t, x, D_{x}\right) u \in C\left([0, T], L^{2}\right)\right\}
$$

This space is called Sobolev space of variable order and has the norm

$$
\|u\|_{M, T}:=\sup _{[0, T]}\left(\left\|\Theta_{M}\left(t, x, D_{x}\right) u(t, x)\right\|_{L^{2}}+\|u(t, x)\|_{L^{2}}\right) .
$$

The next two propositions follow from Theorem 6.1 of [3].

Proposition 2.7. The Banach spaces $B_{\vartheta, M, T}$ satisfy the embeddings

$$
C\left([0, T], H^{M+K_{+}}\right) \subset B_{\vartheta, M, T} \subset C\left([0, T], H^{M+K_{-}}\right)
$$

Proposition 2.8. If $p \in C\left([0, T], S_{\varrho, \delta}^{m}\right)$ with $\min (\varrho, 1-\varepsilon)>\max (\delta, \varepsilon)$, then

$$
P: B_{\vartheta, M+m, T} \rightarrow B_{\vartheta, M, T} \text { continuously. }
$$

2.2. Symbol classes in $Z_{\text {hyp }}(N)$ We split the $(t, \xi)$ space into two zones, the so-called pseudodifferential zone $Z_{p d}(N)$ and the hyperbolic zone $Z_{\text {hyp }}(N)$, where $N$ is some large number:

$$
\begin{array}{rll}
Z_{p d}(N):=\{(t, \xi):|\xi| \geq 1, & \Lambda(t)\langle\xi\rangle \leq N\}, \\
Z_{h y p}(N):=\{(t, \xi):|\xi| \geq 1, & \Lambda(t)\langle\xi\rangle \geq N\} .
\end{array}
$$

Let $t_{\xi} \in \mathbb{R}_{+}$be defined by $\Lambda\left(t_{\xi}\right)\langle\xi\rangle=N$.
In order to describe the $t$-dependence of the pseudodifferential operators referred to in this paper more precisely, we introduce the following classes of symbols in $Z_{\text {hyp }}(N)$. A detailed theory of these symbols can be found in [19].

Definition 2.9. We say that $a(t, x, \xi) \in C^{\infty}\left([0, T] \times \mathbb{R}^{2 n}\right)$ belongs to the symbol class $S_{N}\left\{m_{1}, m_{2}, m_{3}\right\}$ if $\partial_{t}^{j} a \in C\left([0, T], S_{1,0}^{M_{j}}\right)$ for all $j$ and some $M_{j}$ and

$$
\begin{equation*}
\left|\partial_{t}^{k} \partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(t, x, \xi)\right| \leq C_{k \alpha \beta}\langle\xi\rangle^{m_{1}-|\beta|} \lambda(t)^{m_{2}} t^{m_{3}-k} \tag{2.4}
\end{equation*}
$$

for all $k \geq 0, \alpha, \beta \in \mathbb{N}^{n}$, and all $(t, x, \xi) \in Z_{\text {hyp }}(N)$.
Proposition 2.10. The symbols of these classes satisfy

$$
\begin{align*}
& S_{N}\left\{m_{1}, m_{2}, m_{3}\right\} \subset S_{N}\left\{m_{1}+k, m_{2}+k, m_{3}+k\right\} \quad \forall k \geq 0,  \tag{2.5}\\
& a \in S_{N}\left\{m_{1}, m_{2}, m_{3}\right\} \Longrightarrow D_{t}^{k} D_{\xi}^{\beta} a \in S_{N}\left\{m_{1}-|\beta|, m_{2}, m_{3}-k\right\},  \tag{2.6}\\
& a \in S_{N}\left\{m_{1}, m_{2}, m_{3}\right\}, \quad b \in S_{N}\left\{k_{1}, k_{2}, k_{3}\right\} \\
& \quad \Longrightarrow a b, \quad a \circ b \in S_{N}\left\{m_{1}+k_{1}, m_{2}+k_{2}, m_{3}+k_{3}\right\},  \tag{2.7}\\
& a \in S_{N}\left\{m_{1}, m_{2}, m_{3}\right\}, \quad a(t, x, \xi) \equiv 0 \quad \text { if } \quad \Lambda(t)\langle\xi\rangle \notin\left[N, N^{\prime}\right] \\
& \quad \Longrightarrow a \in S_{N}\left\{m_{1}-k, m_{2}-k, m_{3}-k\right\} \quad \forall k \geq 0 . \tag{2.8}
\end{align*}
$$

Proposition 2.11. (a) Assume that $a_{k} \in S_{N}\left\{m_{1 k}, m_{2}, m_{3}\right\}\left(k \in \mathbb{N}_{0}\right)$ with $m_{1 k} \searrow$ $-\infty$ as $k \rightarrow \infty$. Suppose that each $a_{k}$ vanishes in $Z_{p d}(N)$. Then there is a (unique up to $\left.C^{\infty}\left([0, T], S^{-\infty}\right)\right)$ symbol $a \in S_{N}\left\{m_{1(0)}, m_{2}, m_{3}\right\}$ with support in $Z_{h y p}(N)$ and

$$
\begin{equation*}
a-\sum_{j=0}^{k-1} a_{j} \in S_{N}\left\{m_{1 k}, m_{2}, m_{3}\right\} \quad \forall k . \tag{2.9}
\end{equation*}
$$

(b) Suppose that $b_{k} \in S_{N}\left\{m_{1}-k, m_{2}-k, m_{3}-k\right\}\left(k \in \mathbb{N}_{0}\right)$. Assume that each $b_{k}$ vanishes in $Z_{p d}(N)$. Then there is a (unique up to $\cap_{k \geq 0} S_{N}\left\{m_{1}-k, m_{2}-k, m_{3}-k\right\}$ )
symbol $b \in S_{N}\left\{m_{1}, m_{2}, m_{3}\right\}$ with support in $Z_{\text {hyp }}(N)$ and

$$
\begin{equation*}
b-\sum_{j=0}^{k-1} b_{j} \in S_{N}\left\{m_{1}-k, m_{2}-k, m_{3}-k\right\} \quad \forall k \tag{2.10}
\end{equation*}
$$

Sketch of proof. We choose some smooth function $\chi$ with $\chi(s)=1$ for $|s| \leq 1$, $\chi(s)=0$ for $|s| \geq 2$ and set

$$
\begin{aligned}
& a(t, x, \xi):=\sum_{k=0}^{\infty}\left(1-\chi\left(\varepsilon_{k}\langle\xi\rangle\right)\right) a_{k}(t, x, \xi) \\
& b(t, x, \xi):=\sum_{k=0}^{\infty}\left(1-\chi\left(\delta_{k} \Lambda(t)\langle\xi\rangle\right)\right) b_{k}(t, x, \xi)
\end{aligned}
$$

Here $\left\{\varepsilon_{k}\right\},\left\{\delta_{k}\right\}$ are sequences of positive numbers, monotonically converging to zero. If we choose these numbers appropriately, then (2.9), (2.10) can be shown. For details, we refer the reader to [19, Proposition 3.3.2].

Proposition 2.12 (Parametrix of elliptic operators). Assume that the matrix symbol $a \in S_{N}\{0,0,0\}$ is constant in $Z_{p d}(N)$ and satisfies

$$
|\operatorname{det} a(t, x, \xi)| \geq \text { const }>0
$$

for all $(t, x, \xi) \in[0, T] \times \mathbb{R}^{2 n}$. Then a parametrix $B\left(t, x, D_{x}\right)$ exists,

$$
B A-I, \quad A B-I \in C^{\infty}\left([0, T], \Psi^{-\infty}\right)
$$

with the property that $b \in S_{N}\{0,0,0\}$ and

$$
b(t, x, \xi)=a(t, x, \xi)^{-1} \quad \text { in } Z_{p d}(N)
$$

Proof. We set $b_{0}(t, x, \xi):=a(t, x, \xi)^{-1}$ and observe that $b_{0} \in S_{N}\{0,0,0\}$. Now we recursively define symbols $b_{k}(k \geq 1)$ by

$$
\sum_{|\alpha|=1}^{k} \frac{1}{\alpha!}\left(D_{\xi}^{\alpha} a(t, x, \xi)\right)\left(\partial_{x}^{\alpha} b_{k-|\alpha|}(t, x, \xi)\right)=:-a(t, x, \xi) b_{k}(t, x, \xi)
$$

and see that $b_{k}(t, x, \xi) \equiv 0$ in $Z_{p d}(N)$ and $b_{k} \in S_{N}\{-k, 0,0\}$. Proposition 2.11 gives us a symbol $b \in S_{N}\{0,0,0\}$ with

$$
b-\sum_{j=0}^{k-1} b_{j} \in S_{N}\{-k, 0,0\} \quad \forall k, \quad b(t, x, \xi)=a(t, x, \xi)^{-1} \text { in } Z_{p d}(N)
$$

By construction, we have $a \circ b-I \in C^{\infty}\left([0, T], S^{-\infty}\right)$. The statement about the left parametrix can be proved in a similar way.

Finally, we connect the symbol class $S_{N}\{0,0,-1\}$ with the weight symbols from the class $\mathcal{K}_{\varrho, \delta}$.

Proposition 2.13. If the symbol a vanishes in $Z_{p d}(N)$ and satisfies (2.4) with $\left\{m_{1}, m_{2}, m_{3}\right\}=\{0,0,-1\}$ and $k=0$, then

$$
\begin{aligned}
& \exp \left(\int_{0}^{t} a(\tau, x, \xi) d \tau\right) \in \mathcal{K}_{1-\varepsilon, \varepsilon} \quad \forall \varepsilon>0, \quad 0 \leq t \leq T, \\
& \exp \left(\int_{0}^{t} a(\tau, x, \xi) d \tau\right) \in L^{\infty}\left([0, T], S_{1-\varepsilon, \varepsilon}^{K^{+}}\right) \quad \forall \varepsilon>0
\end{aligned}
$$

with $K_{+}:=\sup \left\{|a(t, x, \xi) \Lambda(t) / \lambda(t)|:(t, x, \xi) \in[0, T] \times \mathbb{R}^{2 n}\right\}$.
Proof. We only note that $\left|\partial_{\xi}^{\beta} \int_{0}^{t} a(\tau, x, \xi) d \tau\right| \leq C_{\beta}\langle\xi\rangle^{-|\beta|} \ln \langle\xi\rangle$, due to

$$
\int_{t_{\xi}}^{t}\langle\xi\rangle^{-|\beta|} \frac{1}{\tau} d \tau=C\langle\xi\rangle^{-|\beta|} \ln \frac{\Lambda(t)}{\Lambda\left(t_{\xi}\right)} \leq C\langle\xi\rangle^{-|\beta|} \ln \langle\xi\rangle .
$$

In Section 4, we will fix some symbol $\tilde{\beta}$ which has, basically, the form

$$
\tilde{\beta}(t, x, \xi)=\frac{\lambda(t)}{\Lambda(t)} \beta(x) \in S_{N}\{0,0,-1\}
$$

in the hyperbolic zone. Then we will set (compare (4.2), (4.3))

$$
\vartheta(0, x, \xi):=\exp \left(\int_{t_{\xi}}^{T} \tilde{\beta}(\tau, x, \xi) d \tau\right)=C(T, N, x)\langle\xi\rangle^{\beta(x)} .
$$

This is a symbol whose growth rate will depend on $x$. That is the reason why we studied such weight symbols.

## 3. Fundamental solutions

The solution $w$ to the Cauchy problem $L w=f(t, x), w(0, x)=u_{0}(x), w_{t}(0, x)=$ $u_{1}(x)$, behaves differently in the two zones $Z_{p d}(N)$ and $Z_{h y p}(N)$. In order to study fundamental solutions in each zone separately, we introduce the following cut functions. Fix some $\chi \in C^{\infty}(\mathbb{R})$ with $\chi(s)=1$ for $s \leq 1, \chi(s)=0$ for $s \geq 2$, and $0 \leq \chi(s) \leq 1$ else. Then we set

$$
\chi_{N}^{+}(t, \xi):=\chi\left(\frac{\Lambda(t)\langle\xi\rangle}{N}\right), \quad \chi_{N}^{-}(t, \xi):=1-\chi_{N}^{+}(t, \xi),
$$

define the weight symbol

$$
\begin{equation*}
g(t, \xi)=\lambda(t)|\xi| \chi_{N}^{-}(t, \xi)+t_{\xi}^{-1} \chi_{N}^{+}(t, \xi), \tag{3.1}
\end{equation*}
$$

and consider $W(t, x)=\left(G\left(t, D_{x}\right) w(t, x), D_{t} w(t, x)\right)^{T}$. This vector solves the following pseudodifferential system of first order:

$$
\begin{aligned}
D_{t} W= & \left(\begin{array}{cc}
0 & G \\
\sum \lambda^{2} a_{i j} D_{x_{i} x_{j}} G^{-1} & -2 \sum \lambda c_{j} D_{x_{j}}
\end{array}\right) W \\
& +\left(\begin{array}{cc}
\left(D_{t} G\right) G^{-1} & 0 \\
-\sum\left(D_{t} \lambda\right) b_{j} D_{x_{j}} G^{-1} & 0
\end{array}\right) W+\left(\begin{array}{cc}
0 & 0 \\
0 & -i c_{0}
\end{array}\right) W+\binom{0}{f} \\
= & \left(\mathrm{A}_{N}^{0}+\mathrm{A}_{N}^{1}+\mathrm{A}_{N}^{2}\right) W+F=: \mathrm{A}_{N} W+F .
\end{aligned}
$$

For the further description of the symbols $\mathrm{a}_{N}^{j}$, we define the set $\mathscr{H}\left(Z_{p d}(2 N)\right)$ :

$$
\begin{aligned}
\mathscr{H}\left(Z_{p d}(2 N)\right):= & \left\{p(t, x, \xi) \in C^{\infty}\left(Z_{p d}(2 N)\right):\right. \\
& \left.\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} p(t, x, \xi)\right| \leq C_{\alpha \beta} g(t, \xi)\langle\xi\rangle^{-|\beta|}, \quad(t, x, \xi) \in Z_{p d}(2 N)\right\} .
\end{aligned}
$$

We observe that

$$
\begin{aligned}
& \mathrm{a}_{N}^{0}, \mathrm{a}_{N}^{1}, \mathrm{a}_{N}^{2} \in \mathscr{H}\left(Z_{p d}(2 N)\right), \\
& \mathrm{a}_{N}^{0} \in S_{N}\{1,1,0\}, \quad \mathrm{a}_{N}^{1} \in S_{N}\{0,0,-1\}, \quad \mathrm{a}_{N}^{2} \in S_{N}\{0,0,0\} .
\end{aligned}
$$

In the sequel, we consider the fundamental solution $\mathrm{E}(t, s)$, which satisfies

$$
D_{t} \mathrm{E}(t, s)=\mathrm{A}_{N}(t) \mathrm{E}(t, s), \quad \mathrm{E}(s, s)=I, \quad 0 \leq s, t \leq T .
$$

