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MULTIPLICATIVE OPERATIONS IN BP COHOMOLOGY

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Introduction. In the present work we study multiplicative operations in BP cohomology. In § 1 we show that all multiplicative operations in BP* are automorphisms (Theorem 1.3). Thus they from the group Aut (BP). In §2 we define Adams operations in BP* by the formal group \( \mu_{BP} \) of BP cohomology and study the basic proprties of them. These oprations are primarily defined for units in \( \mathbb{Z}_p \) and then extended to \( p \)-adic units. Thereby we discuss BP* by extending the ground ring \( \mathbb{Z}_p \) to the ring of \( p \)-adic integers \( \mathbb{Z}_p \). To achieve this extension simply by tensoring with \( \mathbb{Z}_p \) we restrict our cohomologies to the category of finite CW-complexes. Correspondingly we consider all multiplicative operations in \( BP^*(\ ) \otimes \mathbb{Z}_p \) whenever it becomes necessary to do so. Adams operations could be defined also for non-units, but we are not interested in such a case in this paper. In §3 we prove that the center of Aut (BP) consists of all Adams operations (Theorem 3.1).

We regard the lecture note [2] as our basic reference and use the results contained there rather freely.

1. Multiplicative operations in BP*. 

Let BP* denote the Brown-Peterson cohomology for a specified prime p. By a multiplicative operation in BP* we understand a stable, linear and degree-preserving cohomology operation

\[ \Theta_a: BP^*(\ ) \to BP^*(\ ) \]

which is multiplicative and \( \Theta_a(1)=1 \). The set of all multiplicative operations in BP* forms a semi-group by composition, which will be denoted by Mult (BP).

With respect to the standard complex orientation of BP* [1], [2], [7], we denote by \( e^{BP}(L) \) the Euler class of a complex line bundle L and by \( \mu_{BP} \) the associated formal group. Let \( \Theta_a \in \text{Mult} (BP) \). Putting

\[ \Theta_a(e^{BP}(L)) = \sum_{i \in \mathbb{Z}_+} \theta_i(e^{BP}(L))^i \]

for an arbitrary line bundle L, by naturality we obtain a well-determined power
series

$$\theta_a(T) = \sum_{i \geq 0} \theta_i T^i, \quad \theta_i \in BP^{2-2i}(T).$$

By naturality $\theta_0 = 0$ and by stability $\theta_1 = 1$. In particular $\theta_a$ is invertible.

Put

$$\phi_a(T) = \theta_a^{-1}(T).$$

Then

(1.2) \hspace{1cm} \Theta_a(pt) \ast \mu_{BP} = \mu_a, \quad \mu_a = \mu_{BP} \ast b.

Recall that $\mu_{BP}$ is typical. Hence $\mu_a$ is a typical formal group and $\phi_a$ is a typical curve over $\mu_{BP}$.

Conversely, given a typical curve $\phi_a$ over $\mu_{BP}$, by the universality of $BP^*$, [2], Theorem 7.2, $\phi_a$ determines uniquely a multiplicative operation $\Theta_a$ in $BP^*$ satisfying

(1.3) \hspace{1cm} \Theta_a(e^{BP}(L)) = \phi_a^{-1}(e^{BP}(L)).

Thus, via (1.3) multiplicative operations $\Theta_a$ in $BP^*$ correspond bijectively with typical curves $\phi_a$ over $\mu_{BP}$ such that

(1.4) \hspace{1cm} \phi_a(T) \equiv T \text{ mod deg 2} \quad \text{and} \quad \dim \phi_a^{-1}(e^{BP}(L)) = 2

for complex line bundles $L$.

Recall that a typical curve $\phi_a$ satisfying (1.4) can be expressed uniquely as a Cauchy series

(1.5) \hspace{1cm} \phi_a(T) = \sum_{k \geq 0} a_k T^k, \quad a_0 = 1, \quad a_k \in BP^{2(1-p)}(pt),

where $\mu = \mu_{BP}$ (cf., [2], [3]). Thus multiplicative operations $\Theta_a$ correspond bijectively with sequences

(1.6) \hspace{1cm} a = (a_0, a_2, \cdots, a_n, \cdots), \quad a_n \in BP^{2(1-p)}(pt),

via (1.3) and (1.5). The identity operation corresponds to the zero sequence $0 = (0, 0, \cdots)$.

First we remark

Proposition 1.1. Let $\Theta_a$ and $\Theta_b$ be multiplicative operations in $BP^*$ such that

$$\Theta_a(pt) = \Theta_b(pt).$$

Then $a = b$ as sequences (1.6). Hence $\Theta_a = \Theta_b$.

Proof. By (1.2) we see that

$$\mu_a = \mu_b.$$
Then, by the uniqueness of logarithm we see that
\[ \log_{\mu_p} = \log_{\mu_p}, \]
or
\[ \log_{BP} \circ \phi_a = \log_{BP} \circ \phi_b. \]

thus \( \phi_a = \phi_b \). q.e.d.

Let \( \Theta_a \in \text{Mult}(BP) \). We have
\[ \Theta_a(pt) \circ \log_{BP}(T) = \log_{BP} \circ \phi_a(T) \]
over \( BP^*(pt) \otimes \mathbb{Q} \). Putting
\[ \log_{BP}(T) = \sum_{k \geq 0} n_k T^{p^k}, \quad n_k = [CP_{p^k}]_p, \]
expanding both sides of the above formula as power series of \( T \) and comparing coefficients of \( T^{p^k} \) we get
\[ (1.7) \quad \epsilon^k = \sum_{j=0}^{k} n_j a_{p^j}^k, \quad k \geq 0. \]

This is a recursive formula to describe \( \Theta_a(n_k) \), hence determines \( \Theta_a(pt) \). We discuss another formula to describe \( \Theta_a(pt) \).

