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# HIGHEST WEIGHT MODULES ASSOCIATED WITH CLASSICAL IRREDUCIBLE REGULAR PREHOMOGENEOUS VECTOR SPACES OF COMMUTATIVE PARABOLIC TYPE

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### Introduction

Let g be a classical complex simple Lie algebra. We assume g has a Z-gradation of the form:

$$\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(1) \,.$$

Let G be a connected complex Lie group with Lie algebra g, and G(0) the connected subgroup of G corresponding to the Lie subalgebra g(0) of g. We further assume that the pairs  $(G(0), g(\pm 1))$  are irreducible regular prehomogeneous vector spaces [6], [8]. Let  $d\lambda$  be a 1-dimensional representation of the parabolic subalgebra  $\mathfrak{p}=\mathfrak{g}(0)+\mathfrak{g}(1)$ , and  $C_{d\lambda}$  its representation space. Let  $U(\mathfrak{g})$  and  $U(\mathfrak{p})$  be the universal enveloping algebras of g and  $\mathfrak{p}$ , respectively. We denote by  $V(d\lambda)$  the generalized Verma module induced from  $d\lambda$ :

$$V(d\lambda) = U(\mathfrak{g}) \otimes_U(\mathfrak{p}) \boldsymbol{C}_{d\lambda},$$

and by  $L(d\lambda)$  its irreducible quotient.

The purpose of this paper is to give a realization of the  $U(\mathfrak{g})$ -module  $L(d\lambda)$ using the irreducible relative invariant polynomial f of the pair  $(G(0), \mathfrak{g}(-1))$ . As an application, we recover the reducibility criterion of  $V(d\lambda)$  (due to Jantzen [2]) and show that it has a natural interpretation in terms of the zeros of the *b*-function [4], [8] of f.

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#### 1. Statement of main results

In this section we state our main results more precisely. Let  $g, g(0), g(\pm 1), p, U(g)$  and U(p) be as in the Introduction. In particular g is a classical complex simple Lie algebra with Z-gradation of the form:

(1.1) 
$$g = g(-1) + g(0) + g(1)$$
.

Let *P* be the normalizer of the parabolic subalgebra  $\mathfrak{p}=\mathfrak{g}(0)+\mathfrak{g}(1)$  in *G*. Let  $\tilde{P}$  be the universal covering group of *P* and  $\pi: \tilde{P} \to P$  the projection homomorphism. We choose an open neighborhood  $V \subset P$  of the identity element so that there exists a section  $\sigma: V \to \tilde{P}$  of  $\pi$ .

We assume that the pairs  $(G(0), \mathfrak{g}(\pm 1))$  are irreducible regular prehomogeneous vector spaces [6], [8]. Let f (resp.  $f^*$ ) be the irreducible relative invariant polynomial on  $\mathfrak{g}(-1)$  (resp.  $\mathfrak{g}(1)$ ). By definition f is an irredicible polynomial on  $\mathfrak{g}(-1)$  satisfying

(1.2) 
$$f(Ad(k) x) = \chi(k) f(x) \quad (k \in G(0), x \in g(-1))$$

for a 1-dimensional character  $\chi$  of G(0). We extend  $\chi$  to P trivially. We also consider  $\chi$  as a character of  $\tilde{P}$  and denote it by the same letter.

Let  $N^-$  be the subgroup of G corresponding to  $\mathfrak{g}(-1)$ . We denote the inverse of the exponential map exp:  $\mathfrak{g}(-1) \rightarrow N^-$  by log:  $N^- \rightarrow \mathfrak{g}(-1)$ . We set  $O = N^- V$ , which is an open subset of G. Let  $\lambda$  be an arbitrary 1-dimensional character of  $\tilde{P}$ . Let

(1.3) 
$$H(\lambda) = \{h: O \rightarrow C: h \text{ is holomorphic}, h(gq) = \lambda(\sigma(q)) h(g), g \in O, q \in V\}$$
.

We can identify  $H(\lambda)$  with the space of holomorphic functions  $H(N^{-})$  on  $N^{-}$ . By differentiating the left G-translation on  $H(\lambda)$ , we get an algebra homomorphism  $\varphi: U(\mathfrak{g}) \rightarrow \mathcal{D}(N^{-})$  from  $U(\mathfrak{g})$  to the algebra  $\mathcal{D}(N^{-})$  of differential operators on  $N^{-}$  with holomorphic coefficients.

Let  $f_x$  be the holomorphic function on O defined by

$$f_{\mathsf{x}}(nq) = \chi^{-1/2}(\sigma(q)) f(\log n) \quad n \in N^-, q \in P.$$

We denote the differentials of  $\lambda$  and  $\chi$  by  $d\lambda$  and  $d\chi$ , respectively. Let  $\mu = \mu(\lambda)$  be the complex number defined by

$$(1.4) d\lambda = \mu d \chi \,.$$

We consider the complex power  $v^{\lambda} = f_{\chi}^{-\mu}$  of  $f_{\chi}$ . Then  $\varphi(U(\mathfrak{g}))$  acts on  $v^{\lambda}$ . Let  $W(\lambda) = \varphi(U(\mathfrak{g}))$ .  $v^{\lambda}$ , a  $U(\mathfrak{g})$ -module generated by  $v^{\lambda}$ . Rigorously speaking,  $v^{\lambda}$  should be defined as follows. Let  $\alpha$  be a variable. Let X be an open ball in  $\{x \in N^-: f(\log(x)) \neq 0\}$ . Let  $\mathcal{D}(N^-)[\alpha]$  be a polynomial ring with coefficients in  $\mathcal{D}(N^-)$ . Then  $N_{\sigma} = \mathcal{D}(N^-)[\alpha] f_{\chi}^{\sigma}$  is a  $\mathcal{D}(N^-)[\alpha]$ -module on X. We define  $v^{\lambda}$  to be the image of  $f_{\chi}^{\sigma}$  in the quotient  $\mathcal{D}(N^-)[\alpha]$ -module  $N_{\sigma}/(2\mu+\alpha) N_{\sigma}$ .

**Theorem 1.1.**  $W(\lambda)$  is an irreducible highest weight U(g)-module with highest weight  $\lambda$ . In other words,  $W(\lambda)$  is isomorphic to  $L(d\lambda)$ .

If  $\lambda$  is the highest weight of a finite dimensional  $U(\mathfrak{g})$ -module, then the above realization is a special case of the Borel-Weil Theorem. (See, for example, [5].)

As an application of Theorem 1.1, we give a reducibility criterion for the generalized Verma modules  $V(d\lambda)$ . To state this, let  $f^*(D_x)$  be the linear differential operator with constant coefficients defined by

$$f^*(D_x) \exp\langle \xi, x \rangle = f^*(\xi) \exp\langle \xi, x \rangle, \quad \xi \in \mathfrak{g}(1), x \in \mathfrak{g}(-1),$$

where  $\langle , \rangle$  is the Killing form on g. It is known [4], [6] that there exists a polynomial b(s) such that

(1.5) 
$$f^*(D_x)f(x)^s = b(s)f(x)^{s-1}, s \in C.$$

The polynomial b(s) is called the *b*-ufficient of the relative invariant f.

**Corollary 1.2.** If  $-2\mu$  is a positive integer or a zero of b(s), then  $V(d\lambda)$  is reducible.

This gives a new interpretation of a result of Jantzen [2] in this special case.

Theorem 1.1 is proved in Sections 4-7 by case-by-case consideration. Corollary 1.2 is proved in Section 8.

## 2. Irreducible Regular Prehomogeneous Vector Spaces of Commutative Parabolic Type

In this section we summarize a part of the results of [6] in a form convenient to our purpose (See also [8]). We also give explicit formulas for the irreducible relative invariant polynomials and the corresponding characters.

We retain the notations in Section 1. If  $(G(0), g(\pm 1))$  are irreducible regular prehomogeneous vector spaces, then the pairs  $(G(0), g(\pm 1))$  are called *irredicible regular prehomogeneous vector spaces of commutative parabolic type*. According to [6], if g is classical, these are classified into the following four cases.

Case I.

g = sl(2n, C).