3.1. Diagonalization Let $\tau^{\mp}(t, x, \xi)$ be the characteristic roots of $\mathrm{a}_{N}^{0}$ (in $Z_{\text {hyp }}(2 N)$ ),

$$
\begin{align*}
\tau^{\mp}(t, x, \xi):= & \left(-c(t, x, \xi) \mp \sqrt{c(t, x, \xi)^{2}+a(t, x, \xi)}\right) \lambda(t)|\xi| \chi_{N}^{-}(t, \xi) \\
& \mp t_{\xi}^{-1} \chi_{N}^{+}(t, \xi),  \tag{3.2}\\
c(t, x, \xi):= & \sum_{j=1}^{n} c_{j}(t, x) \frac{\xi_{j}}{|\xi|}, \quad a(t, x, \xi):=\sum_{i, j=1}^{n} a_{i j}(t, x) \frac{\xi_{i} \xi_{j}}{|\xi|^{2}} \tag{3.3}
\end{align*}
$$

Then we set

$$
\mathrm{m}(t, x, \xi):=\left(\begin{array}{ll}
1 & \left(\left(-c-\sqrt{c^{2}+a}\right) \chi_{N}^{-}\right)(t, x, \xi)-\chi_{N}^{+}(t, \xi) \\
1 & \left(\left(-c+\sqrt{c^{2}+a}\right) \chi_{N}^{-}\right)(t, x, \xi)+\chi_{N}^{+}(t, \xi)
\end{array}\right)^{T} .
$$

From (1.9) it follows that $\operatorname{det} \mathrm{m} \geq \min \left(2,2 \sqrt{\alpha_{0}}\right)>0$. We have $\mathrm{m} \in S_{N}\{0,0,0\}$ and the matrix $m$ is constant in $Z_{p d}(N)$. Then Proposition 2.12 gives us a parametrix $\mathrm{M}^{\sharp}$ for the operator M . Since $\mathrm{a}_{N} \in S_{N}\{1,1,0\}$, we have

$$
\mathrm{m}^{\sharp} \circ \mathrm{a}_{N} \circ \mathrm{~m}=\mathrm{m}^{\sharp} \mathrm{a}_{N} \mathrm{~m}+\mathrm{r}_{1}+\mathrm{r}_{\infty},
$$

with remainders $r_{1} \in S_{N}\{0,1,0\}, r_{1} \equiv 0$ in $Z_{p d}(N)$ and $r_{\infty} \in C^{\infty}\left([0, T], S^{-\infty}\right)$. It remains to consider the product of symbols $\mathrm{m}^{\sharp} \mathrm{a}_{N} \mathrm{~m}$. By the choice of m ,

$$
\mathrm{m}^{\sharp} \mathrm{a}_{N}^{0} \mathrm{~m}= \begin{cases}\operatorname{diag}\left(\tau^{-}, \tau^{+}\right) & : \Lambda(t)\langle\xi\rangle \geq 2 N \\ \in \mathscr{H}\left(Z_{p d}(2 N)\right) & : \Lambda(t)\langle\xi\rangle \leq 2 N .\end{cases}
$$

We introduce the notation $\mathrm{d}:=\operatorname{diag}\left(\tau^{-}, \tau^{+}\right)$. It can be shown that

$$
\left.\begin{array}{rl}
\mathrm{m}^{\sharp} \mathrm{a}_{N}^{1} \mathrm{~m}= & \begin{cases}\mathrm{h}^{(0)} & : \Lambda(t)\langle\xi\rangle \geq 2 N, \\
\in \mathscr{H}\left(Z_{p d}(2 N)\right) & : \Lambda(t)\langle\xi\rangle \leq 2 N,\end{cases} \\
\mathrm{h}^{(0)}(t, x, \xi)= & \frac{D_{t} \lambda(t)}{2 \lambda(t)}\left(\begin{array}{ll}
1-\frac{b(t, x, \xi)+c(t, x, \xi)}{\sqrt{c(t, x, \xi)^{2}+a(t, x, \xi)}} & 1-\frac{b(t, x, \xi)+c(t, x, \xi)}{\sqrt{c(t, x, \xi)^{2}+a(t, x, \xi)}} \\
1+\frac{b(t, x, \xi)+c(t, x, \xi)}{\sqrt{c(t, x, \xi)^{2}+a(t, x, \xi)}} & 1+\frac{b(t, x, \xi)+c(t, x, \xi)}{\sqrt{c(t, x, \xi)^{2}+a(t, x, \xi)}}
\end{array}\right) \\
& +\mathrm{r}_{1}(t, x, \xi), \quad \mathrm{r}_{1} \in S_{N}\{0,0,0\},
\end{array}\right\} \begin{aligned}
& b(t, x, \xi):= \\
& -\sum_{j=1}^{n} b_{j}(t, x) \frac{\xi_{j}}{|\xi|} . \tag{3.4}
\end{aligned}
$$

Finally, $\mathrm{m}^{\sharp} \mathrm{a}_{N}^{2} \mathrm{~m} \in S_{N}\{0,0,0\}$. In the sequel, $\mathrm{R}_{\infty}$ denotes a generic regularizing operator from $C^{\infty}\left([0, T], \Psi^{-\infty}\right)$. Then we obtain

$$
\begin{aligned}
& D_{t}\left(M^{\sharp}(t) \mathrm{E}\right)=\left(D_{t} \mathrm{M}^{\sharp}\right)\left(\mathrm{MM}^{\sharp}+\mathrm{R}_{\infty}\right) \mathrm{E}+\mathrm{M}^{\sharp} \mathrm{A}_{N}\left(\mathrm{MM}^{\sharp}+\mathrm{R}_{\infty}\right) \mathrm{E} \\
&=\left(\mathrm{D}+\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-}+\tilde{\mathrm{H}}^{(1)}+\mathrm{R}_{0}\right) \mathrm{M}^{\sharp} \mathrm{E}+\mathrm{R}_{\infty} \mathrm{E}, \\
& \tilde{\mathrm{~h}}^{(1)} \in S_{N}\{-1,-1,-2\}, \quad \mathrm{r}_{0} \in \mathscr{H}\left(Z_{p d}(2 N)\right), \\
& \operatorname{supp} \tilde{\mathrm{h}}^{(1)} \subset Z_{\text {hyp }}(N), \quad \operatorname{supp} \mathrm{r}_{0} \subset Z_{p d}(2 N) .
\end{aligned}
$$

Here we have modified the term $r_{1}$ in the definition of $h^{(0)}$ in order to include all terms from $S_{N}\{0,0,0\}$.

This was the first step of diagonalization. We shall apply further steps in order to diagonalize $\mathrm{h}^{(0)}$ and $\tilde{\mathrm{h}}^{(1)}$ modulo $\cap_{p} S_{N}\{-p,-p,-p-1\}$. For this purpose, we define

$$
\begin{aligned}
\mathrm{k}^{(0)} & :=\left(\begin{array}{cc}
\mathrm{h}_{11}^{(0)} & 0 \\
0 & \mathrm{~h}_{22}^{(0)}
\end{array}\right) \chi_{N}^{-} \in S_{N}\{0,0,-1\}, \\
\mathrm{n}^{(1)} & :=\left(\begin{array}{cc}
0 & \frac{\mathrm{~h}_{12}^{(0)}}{\tau^{+}-\tau^{-}} \\
\frac{\mathrm{h}_{21}^{(0)}}{\tau^{-}-\tau^{+}} & 0
\end{array}\right) \chi_{N}^{-} \in S_{N}\{-1,-1,-1\}
\end{aligned}
$$

and observe that $\mathrm{N}^{(1)} \mathrm{D}-\mathrm{DN}^{(1)}=\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-}-\mathrm{K}^{(0)}+\mathrm{R}^{(0)}$ with remainder $\mathrm{r}^{(0)} \in$ $S_{N}\{-1,0,-1\} \subset S_{N}\{-1,-1,-2\}$. Then it can be concluded that

$$
\begin{aligned}
\mathrm{H}^{(1)} & :=\left(D_{t}-\mathrm{D}-\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-}-\tilde{\mathrm{H}}^{(1)}\right)\left(I+\mathrm{N}^{(1)}\right)-\left(I+\mathrm{N}^{(1)}\right)\left(D_{t}-\mathrm{D}-\mathrm{K}^{(0)}\right) \\
& =\left(D_{t} \mathrm{~N}^{(1)}\right)+\mathrm{R}^{(0)}-\tilde{\mathrm{H}}^{(1)}-\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-} \mathrm{N}^{(1)}-\tilde{\mathrm{H}}^{(1)} \mathrm{N}^{(1)}+\mathrm{N}^{(1)} \mathrm{K}^{(0)} ;
\end{aligned}
$$

hence $\mathrm{h}^{(1)} \in S_{N}\{-1,-1,-2\}$. We see that $\mathrm{k}^{(0)} \equiv \mathrm{n}^{(1)} \equiv 0$ in $Z_{p d}(N)$. Modulo $C\left([0, T], S^{-\infty}\right)$, we have $\mathrm{h}^{(1)} \equiv 0$ in $Z_{p d}(N)$. Inductively we set (for $\nu \geq 2$ )

$$
\begin{aligned}
& \mathrm{k}^{(\nu-1)}:=\left(\begin{array}{cc}
\mathrm{h}_{11}^{(\nu-1)} & 0 \\
0 & \mathrm{~h}_{22}^{(\nu-1)}
\end{array}\right) \in S_{N}\{-\nu+1,-\nu+1,-\nu\}, \\
& \mathrm{n}^{(\nu)}:=\left(\begin{array}{cc}
0 & \frac{\mathrm{~h}_{12}^{(\nu-1)}}{\tau^{+}-\tau^{-}} \\
\frac{\mathrm{h}_{21}^{(\nu-1)}}{\tau^{-}-\tau^{+}} & 0
\end{array}\right) \in S_{N}\{-\nu,-\nu,-\nu\}, \\
& \mathrm{H}^{(\nu)}:=\left(D_{t}-\mathrm{D}-\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-}-\tilde{\mathrm{H}}^{(1)}\right)\left(I+\sum_{\mu=1}^{\nu} \mathrm{N}^{(\mu)}\right) \\
&-\left(I+\sum_{\mu=1}^{\nu} \mathrm{N}^{(\mu)}\right)\left(D_{t}-\mathrm{D}-\sum_{\mu=0}^{\nu-1} \mathrm{~K}^{(\mu)}\right)
\end{aligned}
$$

From the induction assumption and Proposition 2.10, it follows that $h^{(\nu)} \in$ $S_{N}\{-\nu,-\nu,-\nu-1\}$ and $\mathrm{h}^{(\nu)} \equiv 0$ in $Z_{p d}(N)$, modulo $C\left([0, T], S^{-\infty}\right)$.

Employing Proposition 2.11, we find symbols n and $\tilde{\mathrm{k}}$ that satisfy

$$
\begin{aligned}
& \mathrm{n} \in S_{N}\{0,0,0\}, \quad \tilde{\mathrm{k}} \in S_{N}\{-1,-1,-2\} \\
& \mathrm{n}(t, x, \xi) \equiv I, \quad \tilde{\mathrm{k}}(t, x, \xi) \equiv 0 \text { in } Z_{p d}(N) \\
& \mathrm{n} \sim I+\sum_{\mu=1}^{\infty} \mathrm{n}^{(\mu)} \quad \bmod \bigcap_{p \geq 0} S_{N}\{-p,-p,-p\}, \\
& \tilde{\mathrm{k}} \sim \sum_{\mu=1}^{\infty} \mathrm{k}^{(\mu)} \quad \bmod \bigcap_{p \geq 0} S_{N}\{-p,-p,-p-1\} .
\end{aligned}
$$

Then the operator identity

$$
\left(D_{t}-\mathrm{D}-\mathrm{H}^{(0)} \mathrm{Op} \chi_{N}^{-}-\tilde{\mathrm{H}}^{(1)}\right) \mathrm{N}=\mathrm{N}\left(D_{t}-\mathrm{D}-\mathrm{K}^{(0)}-\tilde{\mathrm{K}}\right)+\mathrm{K}^{(\infty)}
$$

holds, where $K^{(\infty)}$ is an operator with full matrix symbol

$$
\mathbf{k}^{(\infty)} \in \bigcap_{p \geq 0} S_{N}\{-p,-p,-p-1\} \cap \mathscr{H}\left(Z_{p d}(2 N)\right) .
$$

To complete the diagonalization, we need a parametrix for the operator N . The matrix norm of its symbol satisfies

$$
\|\mathrm{n}(t, x, \xi)-I\| \leq \frac{C}{\Lambda(t)\langle\xi\rangle} \leq \frac{1}{3}
$$

if $\Lambda(t)\langle\xi\rangle \geq N_{1}$, where $N_{1} \gg N$ is an appropriate number. We define

$$
\tilde{\mathrm{n}}(t, x, \xi):=\mathrm{n}(t, x, \xi)-\chi_{N_{1}}^{+}(t, \xi)(\mathrm{n}(t, x, \xi)-I)
$$

and observe that $\mathrm{n}-\tilde{\mathrm{n}} \in \cap_{p \geq 0} S_{N}\{-p,-p,-p\}$ and

$$
\begin{equation*}
\|\tilde{\mathrm{n}}(t, x, \xi)-I\| \leq \frac{1}{3} \quad \forall(t, x, \xi) . \tag{3.5}
\end{equation*}
$$

Since n is uniquely determined only modulo $\cap_{p \geq 0} S_{N}\{-p,-p,-p\}$, we may drop the tilde and assume that n satisfies (3.5), too. Proposition 2.12 shows that a parametrix $N^{\sharp}$ to the operator $N$ exists, whose symbol belongs to $S_{N}\{0,0,0\}$ and coincides with the identity matrix in $Z_{p d}(N)$.

If we allow some modifications in the term $\mathrm{K}^{(\infty)}$, we can show

$$
\begin{equation*}
\left(D_{t}-\mathrm{D}-\mathrm{K}^{(0)}-\tilde{\mathrm{K}}-\mathrm{K}^{(\infty)}\right)\left(\mathrm{N}^{\sharp} \mathrm{M}^{\sharp} \mathrm{E}\right)=\mathrm{R}_{\infty} \mathrm{E} . \tag{3.6}
\end{equation*}
$$

Thus we have proved:
Proposition 3.1. The fundamental solution $\mathrm{E}(t, s)$ to the operator $D_{t}-\mathrm{A}_{N}(t)$ satisfies (3.6), where $N^{\sharp}$ and $M^{\sharp}$ are elliptic operators with symbols from $L^{\infty}\left([0, T], S_{1,0}^{0}\right)$; and $\mathrm{D}, \mathrm{K}^{(0)}, \tilde{\mathrm{K}}$ are diagonal operators with symbols from $S_{N}\{1,1,0\}, S_{N}\{0,0,-1\}$, $S_{N}\{-1,-1,-2\}$, respectively. The (full matrix) symbol of $K^{(\infty)}$ belongs to $\cap_{p \geq 0} S_{N}\{-p,-p,-p-1\}$ and $\mathscr{H}\left(Z_{p d}(2 N)\right)$. In $Z_{p d}(N)$, the symbol of D is independent of $(t, x)$ and the symbols of $\mathrm{K}^{(0)}, \tilde{\mathrm{K}}$ vanish. Moreover, the symbols of D and $\mathrm{K}^{(0)}$ are given by

$$
\begin{align*}
& \mathrm{d}=\operatorname{diag}\left(\tau^{-}, \tau^{+}\right), \quad \mathrm{k}^{(0)}=\operatorname{diag}\left(\mathrm{k}_{-}^{(0)}, \mathrm{k}_{+}^{(0)}\right), \\
& \mathrm{k}_{\mp}^{(0)}=\frac{D_{t} \lambda}{2 \lambda}\left(1 \mp \frac{b+c}{\sqrt{c^{2}+a}}\right) \chi_{N}^{-}+\mathrm{r}_{\mp} \chi_{N}^{-}, \quad \mathrm{r}_{\mp} \in S_{N}\{0,0,0\}, \tag{3.7}
\end{align*}
$$

see (3.2), (3.3) and (3.4).
3.2. Construction of the fundamental solution Now we are going to construct $\mathrm{E}(t, s)$. Let $\mathrm{P}_{2}, \mathrm{P}_{1}, \mathrm{P}_{0}$ denote the operators

$$
P_{2}:=D+K^{(0)}+\tilde{K}+K^{(\infty)}, \quad P_{1}:=D+K^{(0)}+\tilde{K}, \quad P_{0}:=D
$$

and $\mathrm{E}_{j}(t, s),(j=0,1,2)$, the fundamental solutions to $D_{t}-\mathrm{P}_{j}$ :

$$
\left(D_{t}-\mathrm{P}_{j}(t)\right) \mathrm{E}_{j}(t, s)=0, \quad \mathrm{E}_{j}(s, s)=I
$$

Since $\mathrm{E}(t, s)=\mathrm{M}(t) \mathrm{N}(t) \mathrm{E}_{2}(t, s) \mathrm{N}^{\sharp}(s) \mathrm{M}^{\sharp}(s)+\mathrm{R}_{\infty}(t, s)$, it suffices to construct $\mathrm{E}_{2}$. This task is done in three steps. First, we construct $\mathrm{E}_{0}$, which is a diagonal Fourier integral operator of order zero. Then we compute $E_{1}=E_{0} Q_{0}$ by means of Egorov's Theorem. It turns out that $Q_{0}$ is a diagonal pseudodifferential operator which describes the loss of regularity. Finally, $E_{2}=E_{1} Q_{1}$, where $Q_{1}$ is some matrix pseudodifferential operator of order zero.

Since these constructions are quite standard, we only sketch them. For an exhaustive representation and related problems, see [19].

### 3.2.1. Fundamental solutions to scalar first order operators

Definition 3.2. Let $\tau=\tau(t, x, \xi)$ be either $\tau^{-}$or $\tau^{+}$. The Hamilton flow $\mathscr{H}_{s, t}(y, \eta):=(x, \xi)=(x, \xi)(t, s, y, \eta)$ is defined as the solution to the system of ODEs

$$
\begin{aligned}
& \frac{d x}{d t}=-\nabla_{\xi} \tau(t, x, \xi), \quad x(s, s, y, \eta)=y, \\
& \frac{d \xi}{d t}=\nabla_{x} \tau(t, x, \xi), \quad \xi(s, s, y, \eta)=\eta .
\end{aligned}
$$

It is known that for small $T, 0<T \leq T_{0}$, the solution $(x, \xi)$ exists (uniformly with respect to $(y, \eta))$ for $0 \leq s, t \leq T$ and that

$$
\begin{aligned}
& \left\{\frac{x(t, s)-y}{t-s}, \partial_{t} x(t, s), \partial_{s} x(t, s)\right\}_{0 \leq s, t \leq T} \text { is bounded in } S_{1,0}^{0}\left(\mathbb{R}_{y}^{n} \times \mathbb{R}_{\eta}^{n}\right), \\
& \left\{\frac{\xi(t, s)-\eta}{t-s}, \partial_{t} \xi(t, s), \partial_{s} \xi(t, s)\right\}_{0 \leq s, t \leq T} \text { is bounded in } S_{1,0}^{1}\left(\mathbb{R}_{y}^{n} \times \mathbb{R}_{\eta}^{n}\right) .
\end{aligned}
$$

If $T$ is sufficiently small, then an inverse function $y=y(t, s, x, \eta)$ to the mapping $x=x(t, s, y, \eta)$ can be found. Then the set

$$
\left\{\frac{y(t, s)-x}{t-s}, \partial_{t} y(t, s), \partial_{s} y(t, s)\right\}_{0 \leq s, t \leq T}
$$

is bounded in $S_{1,0}^{0}\left(\mathbb{R}_{x}^{n} \times \mathbb{R}_{\eta}^{n}\right)$.
Now we can construct the phase function $\phi$ which solves the eikonal equation

$$
\partial_{t} \phi(t, x ; s, \xi)-\tau\left(t, x, \nabla_{x} \phi(t, x ; s, \xi)\right)=0
$$

with the initial condition $\phi(s, x ; s, \xi)=x \cdot \xi$. We set

$$
v(t, y, s, \eta)=y \cdot \eta-\int_{s}^{t}\left(\xi \cdot \nabla_{\xi} \tau-\tau\right)(\sigma, x(\sigma, s, y, \eta), \xi(\sigma, s, y, \eta)) d \sigma
$$

and can express the phase function in the form

$$
\phi\left(t, x_{0} ; s, \xi_{0}\right):=v\left(t, y\left(t, s, x_{0}, \xi_{0}\right), s, \xi_{0}\right)
$$

The representation of $v$ and the properties of $y(t, s, x, \xi)$ imply

$$
\begin{equation*}
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta}(\phi(t, x ; s, \xi)-x \cdot \xi)\right| \leq C_{\alpha \beta}\langle\xi\rangle^{1-|\beta|} \Lambda(s \vee t), \quad s \vee t \geq t_{\xi}, \tag{3.8}
\end{equation*}
$$

where $s \vee t:=\max (s, t)$. The function $\tau(t, x, \xi)$ does not depend on $(t, x)$ if $(t, x, \xi) \in$ $Z_{p d}(N)$. Consequently,

$$
\begin{equation*}
\phi(t, x ; s, \xi)=x \cdot \xi-(t-s) \tau(\xi), \quad s, t \leq t_{\xi} . \tag{3.9}
\end{equation*}
$$

In order to formulate the transport equations for the amplitude functions of $E_{0}$, we recall a theorem about compositions of pseudodifferential operators and Fourier integral operators, see e.g., [12].

Theorem 3.3. Let $P_{1}$ be a pseudodifferential operator with symbol $p_{1}(x, \xi) \in$ $S_{1,0}^{m_{1}}$ and let $P_{2 \phi}$ be a Fourier integral operator with phase function $\phi(x, \xi)$ and amplitude $p_{2}(x, \xi) \in S_{1,0}^{m_{2}}$. We assume that $\pi_{\alpha, \beta}^{1}(\phi(x, \xi)-x \cdot \xi) \leq \delta<1$ for $|\alpha|+|\beta| \leq 2$. Then the composition $P_{1} P_{2 \phi}$ is a Fourier integral operator with phase function $\phi$ and amplitude $r(x, \xi) \in S_{1,0}^{m_{1}+m_{2}}$ which can be written as

$$
\begin{aligned}
r(x, \xi)= & p_{1}\left(x, \nabla_{x} \phi(x, \xi)\right) p_{2}(x, \xi)+\sum_{j=1}^{n}\left(\partial_{\xi_{j}} p_{1}\right)\left(x, \nabla_{x} \phi(x, \xi)\right) D_{x_{j}} p_{2}(x, \xi) \\
& -\frac{i}{2} \sum_{j, k=1}^{n}\left(\partial_{\xi_{j} \xi_{k}}^{2} p_{1}\right)\left(x, \nabla_{x} \phi(x, \xi)\right)\left(\partial_{x_{j} x_{k}}^{2} \phi(x, \xi)\right) p_{2}(x, \xi)+r_{2}(x, \xi),
\end{aligned}
$$

with some $r_{2} \in S_{1,0}^{m_{1}+m_{2}-2}$. We have the asymptotic expansion

$$
r(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} D_{y}^{\alpha}\left(\left(\partial_{\xi}^{\alpha} p_{1}\right)\left(x, \tilde{\nabla}_{x} \phi(x, y, \xi)\right) p_{2}(y, \xi)\right)_{y=x}
$$

where $\tilde{\nabla}_{x} \phi(x, y, \xi):=\int_{0}^{1}\left(\nabla_{x} \phi\right)(y+s(x-y), \xi) d s$.
This theorem is now employed to find the fundamental solution to $D_{t}-\mathrm{D}$.

Proposition 3.4. The fundamental solution $\mathrm{E}_{0}(t, s)$ to the operator $D_{t}-\mathrm{D}$ is a diagonal Fourier integral operator,

$$
\begin{aligned}
& \mathrm{E}_{0}(t, s)=\operatorname{diag}\left(\mathrm{E}_{0}^{-}(t, s), \mathrm{E}_{0}^{+}(t, s)\right), \\
& \left(\mathrm{E}_{0}^{\mp}(t, s) w\right)(x)=\int e^{i \phi^{\mp}(t, x ; s, \xi)} e^{\mp}(t, x ; s, \xi) \hat{w}(\xi) d \xi, \\
& \phi^{\mp}(s, x ; s, \xi)=x \cdot \xi, \quad e^{\mp}(s, x ; s, \xi)=1 .
\end{aligned}
$$

The phase functions $\phi^{+}, \phi^{-}$satisfy (3.8), (3.9), and the amplitude functions $e^{\mp}(t, x ; s, \xi)$ belong to $C\left([0, T]^{2}, S_{1,0}^{0}\right)$.

Proof. We shall express $e^{\mp}$ as an asymptotic series,

$$
e^{\mp}(t, x ; s, \xi) \sim \sum_{j=0}^{\infty} e_{j}^{\mp}(t, x ; s, \xi), \quad e_{j}^{\mp} \in C\left([0, T]^{2}, S_{1,0}^{-j}\right),
$$

with the initial conditions $e_{j}^{\mp}(s, x ; s, \xi)=0$ for $j \geq 1$. Obviously,

$$
\left(D_{t} \mathrm{E}_{0}^{\mp}(t, s) w\right)(x)=\int e^{i \phi^{\mp}(t, x ; s, \xi)}\left(\left(\phi_{t}^{\mp} e^{\mp}\right)+i^{-1} e_{t}^{\mp}\right)(t, x ; s, \xi) \hat{w}(\xi) d \xi .
$$

Let $\mathrm{E}_{0 m}^{\mp}(t, s)$ denote the Fourier integral operator with phase function $\phi^{\mp}(t, x ; s, \xi)$ and amplitude $e_{m}^{\mp}(t, x ; s, \xi)$. Theorem 3.3 yields

$$
\begin{aligned}
& \left(\tau^{\mp}\left(t, x, D_{x}\right) \mathrm{E}_{0 m}^{\mp}(t, s) w\right)(x)= \\
& \quad \int e^{i \phi^{\mp}(t, x ; s, \xi)}\left\{\tau^{\mp}\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right) e_{m}^{\mp}(t, x ; s, \xi)\right. \\
& \quad+i^{-1}\left(\nabla_{\xi} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right) \nabla_{x} e_{m}^{\mp}(t, x ; s, \xi) \\
& \quad-\frac{i}{2} \sum_{k, l}\left(\partial_{\xi_{k} \xi_{l}}^{2} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right)\left(\left(\partial_{x_{k} x_{l}}^{2} \phi^{\mp}\right) e_{m}^{\mp}\right)(t, x ; s, \xi) \\
& \left.\quad+r_{m, 2}(t, x ; s, \xi)\right\} \hat{w}(\xi) d \xi, \quad \operatorname{ord}\left(r_{m, 2}\right)=-1-m .
\end{aligned}
$$

This leads us to the eikonal equation

$$
\phi_{t}^{\mp}(t, x ; s, \xi)-\tau^{\mp}\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right)=0
$$

and the transport equations ( $m=0,1,2, \ldots$ )

$$
\begin{align*}
& \left(\partial_{t}-\left(\nabla_{\xi} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right) \nabla_{x}\right) e_{m}^{\mp}(t, x ; s, \xi)  \tag{3.10}\\
& \quad-\frac{1}{2} \sum_{k, l}\left(\partial_{\xi_{k} \xi_{l}}^{2} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right)\left(\partial_{x_{k} x_{l}}^{2} \phi^{\mp}(t, x ; s, \xi)\right) e_{m}^{\mp}(t, x ; s, \xi)
\end{align*}
$$

$$
=\sum_{k=0}^{m-1} \sum_{|\alpha|=m+1-k} \frac{i}{\alpha!} D_{y}^{\alpha}\left(\left(\partial_{\xi}^{\alpha} \tau^{\mp}\right)\left(t, x, \tilde{\nabla}_{x} \phi^{\mp}(t, x, y ; s, \xi)\right) e_{k}^{\mp}(t, y ; s, \xi)\right)_{y=x}
$$

To solve the transport equations, we recall a well-known result from the theory of first order PDEs. Consider the Cauchy problem with parameter $p$,

$$
\begin{aligned}
& \left(\partial_{t}-\sum_{j=1}^{n} a_{j}(t, x ; p) \partial_{x_{j}}\right) w(t, x, s ; p)-a_{0}(t, x ; p) w(t, x, s ; p)=f(t, x ; p) \\
& w(s, x, s ; p)=w_{0}(x ; p)
\end{aligned}
$$

Let $\gamma=\gamma\left(t, s, x_{0} ; p\right):[0, T]^{2} \times \mathbb{R}^{n} \times \mathbb{R}_{p}^{k} \rightarrow \mathbb{R}^{n}$ be the solution to

$$
\begin{aligned}
& \partial_{t} \gamma_{j}\left(t, s, x_{0} ; p\right)=-a_{j}\left(t, \gamma\left(t, s, x_{0} ; p\right) ; p\right), \quad j=1, \ldots, n \\
& \gamma\left(s, s, x_{0} ; p\right)=x_{0}
\end{aligned}
$$

Then the solution $w$ satisfies

$$
\begin{align*}
& w(t, x, s ; p)=w_{0}(\gamma(s, t, x ; p) ; p) \exp \left(\int_{s}^{t} a_{0}(r, \gamma(r, t, x ; p) ; p) d r\right)  \tag{3.11}\\
& \quad+\int_{s}^{t} \exp \left(\int_{\sigma}^{t} a_{0}(r, \gamma(r, t, x ; p) ; p) d r\right) f(\sigma, \gamma(\sigma, t, x ; p) ; p) d \sigma
\end{align*}
$$