We define the gradation g=g(-1)+g(0)+g(1) by

$$g(-1) = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in M_n(C) \right\},$$
$$g(0) = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A, D \in M_n(C), \operatorname{tr} A + \operatorname{tr} D = 0 \right\},$$

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$$g(1) = \{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} : B \in M_n(C) \} .$$

We set G = SL(2n, C). Then

$$G(0) = \{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} : A, D \in GL(n, C), \det A \det D = 1 \}$$
  

$$\simeq S(GL(n, C) \times GL(n, C)).$$

The irreducible relative invariant f is given by

$$f(x) = \det C, x = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathfrak{g}(-1).$$

The character  $\chi$  defined by (1.2) is given by

$$\chi(g) = (\det A)^{-2}, \quad g = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in G(0).$$

## Case II.

$$g = sp(2n, C) = \{X \in M_{2n}(C): {}^{t}XA_{2n} + A_{2n}X = 0\},\$$
where  $A_{2n} = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}$  with the  $n \times n$  identity matrix  $1_n$ .  
 $g(-1) = \{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}: C \in M_n(C), {}^{t}C = C\},\$ 
 $g(0) = \{\begin{bmatrix} A & 0 \\ 0 & -{}^{t}A \end{bmatrix}: A \in M_n(C)\},\$ 
 $g(1) = \{\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}: B \in M_n(C), {}^{t}B = B\}.$ 
We set  $G = Sp(2n, C) = \{g \in GL(2n, C): {}^{t}g A_{2n}g = A_{2n}\}.$  Then  $G(0) = \{\begin{bmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{bmatrix}: A \in GL(n, C)\} \cong GL(n, C),\$ 
 $f(x) = \det C \quad \text{for} \quad x = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in g(-1),\$ 
 $\chi(g) = (\det A)^{-2} \quad \text{tor} \quad g = \begin{bmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{bmatrix} \in G(0).$ 

Case III.

$$g = so(4n, C) = \{X \in M_{4n}(C): {}^{t}XS_{4n} + S_{4n}X = 0, trX = 0\},\$$

where 
$$S_{4n} = \begin{bmatrix} 0 & 1_{2n} \\ 1_{2n} & 0 \end{bmatrix}$$
.  
 $g(-1) = \{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in M_{2n}(C), \ {}^{t}C = -C \}$ ,  
 $g(0) = \{ \begin{bmatrix} A & 0 \\ 0 & -{}^{t}A \end{bmatrix} : A \in M_{2n}(C) \}$ ,  
 $g(1) = \{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} : B \in M_{2n}(C), \ {}^{t}B = -B \}$ .

We set  $G = SO(4n, \mathbb{C}) = \{g \in GL(4n, \mathbb{C}): {}^{t}g S_{4n} g = S_{4n}, \text{ det } g = 1\}$ . Then  $G(0) = \{ \begin{bmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{bmatrix} : A \in GL(2n, \mathbb{C}) \} \cong GL(2n, \mathbb{C}) .$ 

The irreducible relative invariant f is the Pfaffian defined by:

$$f^{2}(x) = \det C, \ x = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathfrak{g}(-1) .$$
$$\mathfrak{X}(g) = (\det A)^{-1}, \ g = \begin{bmatrix} A & 0 \\ 0 & {}^{t}A^{-1} \end{bmatrix} \in G(0) .$$

Case IV.

$$g = so(n+2, C) = \{X \in M_{n+2}(C) : {}^{t}Xs_{n+2} + s_{n+2}X = 0, trX = 0\},\$$
where  $s_{n+2} = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1_{n} & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$ .
$$g(-1) = \{\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2^{t}x & 0 \end{bmatrix} : {}^{t}x = (x_{1}, \dots, x_{n}) \in C^{n}\},\$$

$$g(0) = \{\begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 - a \end{bmatrix} : a \in C, A \in M_{n}(C), A + {}^{t}A = 0, trA = 0\},\$$

$$g(1) = \{\begin{bmatrix} 0 & 2^{t}x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix} : {}^{t}x = (x_{1}, \dots, x_{n}) \in C^{n}\}.$$

We set

$$G = SO(n+2, C) = \{g \in GL(n+2, C): tg s_{n+2}g = s_{n+2}, \det g = 1\}.$$

Then

$$G(0) = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C}^{\times}, A \in SO(n, \mathbb{C}) \right\} \simeq SO(n, \mathbb{C}) \times GL(1, \mathbb{C}) .$$

$$f(y) = {}^{t}xx = x_{1}^{2} + \dots + x_{2}^{n} \quad \text{for} \quad y = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2^{t}x & 0 \end{bmatrix} \in g(-1) ,$$

$$\chi(g) = a^{-2} \quad \text{for} \quad g = \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} \in G(0) .$$

Remark 2.1. Besides these four "classical" ones there exists an "exceptional" irrecucible regular prehomogeneous vector space of commutative parabolic type: g is the simple Lie algebra of type  $E_7$ , g(0) is of type  $E_6$  and dim  $g(\pm 1)$ =27.

See [6] for the details.

### 3. $W(\lambda)$ is a highest weight module

In this section we show that the  $U(\mathfrak{g})$ -module  $W(\lambda)$  defined in Section 1 is a highest weight module with highest weight  $\lambda$ . We retain the notations in the previous sections.

**Lemma 3.1.** For  $a \in G(0) \cap V$  sufficiently near to the identity, we have

$$a \cdot v^{\lambda} = \lambda(\sigma(a)) v^{\lambda}$$
.

**Proof.** For  $n \in N^-$  and  $q \in V$ , we have

$$\begin{aligned} (a \cdot v^{\lambda}) (nq) &= v^{\lambda} (a^{-1} nq) \\ &= f_{x}^{-2^{\mu}} (a^{-1} n a a^{-1} q) \\ &= \chi(\sigma(a^{-1} q))^{\mu} f^{-2^{\mu}} (Ad(a^{-1}) (\log(n))) \\ &= \chi(\sigma(q))^{\mu} \chi(\sigma(a))^{-\mu} \chi(\sigma(a))^{2^{\mu}} f^{-2^{\mu}} (\log(n)) \\ &= \lambda(\sigma(a)) v^{\lambda} (nq) . \end{aligned}$$

#### **Lemma 3.2.** The Lie subalgebra g(1) annihilates $v^{\lambda}$ .

We first consider cases I–III simultaneously. In these cases the Lie subalgebras g(-1), g(0) and g(1) are given in the following form:

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$$\mathfrak{g}(-1) = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathfrak{g} \right\}, \, \mathfrak{g}(0) = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in \mathfrak{g} \right\} \text{ and } \mathfrak{g}(1) = \left\{ \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \in \mathfrak{g} \right\}.$$

The subgroups G(0) and  $N^-$  are

$$G(0) = \{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \in G \}, N^{-} = \{ \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \in G \}.$$

Let  $\nu$  be any complex number, we define a 1-dimensional character  $\lambda = \lambda_{\nu}$  of  $\tilde{P}$  by

$$\lambda(\tilde{g}) = (\det A)^{\nu} \text{ for } \pi(\tilde{g}) = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \in P, \ \tilde{g} \in \pi^{-1}(V) \cap \tilde{P}.$$

Then  $v^{\lambda}$  is given by

$$v^{\lambda}\left(\begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} q\right) = \lambda(\sigma(q)) (\det(C))^{\nu}, q \in V.$$

(If we define the complex number  $\mu$  by (1.4), then  $\mu = -\nu/2$  (case I,II) or  $\mu = -\nu$  (case III).)

Proof of Lemma 3.2 (cases I-III).

For 
$$A = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in \mathfrak{g}(1), n = \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} \in N^-$$
 and  $q \in V$ , we have  
 $\exp(-sA) nq = \begin{bmatrix} 1 & -sX \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C & 1 \end{bmatrix} q$   
 $= \begin{bmatrix} 1 & 0 \\ C(1-sXC)^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1-sXC & -sX \\ 0 & 1+sC(1-sXC)^{-1}X \end{bmatrix} q$ .