In our situation, $p=(s, \xi), a_{i}(t, x ; p)=a_{i}^{\mp}(t, x ; p),(0 \leq i \leq n)$, with

$$
\begin{aligned}
& a_{i}^{\mp}(t, x ; p)=\left(\partial_{\xi_{i}} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right), \quad 1 \leq i \leq n \\
& a_{0}^{\mp}(t, x ; p)=\frac{1}{2} \sum_{k, l}\left(\partial_{\xi_{k} \xi_{l}}^{2} \tau^{\mp}\right)\left(t, x, \nabla_{x} \phi^{\mp}(t, x ; s, \xi)\right) \partial_{x_{k} x_{l}}^{2} \phi^{\mp}(t, x ; s, \xi),
\end{aligned}
$$

and $f=f_{m}^{\mp}(t, x ; p)$ is given by the right side of (3.10). Consequently,

$$
\begin{aligned}
& e_{0}^{\mp}(t, x ; s, \xi)=\exp \left(\int_{s}^{t} a_{0}^{\mp}(r, \gamma(r, t, x ; s, \xi) ; s, \xi) d r\right), \\
& e_{m}^{\mp}(t, x ; s, \xi) \\
& \quad=\int_{s}^{t} \exp \left(\int_{\sigma}^{t} a_{0}^{\mp}(r, \gamma(r, t, x ; s, \xi) ; s, \xi) d r\right) f_{m}^{\mp}(\sigma, \gamma(\sigma, t, x ; s, \xi) ; s, \xi) d \sigma .
\end{aligned}
$$

The coefficients $a_{i},(0 \leq i \leq n)$, belong to $C\left([0, T]^{2}, S_{1,0}^{0}\right)$; more precisely,

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a_{i}^{\mp}(t, x ; s, \xi)\right| \leq\left\{\begin{array}{ll}
C_{\alpha \beta} \lambda(t)\langle\xi\rangle^{-|\beta|} & : t_{\xi} \leq t, \\
C_{\alpha \beta} t_{\xi}^{-1}\langle\xi\rangle^{-|\beta|-1} & : t \leq t_{\xi}
\end{array} \quad 1 \leq i \leq n\right.
$$

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} a_{0}^{\mp}(t, x ; s, \xi)\right| \leq \begin{cases}C_{\alpha \beta} \lambda(t)\langle\xi\rangle^{-|\beta|} \Lambda(t \vee s) & : t_{\xi} \leq t \\ C_{\alpha \beta} t_{\xi}^{-1}\langle\xi\rangle^{-|\beta|-1} \Lambda(s) & : t \leq t_{\xi} \leq s \\ 0 & : t, s \leq t_{\xi}\end{cases}
$$

see (3.8), (3.9). Then we obtain $\gamma \in C\left([0, T]^{3}, S_{1,0}^{0}\right)$ with $|\gamma(r, t, x ; s, \xi)-x| \leq C|r-t|$. The representation of $e_{0}^{\mp}$ implies $e_{0}^{\mp}(t, x ; s, \xi)=1$ if $s \vee t \leq t_{\xi}$ and

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} e_{0}^{\mp}(t, x ; s, \xi)\right| \leq C_{\alpha \beta}\langle\xi\rangle^{-|\beta|}
$$

if $s \vee t \geq t_{\xi}$. Inductively we conclude that

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} f_{m}^{\mp}(t, x ; s, \xi)\right| \leq \begin{cases}C_{\alpha \beta m} \lambda(t)\langle\xi\rangle^{-m-|\beta|} & : t_{\xi} \leq t, \\ C_{\alpha \beta m} t_{\xi}^{-1}\langle\xi\rangle^{-m-1-|\beta|} & : t \leq t_{\xi} \leq s, \\ 0 & : t, s \leq t_{\xi}\end{cases}
$$

and

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} e_{m}^{\mp}(t, x ; s, \xi)\right| \leq \begin{cases}C_{\alpha \beta m} \Lambda(t)\langle\xi\rangle^{-m-|\beta|} & : t_{\xi} \leq t \\ C_{\alpha \beta m} \Lambda(s)\langle\xi\rangle^{-m-|\beta|} & : t \leq t_{\xi} \leq s, \\ 0 & : t, s \leq t_{\xi} .\end{cases}
$$

We see that $e_{m}^{\mp} \in C\left([0, T]^{2}, S_{1,0}^{-m}\right)$. If we set

$$
\tilde{e}^{\mp}(t, x ; s, \xi)=e_{0}^{\mp}(t, x ; s, \xi)+\sum_{j=1}^{\infty}\left(1-\chi\left(\varepsilon_{j}\langle\xi\rangle\right)\right) e_{j}^{\mp}(t, x ; s, \xi)
$$

and choose the sequence $\left\{\varepsilon_{j}\right\}$ with $\varepsilon_{j} \searrow 0$ suitably, then

$$
\tilde{e}^{\mp}(t, x ; s, \xi) \sim \sum_{j=0}^{\infty} e_{j}^{\mp}(t, x ; s, \xi) \quad \bmod C\left([0, T]^{2}, S^{-\infty}\right)
$$

and $\tilde{e}^{\mp}(t, x ; s, \xi)=1$ for $s \vee t \leq t_{\xi}$. The function $\tilde{e}$ is the amplitude to an approximate fundamental solution $\tilde{E}_{0}^{\mp}$ which satisfies

$$
\left(D_{t}-\tau^{\mp}\left(t, x, D_{x}\right)\right) \tilde{E}_{0}^{\mp}\left(t, x ; s, D_{x}\right)=R_{\infty}^{\mp}\left(t, x ; s, D_{x}\right)
$$

with $r_{\infty}^{\mp} \in C\left([0, T]^{2}, S^{-\infty}\right)$. In order to find the exact solution $\mathrm{E}_{0}^{\mp}$, we set

$$
W_{1}(t, s):=-i R_{\infty}^{\mp}(t, s), \quad W_{\nu+1}(t, s):=\int_{s}^{t} W_{1}(t, \sigma) W_{\nu}(\sigma, s) d \sigma .
$$

Then the desired fundamental solution to $D_{t}-\mathrm{D}$ is given by

$$
\mathrm{E}_{0}^{\mp}(t, s):=\tilde{\mathrm{E}}_{0}^{\mp}(t, s)+\int_{s}^{t} \tilde{\mathrm{E}}_{0}^{\mp}(t, \sigma) \sum_{\nu=1}^{\infty} W_{\nu}(\sigma, s) d \sigma .
$$

Now we construct $\mathrm{E}_{1}(t, s)$.
Proposition 3.5. The fundamental solution $\mathrm{E}_{1}(t, s)$ to the operator $D_{t}-\mathrm{D}-$ $\mathrm{K}^{(0)}-\tilde{\mathrm{K}}$ can be written as $\mathrm{E}_{1}(t, s)=\mathrm{E}_{0}(t, s) \mathrm{Q}_{0}(t, s)$, where $\mathrm{Q}_{0}$ is a diagonal pseudodifferential operator, $\mathrm{Q}_{0} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{K_{0}}\right)$ for some $K_{0}$. With some $\tilde{\mathrm{q}}_{0} \in$ $L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{0}\right)$, its symbol can be written in the form

$$
\begin{aligned}
& \mathrm{q}_{0}(t, s, x, \xi)=\exp \left(\int_{s}^{t} i \mathrm{k}^{(0)}(\tau, s, x, \xi) d \tau\right) \tilde{\mathrm{q}}_{0}(t, s, x, \xi), \\
& \mathrm{k}^{(0)}(t, s, x, \xi)=\operatorname{diag}\left(\mathrm{k}_{-}^{(0)}\left(t, \mathscr{H}_{s, t}^{-}(x, \xi)\right), \mathrm{k}_{+}^{(0)}\left(t, \mathscr{H}_{s, t}^{+}(x, \xi)\right)\right),
\end{aligned}
$$

where the Hamilton flows $\mathscr{H}_{s, t}^{\mp}(x, \xi)$ have been defined in Definition 3.2; and $\mathrm{k}_{ \pm}^{(0)}$ is given by (3.7).

Proof. We look for $E_{1}$ having the form $E_{1}=E_{0} Q_{0}$ and obtain

$$
D_{t} \mathrm{Q}_{0}(t, s)=\mathrm{E}_{0}(s, t)\left(\mathrm{K}^{(0)}(t)+\tilde{\mathrm{K}}(t)\right) \mathrm{E}_{0}(t, s) \mathrm{Q}_{0}(t, s) .
$$

According to Egorov's Theorem (see [19]), $\mathrm{E}_{0}(s, t)\left(\mathrm{K}^{(0)}(t)+\tilde{\mathrm{K}}(t)\right) \mathrm{E}_{0}(t, s)$ is a pseudodifferential operator $\mathrm{K}(t, s)$, whose diagonal principal symbol $\mathrm{k}^{(0)}(t, s)+\mathrm{k}^{(1)}(t, s)$ is given by

$$
\mathrm{k}_{\mp}^{(0)}(t, x ; s, \xi)+\mathrm{k}_{\mp}^{(1)}(t, x ; s, \xi)=\mathrm{k}_{\mp}^{(0)}\left(t, \mathscr{H}_{s, t}^{\mp}(x, \xi)\right)+\tilde{\mathrm{k}}_{\mp}\left(t, \mathscr{H}_{s, t}^{\mp}(x, \xi)\right),
$$

see also Definition 3.2. Therefore, we write

$$
\operatorname{sym}(\mathrm{K}(t, s))=: \mathrm{k}(t, s)=: \mathrm{k}^{(0)}(t, s)+\mathrm{k}^{(1)}(t, s)+\mathrm{k}^{(2)}(t, s)
$$

with lower order terms $\mathrm{k}^{(2)}(t, s)$ and look for $\mathrm{Q}_{0}$ having the form

$$
\left(\mathrm{Q}_{0}(t, s) w\right)(x)=\int e^{i x \xi} \mathrm{q}_{0}(t, s, x, \xi) \hat{w}(\xi) d \xi, \quad \mathrm{q}_{0}(s, s, x, \xi)=I .
$$

We expand $\mathrm{q}_{0}$ into an asymptotic series $\sum_{j=0}^{\infty} \mathrm{q}_{0 j}$ with $\operatorname{ord}\left(\mathrm{q}_{0 j}\right)=-j$ and

$$
D_{t} \mathrm{q}_{0 m}(t, s)=\sum_{|\alpha|=0}^{m} \frac{1}{\alpha!}\left(D_{\xi}^{\alpha} \mathrm{k}(t, s)\right)\left(\partial_{x}^{\alpha} \mathrm{q}_{0(m-|\alpha|)}(t, s)\right) .
$$

Recalling that all matrices are diagonal, we see that

$$
\begin{aligned}
& \mathrm{q}_{00}(t, s)=\exp \left(\int_{s}^{t} i \mathrm{k}(\tau, s) d \tau\right), \\
& \mathrm{q}_{0 m}(t, s)=i \int_{s}^{t} \exp \left(\int_{\sigma}^{t} i \mathrm{k}(\tau, s) d \tau\right) \sum_{|\alpha|=1}^{m} \frac{1}{\alpha!}\left(D_{\xi}^{\alpha} \mathrm{k}\right)\left(\partial_{x}^{\alpha} \mathrm{q}_{0(m-|\alpha|)}\right)(\sigma, s) d \sigma .
\end{aligned}
$$

We know that the $\mathrm{k}^{(j)}$ vanish in $Z_{p d}(N)$ and that

$$
\begin{aligned}
& \mathrm{k}^{(0)} \in \frac{\lambda}{\Lambda} C\left([0, T], S_{1,0}^{0}\right), \quad \mathrm{k}^{(1)} \in \frac{\lambda}{\Lambda^{2}} C\left([0, T], S_{1,0}^{-1}\right), \\
& \mathrm{k}^{(2)} \in \frac{\lambda}{\Lambda^{2}} C\left([0, T], S_{1,0}^{-2}\right) \subset C\left([0, T], S_{1,0}^{0}\right),
\end{aligned}
$$

uniformly with respect to $s \in[0, T]$. Consequently,

$$
\int_{s \wedge t}^{s \vee t}\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} k^{(j)}(\tau, s, x, \xi)\right| d \tau \leq C_{\alpha \beta}\langle\xi\rangle^{-|\beta|}, \quad j=1,2 .
$$

The proof of Proposition 2.13 gives the rough estimate

$$
\int_{s \wedge t}^{s \vee t}\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathbf{k}^{(0)}(\tau, s, x, \xi)\right| d \tau \leq C_{\alpha \beta \varepsilon}\langle\xi\rangle^{-|\beta|+\varepsilon} \quad \forall(t, s, x, \xi)
$$

for every positive $\varepsilon$. By induction we obtain

$$
\begin{aligned}
& \left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathrm{q}_{0 m}(t, s, x, \xi)\right|\left\|\exp \left(-\int_{s}^{t} i \mathrm{k}^{(0)}(\tau, s, x, \xi) d \tau\right)\right\| \\
& \quad \leq C_{\alpha \beta \varepsilon m}\langle\xi\rangle^{-(1-\varepsilon) m+\varepsilon|\alpha|-(1-\varepsilon)|\beta|} .
\end{aligned}
$$

There is a positive real number $K_{+}$with

$$
\exp \left(\int_{s}^{t}\left\|\mathrm{k}^{(0)}(\tau, s, x, \xi)\right\| d \tau\right) \leq \begin{cases}C\left(\frac{\Lambda(t \vee s)}{\Lambda(t \wedge s)}\right)^{K_{+}} & : t_{\xi}^{\prime} \leq s, t \\ C\left(\frac{\Lambda(t \vee s)}{\Lambda\left(t_{\xi}\right)}\right)^{K_{+}} & : s \wedge t \leq t_{\xi}^{\prime} \leq s \vee t \\ 1 & : t, s \leq t_{\xi}^{\prime} .\end{cases}
$$

Here $t_{\xi}^{\prime}$ is given by $\Lambda\left(t_{\xi}^{\prime}\right)\langle\xi\rangle=N / 2$. We shall see that the number $K_{+}$, which gives a bound for the loss of regularity, is the same number as in Section 2.1. Since $\mathrm{q}_{0 m} \in$ $L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{K_{+}-(1-\varepsilon) m}\right)$, the series $\sum \mathrm{q}_{0 m}$ defines in a canonical way the symbol $\tilde{\mathrm{q}}_{0}$ of an approximative solution $\tilde{Q}$. The solution $\mathrm{Q}_{1}$ can be constructed from $\tilde{Q}$ in the same way as in the proof of Proposition 3.4.
3.2.2. Fundamental solutions to matrix first order operators Finally, we consider $\mathrm{E}_{2}$. If $\mathrm{E}_{2}(t, s)=\mathrm{E}_{1}(t, s) \mathrm{Q}_{1}(t, s)$, then

$$
\begin{equation*}
D_{t} \mathrm{Q}_{1}(t, s)=\mathrm{E}_{1}(s, t) \mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s) \mathrm{Q}_{1}(t, s), \quad \mathrm{Q}_{1}(s, s)=I . \tag{3.12}
\end{equation*}
$$