In the above calculations we assumed that s is small enough so that  $(1-sXC)^{-1}$  exists. Now from the definition of  $v^{\lambda}$ , we have

$$\begin{aligned} v^{\lambda}(\exp(-sA) nq) &= \lambda(\sigma(q)) \det\{C(1-sXC)^{-1}\}^{\nu} \det(1-sXC)^{\nu} \\ &= \lambda(\sigma(q)) \det(C)^{\nu} \\ &= v^{\lambda}(nq) \end{aligned}$$

Differentiating this at s=0, we get the assertion of the lemma in cases I-III. Q.E.D.

We consider the remaining case IV. In this case the subgroups G(0) and  $N^-$  are given by

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$$G(0) = \{ \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} : a \in \mathbf{C}^{\times}, A \in SO(n, \mathbf{C}) \} \simeq SO(n, \mathbf{C}) \times GL(1, \mathbf{C})$$

and

$$N^{-} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1_{n} & 0 \\ {}^{t}xx & 2^{t}x & 1 \end{bmatrix} : {}^{t}x = (x_{1}, \cdots, x_{n}) \in \mathbb{C}^{n} \right\}.$$

Let  $\nu$  be any complex number, we define a 1-dimensional character  $\lambda = \lambda_{\nu}$  of  $\tilde{P}$  by

$$\lambda(\tilde{g}) = a^{\nu} \text{ for } \pi(\tilde{g}) = g = \begin{bmatrix} a & * & * \\ 0 & A & * \\ 0 & 0 & a^{-1} \end{bmatrix} \in G(0), \ \tilde{g} \in \pi^{-1}(V) \cap \tilde{P}.$$

Then  $v^{\lambda}$  is given by

$$v^{\lambda}\left(egin{array}{cccc} 1 & 0 & 0 \ x & 1_{\pi} & 0 \ ^{t}xx & 2^{t}x & 1 \end{array}
ight]q) = \lambda(\sigma(q)) (x_{1}^{2} + \cdots + x_{n}^{2})^{
u}, q \in V.$$

(If we define the complex number  $\mu$  by (1.4), then  $\mu = -\nu/2$ ).

Proof of Lemma 3.2 (case IV).

For 
$$A = \begin{bmatrix} 0 & 2^{t}z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \in g(1), n = \begin{bmatrix} 1 & 0 & 0 \\ x & 1_{n} & 0 \\ xx^{t} & 2^{t}x & 1 \end{bmatrix} \in N^{-} \text{ and } q \in V,$$

we have

$$\exp(-sA) nq = \begin{bmatrix} 1 & 0 & 0 \\ \frac{(x-s(^{t}xx) z)}{(1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx)))} & 1_{n} & 0 \\ \frac{t}{(1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx)))} & 1_{n} & 0 \\ \frac{t}{(1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx)))} & * & 1 \end{bmatrix} \begin{bmatrix} 1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx) & -2s^{t}z s^{2}(^{t}zz) \\ 0 & * & -sz \\ 0 & 0 & * \end{bmatrix} q.$$

In the above calculations we assumed that s is small enough so that  $(1-2s^tzx+s^2(tzz)(txz))^{-1}$  exists. From the definition of  $v^{\lambda}$ , we have

$$\begin{aligned} v^{\lambda}(\exp\left(-sA\right)nq) &= \lambda(\sigma(q)) \Big\{ \frac{t(x-s(^{t}xx) z) (x-s(^{t}xx) z)}{(1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx)))^{2}} \Big\}^{\nu} (1-2s^{t}zx+s^{2}(^{t}zz) (^{t}xx)) \\ &= \lambda(\sigma(q)) (^{t}xx)^{\nu} \\ &= v^{\lambda}(nq) , \end{aligned}$$

from which the lemma follows.

From Lemma 3.1 and Lemma 3.2 we have:

**Proposition 3.4.** The  $U(\mathfrak{g})$ -module  $W(\lambda) = \varphi(U(\mathfrak{g}))$ .  $v^{\lambda}$  is a highest weight module with highest weight  $\lambda$ .

#### 4. The irreducibility of $W(\lambda)$ (Case I)

In this section we prove Theorem 1.1 in case I. We retain the notations in the previous sections. First we analyze the representation space  $\varphi(U(\mathfrak{g}))$ .  $v^{\lambda} = W(\lambda)$ . From the definition (1.3), we can identify  $H(\lambda)$  with the space  $H(N^{-})$ of holomorphic functions on  $N^{-}$ . The latter space can be identified with the space  $H(\mathfrak{g}(-1))$  of holomorphic functions on  $\mathfrak{g}(-1)$  via the exponential map. Let  $(e_{jk})_{j,k=1,\dots,n}$  be the  $n \times n$  matrix units, and  $(x_{jk})_{j,k=1,\dots,n}$  the standard coordinate system on

$$\mathfrak{g}(-1) = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in M_n(C) \right\}.$$

Then the set of matrices

(4.1) 
$$E_{jk} = \begin{bmatrix} 0 & 0 \\ e_{jk} & 0 \end{bmatrix}, \quad j, k = 1, 2, ..., n$$

gives a basis of  $\mathfrak{g}(-1)$ . It is easy to see that  $E_{jk}$  acts on  $H(\lambda)=H(\mathfrak{g}(-1))$  as  $-\frac{\partial}{\partial x_{jk}}$ . By the Poincaré-Birkhoff-Witt theorem, we have  $\varphi(U(\mathfrak{g}))\cdot v^{\lambda} = \varphi(U(\mathfrak{g}(-1)) U(\mathfrak{g}(0)) U(\mathfrak{g}(1)))\cdot v^{\lambda} = \varphi(U(\mathfrak{g}(-1)))\cdot v^{\lambda}$ 

$$= \{ Dv^{\lambda} \colon D \in \mathscr{D}_{\text{const}}(\mathfrak{g}(-1)) \},$$

where  $\mathcal{D}_{\text{const}}(\mathfrak{g}(-1))$  is the set of constant coefficient differential operators on  $\mathfrak{g}(-1)$ .

For the proof of Theorem 1.1, we introduce simple root vectors of g explicitly. Let

$$E_{\alpha_j} = \begin{bmatrix} 0 & 0 & & & & \\ \ddots & \ddots & & 0 & \\ & 0 & 1 & & \\ & 0 & 0 & & \\ & & 0 & \ddots & \\ 0 & & 0 & 0 & \\ & & & \ddots & \ddots & \\ 0 & & 0 & 0 & \\ & & & 0 & 0 \end{bmatrix}, 1 \le j \le 2n-1,$$

and let  $E_{\beta} = E_{\sigma_n}$ .

Since  $W(\lambda)$  is a highest weight module by Proposition 3.4, the following

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theorem yields its irreducibility.

**Theorem 4.1.** Suppose  $w \in W(\lambda)$  is annihilated by every  $\varphi(E_{\alpha_j}), j=1, \cdots, 2n-1$ . Then w is a scalar multiple of the highest weight vector  $v^{\lambda}$ .

We need some lemmas for the proof of this Theorem. We can assume w is a weight vector. In particular, w may be considered as a weight vector with respect to the center of g(0).

Let

$$z = \frac{1}{2} \begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

which is an element of the 1-dimensional center of g(0). It is easy to check that  $d\chi(z) = -n$ .

**Lemma 4.2.** Let  $\mu$  be the complex number defined by (1.4). Then the element z acts on  $H(\lambda) = H(g(-1))$ 

(4.2) 
$$\varphi(z) = \sum_{j,k=1}^{n} x_{jk} \frac{\partial}{\partial x_{jk}} + \mu n.$$

In other words,  $\varphi(z)$  is essentially the Euler's differential operator.

Proof. For  $q \in V$  and

$$m = \begin{bmatrix} 1_n & 0 \\ C & 1_n \end{bmatrix} \in N^-, C = (x_{jk}),$$

we have

$$\exp(-sz) mq = \begin{bmatrix} 1_n & 0 \\ e^s C & 1_n \end{bmatrix} \begin{bmatrix} e^{-s/2} & 1_n & 0 \\ 0 & e^{s/2} & 1_n \end{bmatrix} q.$$

Hence, for  $h \in H(\lambda) = H(\mathfrak{g}(-1))$ , we have

$$\{\exp(sz) h\}(mq) = h(\exp(-sz) mq)$$
  
=  $\lambda(\exp(-sz)) h(e^{s}C)$   
=  $\chi(\exp(-sz))^{\mu}h(e^{s}C)$ .

Differentiating this at s=0, we get the lemma.

As a function on g(-1),  $v^{\lambda}$  has homogeneous degree  $-2n\mu$ . By Lemma 4.2 if the vector w in Theorem 4.1 is a weight vector with respect to  $\varphi(z)$ , then it must be a homogeneous function on g(-1). Since w is a linear combination of various partial derivatives of  $v^{\lambda}$ , we can assume that it is a linear combination

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of the *i*-th derivatives of  $v^{\lambda}$  some fixed nonnegative integer *i*. Hence we can assume that

(4.3) 
$$w(x) = a(C) (\det C)^{-2\mu-i} \quad \text{for} \quad x = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \in \mathfrak{g}(1) \quad (C = (x_{jk}))$$

where a(C) is a homogeneous polynomial in  $x_{jk}$  whose degree is (n-1)i.

Let U be the maximal unipotent subgroup of G(0) generated by  $\{\exp(tE_{\omega_j})\}_{j\neq n}$ . Since  $\omega$  is annihilated by  $\{\varphi(E_{\omega_j})\}_{j\neq n}$ ,  $\omega$  is an Ad(U)-invariant function on g(-1). Since  $(\det C)^{-2^{\mu-i}}$  is Ad(U)-invariant, a(C) is also an Ad(U)-invariant polynomial. The following proposition on the ring of Ad(U)-invariant polynomials on g(-1) is due to Johnson [3]. (See also Muller, Rubenthaler and Schiffmann [6].)

**Proposition 4.3.** Let  $C[g(-1)]^{U}$  be the ring of Ad(U)-invariant polynomials on g(-1). Then it is isomorphic to the polynomial ring with n indeterminates:

$$\boldsymbol{C}[\boldsymbol{\mathfrak{g}}(-1)]^{\boldsymbol{v}} = \boldsymbol{C}[I_1, \cdots, I_n].$$

Here I, is given by the following formulas:

(4.4)  $I_{j} = \det \begin{bmatrix} x_{n-j+11} & \cdots & x_{n-j+1j} \\ \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nj} \end{bmatrix}, \quad j = 1, 2, \cdots, n.$ 

We also need:

**Lemma 4.4.** The simple root vector  $E_{\beta}$  acts on  $H(\lambda) = H(\mathfrak{g}(-1))$  as

(4.5) 
$$\varphi(E_{\beta}) = \sum_{j,k=1}^{n} x_{jn} x_{1k} \frac{\partial}{\partial x_{jk}} + 2\mu x_{1n},$$

where  $2\mu x_{1n}$  means a multiplication operator.

Proof. Using  $E_{\beta}$  instead of A in the calculations of the proof of Lemma 3.2, for  $h \in H(\mathfrak{g}(-1))$ , we have

$$\{\exp(sE_{\beta})h\}(x_{jk}) = (1-sx_{1n})^{-2\mu}h(x_{jk}+\frac{sx_{jn}x_{1k}}{1-tx_{1n}}).$$

Differentiating this at s=0, we get the lemma.

**Lemma 4.5.** If we write  $\varphi(E_{\beta}) = D_{\beta} + 2\mu x_{1n}$ , then  $D_{\beta}$  acts on  $det(x_{jk})$ 

(4.6) 
$$D_{\beta}(\det(x_{jk})) = x_{1n} \det(x_{jk}).$$

Proof. By Lemma 3.2 we have

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$$D_{\beta}(\det(x_{jk}))^{-2\mu} = -2\mu x_{1n} \det(x_{jk})^{-2\mu}.$$

Since  $D_{\beta}$  is a first-order differential operator, the left hand side of the above equation is

$$-2\mu \det(x_{jk})^{-2\mu-1} D_{\beta}(\det(x_{jk}))$$
,

from which the lemma follows.

Proof of Theorem 4.1.

If  $d\lambda=0$ , then  $v^{\lambda}=v^{0}$  corresponds to the function whose value is identically 1 on  $\mathfrak{g}(-1)$ . Hence  $W(\lambda)=\varphi(U(\mathfrak{g}))\cdot v^{\lambda}$  is the 1-dimensional trivial g-module and Theorem is obvious in this case.

Suppose  $d\lambda \neq 0$  and assume w(x) is annihilated by all the  $\varphi(E_{\sigma_j}), j=1, \dots, 2n-1$ . Recall (4.3) that we can assume

$$w = w(x) = a(C) (\det C)^{-2\mu-i}.$$

Here a(C) is a homogeneous polynomial in  $\{x_{jk}\}$  with homogeneous degree (n-1)i. If i=0, we have nothing to prove. Hence we assume i>0. By Proposition 4.3 we conclude a(C) is an element of  $C[I_1, \dots, I_n]$ . By Lemma 4.5 we have

$$\begin{split} \varphi(E_{\beta}) w(x) &= (D_{\beta} + 2\mu x_{1n}) \left( a(C) \left( \det C \right)^{-2\mu - i} \right. \\ &= (D_{\beta} a) \left( C \right) \left( \det C \right)^{-2\mu - i} - (2\mu + i) a(C) \left( \det C \right)^{-2\mu - i - 1} D_{\beta} \left( \det C \right) \\ &+ 2\mu x_{1n} a(C) \left( \det C \right)^{-2\mu - i} \\ &= (D_{\beta} a) \left( C \right) \left( \det C \right)^{-2\mu - i} - (2\mu + i) x_{1n} a(C) \left( \det C \right)^{-2\mu - i} + 2\mu x_{1n} a(C) \left( \det C \right)^{-2\mu - i} \\ &= \{ (D_{\beta} a) \left( C \right) - i x_{1n} a(C) \} \left( \det C \right)^{-2\mu - i} . \end{split}$$

Hence  $\varphi(E_{\beta}) w(x) = 0$  implies  $(D_{\beta} a) (C) - ix_{1n} a(C) = 0$ . Since a(C) has homogeneous degree ni - i, we write a(C) in the following form:

$$a(C) = \sum_{\substack{0 \leq m < i}} a_m(I_1, \cdots, I_{n-1}) I_n^m,$$

where  $a_m(I_1, \dots, I_{n-1}) \in \mathbb{C}[I_1, \dots, I_{n-1}]$ . Then by Lemma 4.5

$$(D_{\beta} - ix_{1n}) a(C) = \sum_{0 \le m < i} \{ D_{\beta} a_m + (m - i) x_{1n} a_m \} I_n^m .$$

Hence we have

$$(D_{\beta} a)(I_{1}, \dots, I_{n-1}) + (m-i) x_{1n} a_{m}(I_{1}, \dots, I_{n-1}) = 0 \quad (0 \le m < i).$$

We consider the coefficients of  $x_{1n}$  in the above equation. We can write  $D_{\beta}$  as follows:

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$$D_{\beta} = x_{1n}^2 \frac{\partial}{\partial x_{1n}} + \sum_{k=1}^{n-1} x_{1n} x_{1k} \frac{\partial}{\partial x_{1k}} + \sum_{j=2}^n x_{jn} x_{1n} \frac{\partial}{\partial x_{jn}} + \sum_{j\neq 1}^{j\neq 1} x_{jn} x_{1k} \frac{\partial}{\partial x_{jk}}$$

From the definition of  $I_1, \dots, I_{n-1}$  (4.4),  $a_m(I_1, \dots, I_{n-1})$  contains no  $x_{11}, \dots, x_{1n}$ ,  $x_{2n}, \dots, x_{nn}$ . Hence the above description of  $D_{\beta}$  shows that the coefficient of  $x_{1n}$  in  $(D_{\beta} a)$   $(I_1, \dots, I_{n-1})$  is equal to zero. This implies  $a_m(I_1, \dots, I_{n-1})$  is equal to zero for any m. Hence w(x) must be zero. This completes the proof of Theorem 4.1. Q.E.D.