The following proposition describes the factor on the right-hand side.
Proposition 3.6. The operator $\mathrm{K}^{(\infty)}(t, s):=\mathrm{E}_{1}(s, t) \mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s)$, $s \leq t$, is a pseudodifferential operator whose (matrix) symbol satisfies, for every $p \geq 0$,

$$
\begin{align*}
& \left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathbf{k}^{(\infty)}(t, s, x, \xi)\right\|  \tag{3.13}\\
& \quad \leq \begin{cases}C_{\alpha \beta \varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{2 K_{+}} \frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : t_{\xi}^{\prime} \leq s \leq t \\
C_{\alpha \beta \varepsilon p}\left(\frac{\Lambda(t)}{\Lambda\left(t_{\xi}\right)}\right)^{2 K_{+}} \frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{||\alpha|-(1-\varepsilon)| \beta \mid} & : s \leq t_{\xi}^{\prime} \leq t \\
C_{\alpha \beta \varepsilon} t_{\xi}^{-1}\langle\xi\rangle^{|\alpha|-(1-\varepsilon)|\beta|} & : s \leq t \leq t_{\xi}^{\prime}\end{cases}
\end{align*}
$$

Proof. We start with the composition $\mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s)$, which can be written as

$$
\begin{aligned}
\left(\mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s) W\right)(x)= & \int e^{i \phi^{-}(t, s, x, \xi)} \mathrm{k}^{1-}(t, s, x, \xi) \hat{W}(\xi) d \xi \\
& +\int e^{i \phi^{+}(t, s, x, \xi)} \mathrm{k}^{1+}(t, s, x, \xi) \hat{W}(\xi) d \xi
\end{aligned}
$$

The amplitudes $\mathrm{k}^{1-}, \mathrm{k}^{1+}$ are given by expansions of the form

$$
\sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} D_{y}^{\alpha}\left(\left(\partial_{\xi}^{\alpha} \mathrm{k}^{(\infty)}\right)\left(t, x, \tilde{\nabla}_{x} \phi(t, s, x, y, \xi)\right) \mathrm{e}(t, s, y, \xi)\right)_{y=x}
$$

and satisfy, for every $p \geq 0$,

$$
\begin{align*}
& \left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathrm{k}^{1 \pm}(t, s, x, \xi)\right\|  \tag{3.14}\\
& \quad \leq \begin{cases}C_{\alpha \beta \varepsilon p}\left(\frac{\Lambda(t)}{\Lambda(s)}\right)^{K_{+}} \frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : t_{\xi}^{\prime} \leq s \leq t \\
C_{\alpha \beta \varepsilon p}\left(\frac{\Lambda(t)}{\Lambda\left(t_{\xi}\right)}\right)^{K_{+}} \frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \leq t_{\xi}^{\prime} \leq t \\
C_{\alpha \beta \varepsilon} t_{\xi}^{-1}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} & : s \leq t \leq t_{\xi}^{\prime}\end{cases}
\end{align*}
$$

On the other hand,

$$
\left|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} e^{i \phi^{ \pm}(t, s, x, \xi)-i x \xi}\right| \leq \begin{cases}C_{\alpha \beta} \Lambda(t)^{|\alpha|+|\beta|}\langle\xi\rangle^{|\alpha|} & : t_{\xi}^{\prime} \leq t, \quad s \leq t, \\ C_{\beta}\langle\xi\rangle^{-|\beta|} & : s \leq t \leq t_{\xi} .\end{cases}
$$

Since (3.14) holds for every $p \geq 0$, the operator $\mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s)$ is a pseudodifferential operator. The symbol of that operator (and its derivatives) can be bounded by the right-hand side of (3.14), for every $p \geq 0$. In a similar way we can consider the operator $\mathrm{E}_{1}(s, t)\left(\mathrm{K}^{(\infty)}(t) \mathrm{E}_{1}(t, s)\right)$ and obtain (3.13).

Remark 3.7. We can write (3.13) in the form

$$
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathbf{k}^{(\infty)}(t, s, x, \xi)\right\| \leq C_{\alpha \beta \varepsilon p} g_{p}(t, x, \xi)\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \quad \forall(t, s, x, \xi)
$$

with

$$
g_{p}(t, x, \xi):= \begin{cases}\left(\frac{\Lambda(t)}{\Lambda\left(t_{\xi}\right)}\right)^{2 K_{+}} \frac{\lambda(t)}{\Lambda(t)}(\Lambda(t)\langle\xi\rangle)^{-p} & : t_{\xi}^{\prime} \leq t \\ t_{\xi}^{-1} & : t \leq t_{\xi}^{\prime}\end{cases}
$$

By direct computation, $\int_{0}^{T} g_{p}(t, x, \xi) d t \leq C$ if $2 K_{+}-p<0$.
Now we are in a position to describe $Q_{1}$.

Proposition 3.8. There is a pseudodifferential operator $\mathrm{Q}_{1}(t, s)$ that solves (3.12) and whose matrix symbol belongs to $L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{0}\right)$.

Proof. We look for $\mathrm{a}_{1}$ in the form $\mathrm{a}_{1} \sim \sum_{j=0}^{\infty} \mathrm{a}_{1 j} \bmod L^{\infty}\left([0, T]^{2}, S^{-\infty}\right)$ where

$$
D_{t} \mathrm{q}_{1 m}(t, s, x, \xi)=\sum_{|\alpha|=0}^{m} \frac{1}{\alpha!}\left(D_{\xi}^{\alpha} \mathrm{k}^{(\infty)}(t, s, x, \xi)\right)\left(\partial_{x}^{\alpha} \mathrm{a}_{1(m-|\alpha|)}(t, s, x, \xi)\right)
$$

with the initial conditions $\mathrm{q}_{10}(s, s)=I$ and $\mathrm{q}_{1 m}(s, s)=0$ for $m \geq 1$. We write this matrix ODE in the form

$$
\partial_{t} \mathrm{a}_{1 m}(t, s)=i \mathrm{k}^{(\infty)}(t, s) \mathrm{q}_{1 m}(t, s)+\mathrm{r}_{m}(t, s)
$$

and introduce the notations

$$
\begin{aligned}
& b_{0}(t, s):=I, \quad b_{\nu+1}(t, s):=\int_{s}^{t} i \mathrm{k}(\tau, s) b_{\nu}(\tau, s) d \tau \\
& d_{0 m}(t, s):=\int_{s}^{t} \mathrm{r}_{m}(\tau, s) d \tau, \quad d_{(\nu+1) m}(t, s):=\int_{s}^{t} i \mathrm{k}(\tau, s) d_{\nu m}(\tau, s) d \tau
\end{aligned}
$$

Then the representations

$$
\mathrm{q}_{10}(t, s)=\sum_{\nu=0}^{\infty} b_{\nu}(t, s), \quad \mathrm{q}_{1 m}(t, s)=\sum_{\nu=0}^{\infty} d_{\nu m}(t, s) \quad(m \geq 1)
$$

hold. By means of induction, we show that

$$
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} b_{\nu}(t, s, x, \xi)\right\| \leq C_{\alpha \beta \varepsilon}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{\nu!}\left(\int_{s}^{t} g_{p}(\tau, x, \xi) d \tau\right)^{\nu}
$$

for all $(t, s, x, \xi)$ with $t \geq s$ and all $\nu$. This is true for $\nu=0$. Then we have

$$
\begin{aligned}
& \left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} b_{\nu+1}(t, s, x, \xi)\right\| \\
& \quad \leq C_{\alpha \beta \varepsilon}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{\nu!} \int_{s}^{t} g_{p}(\tau, x, \xi)\left(\int_{s}^{\tau} g_{p}(\sigma, x, \xi) d \sigma\right)^{\nu} d \tau \\
& \quad=C_{\alpha \beta \varepsilon}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \frac{1}{(\nu+1)!}\left(\int_{s}^{t} g_{p}(\tau, x, \xi) d \tau\right)^{\nu+1} .
\end{aligned}
$$

Now we prove that

$$
\begin{equation*}
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathrm{q}_{1 m}(t, s, x, \xi)\right\| \leq C_{\alpha \beta \varepsilon m}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|-(1-2 \varepsilon) m} \quad \forall(t, s, x, \xi) . \tag{3.15}
\end{equation*}
$$

Due to $\int_{0}^{T} g_{p}(t, x, \xi) d t \leq C$ for large $p$, this holds for $m=0$. Assuming that (3.15) is true for $m-1$, we show (3.15) for $m$. Clearly,

$$
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \mathrm{r}_{m}(t, s, x, \xi)\right\| \leq C_{\alpha \beta \varepsilon m}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} g_{p}(t, x, \xi) .
$$

Then it follows immediately that

$$
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} d_{0 m}(t, s, x, \xi)\right\| \leq C_{\alpha \beta \varepsilon m}\langle\xi\rangle^{\varepsilon|\alpha|-(1-\varepsilon)|\beta|} \int_{s}^{t} g_{p}(\tau, x, \xi) d \tau .
$$

Similarly to the estimate of $b_{\nu}$, we can show that

$$
\left\|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} d_{\nu m}(t, s, x, \xi)\right\| \leq \frac{C_{\alpha \beta \varepsilon m}\langle\xi\rangle| | \alpha|-(1-\varepsilon)| \beta \mid-(1-2 \varepsilon) m}{(\nu+1)!}\left(\int_{s}^{t} g_{p}(\tau, x, \xi) d \tau\right)^{\nu+1}
$$

which implies (3.15).
So far, we have constructed $\mathrm{q}_{1 m} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{-(1-2 \varepsilon) m}\right)$. Therefore, we find a symbol $\tilde{\mathrm{q}}_{1} \sim \sum_{m=0}^{\infty} \mathrm{q}_{1 m} \bmod L^{\infty}\left([0, T]^{2}, S^{-\infty}\right)$ which satisfies

$$
D_{t} \tilde{\mathrm{q}}_{1}(t, s)-\mathrm{k}^{(\infty)}(t, s) \circ \tilde{\mathrm{q}}_{1}(t, s)=\mathrm{r}_{\infty}(t, s) \in L^{\infty}\left([0, T]^{2}, S^{-\infty}\right) .
$$

The solution $Q_{1}$ can be constructed from the approximative solution $\tilde{Q}_{1}$ in a standard way, compare the proof of Proposition 3.4.

## 4. The linear Cauchy problem

Let us consider the solution $w$ to

$$
\begin{equation*}
L w=f(t, x), \quad w(0, x)=u_{0}(x), \quad w_{t}(0, x)=u_{1}(x) \tag{4.1}
\end{equation*}
$$

We define the weight symbol $\vartheta(t, x, \xi)$ by

$$
\begin{align*}
& \vartheta(t, x, \xi)=\exp \left(\int_{t}^{T_{0}} \frac{\lambda(\tau)}{\Lambda(\tau)} \chi_{N}^{-}(\tau, \xi) \beta(x) d \tau\right)  \tag{4.2}\\
& \beta(x):=\frac{l_{*}}{2\left(l_{*}+1\right)} \max _{\mp, \xi}\left|\left(1 \mp \frac{b+c}{\sqrt{c^{2}+a}}\right)(0, x, \xi)\right|, \tag{4.3}
\end{align*}
$$

compare (3.7). The number $\beta(x)$ tells us the amount of Sobolev regularity that was lost at the point $x$ when we passed from $W(0, x)$ to $W(t, x)(t>0)$, where $W(t, x)=$ $\left(G\left(t, D_{x}\right) w(t, x), D_{t} w(t, x)\right)$, see (3.1). By computation,

$$
\beta(x)=\frac{l_{*}}{2\left(l_{*}+1\right)}\left(1+\left(l(x)^{T}(M(x))^{-1} l(x)\right)^{1 / 2}\right)
$$

where $l(x)(M(x))$ is the uniquely determined vector (symmetric matrix, respectively) such that

$$
l(x)^{T} \xi=(b(0, x, \xi)+c(0, x, \xi))|\xi|, \quad \xi^{T} M(x) \xi=\left(c(0, x, \xi)^{2}+a(0, x, \xi)\right)|\xi|^{2}
$$

Observe that $M(x)$ is positively definite, uniformly in $x$, due to (1.9).

Assumption A. The vector $\left(c_{j}(0, x)-b_{j}(0, x)\right)_{j=1}^{n}$ either vanishes identically on $\mathbb{R}_{x}^{n}$, or it never vanishes.

Under this assumption, $\beta(x) \in C^{\infty}\left(\mathbb{R}^{n}\right)$. Now we choose the Sobolev space of variable order $B_{\vartheta, M, T}$ and space $C_{\vartheta, M}$ of traces of functions of $B_{\vartheta, M, T}$ at $t=0$,

$$
\begin{aligned}
& \|W\|_{M, T}:=\sup _{[0, T]}\left(\left\|\Theta_{M}\left(t, x, D_{x}\right) W(t, x)\right\|_{L^{2}}+\|W(t, x)\|_{L^{2}}\right) \\
& \|W(0)\|_{C_{\vartheta, M}}:=\left\|\Theta_{M}\left(0, x, D_{x}\right) W(0, x)\right\|_{L^{2}}+\|W(0, x)\|_{L^{2}}
\end{aligned}
$$

The main result of this section is the following a priori estimate:

Theorem 4.1. Suppose Assumption $\mathrm{A}, u_{0} \in C_{\vartheta, M+1 /\left(l_{*}+1\right)}$ and $u_{1} \in C_{\vartheta, M}$ with $M \geq 0$. Then there are a time interval $[0, T]$ and a constant $C_{a p r}$ with the property that a solution $w \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$ to (4.1) exists and satisfies

$$
\|W\|_{M, T} \leq C_{a p r}\left(\|W(0)\|_{C_{\vartheta, M}}+T\|f\|_{M, T}\right),
$$

where $W(t, x)=\left(G\left(t, D_{x}\right) w(t, x), D_{t} w(t, x)\right)^{T}$, see (3.1).
The proof is split into several lemmas.
Lemma 4.2. If $\mathscr{H}_{s, t}^{\mp}(x, \xi)$ denotes the Hamilton flow of Definition 3.2 and $0 \leq s$, $\sigma \leq t \leq T$, then

$$
\frac{\vartheta\left(t, \mathscr{H}_{s, \sigma}^{\mp}(x, \xi)\right)}{\vartheta(t, x, \xi)} \in L^{\infty}\left([0, T]^{3}, S_{1-\varepsilon, \varepsilon}^{0}\right) .
$$

Proof. Due to Hadamard's formula, we can write

$$
\begin{aligned}
& \frac{\lambda(\tau)}{\Lambda(\tau)}\left(\chi_{N}^{-} \beta\right)\left(\tau, \mathscr{H}_{s, \sigma}^{\mp}(x, \xi)\right) \\
& \quad=\frac{\lambda(\tau)}{\Lambda(\tau)}\left(\chi_{N}^{-} \beta\right)(\tau, x, \xi)+\frac{\lambda(\tau)}{\Lambda(\tau)} r(\tau, s, \sigma, x, \xi)(s-\sigma)
\end{aligned}
$$

with some remainder $r \in L^{\infty}\left([0, T]^{3}, S_{1,0}^{0}\right)$. Our assumptions and $t \leq \tau$ show that $\lambda(\tau)|s-\sigma| / \Lambda(\tau) \leq C$, which yields the assertion.