#### 5. The irreducibility of $W(\lambda)$ (Case II)

In this section we set g=sp(2n, C) and prove the irreducibility of  $W(\lambda)$  in case II. We use the notations in Section 2. In particular the Lie subalgebra g(-1) is given by

$$\mathfrak{g}(-1) = \{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in M_{\mathfrak{n}}(C), \, {}^{t}C = C \} .$$

As in the previous section we identify  $H(\lambda)$  with  $H(\mathfrak{g}(-1))$ . Let  $\{e_{jk}\}_{j \ k=1,\dots,n}$  be the matrix units of  $n \times n$  matrices and  $(x_{jk}) \ 1 \le j \le k \le n$  the standard coordinate system on  $\mathfrak{g}(-1)$ . Then the set of matrices

$$E_{jk} = \begin{bmatrix} 0 & 0 \\ e_{jk} + e_{kj} & 0 \end{bmatrix}, \quad 1 \le j \le k \le n$$

gives a basis of  $\mathfrak{g}(-1)$ . It is easy to see that  $E_{jk}$  acts on  $H(\lambda)=H(\mathfrak{g}(-1))$  as  $-\frac{\partial}{\partial x_{jk}}$  for j < k, and as  $-2\frac{\partial}{\partial x_{jj}}$  for j=k. Hence by the Poincaré-Birkhoff-Witt theorem, we have

$$arphi(U(\mathfrak{g})) v^{\lambda} = \{Dv^{\lambda} \colon D \in \mathscr{D}_{\mathrm{const}}(g(-1))\}$$

Let

Let

$$E_{\sigma_j} = \begin{bmatrix} e_j & 0 \\ 0 & -te_j \end{bmatrix}, \ 1 \le j \le n-1, \quad \text{where} \quad e_j = e_{j \ j+1}.$$
$$E_{\beta} = \begin{bmatrix} 0 & e_{nn} \\ 0 & 0 \end{bmatrix}.$$

We prove the following Theorem which yields the irreducibility of  $W(\lambda)$ .

**Theorem 5.1.** Suppose  $w \in W(\lambda)$  is annihilated by every  $\varphi(E_{\sigma_j}), j=1, \dots, n-1$  and  $\varphi(E_{\beta})$ . Then w is a scalar multiple of the highest weight vector  $v^{\lambda}$ .

We can assume that w is a weight vector and consider the weight of the central element

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$$z=\frac{1}{2}\begin{bmatrix}1_n&0\\0&-1_n\end{bmatrix},$$

of g(0). The following lemma can be proved by a calculation similar to that in the proof of Lemma 4.2.

**Lemma 5.2.** Let  $\mu$  be the complex number defined by (1.4). Then the element z acts on  $H(\lambda)=H(g(-1))$  as the following differential operator:

(5.1) 
$$\varphi(z) = \sum_{j \leq k} x_{jk} \frac{\partial}{\partial x_{jk}} + \mu n.$$

As a function on g(-1),  $v^{\lambda}$  has homogeneous degree  $-2n\mu$ . Hence if the vector w in Theorem 5.1 is a weight vector with respect to  $\varphi(z)$ , then it must be a homogeneous function on g(-1). Hence w must be a linear combination of the *i*-th derivatives of  $v^{\lambda}$  for some fixed nonnegative integer *i*. Hence we can assume that

(5.2) 
$$w(x) = a(C) (\det C)^{-2\mu - i}$$
 for  $x = \begin{bmatrix} 1_n & 0 \\ C & 1_n \end{bmatrix}$ ,  ${}^tC = C, C = (x_{jk})$ 

where a(C) is a homogeneous polynomial in  $\{x_{jk}\}_{j \le k}$  whose degree is (n-1)i.

Let U be the maximal unipotent subgroup of G(0) determined by the simple root vectors  $\{E_{\omega_j}\}_{j=1,\dots,n-1}$ . Since w is annihilated by  $\{\varphi(E_{\omega_j})\}_{j=1,\dots,n-1}$ , then w is an Ad(U)-invariant function on g(0). Since  $(\det C)^{-2\mu-i}$  is Ad(U)-invariant, we conclude that a(C) is also an Ad(U)-invariant polynomial. Here we need Johnson's result [3].

**Proposition 5.3.** The ring of Ad(U)-invariant polynomials  $C[g(-1)]^U$  is isomorphic to the polynomial ring with n indeterminates:

$$\boldsymbol{C}[\boldsymbol{g}(-1)]^{\boldsymbol{v}} = \boldsymbol{C}[I_1, \cdots, I_n].$$

Here I; are given by the following formulas:

(5.3) 
$$I_j = \det \begin{bmatrix} x_{11} \cdots x_{1j} \\ \cdots \cdots \\ x_{1j} \cdots x_{jj} \end{bmatrix}, \quad j = 1, 2, \cdots, n.$$

**Lemma 5.4.** The simple root vector  $E_{\beta}$  acts on  $H(\lambda)=H(\mathfrak{g}(-1))$  as the differential operator:

(5.4) 
$$\varphi(E_{\beta}) = \sum_{j \leq k} x_{jn} x_{kn} \frac{\partial}{\partial x_{jk}} + 2\mu x_{nn}.$$

**Lemma 5.5.** If we write  $E_{\beta}=D_{\beta}+2\mu x_{nn}$ , then

$$(5.5) D_{\beta}(\det C) = x_{nn}(\det C) for C = {}^{t}C, C = (x_{jk}).$$

We omit the proof.

Proof of the Theorem 5.1.

If  $d\lambda = 0$ , then  $W(\lambda)$  is the 1-dimensional trivial g-module and Theorem is obvious.

Suppose  $d\lambda \neq 0$  and

$$w = w(x) = a(c) (\det C)^{-2^{\mu-i}} \in W(\lambda), C = {}^tC = (x_{jk})$$

is annihilated by  $\varphi(E_{\alpha_j}) j=1, \dots, n-1$  and  $\varphi(E_{\beta})$ . Here a(C) is a homogeneous polynomial in  $\{x_{jk}\}_{j\leq k}$  with homogeneous degree (n-1)i. We can assume i>0. By Proposition 5.3 we conclude a(C) is an element of  $C[I_1, \dots, I_n]$ . We consider the action of the simple root vector  $E_{\beta}$ . By Lemma 5.6 we have

$$\begin{split} \varphi(E_{\beta}) & w(x) = (D_{\beta} + 2\mu x_{nn}) a(C) (\det C)^{-2^{\mu-i}} \\ &= (D_{\beta} a) (C) (\det C)^{-2^{\mu-i}} - (2\mu+1) a(C) (\det C)^{-2^{\mu-i-1}} D_{\beta} (\det C) \\ &+ 2\mu x_{nn} a(C) (\det C)^{-2^{\mu-i}} \\ &= (D_{\beta} a) (C) (\det C)^{-2^{\mu-i}} - (2\mu+i) x_{nn} a(C) (\det C)^{-2^{\mu-i}} + 2\mu x_{nn} a(C) (\det C)^{-2^{\mu-i}} \\ &= \{ (D_{\beta} a) (C) - i x_{nn} a(C) \} (\det C)^{-2^{\mu-i}} . \end{split}$$

Hence  $\varphi(E_{\beta}) w(x)=0$  imples  $(D_{\beta} a)(C)-ix_{nn} a(C)=0$ . Since a(C) has homogeneous degree ni-i, we write a(C) in the following form:

$$a(C) = \sum_{0 \leq m < i} a_m(I_1, \cdots, I_{n-1}) I_n^m,$$

where  $a_m(I_1, \dots, I_{n-1}) \in C[I_1, \dots, I_n]$ . Then by Lemma 5.5

$$(D_{\beta} - ix_{nn}) a(C) = \sum_{0 \le m < i} \{ D_{\beta} a_m + (m - i) x_{nn} a_m \} I_n^m.$$

Hence we have

$$(D_{\beta} a_{m})(I_{1}, \cdots, I_{n-1}) + (m-i) x_{nn} a_{m}(I_{1}, \cdots, I_{n-1}) = 0 \quad 0 \leq m < i.$$

We consider the coefficients of  $x_{nn}$  in the above equation. We can write  $D_{\beta}$  as follows:

$$D_{\beta} = x_{nn}^2 \frac{\partial}{\partial x_{nn}} + \sum_{j=1}^{n-1} x_{jn} x_{nn} \frac{\partial}{\partial x_{jn}} + \sum_{1 \le j \le k < n} x_{jn} x_{kn} \frac{\partial}{\partial x_{jk}}$$

From the definition (5.3) of  $I_1, \dots, I_{n-1}, a_m(I_1, \dots, I_{n-1})$  contains no  $x_{1n}, x_{2n}, \dots, x_{nn}$ . Hence the above description of  $D_\beta$  shows that the coefficient of  $x_{nn}$  in  $(D_\beta a)$   $(I_1, \dots, I_{n-1})$  is equal to zero. This implies  $a_m(I_1, \dots, I_{n-1})$  is equal to zero for any m. Hence w(x) is zero. This completes the proof of Theorem 5.1. Q.E.D.