The same argument gives the following improvement of Proposition 3.5.
Lemma 4.3. The symbol $\operatorname{diag}\left(\mathrm{q}_{0}^{-}, \mathrm{q}_{0}^{+}\right)$of the operator $\mathrm{Q}_{0}(t, s)$ has the form

$$
\begin{aligned}
& \mathrm{q}_{0}^{\mp}(t, s, x, \xi) \\
& \quad=\exp \left(\int_{s}^{t} \frac{\lambda^{\prime}(\tau)}{2 \lambda(\tau)} \chi_{N}^{-}(\tau, \xi)\left(1 \mp \frac{b+c}{\sqrt{c^{2}+a}}\right)(0, x, \xi) d \tau\right) \circ \tilde{\mathbf{q}}_{0}^{\mp}(t, s, x, \xi),
\end{aligned}
$$

where $\tilde{\mathrm{q}}_{0}^{\mp} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{0}\right)$.
The next lemma is a variant of Egorov's theorem.
Lemma 4.4. There are operators $\mathrm{R}(t, s) \in L^{\infty}\left([0, T]^{2}, \Psi_{1-\varepsilon, \varepsilon}^{0}\right)$ and $\mathrm{R}_{\infty}(t, s) \in$ $L^{\infty}\left([0, T]^{2}, \Psi^{-\infty}\right)$ with the property that

$$
\mathrm{E}_{0}(s, t) \Theta_{M}(t) \mathrm{E}_{0}(t, s)=\mathrm{R}(t, s) \Theta_{M}(t)+\mathrm{R}_{\infty}(t, s), \quad 0 \leq s \leq t \leq T .
$$

Proof. In the sequel, we consider $t$ as fixed and let $s$ be running in $[0, t]$. We write $\tilde{\Theta}(t, s):=\mathrm{E}_{0}(s, t) \Theta_{M}(t) \mathrm{E}_{0}(t, s)$ and obtain

$$
D_{s} \tilde{\Theta}(t, s)=[\mathrm{D}(s), \tilde{\Theta}(t, s)]=\mathrm{D}(s) \tilde{\Theta}(t, s)-\tilde{\Theta}(t, s) \mathrm{D}(s) .
$$

We look for $\tilde{\Theta}$ having the form

$$
(\tilde{\Theta}(t, s) W)(x)=\int e^{i x \xi} \tilde{\vartheta}(t, s, x, \xi) \hat{W}(\xi) d \xi, \quad \tilde{\vartheta} \sim \sum_{m=0}^{\infty} \tilde{\vartheta}_{m}
$$

where $\tilde{\vartheta}_{m}=\operatorname{diag}\left(\tilde{\vartheta}_{m}^{-}, \tilde{\vartheta}_{m}^{+}\right)$are diagonal symbols satisfying

$$
\begin{aligned}
& \partial_{s} \tilde{\vartheta}_{m}(t, s, x, \xi) \\
& \quad=\sum_{|\alpha|=1}\left(\partial_{\xi}^{\alpha} \mathrm{D}(s, x, \xi)\right)\left(\partial_{x}^{\alpha} \tilde{\vartheta}_{m}(t, s, x, \xi)\right)-\left(\partial_{x}^{\alpha} \mathrm{D}(s, x, \xi)\right)\left(\partial_{\xi}^{\alpha} \tilde{\vartheta}_{m}(t, s, x, \xi)\right) \\
& \quad+\sum_{l=0}^{m-1} \sum_{|\alpha|=m+1-l} \frac{i}{\alpha!}\left(\left(D_{\xi}^{\alpha} \mathrm{D}(s, x, \xi)\right)\left(\partial_{x}^{\alpha} \tilde{\vartheta}_{l}(t, s, x, \xi)\right)\right. \\
& \\
& \left.\quad-\left(\partial_{x}^{\alpha} \mathrm{D}(s, x, \xi)\right)\left(D_{\xi}^{\alpha} \tilde{\vartheta}_{l}(t, s, x, \xi)\right)\right)
\end{aligned}
$$

with the initial conditions $\tilde{\vartheta}_{0}^{\mp}(t, t, x, \xi)=\vartheta_{M}(t, x, \xi), \tilde{\vartheta}_{m}^{\mp}(t, t, x, \xi)=0(m \geq 1)$. This first order PDE can be written as

$$
\partial_{s} \tilde{\vartheta}_{m}^{\mp}(t, s, x, \xi)=\mathscr{H}^{\mp} \tilde{\vartheta}_{m}^{\mp}(t, s, x, \xi)+f_{m}^{\mp}(t, s, x, \xi), \quad f_{0}^{\mp} \equiv 0,
$$

where $\mathscr{H}^{\mp}=\sum_{|\alpha|=1}\left(\partial_{\xi}^{\alpha} \tau^{\mp}\right) \partial_{x}^{\alpha}-\left(\partial_{x}^{\alpha} \tau^{\mp}\right) \partial_{\xi}^{\alpha}$ is the Hamilton vector field. Then the solutions $\tilde{\vartheta}_{m}^{\mp}$ are given by

$$
\begin{aligned}
& \tilde{\vartheta}_{0}^{\mp}(t, s, x, \xi)=\vartheta_{M}\left(t, \mathscr{H}_{s, t}^{\mp}(x, \xi)\right), \\
& \tilde{\vartheta}_{m}^{\mp}(t, s, x, \xi)=\int_{t}^{s} f_{m}^{\mp}\left(t, \sigma, \mathscr{H}_{s, \sigma}^{\mp}(x, \xi)\right) d \sigma, \quad 0 \leq s \leq t \leq T, \quad m \geq 1 .
\end{aligned}
$$

From Lemma 4.2, we deduce that $\tilde{\vartheta}_{0}^{\mp}(t, s, x, \xi)=r_{0}^{\mp}(t, s, x, \xi) \vartheta_{M}(t, x, \xi)$ with $r_{0}^{\mp} \in$ $L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{0}\right)$. By $\vartheta_{M}(t, .,.) \in \mathcal{K}_{1-\varepsilon, \varepsilon}$, it can be concluded that $f_{1}^{\mp}(t, s, x, \xi)=$ $\tilde{f}_{1}^{\mp}(t, s, x, \xi) \vartheta_{M}(t, x, \xi)$, with $\tilde{f}_{1}^{\mp} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{-(1-2 \varepsilon)}\right)$. Applying Lemma 4.2 again shows $\tilde{\vartheta}_{1}^{\mp}(t, s, x, \xi)=r_{1}^{\mp}(t, s, x, \xi) \vartheta_{M}(t, x, \xi)$, where $r_{1}^{\mp} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{-(1-2 \varepsilon)}\right)$. Following this procedure we get

$$
\tilde{\vartheta}_{m}^{\mp}(t, s, x, \xi)=r_{m}^{\mp}(t, s, x, \xi) \vartheta_{M}(t, x, \xi), \quad r_{m}^{\mp} \in L^{\infty}\left([0, T]^{2}, S_{1-\varepsilon, \varepsilon}^{-(1-2 \varepsilon) m}\right) .
$$

We define $\vartheta^{*, \mp} \sim\left(\sum_{m=0}^{\infty} r_{m}^{\mp}\right) \vartheta_{M}$, which gives us an approximative solution,

$$
D_{s} \Theta^{*}(t, s)-\left[\mathrm{D}(s), \Theta^{*}(t, s)\right]=\mathrm{R}_{0, \infty}(t, s) \in L^{\infty}\left([0, T]^{2}, \Psi^{-\infty}\right) .
$$

It remains to show that $\tilde{\Theta}-\Theta^{*}$ is a smoothing operator. This is equivalent to prove that $W(t, s):=\mathrm{E}_{0}(s, t) \Theta_{M}(t)-\Theta^{*}(t, s) \mathrm{E}_{0}(s, t)$ smooths. We have

$$
\left(D_{s}-\mathrm{D}(s)\right) W(t, s)=-\mathrm{R}_{0, \infty}(t, s) \mathrm{E}_{0}(s, t)
$$

and $W(t, t)=0$. Then Duhamel's principle yields

$$
W(t, s)=-\int_{t}^{s} \mathrm{E}_{0}(s, \tau) \mathrm{R}_{0, \infty}(t, \tau) \mathrm{E}_{0}(\tau, t) d \tau \in \Psi^{-\infty}
$$

This completes the proof.
Proof of Theorem 4.1. The vector $W$ can be written as

$$
\begin{aligned}
& W(t, x)=\mathrm{E}(t, 0) W(0, x)+\int_{0}^{t} \mathrm{E}(t, s) F(s, x) d s \\
&=\mathrm{M}(t) \mathrm{N}(t) \mathrm{E}_{0}(t, 0) \mathrm{Q}_{0}(t, 0) \mathrm{Q}_{1}(t, 0) \mathrm{N}^{\sharp}(0) \mathrm{M}^{\sharp}(0) W(0, x) \\
&+\int_{0}^{t} \mathrm{M}(t) \mathrm{N}(t) \mathrm{E}_{0}(t, s) \mathrm{Q}_{0}(t, s) \mathrm{Q}_{1}(t, s) \mathrm{N}^{\sharp}(s) \mathrm{M}^{\sharp}(s) F(s, x) d s \\
&+\mathrm{R}_{\infty}(t, 0) W(0, x)+\int_{0}^{t} \mathrm{R}_{\infty}(t, s) F(s, x) d s .
\end{aligned}
$$

Lemma 2.5 shows that there are $\tilde{\mathrm{M}}(t), \tilde{\mathrm{N}}(t) \in L^{\infty}\left([0, T], \Psi_{1-\varepsilon, \varepsilon}^{0}\right)$, and $\tilde{\mathrm{Q}}_{1}(t, s) \in$ $L^{\infty}\left([0, T]^{2}, \Psi_{1-\varepsilon, \varepsilon}^{0}\right)$, such that modulo regularizing operators we have

$$
\Theta_{M}(t) \mathrm{M}(t) \mathrm{N}(t)=\tilde{\mathrm{M}}(t) \tilde{\mathrm{N}}(t) \Theta_{M}(t), \quad \Theta_{M}(s) \mathrm{Q}_{1}(t, s)=\tilde{\mathrm{Q}}_{1}(t, s) \Theta_{M}(s)
$$

Then we have, modulo smoothing operators,

$$
\Theta_{M}(t) \mathrm{M}(t) \mathrm{N}(t) \mathrm{E}_{0}(t, s) \mathrm{Q}_{0}(t, s)=\tilde{\mathrm{M}}(t) \tilde{\mathrm{N}}(t) \mathrm{E}_{0}(t, s) \mathrm{R}(t, s) \Theta_{M}(t) \mathrm{Q}_{0}(t, s),
$$

see Lemma 4.4. From Lemma 4.3 and the choice of $\vartheta$, it follows that

$$
\Theta_{M}(t) \mathrm{Q}_{0}(t, s)=\mathrm{R}_{0}(t, s) \Theta_{M}(s)+\mathrm{R}_{\infty}(t, s),
$$

with some $\mathrm{R}_{0} \in L^{\infty}\left([0, T]^{2}, \Psi_{1-\varepsilon, \varepsilon}^{0}\right)$. As a summary, we have

$$
\begin{aligned}
& \Theta_{M}(t) \mathrm{M}(t) \mathrm{N}(t) \mathrm{E}_{0}(t, s) \mathrm{Q}_{0}(t, s) \mathrm{Q}_{1}(t, s) \mathrm{N}^{\sharp}(s) \mathrm{M}^{\sharp}(s) \\
& \quad=\mathrm{R}(t, s) \Theta_{M}(s)+\mathrm{R}_{\infty}(t, s), \quad 0 \leq s \leq t \leq T,
\end{aligned}
$$

with some $\mathrm{R}(t, s) \in L^{\infty}\left([0, T]^{2}, \mathcal{L}\left(L^{2}\right)\right)$. Then it follows that

$$
\begin{aligned}
& \left\|\Theta_{M}(t) W(t, x)\right\|^{2} \leq C\left\|\mathrm{R}(t, 0) \Theta_{M}(0) W(0, x)\right\|^{2}+C\left\|\mathrm{R}_{\infty}(t, 0) W(0, x)\right\|^{2} \\
& \quad+C t \int_{0}^{t}\left\|\mathrm{R}(t, s) \Theta_{M}(s) F(s, x)\right\|^{2} d s+C t \int_{0}^{t}\left\|\mathrm{R}_{\infty}(t, s) F(s, x)\right\|^{2} d s .
\end{aligned}
$$

Integration over $\mathbb{R}_{x}^{n}$ gives the desired inequality.
The weight symbol $g(t, \xi)$ of (3.1) satisfies $g(t, \xi) \geq C\langle\xi\rangle^{1 /\left(l_{*}+1\right)}, C>0$. Then we have $\|w\|_{M+1 /\left(l_{*}+1\right), T} \leq C\|G w\|_{M, T}$; hence $w \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$.

## 5. The algebra property

One variant of the extension of the result of Theorem 4.1 to semilinear Cauchy problems relies on the algebra property of the space $B_{\vartheta, M, T}$. This idea was used in [7], where Cauchy problems similar to (1.6) and (1.7) were studied. In this section, the algebra property of the spaces $B_{\vartheta, M, T}$ is proved.

Theorem 5.1. Assume $M>n / 2+1$. Then there is a constant $C_{0}$ such that

$$
\left\|w_{1} w_{2}\right\|_{M, T} \leq C_{0}\left\|w_{1}\right\|_{M, T}\left\|w_{2}\right\|_{M, T}
$$

for all $w_{1}, w_{2} \in B_{\vartheta, M, T}$.
Corollary 5.2. Let $f(t, x, u)$ satisfy (1.10), (1.11) and suppose $M>n / 2+1$. Then there is, for each $K>0$, a constant $C_{1}(K)$ with the property that

$$
\|f(t, x, u)-f(t, x, v)\|_{M, T} \leq C_{1}(K)\|u-v\|_{M, T}
$$

provided that $u, v \in B_{\vartheta, M, T}$ and $\|u\|_{M, T},\|v\|_{M, T} \leq K$.
The proof is split into several parts. In Lemma 5.3 and Corollary 5.4, we replace the operator $\Theta_{M}$ by a new operator whose Schwartz Kernel has support close to the diagonal $\{x=y\}$ of $\mathbb{R}_{x}^{n} \times \mathbb{R}_{y}^{n}$. Then this new localized operator is locally decomposed into a product of two operators; the first is an operator of small order, the second has a symbol independent of $x$. Exploiting an estimate given in Proposition 5.5, the desired inequality is proved locally in Lemma 5.6. Finally, all these local estimates are glued together in Lemma 5.7.