#### 6. The irredicibility of $W(\lambda)$ (Case III)

In this section we set g=so(4n, C) and prove the irreducibility of  $W(\lambda)$  in case III. We use the notations in Section 2. In particular the Lie subalgebra g(-1) is given by

$$\mathfrak{g}(-1) = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} | C \in M_{2n}(C), C = -C \right\}$$

We identify  $H(\lambda)$  with H(g(-1)) as in the previous sections. Let  $\{e_{jk}\}_{j,k=1,\cdots,2n}$  be the matrix units of  $2n \times 2n$  matrices, and  $(x_{jk})$   $1 \le j < k \le 2n$  the standard coordinate system on g(-1). Then the set of matrices

$$E_{jk} = \begin{bmatrix} 0 & 0 \\ e_{jk} - e_{kj} & 0 \end{bmatrix}, 1 \le j < k \le 2n$$

gives a basis of  $\mathfrak{g}(-1)$ . It is easy to see that  $E_{jk}$  acts on  $H(\lambda)=H(\mathfrak{g}(-1))$  as  $-\frac{\partial}{\partial x_{jk}}$ . Hence by the Poincaré-Birkhoff-Witt theorem, we have

$$\varphi(U(\mathfrak{g})) v^{\lambda} = \{Dv^{\lambda} \colon D \in \mathscr{D}_{\operatorname{const}}(\mathfrak{g}(-1))\}$$

Let

$$E_{\omega_{j}} = \begin{bmatrix} e_{j} & 0 \\ 0 & -{}^{t}e_{j} \end{bmatrix}, \ 1 \le j \le 2n - 1 \quad \text{where} \quad e_{i} = e_{ii+1} \, .$$
  
Let 
$$E_{\beta} = \begin{bmatrix} 0 & e_{2n \, 2n-1} - e_{2n-1 \, 2n} \\ 0 & 0 \end{bmatrix} \, .$$

We prove the following Theorem which yields the irreducibility of  $W(\lambda)$ .

**Theorem 6.1.** Suppose  $w \in W(\lambda)$  is annihilated by every  $\varphi(E_{\sigma_j}), j=1, \dots, 2n-1$  and  $\varphi(E_{\beta})$ , then w is a scalar multiple of the highest weight vector  $v^{\lambda}$ .

We can assume that w is a weight vector and consider the weight of the central element

$$z = \frac{1}{2} \begin{bmatrix} 1_{2n} & 0 \\ 0 & -1_{2n} \end{bmatrix},$$

of g(0).

**Lemma 6.2.** Let  $\mu$  be the complex number defined by (1.4). Then the element z acts on  $H(\lambda) = H(g(-1))$  as the following differential operator:

(6.1) 
$$\varphi(z) = \sum_{1 \leq j < k \leq 2n} x_{jk} \frac{\partial}{\partial x_{jk}} + \mu n.$$

As a function on g(-1),  $v^{\lambda}$  has homogeneous degree  $-2n\mu$ . Hence if the

vector w in Theorem 6.1 is a weihgt vector with respect to the  $\varphi(z)$ , then it must be a homogeneous function on g(-1). Hence w is a linear combination of the *i*-th derivatives of  $v^{\lambda}$  for some fixed nonnegative integer *i*. Hence we can assume that

(6.2) 
$$w(x) = a(C) (\det C)^{-\mu-i}, x = \begin{bmatrix} 1_{2n} & 0 \\ C & 1_{2n} \end{bmatrix}, \, {}^{i}C = -C, C = (x_{jk})$$

where a(C) is a homogeneous polynomial in  $\{x_{jk}\}_{j \le k}$  whose degree is (2n-1)i.

Let U be the maximal unipotent subgroup of G(0) determined by  $\{E_{\omega_j}\}_{j=1,\dots,2n-1}$ . Since w is annihilated by  $\{\varphi(E_{\omega_j})\}_{j=1,\dots,2n-1}$ , w is an Ad(U)-invariant function on g(-1). Since  $(\det C)^{-\mu-i}$  is Ad(U)-invariant, we conclude that a(C) is an Ad(U)-invariant polynomial. Here we need Johnson's result [3].

**Proposition 6.3.** The ring of Ad(U)-invariant polynomials  $C[g(-1)]^{U}$  is isomorphic to the polynomial ring with n indeterminates:

$$C \mathfrak{g}(-1)^{U} = C[I_1, \cdots, I_n].$$

Here  $I_i$  are given by the following formulas:

(6.3) 
$$I_{j} = \det \begin{bmatrix} 0 & -x_{12} \cdots & -x_{12j} \\ x_{12} & 0 & \cdots & -x_{22j} \\ \cdots & \cdots & \cdots & \cdots \\ x_{12j} & \cdots & \cdots & 0 \end{bmatrix}, j = 1, 2, \cdots, n.$$

**Lemma 6.4.** The simple root vector  $E_{\beta}$  acts on  $H(\lambda) = Hg(-(-1))$  as:

$$\varphi(E_{\beta}) = \sum_{1 \le j < k \le 2n-2} (x_{j \ 2n-1} \ x_{k \ 2n} - x_{j \ 2n} \ x_{k \ 2n-1}) \frac{\partial}{\partial x_{jk}} + \sum_{j=1}^{2n-2} x_{j \ 2n-1} \ x_{2n-1 \ 2n} \ \frac{\partial}{\partial x_{j \ 2n-1}} + \sum_{j=1}^{2n-1} x_{j \ 2n} \ x_{2n-1 \ 2n} \ \frac{\partial}{\partial x_{j \ 2n}} + 2\mu x_{1n-1 \ 2n} \ .$$

**Lemma 6.5.** If we write  $\varphi(E_{\beta}) = D_{\beta} + 2\mu x_{2n-12n}$ , then  $D_{\beta}(detC) = 2x_{2n-12n}$  (detC) for  $C \in M_{2n}(C)$ ,  ${}^{t}C = -C$ .

Proof of Theorem 6.1.

If  $d\lambda=0$ , then  $W(\lambda)$  is the 1-dimensional trivial g-module and Theorem is obvious.

Suppose  $d\lambda \neq 0$  and

$$w = w(x) = a(C) (\det C)^{-\mu-i} \in W(\lambda), \quad C = -{}^t C = (x_{jk})$$

is annihilated by  $\varphi(E_{\alpha_j}) j=1, \dots, n-1$  and  $\varphi(E_{\beta})$ . Here a(C) is a homogeneous polynomial in  $\{x_{jk}\}_{j < k}$  with homogeneous degree (2n-1)i. We can assume i > 0. By Proposition 6.3, a(C) is an element of  $C[I_1, \dots, I_n]$ . By Lemma 5.6, we have

$$\begin{split} \varphi(E_{\beta}) w(x) &= (D_{\beta} + 2\mu x_{2n-12n}) a(C) (\det C)^{-\mu-i} \\ &= (D_{\beta}a) (C) (\det C)^{-\mu-i} - (\mu+i) a(C) (\det C)^{-\mu-i} D_{\beta} (\det C) \\ &+ 2\mu x_{2n-12n} a(C) (\det C)^{-\mu-i} \\ &= (D_{\beta}a) (C) (\det C)^{-\mu-i} - 2(\mu+i) x_{2n-12n} a(C) (\det C)^{-\mu-1} \\ &+ 2\mu x_{2n-12n} a(C) (\det C)^{-\mu-i} \\ &= \{(D_{\beta}a) (C) - 2ix_{2n-12n} a(C)\} (\det C)^{-\mu-i} \end{split}$$