Let $\left\{B_{\varrho}\left(x_{k}^{0}\right)\right\}_{k=0}^{\infty}$ be a locally finite covering of $\mathbb{R}^{n}$ with balls of radius $\varrho$ and center $x_{k}^{0}$ and let $\left\{\varphi_{k}(x)\right\}_{k=0}^{\infty}$ be its associated partition of unity. Fix some $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ with $\psi(x)=1$ for $|x| \leq 2, \psi(x)=0$ for $|x| \geq 3$ and define $\psi_{\varrho}(x)=\psi(x / \varrho)$.

Lemma 5.3. For $p \in S_{1, \varepsilon}^{m}$, define $P_{\varrho}=P_{\varrho}\left(x, D_{x}\right)$ by

$$
\left(P_{\varrho} w\right)(x)=\sum_{k=0}^{\infty} \int_{\mathbb{R}_{\xi}^{n}} \int_{\mathbb{R}_{y}^{n}} e^{i(x-y) \xi} \varphi_{k}(x) p(x, \xi)\left(1-\psi_{\varrho}\right)\left(y-x_{k}^{0}\right) w(y) d y d \xi
$$

Then $P_{\varrho}$ is a smoothing operator, $P_{\varrho} \in \Psi^{-\infty}$.
Proof. The Schwartz Kernel of $P_{\varrho}$,

$$
\sum_{k=0}^{\infty} \int_{\mathbb{R}_{\xi}^{n}} e^{i(x-y) \xi} \varphi_{k}(x) p(x, \xi)\left(1-\psi_{\varrho}\right)\left(y-x_{k}^{0}\right) d \xi
$$

vanishes for $|x-y| \leq \varrho$.
Corollary 5.4. The following norm is an equivalent norm for $B_{\vartheta, M, T}$ :

$$
\|w\|_{M, T}^{2} \sim \sup _{[0, T]}\left(\sum_{k=0}^{\infty}\left\|\varphi_{k}(x) \Theta_{M}\left(t, x, D_{x}\right)\left(\psi_{\varrho}\left(x-x_{k}^{0}\right) w(t, x)\right)\right\|_{L^{2}}^{2}+\|w\|_{L^{2}}^{2}\right) .
$$

Proposition 5.5. Suppose that $p(t, \xi) \in L^{\infty}\left([0, T], S_{1,0}^{m}\right)$ satisfies

$$
\begin{align*}
& p(t, \xi)=p(t,|\xi|) \quad \text { is monotonically increasing in }|\xi|,  \tag{5.1}\\
& p(t, 2 \xi) \leq C_{0} p(t, \xi),  \tag{5.2}\\
& C_{0} p(t, \xi) \geq\langle\xi\rangle^{n / 2+1+\gamma}, \quad \gamma>0,  \tag{5.3}\\
& \left|\nabla_{\xi} p(t, \xi)\right| \leq C_{0} p(t, \xi)\langle\xi\rangle^{-1} \tag{5.4}
\end{align*}
$$

for all $(t, \xi) \in[0, T] \times \mathbb{R}^{n}$ and some $C_{0}>0$. If $\delta \geq 0$, then the estimate

$$
\|P(f g)-f P g-g P f\|_{H^{\delta}} \leq C_{1}\|P f\|_{H^{\delta-1 / 2}}\|P g\|_{H^{\delta-1 / 2}}
$$

holds for all $f, g$ with $P f, P g \in H^{\delta-1 / 2}$; and $C_{1}=C_{1}\left(C_{0}, n, \gamma, \delta\right)$.
Proof. If $w$ is an arbitrary function of $L^{2}\left(\mathbb{R}_{\xi}^{n}\right)$, then

$$
\begin{aligned}
& \int_{\mathbb{R}_{\xi}^{n}}\langle\xi\rangle^{\delta}(P(f g)-f P g-g P f) \curlyvee(\xi) w(\xi) d \xi \\
&= \int_{\mathbb{R}_{\xi}^{n}} \int_{\mathbb{R}_{\eta}^{n}}\left(\langle\xi-\eta\rangle^{\delta-1 / 2} p(\xi-\eta) \hat{f}(\xi-\eta)\right)\left(\langle\eta\rangle^{\delta-1 / 2} p(\eta) \hat{g}(\eta)\right) \times \\
& \quad \times m(\xi, \eta) w(\xi) d \eta d \xi
\end{aligned}
$$

holds, where we have neglected the variable $t$ and

$$
m(\xi, \eta):=\frac{|p(\xi)-p(\xi-\eta)-p(\eta)|\langle\xi\rangle^{\delta}\langle\eta\rangle^{1 / 2}\langle\xi-\eta\rangle^{1 / 2}}{p(\xi-\eta) p(\eta)\langle\eta\rangle^{\delta}\langle\xi-\eta\rangle^{\delta}}
$$

Assume that we had shown

$$
\begin{equation*}
\sup _{\xi, t} \int_{\mathbb{R}_{\eta}^{n}} m(\xi, \eta)^{2} d \eta=C_{1}^{2}<\infty . \tag{5.5}
\end{equation*}
$$

Then the Cauchy-Schwarz inequality implies

$$
\left|\int_{\mathbb{R}_{\xi}^{n}}\langle\xi\rangle^{\delta}(P(f g)-f P g-g P f)^{\curlyvee}(\xi) w(\xi) d \xi\right|
$$

$$
\begin{aligned}
& \leq\left(\int_{\mathbb{R}_{\eta}^{n}} \int_{\mathbb{R}_{\xi}^{n}}\left|\langle\xi-\eta\rangle^{\delta-1 / 2} p(\xi-\eta) \hat{f}(\xi-\eta)\langle\eta\rangle^{\delta-1 / 2} p(\eta) \hat{g}(\eta)\right|^{2} d \xi d \eta\right)^{1 / 2} \\
& \times\left(\int_{\mathbb{R}_{\xi}^{n}} \int_{\mathbb{R}_{\eta}^{n}} m(\xi, \eta)^{2}|w(\xi)|^{2} d \eta d \xi\right)^{1 / 2} \\
& \leq C_{1}\|w\|_{L^{2}\left(\mathbb{R}_{\xi}^{n}\right)}\|P f\|_{H^{\delta-1 / 2}}\|P g\|_{H^{\delta-1 / 2}},
\end{aligned}
$$

which completes the proof. Therefore, it remains to show (5.5).
For each $\xi$, we split $\mathbb{R}_{\eta}^{n}$ into four parts:

$$
\begin{aligned}
& A=\{\eta:|\eta| \geq 2|\xi|\}, \quad B=\left\{\eta:|\eta| \leq 2|\xi|,|\xi-\eta| \leq \frac{|\eta|}{2}\right\}, \\
& C=\{\eta:|\eta| \leq 2|\xi|,|\xi-\eta| \geq 2|\eta|\}, \\
& D=\left\{\eta:|\eta| \leq 2|\xi|, \frac{|\eta|}{2} \leq|\xi-\eta| \leq 2|\eta|\right\} .
\end{aligned}
$$

If $\eta \in A$, then $|\xi-\eta| \geq|\eta| / 2$; hence

$$
m(\xi, \eta) \leq C \frac{p(\eta)\langle\eta\rangle^{1 / 2}\langle\xi-\eta\rangle^{1 / 2}}{p(\xi-\eta) p(\eta)} \leq C \frac{\langle\xi-\eta\rangle}{p(\xi-\eta)} \leq \frac{C}{\langle\eta\rangle^{n / 2+\gamma}}
$$

This implies $\int_{A} m(\xi, \eta)^{2} d \eta \leq C$.
If $|\xi-\eta| \leq|\xi| / 2$, then $|\xi| / 2 \leq|\eta| \leq 3|\xi| / 2$ and $|\xi-\eta| \leq|\eta|$. Consequently, $(\xi, \eta)$ is the shortest side in the triangle with the corners $0, \xi, \eta$. Then we can find a positive number $C$ with

$$
C \min (|\eta|,|\xi|) \leq|\alpha \eta+(1-\alpha) \xi| \leq \max (|\eta|,|\xi|), \quad \forall 0 \leq \alpha \leq 1
$$

From this and (5.4), we deduce that

$$
\begin{align*}
|p(\xi)-p(\eta)| & \leq \max _{0 \leq \alpha \leq 1}|\nabla p(\alpha \xi+(1-\alpha) \eta)| \cdot|\xi-\eta| \\
& \leq C \max _{0 \leq \alpha \leq 1} \frac{p(\alpha \xi+(1-\alpha) \eta)}{\langle\alpha \xi+(1-\alpha) \eta\rangle} \cdot\langle\xi-\eta\rangle, \\
|p(\xi)-p(\eta)| & \leq C \frac{p(\xi)}{\langle\xi\rangle}\langle\xi-\eta\rangle \quad \forall|\xi-\eta| \leq \frac{|\xi|}{2} . \tag{5.6}
\end{align*}
$$

Now let $\eta \in B$. From (5.6), we get

$$
\begin{aligned}
|p(\xi)-p(\eta)-p(\xi-\eta)| & \leq C \frac{p(\eta)}{\langle\eta\rangle}\langle\xi-\eta\rangle+p(\xi-\eta), \\
m(\xi, \eta) & \leq C \frac{p(\eta)\langle\xi-\eta\rangle^{3 / 2}\langle\eta\rangle^{1 / 2}}{p(\xi-\eta) p(\eta)\langle\eta\rangle}+C \frac{p(\xi-\eta)\langle\eta\rangle^{1 / 2}\langle\xi-\eta\rangle^{1 / 2}}{p(\xi-\eta) p(\eta)}
\end{aligned}
$$

$$
\leq C \frac{\langle\xi-\eta\rangle}{p(\xi-\eta)}+C \frac{\langle\eta\rangle}{p(\eta)}
$$

This implies $\int_{B} m(\xi, \eta)^{2} d \eta \leq C$.
The case $\eta \in C$ is treated similarly; we just change the roles of $\eta$ and $\xi-\eta$.
Finally, if $\eta \in D$, then

$$
m(\xi, \eta) \leq C \frac{p(\eta)\langle\eta\rangle^{1 / 2}\langle\xi-\eta\rangle^{1 / 2}}{p(\xi-\eta) p(\eta)} \leq C \frac{\langle\xi-\eta\rangle}{p(\xi-\eta)} \leq \frac{C}{\langle\eta\rangle^{n / 2+\gamma}}
$$

The proof is complete.
We define two additional cut-off functions,

$$
\zeta^{(1)}(x)=\left\{\begin{array}{ll}
1 & :|x| \leq 3, \\
0 & :|x| \geq 4,
\end{array} \quad \zeta^{(2)}(x)= \begin{cases}1 & :|x| \leq 5 \\
0 & :|x| \geq 6\end{cases}\right.
$$

and set $\zeta_{\varrho}^{(j)}(x):=\zeta^{(j)}(x / \varrho), j=1,2$. Then we introduce the notations $w_{1, k}(x)=$ $\zeta_{\varrho}^{(1)}\left(x-x_{k}^{0}\right) w_{1}(x), \quad w_{2, k}(x)=\psi_{\varrho}\left(x-x_{k}^{0}\right) w_{2}(x)$, and it follows that $\psi_{\varrho}(x-$ $\left.x_{k}^{0}\right) w_{1}(x) w_{2}(x)=w_{1, k}(x) w_{2, k}(x)$. We define the symbol

$$
\vartheta_{(k)}(t, x, \xi)=\exp \left(\int_{t}^{T} \frac{\lambda(\tau)}{\Lambda(\tau)} \chi_{N}^{-}(\tau, \xi)\left(\beta(x)-\beta\left(x_{k}^{0}\right)\right) d \tau\right)
$$

and conclude that $\Theta_{M}\left(t, x, D_{x}\right)=\Theta_{(k)}\left(t, x, D_{x}\right) \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right)$.
Lemma 5.6. Let $\delta$ be a positive number with

$$
\begin{equation*}
\sup _{k} \sup _{\left|x-x_{k}^{0}\right| \leq 8 \varrho}\left|\beta(x)-\beta\left(x_{k}^{0}\right)\right| \leq \delta \leq \frac{1}{4} . \tag{5.7}
\end{equation*}
$$

Then there is some constant $C_{\delta, \varrho}$, independent of $t, k$, and $w_{j}$, such that

$$
\begin{aligned}
& \left\|\varphi_{k}(x) \Theta_{M}\left(t, x, D_{x}\right)\left(\psi_{\varrho}\left(x-x_{k}^{0}\right) w_{1}(x) w_{2}(x)\right)\right\|_{L^{2}} \\
& \leq C_{\delta, \varrho}\left(\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M}\left(t, x, D_{x}\right) w_{1, k}(x)\right\|_{L^{2}}+\left\|w_{1, k}(x)\right\|_{L^{2}}\right) \times \\
& \quad \times\left(\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M}\left(t, x, D_{x}\right) w_{2, k}(x)\right\|_{L^{2}}+\left\|w_{2, k}(x)\right\|_{L^{2}}\right) .
\end{aligned}
$$

Proof. We have the decomposition

$$
\begin{aligned}
& \varphi_{k} \Theta_{M}\left(t, x, D_{x}\right)\left(w_{1, k} w_{2, k}\right)=I_{k, 1}+I_{k, 2}+I_{k, 3} \\
& =\varphi_{k} \Theta_{(k)}\left(\Theta_{M}\left(t, x_{k}^{0}, D_{x}\right)\left(w_{1, k} w_{2, k}\right)-w_{1, k} \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{2, k}\right. \\
& \left.\quad-w_{2, k} \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{1, k}\right)
\end{aligned}
$$

$$
+\varphi_{k} \Theta_{(k)}\left(w_{1, k} \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{2, k}\right)+\varphi_{k} \Theta_{(k)}\left(w_{2, k} \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{1, k}\right)
$$

By the choice of $\delta$, we deduce that $\vartheta_{(k)}(t, x, \xi) \leq C\langle\xi\rangle^{\delta}$, as long as $\left|x-x_{k}^{0}\right| \leq 8 \varrho$. Then it follows that $\varphi_{k} \vartheta_{(k)} \in L^{\infty}\left([0, T], S_{1, \varepsilon}^{\delta}\right)$. Proposition 5.5 gives

$$
\left\|I_{k, 1}\right\|_{L^{2}} \leq C\left\|\Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{1, k}\right\|_{H^{\delta-1 / 2}}\left\|\Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{2, k}\right\|_{H^{\delta-1 / 2}}
$$

Now we show that

$$
\begin{align*}
& \left\|\Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{j, k}\right\|_{H^{\delta-1 / 2}} \\
& \quad \leq C\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M}\left(t, x, D_{x}\right) w_{j, k}\right\|_{L^{2}}+C\left\|w_{j, k}\right\|_{L^{2}} . \tag{5.8}
\end{align*}
$$

According to Proposition 2.4, the operator $\Theta_{(k)}$ has a parametrix $\Theta_{(k)}^{\sharp}$. Then we can write

$$
\begin{aligned}
& \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{j, k}=\left(\Theta_{(k)}^{\sharp} \Theta_{M}\right)\left(t, x, D_{x}\right) w_{j, k}+R_{\infty} w_{j, k} \\
& \quad=\Theta_{(k)}^{\sharp}\left(\left(1-\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)+\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)\right) \Theta_{M} w_{j, k}\right)+R_{\infty} w_{j, k} .
\end{aligned}
$$

Since the symbol of the operator $\Theta_{(k)}^{\sharp}\left(1-\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)\right) \Theta_{M}$ vanishes on the support of $w_{j, k}$ (modulo smoothing operators), we see that

$$
\begin{equation*}
\left\|\Theta_{(k)}^{\sharp}\left(\left(1-\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)\right) \Theta_{M} w_{j, k}\right)\right\|_{H^{\delta-1 / 2}} \leq C\left\|w_{j, k}\right\|_{L^{2}} . \tag{5.9}
\end{equation*}
$$

The order of the operator $\Theta_{(k)}^{\sharp}$ is at most $\delta$ in a neighbourhood of the support of $\zeta_{\varrho}^{(2)}\left(\cdot-x_{k}^{0}\right)$. Therefore,

$$
\left\|\Theta_{(k)}^{\sharp}\left(\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M} w_{j, k}\right)\right\|_{H^{\delta-1 / 2}} \leq C\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M} w_{j, k}\right\|_{L^{2}} .
$$

This proves (5.8) and yields the estimate of $I_{k, 1}$.
For each operator $P \in \Psi_{1, \varepsilon}^{\delta}(\delta<1)$, there is a constant $C$ such that for all Lipschitz continuous functions $f$ and all $g \in H^{\delta-1}$ the estimate

$$
\begin{equation*}
\|[P, f] g\|_{L^{2}} \leq C\|\nabla f\|_{L^{\infty}}\|g\|_{H^{\delta-1}} \tag{5.10}
\end{equation*}
$$

holds. This is a special case of Theorem 5.1 in [13]. Using (5.10) and

$$
I_{k, 2}=w_{1, k} \varphi_{k} \Theta_{M}\left(t, x, D_{x}\right) w_{2, k}+\left[\varphi_{k} \Theta_{(k)}, w_{1, k}\right] \Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{2, k},
$$

we deduce that

$$
\left\|I_{k, 2}\right\|_{L^{2}} \leq C\left\|w_{1, k}\right\|_{C^{1}}\left(\left\|\varphi_{k} \Theta_{M} w_{2, k}\right\|_{L^{2}}+\left\|\Theta_{M}\left(t, x_{k}^{0}, D_{x}\right) w_{2, k}\right\|_{H^{\delta-1}}\right) .
$$

From $\langle\xi\rangle^{M} \leq \vartheta_{M}(t, x, \xi)$ and $M>n / 2+1$, we obtain

$$
\begin{aligned}
\left\|w_{1, k}\right\|_{C^{1}} \leq C \| & \left(1-\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)+\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right)\right) \Theta_{M}\left(t, x, D_{x}\right) w_{1, k} \|_{L^{2}} \\
& +C\left\|w_{1, k}\right\|_{L^{2}} .
\end{aligned}
$$

The application of the reasoning that led to (5.9) implies

$$
\left\|w_{1, k}\right\|_{C^{1}} \leq C\left\|\zeta_{\Omega}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M}\left(t, x, D_{x}\right) w_{1, k}\right\|_{L^{2}}+C\left\|w_{1, k}\right\|_{L^{2}} .
$$

The term $I_{k, 3}$ can be treated similarly.
Lemma 5.7. Under the conditions of Lemma 5.6, we have

$$
\begin{aligned}
& \sum_{k=0}^{\infty}\left\|\varphi_{k}(x) \Theta_{M}\left(t, x, D_{x}\right)\left(\psi_{\varrho}\left(x-x_{k}^{0}\right) w_{1}(x) w_{2}(x)\right)\right\|_{L^{2}}^{2} \\
& \quad \leq C_{\delta, \varrho}\left\|w_{1}\right\|_{M, T}^{2}\left\|w_{2}\right\|_{M, T}^{2} \quad \forall w_{1}, w_{2} \in B_{\vartheta, M, T}
\end{aligned}
$$

Proof. It suffices to show

$$
\begin{equation*}
\sum_{k=0}^{\infty}\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) \Theta_{M}\left(t, x, D_{x}\right) w_{j, k}\right\|_{L^{2}}^{2} \leq C\left\|w_{j}\right\|_{M, T}^{2} \tag{5.11}
\end{equation*}
$$

Applying Lemma 2.5 twice, we find an operator $R_{1, k} \in L^{\infty}\left([0, T], \Psi_{1-\varepsilon, \varepsilon}^{0}\right)$ with

$$
\begin{aligned}
& \Theta_{M}\left(t, x, D_{x}\right) \circ \zeta_{\varrho}^{(1)}\left(x-x_{k}^{0}\right)=R_{1, k}\left(t, x, D_{x}\right) \Theta_{M}\left(t, x, D_{x}\right)+R_{\infty} \\
& \quad=R_{1, k}\left(1-\zeta_{\varrho}^{(3)}\left(x-x_{k}^{0}\right)+\zeta_{\varrho}^{(3)}\left(x-x_{k}^{0}\right)\right) \Theta_{M}\left(t, x, D_{x}\right)+R_{\infty}
\end{aligned}
$$

where $\zeta_{\varrho}^{(3)}(x)=\zeta^{(3)}(x / \varrho)$ and $\zeta^{(3)}(x)=1$ for $|x| \leq 7, \zeta^{(3)}(x)=0$ for $|x| \geq 8$. Exploiting the idea behind Lemma 5.3 and Corollary 5.4, we get

$$
\sum_{k=0}^{\infty}\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) R_{1, k}\left(\left(1-\zeta_{\varrho}^{(3)}\left(x-x_{k}^{0}\right)\right) \Theta_{M} w_{1}\right)\right\|_{L^{2}}^{2} \leq C_{\varrho}\left\|w_{1}\right\|_{L^{2}}^{2}
$$

On the other hand, $\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) R_{1, k}$ is an operator of order zero, uniformly bounded with respect to $k$; hence

$$
\begin{aligned}
& \sum_{k=0}^{\infty}\left\|\zeta_{\varrho}^{(2)}\left(x-x_{k}^{0}\right) R_{1, k}\left(\zeta_{\varrho}^{(3)}\left(x-x_{k}^{0}\right) \Theta_{M} w_{1}\right)\right\|_{L^{2}}^{2} \\
& \quad \leq C \sum_{k=0}^{\infty}\left\|\zeta_{\varrho}^{(3)}\left(x-x_{k}^{0}\right) \Theta_{M} w_{1}\right\|_{L^{2}}^{2} \leq C\left\|w_{1}\right\|_{M, T}^{2}
\end{aligned}
$$

This gives (5.11) for $j=1$. The case $j=2$ runs similarly.

Proof of Theorem 5.1. We apply Corollary 5.4 and Lemma 5.7.

## 6. The semilinear Cauchy problem

Theorem 4.1 gives us the existence of a solution $v$ to the Cauchy problem (1.7) in some space $B_{\vartheta, M+1\left(l_{*}+1\right), T}$ provided $u_{0} \in C_{\vartheta, M+1 /\left(l_{*}+1\right)}, u_{1} \in C_{\vartheta, M}$. The main goal of this section is to show that, for small time $T$, a solution $u$ to (1.6) exists and belongs to the same space as $v$. Moreover, we prove that the difference $u-v$ has higher regularity than $u$ and $v$, see Theorem 6.2. This implies that the strongest singularities of $u$ and $v$ are the same.

Theorem 6.1. Suppose Assumption A and $u_{0} \in C_{\vartheta, M+1 /\left(l_{*}+1\right)}, u_{1} \in C_{\vartheta, M}$, and $M>n / 2+1$. Then there is a $T, 0<T \leq T_{0}$, such that a solution $u \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$ to (1.6) exists with $U \in B_{\vartheta, M, T}$, where $U(t, x)=\left(G\left(t, D_{x}\right) u(t, x), D_{t} u(t, x)\right)^{T}$.

Proof. We consider the mapping $\mathcal{A}: w \mapsto u$ defined by

$$
L u=f(t, x, w), \quad u(0, x)=u_{0}(x), \quad u_{t}(0, x)=u_{1}(x),
$$

and show that it has a fixed point in $B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$ for small $T, 0<T \leq T_{0}$. Fix a constant $C_{2}$ such that

$$
\begin{equation*}
\|w(t, x)\|_{M, T} \leq C_{2}\left\|G\left(t, D_{x}\right) w(t, x)\right\|_{M, T} \tag{6.1}
\end{equation*}
$$

for all $w$ with $G w \in B_{\vartheta, M, T}$ and all $T$. Then we fix some positive number $K$,

$$
K:=2 C_{2} C_{a p r}\left(\left\|G\left(0, D_{x}\right) u_{0}(x)\right\|_{C_{\vartheta, M}}+\left\|u_{1}(x)\right\|_{C_{\vartheta, M}}+1\right),
$$

where $C_{a p r}$ is the constant of Theorem 4.1, and choose $0<T \leq T_{0}$ such that

$$
\begin{equation*}
T C_{a p r} C_{1}(K) \leq \frac{1}{2 C_{2}} \tag{6.2}
\end{equation*}
$$

where $C_{1}(K)$ is from Corollary 5.2. We fix some set $X \subset B_{\vartheta, M, T}$,

$$
X=\left\{w \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}:\|G w\|_{M, T}+\left\|w_{t}\right\|_{M, T} \leq \frac{K}{C_{2}}\right\} \subset B_{\vartheta, M, T}
$$

Let $\bar{X}$ denote the closure of $X$ in $B_{\vartheta, M, T}$. If $w, \tilde{w} \in X$, then we have $\|w\|_{M, T} \leq K$, see (6.1); and, consequently, $\|f(t, x, w)\|_{M, T} \leq C_{1}(K)\|w\|_{M, T}$, $\|f(t, x, w)-f(t, x, \tilde{w})\|_{M, T} \leq C_{1}(K)\|w-\tilde{w}\|_{M, T}$. By (6.2) and Theorem 4.1, $\mathcal{A}$ maps $\bar{X}$ into $X$ and is a contraction. Then Banach's fixed point theorem gives us a fixed point $u$ of $\mathcal{A}$.

Finally, we show that the difference $u-v$ has higher regularity than $u$ and $v$.
Theorem 6.2. Let $u$ and $v$ be the solutions to (1.6), (1.7) and let the assumptions of Theorem 6.1 be valid. Then

$$
U-V \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}, \quad u-v \in B_{\vartheta, M+2 /\left(l_{*}+1\right), T} .
$$

Proof. From $u \in B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$ and Corollary 5.2, we deduce that $f(t, x, u) \in$ $B_{\vartheta, M+1 /\left(l_{*}+1\right), T}$. The function $u-v$ solves $L(u-v)=f(t, x, u)$ and has vanishing initial data. Then Theorem 4.1 gives the assertion.

## 7. An example

Let us illustrate the results of this paper by an example. In [6], the example of Qi Min-You [15] has been extended to Cauchy problems of the type

$$
\begin{aligned}
& L v=v_{t t}+2 c t^{l_{*}} v_{x t}-a t^{2 l_{*}} v_{x x}-b l_{*} t^{l_{*}-1} v_{x}=0, \quad a, b, c \in \mathbb{R}, \\
& v(0, x)=u_{0}(x), \quad v_{t}(0, x)=0
\end{aligned}
$$

Looking for a solution $v(t, x)=\sum_{k=0}^{m} C_{k m} t^{\left(l_{*}+1\right) k}\left(\partial_{x}^{k} u_{0}\right)\left(x+\mu t^{l_{*}+1}\right)$ we obtain

$$
\mu_{1,2}=\frac{1}{l_{*}+1}\left(-c \pm \sqrt{c^{2}+a}\right), \quad m_{1,2}=\frac{l_{*}}{2\left(l_{*}+1\right)}\left(-1 \pm \frac{b+c}{\sqrt{c^{2}+a}}\right) .
$$

Assuming that $m_{1} \in \mathbb{N}\left(m_{2} \in \mathbb{N}\right)$, we see a loss of $m_{1}\left(m_{2}\right)$ derivatives and propagation of singularities along one characteristic only. We find that the loss of regularity given by (4.2), (4.3) and Theorem 4.1 is sharp for this example.

Now suppose that $a, b, c$ are functions of $x$. We choose two distinct points $x_{1}^{0}, x_{2}^{0}$ on the initial line and neighbourhoods $V_{1}, V_{2}$ of $x_{1}^{0}$ and $x_{2}^{0}$. Moreover, assume that $a, b, c$ are constant in $V_{1}, V_{2}$, and that their values are arranged in such a way that $m_{1}=m_{1}(x) \in \mathbb{N}$ for $x \in V_{1}$ and $m_{2}=m_{2}(x) \in \mathbb{N}$ for $x \in V_{2}$ with $m_{1} \neq m_{2}$. Then, locally, $v$ has a loss of $m_{j}$ derivatives in a neighbourhood of $x_{j}^{0}$. This can be seen as follows. The solution to a weakly hyperbolic Cauchy problem is unique if the initial data have high Sobolev smoothness, see [11], [14], [16]. Since unique solutions have finite speed of propagation, the solution $v$ coincides with a function $v_{j}$ in a neighbourhood of $\left(x_{j}^{0}, 0\right)$, where $v_{j}$ solves

$$
\begin{aligned}
& v_{j, t t}+2 c\left(x_{j}^{0}\right) t^{l_{*}} v_{j, x t}-a\left(x_{j}^{0}\right) t^{l_{*}} v_{j, x x}-b\left(x_{j}^{0}\right) l_{*} t^{l_{*}-1} v_{j, x}=0, \\
& v_{j}(0, x)=\varphi_{j}(x) u_{0}(x), \quad v_{j, t}(0, x)=0
\end{aligned}
$$

with $\operatorname{supp} \varphi_{j} \subset V_{j}, \varphi_{j} \equiv 1$ near $x_{j}^{0}$. On the other hand, $v_{j}$ is given by

$$
v_{j}(t, x)=\sum_{k=0}^{m_{j}} C_{k m_{j}} t^{\left(l_{*}+1\right) k}\left(\partial_{x}^{k}\left(\varphi_{j} u_{0}\right)\right)\left(x+\mu_{j} t^{l_{*}+1}\right)
$$

This proves that the solution $v$ suffers from a loss of $m_{1}$ derivatives near $x_{1}^{0}$ and of $m_{2}$ derivatives near $x_{2}^{0}$, which is exactly the loss predicted in (4.3). Furthermore, we observe that singularities of the datum $u_{0}$ in $V_{1}$ or $V_{2}$ propagate along one characteristic only. Now let $u$ be the solution to $L u=f(t, x, u), u(0)=u_{0}, u_{t}(0)=0$, where $f$ satisfies (1.11). Then Theorem 6.2 states that $u-v$ has higher regularity than $v$. This means that the strongest singularities of $u$ and $v$ coincide. The function $u$ may have additional singularities produced by nonlinear interaction, but these additional singularities are weaker, at least by the Sobolev order $1 /\left(l_{*}+1\right)$.

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