Hence  $\varphi(E_{\beta})w(x)=0$  implies  $(D_{\beta}a)(C)-2ix_{2n-12n}a(C)=0$ . Since a(C) has homogeneous degree 2ni-i, we write a(C) in the following form:

$$a(C) = \sum_{0 \leq m < i} a_m(I_1, \cdots, I_{n-1}) I_n^m,$$

where  $a_m(I_1, \dots, I_{n-1}) \in \mathbb{C}[I_1, \dots, I_n]$ . Then by Lemma 6.5,

$$(D_{\beta}-2ix_{2n-12n}) a(C) = \sum_{0 \le m < 1} \{D_{\beta} a_m + 2(m-i) x_{2n-12n} a_m\} I_m^m.$$

Hence we have

$$(D_{\beta} a_{m})(I_{1}, \cdots, I_{n-1}) + 2(m-i) x_{2n-12n} a_{m}(I_{1}, \cdots, I_{n-1}) = 0, \quad 0 \leq m < i.$$

We consider the coefficients of  $x_{2n-12n}$  in the above equation. Since

$$D_{\beta} = \sum_{1 \le j < k \le 2n-2} (x_{j 2n-1} x_{k 2n} - x_{j 2n} x_{k 2n-1}) \frac{\partial}{\partial x_{jk}} + \sum_{j=1}^{2n-2} x_{j 2n-1} x_{2n-12n} \frac{\partial}{\partial x_{j 2n-1}} + \sum_{j=1}^{2n-1} x_{j 2n} x_{2n-12n} \frac{\partial}{\partial x_{j 2n}}$$

and from the definition of  $I_1, \dots, I_{n-1}$  (6.3),  $a_m$  contains no  $x_{12n-1}, \dots, x_{2n-22n-1}$ ,  $x_{12n}, \dots, x_{2n-12n}$ . Hence the above description of  $D_{\beta}$  shows that the coefficient of  $x_{2n-12n}$  in  $(D_{\beta} a) (I_1, \dots, I_{n-1})$  is equal to zero. This implies  $a_m(I_1, \dots, I_{n-1})$  is equal to zero for any m. Hence w(x) is in fact zero. This completes the proof of Theorem 6.1. Q.E.D.

## 7. Proof of the irredicibility of $W(\lambda)$ (Case IV)

In this section we set g=so(n+2, C) and prove the irreducibility of  $W(\lambda)$  in case IV. We use the notations in Section 2. In particular the Lie subalgebra g(-1) is given by

$$\mathfrak{g}(-1) = \{ \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2^s x & 0 \end{bmatrix} | x \in \mathbb{C}^n \} .$$

We identify  $H(\lambda)$  with  $H(\mathfrak{g}(-1))$ . Let  $(x_j) \ 1 \le j \le n$  be the standard coordinate system on  $\mathfrak{g}(-1) \cong \mathbb{C}^n$  and  $\{E_j\}_{j=1,\dots,n}$  the standard basis of  $\mathbb{C}^n$ . It is easy to

see that  $E_j$  acts on  $H(\lambda) = H(g(-1))$  as  $-\frac{\partial}{\partial x_j}$ . By the Poincaré-Birkhoff-Witt theorem, we have

$$\varphi(U(\mathfrak{g})) v^{\lambda} = \{Dv^{\lambda} \colon D \in \mathcal{D}_{\text{const}}(g(-1))\}$$
.

We introduce the following notations of matrices:

$$\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, E' = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \text{ and } e = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

We set l=[n/2] and, for  $j=1, \dots, l-1$ , wet

$$E_{\sigma_{j}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots \\ 0 & E \\ -tE & 0 \\ 0 & 0 \end{bmatrix}.$$

If n is odd, then we set

$$E_{\sigma_{I}} = \begin{bmatrix} 0 & 0 & & \\ \mathbf{O} & & & \\ & \ddots & & & \\ & \mathbf{O} & e & & \\ & -^{t}e & 0 & & \\ & 0 & & 0 \end{bmatrix}$$

If n is even, then we set

$$E_{\boldsymbol{\sigma}_{l}} = \begin{bmatrix} 0 & 0 \\ \mathbf{O} & \\ & \ddots & \\ & \mathbf{O} & E' \\ & -^{t}E' & \mathbf{O} \\ 0 & & 0 \end{bmatrix}.$$

Let

$$E_{\boldsymbol{\beta}} = \begin{bmatrix} 0 & t_{\boldsymbol{\theta}} & 0 \\ \mathbf{0} & \\ 0 & \ddots \\ 0 & & 0 \end{bmatrix}.$$

We prove the following Theorem.

**Theorem 7.1.** Suppose  $w \in W(\lambda)$  is annihilated by every  $\varphi(E_{\sigma_j}), j=1, \dots, l$ , and  $\varphi(E_{\beta})$ , then w is a scalar miltiple of the highest weight vector  $v^{\lambda}$ .

We can assume that w is a weight vector and consider the weight of the central element

$$z = \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix},$$

of g(0).

**Lemma 7.2.** Let  $\mu$  be the complex number defined by (1.4). Then the element z acts on  $H(\lambda)=H(g(-1))$  as the following differential operator:

(7.1) 
$$\varphi(z) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + 2\mu.$$

As a function on g(-1),  $v^{\lambda}$  has homogeneous degree  $-4\mu$ . Hence if the vector w in Theorem 7.1 is a weight vector with respect to the  $\varphi(z)$ , then it is a homogeneous function on g(-1). Hence w is a linear combination of the *i*-th derivatives of  $v^{\lambda}$  for some nonnegative integer *i*. Hence we assume that

(7.2) 
$$w(y) = a(x) ({}^{t}xx)^{-2^{\mu-i}}$$
 for  $y = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2^{t}x & 0 \end{bmatrix}$ ,  $x = (x_j)$ 

where a(x) is a homogeneous polynomial in  $\{x_i\}$  whose homogeneous degree is *i*.

Let U be the maximal unipotent subgroup of G(0) determined by the simple root vectors  $\{E_{\sigma_j}\}_{j=1,\dots,l}$ . Since w is annihilated by  $\{\varphi(E_{\sigma_j})\}_{j=1,\dots,l-1}$ , w is an Ad(U)-invariant function on g(-1). Since  $({}^{i}xx)^{-2\mu-i}$  is Ad(U)-invariant, we conclude that a(x) is also an Ad(U)-invariant polynomial. Here we need Johnson's result [3].

**Proposition 7.3.** The ring of Ad(U)-invariant polynomials  $C[\mathfrak{g}(-1)]^U$  is isomorphic to the polynomial ring with two indeterminates;

$$\boldsymbol{C}[\mathfrak{g}(-1)]^{\boldsymbol{v}} = \boldsymbol{C}[I_1, I_2].$$

Here  $I_1$  and  $I_2$  are given by the following formulas:

(7.3) 
$$I_1 = x_1 + \sqrt{-1} x_1, I_2 = {}^t x x.$$

**Lemma 7.4.** The simple root vector  $E_{\beta}$  acts on  $H(\lambda) = H(\mathfrak{g}(-1))$  as the following differential operator:

(7.4) 
$$\varphi(E_{\beta}) = 2(x_1 - \sqrt{-1} x_2) \left(\varphi(z) + 2\mu\right) - I_2\left(\frac{\partial}{\partial x_1} - \sqrt{-1} \frac{\partial}{\partial x_2}\right)$$

where  $\varphi(z)$  is given by Lemma 7.2 and  $I_2 = txx$ .

If we set 
$$J=x_1-\sqrt{-1} x_2$$
 and  $D=\frac{\partial}{\partial x_1}-\sqrt{-1} \frac{\partial}{\partial x_2}$ , then  $DI_2=2J$ .

Proof of Theorem 7.1.

If  $d\lambda=0$ , then  $W(\lambda)$  is the 1-dimensional trivial g-module and Theorem is obvoius.

Suppose  $d\lambda \neq 0$  and  $w = w(y) = a(x) ({}^{t}xx)^{-2^{\mu-i}}$  is annihilated by all  $\varphi(E_{\sigma_{j}})$  $j=1, \dots, l$  and  $\varphi(E_{\beta})$ . Here a(x) is a homogeneous polynomial in  $\{x_{j}\}$  with homogeneous degree *i*. We can assume i > 0. By Proposition 7.3 we conclude a(x) is an element of  $C[I_{1}, I_{2}]$ . By Lemma 7.4

$$\begin{aligned} \varphi(E_{\beta}) w(y) &= \{2J(\varphi(z)+2\mu)-I_{2} D\} a(x) I_{2}^{-2\mu-i} \\ &= 2J(-4\mu-i+2\mu) a(x) I_{2}^{-2\mu-i} - (Da) (x) I_{2}^{-2\mu-i+1} + 2(2\mu+i) JI_{2}^{-2\mu-1} a(x) \\ &= (Da) (x) I_{2}^{\lambda-i+1} \end{aligned}$$

Hence  $\varphi(E_{\beta}) w(x) = 0$  implies (Da)(x) = 0. Since a(x) has homogeneous degree *i*, we can write it as the following form:

$$a(x) = \sum_{1 \le k \le [i/2]} a_k I_2^k I_1^{i-2k}$$

Then

$$(Da)(x) = \sum_{1 \le k \le [i/2]} a_k \{ D(I_2^k) I_1^{i-2k} + I_2^k D(I_1^{i-2k}) \\ = \sum_{1 \le k \le [i/2]} 2a_k I_2^{k-1} I_1^{i-2k-1} (k J I_1 + i I_2 - 2k I_2) \\ = \sum_{1 \le k \le [i/2]} 2a_k I_2^{k-1} I_1^{i-2k-1} \{ (i-k) (x_1^2 + x_2^2) + (i-2k) (x_3^2 + \dots + x_n^2) \} .$$

Since i > k in the summation of the above equation, we conclude that (Da)(x) = 0 implies a(x)=0. This completes the proof of Theorem 7.1. Q.E.D.

#### 8. Reducibilities of generalized Verma modules

In this section we discuss the reducibilities of Verma Modules induced from the maximal parabolic subalgebra  $\mathfrak{p}$  (Corollary 1.2). This gives a representation theoretic interpretation of the zeros of the *b*-function.

Let  $d\lambda$  be a one dimensional representation of  $\mathfrak{p}$ . Let  $C_{d\lambda}$  be the representation space of  $d\lambda$ . We define generalized Verma module  $V(d\lambda)$  by

$$V(d\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} \boldsymbol{C}_{d\lambda}$$
 .

Jantzen [2] gave a reducibility criterion for  $V(d\lambda)$  using his formulas for determinants of contravariant forms. (Jantzen's result is, in fact, much more general.)

Let  $\mu$  be the complex number defined by (1.4). By our construction of the irreducible highest weight module  $W(\lambda)$ , we can recover Jantzen's result in the following form:

**Coroltary 8.1.** If  $-2\mu$  is a positive integer or a zero of the b-function b(s) of f, then  $V(d\lambda)$  is reducible.

Proof. If  $-2\mu$  is a positive integer, then the highest weight vector  $v^{\lambda} = f^{-2\mu}$  of  $W(\lambda)$  is a polynomial function on g(1). Hence  $W(\lambda)$  becomes finite dimensional and  $V(d\lambda)$  is reducible.

Now we consider the case when  $b(-2\mu)=0$ . In case I the equation (1.5) is given explicitly by the following Capelli's identity (See Weyl [10]):

(8.1) 
$$(\det(\partial/\partial x_{jk})) \det(x_{jk})^s = s(s+1)\cdots(s+n-1) \det(x_{jk})^{s-1}$$
  
(i.e.  $b(s) = s(s+1)\cdots(s+n-1)$ ).

Suppose  $V(d\lambda)$  is irreducible. Then  $W(\lambda)$  and  $V(d\lambda)$  are isomorphic. But then by the Poincaré-Birkhoff-Witt theorem,  $W(\lambda)$  is isomorphic to  $U(\mathfrak{g}(-1))$  as vector spaces. Then

$$(-1)^{n} \varphi(E_{1\sigma(1)}) \varphi(E_{2\sigma(2)}) \cdots \varphi(E_{n\sigma(n)}) v^{\lambda} = \frac{\partial^{n}}{\partial x_{1\sigma(1)} \partial x_{2\sigma(2)} \cdots \partial x_{n\sigma(n)}} \det(x_{jk})^{-2\mu}$$

must be linearly independent, where  $\sigma$  runs over the set  $S_n$  of all permutations of  $\{1, 2, \dots, n\}$ . (Recall (4.1) that  $E_{ij}$ 's are basis elements of  $\mathfrak{g}(-1)$ .) Since  $b(-2\mu)=0$ , this contradicts to the Capelli's identity. Hence  $V(d\lambda)$  is also reducible in this case.

Case II-IV can be treated completely analogously, the role of (8.1) being played by the following formulas:

Case II

$$\det \begin{bmatrix} \frac{\partial}{\partial x_{11}} & \frac{1}{2} & \frac{\partial}{\partial x_{12}} & \cdots & \frac{1}{2} & \frac{\partial}{\partial x_{1n}} \\ \frac{1}{2} & \frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & & \vdots \\ \vdots & \ddots & \vdots \\ \frac{1}{2} & \frac{\partial}{\partial x_{1n}} & \cdots & \cdots & \frac{\partial}{\partial x_{nn}} \end{bmatrix} f(x_{ij})^s = s(s+1/2)\cdots(s+(n-1)/2)f(x_{ij})^{s-1}$$

where  $f(x_{ij})$  is the determinant of the  $n \times n$  symmetric matrix  $(x_{ij})$ .

Case III

$$f(\partial/\partial x_{ij})f(x_{ij})^s = s(s+2)\cdots(s+2n-2)f(x_{ij})^{s-1},$$

where  $f(x_{ij})$  is the Pfaffian of the  $2n \times 2n$  antisymmetric matrix  $(x_{ij})$  and given explicitly by the following formula:

$$f(x_{ij}) = \sum_{\substack{\sigma \in S_{2n} \\ \sigma(2i-1) < \sigma(2i) \\ \sigma(2i-1) < \sigma(2i+1)}} \operatorname{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdots x_{\sigma(2n-1)\sigma(2n)}.$$

Case IV

$$\left(\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2\right) \left(\sum_{i=1}^{n} x_i^2\right)^s = 4s(s+(n-2)/2) \left(\sum_{i=1}^{n} x_i^2\right)^{s-1}.$$
 Q.E.D.

#### 9. Final remarks

1. Using the result of Jantzen [2] (see also Enright, Howe and Wallach [1]), one can verify that the statement of Corollary 8.1 is true in the exceptional case (mentioned in Remark 2.1) also.

2. Let  $(G_0, K_0)$  be an irreducible Hermitian symmetric pair of tube type. Let  $g_0$  (resp.  $\mathbf{t}_0$ ) be the Lie algebra of  $G_0$  (resp.  $K_0$ ), and  $g_0 = \mathbf{t}_0 \oplus \mathbf{p}_0$  the Cartan decomposition of  $g_0$ . By convention we delete the subscript o to denote complexified Lie algebras. So we have the decomposition  $g = \mathbf{t} \oplus \mathbf{p}$  of the compexified Lie algebra  $\mathbf{g}$ . The Lie algebra  $\mathbf{t}_0$  has the 1-dimensional center  $Z = \mathbf{R}z$ , where the eigenvalues of z under the adjoint action on  $\mathbf{p}$  are  $\pm i$ . Let

$$\mathfrak{p}^+ = \{x \in \mathfrak{p} | [z, x] = ix\}$$
 and  $\mathfrak{p}^- = \{x \in \mathfrak{p} | [z, x] = -ix\}$ .

If we set  $g(-1)=\mathfrak{p}^-$ ,  $g(0)=\mathfrak{k}$  and  $g(1)=\mathfrak{p}^+$ , then we have a Z-gradation

$$\mathfrak{g} = \mathfrak{g}(-1) + \mathfrak{g}(0) + \mathfrak{g}(-1)$$
.

Then the pairs  $(G(0), g(\pm 1))$  are irreducible regular prehomogeneous vector spaces of commutative parabolic type. If  $\lambda$ , a 1-dimensional character of g(0), corresponds to a zero of the *b*-function, then it is known that the  $g_0$ -module  $W(\lambda)$  is unitarizable. See [1], [7] and [9].

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