Denote by \( f_p \) and \( f'_p \) the Frobenius operators for the prime \( p \) on curves over \( \mu_{BP} \) and \( \mu_a \) respectively. Recall that, if we put
\[ (1.8) \quad (f_p \gamma_0)(T) = f_{f_p\gamma_0}(T) \quad \mu = \mu_{BP}, \quad \gamma_0(T) = T, \]
then \( v_k \in BP^{2i-1} = \mathbb{Q}(pt) \) and the sequence \( (v_1,v_2, \cdots, v_n, \cdots) \) forms a polynomial basis of \( BP^*(pt) \), [2].

Since \( \Theta_a(pt) \ast \mu_{BP} = \mu_a \), we have
\[ (f_p \gamma_0)(T) = (\Theta_a(pt) \ast f_p \gamma_0)(T) = \sum_{i \geq 1} \Theta_a(v_k) T^{p^i}. \]

Using the fact that \( \phi_a : \mu_a \simeq \mu_{BP} \), a strict isomorphism, we compute \( (\phi_a f_p \gamma_0)(T) \) in two ways as follows:
\[ (\phi_a f_p \gamma_0)(T) = (f_p \phi_a \gamma_0)(T) \]
\[ = (f_p \phi_a)(T) = \sum_{k \geq 0} f_{p_k}(a_k T^{p^k}) \]
\[ = (f_p \gamma_0)(T)^{p^i} \sum_{k \geq 1} \sum_{\nu \leq i} [p]_{BP}(a_k T^{p^k}) \]
\[ = \sum_{k \geq 1} \nu_k T^{p^k} + \sum_{k \geq 1} \sum_{\nu \leq i} \nu_k a_k \rho^i T^{p^k+i-1} \]

by [2], Propositions 2.4, 2.5 and 2.9, on one hand, where
\[ [p]_{BP}(T) = \sum_{i \geq 0} w_i T^{p^i}, \quad w_0 \neq p, \quad w_k \in BP^{p^i-1}(pt); \]
on the other hand
Thus we obtain

\[(1.9) \quad \sum_{k \geq 1} \sum_{i \geq 0} a_i \Theta_a(v_k) T^{p^k-1} = \sum_{k \geq 1} \sum_{i \geq 0} a_i \Theta_a(v_k) T^{p^k+i-1}.\]

This is a recursive formula to describe \(\Theta_a(v_k)\).

Let \(I = BP^*(pt)\), the kernel of the augmentation \(\varepsilon: BP^*(pt) \to \mathbb{Z}_p\). By [2], §10, we see that "the left hand side of (1.9)"

\[\equiv \sum_{k \geq 1} \Theta_a(v_k) T^{p^k-1} \mod I^2\]
\[= \Theta_a(v_1) T + \Theta_a(v_2) T^p + \ldots \mod I^2,
\]

and

"the right hand side of (1.9)"

\[\equiv \sum_{k \geq 1} v_k T^{p^k-1} + \sum_{i \geq 1} a_i v_k T^{p^k+i-1} \mod I^2\]
\[= (v_1 + p a_1) T + (v_2 + p a_2) T^p + \ldots \mod I^2\]

Hence (1.9) implies

\[(1.10) \quad \Theta_a(v_k) = v_k + p a_k \mod I^2\]

for all \(k \geq 1\). In particular

\[\Theta_a(v_k) \equiv v_k \mod (p) + I^2\]

for \(k \geq 1\). This shows that \(\{\Theta_a(v_k), k \geq 1\}\) forms a polynomial basis of \(BP^*(pt)\). Thus we obtain

**Proposition 1.2.** For any \(\Theta_a \in \text{Mult}(BP)\)

\[\Theta_a(pt): BP^*(pt) \cong BP^*(pt), \text{ an isomorphism.}\]

Let \(\Theta_a\) and \(\Theta_b\) be two multiplicative operations in \(BP^*\) with corresponding sequences \(a = (a_1, a_2, \ldots)\) and \(b = (b_1, b_2, \ldots)\). Putting

\[\Theta_c = \Theta_a \circ \Theta_b, \quad c = (c_1, c_2, \ldots),\]

we shall discuss the sequence \(c\). Put

\[\tilde{\phi}_b(T) = \Theta_a(pt) \ast \phi_b(T) = \sum_{i \geq 0} \Theta_a(b_k) T^{p^k}.\]

Then
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$$\Theta_a(pt) \ast \mu_b = \Theta_a(pt) \ast (\phi^{-1}_b \circ \mu_a \circ \phi_b)$$
$$= \phi_a^{-1} \circ \mu_a \circ \phi_b = \mu^\Theta_{BP}.$$ 

On the other hand

$$\Theta_a(pt) \ast \mu_b = \Theta_a(pt) \ast \Theta_b(pt) \ast \mu_{BP} = \Theta_c(pt) \ast \mu_{BP} = \mu_c.$$ 

Thus, likewise in the proof of Proposition 1.1, we have

$$(1.11) \quad \phi_c = \phi_a \circ \phi_b,$$

or equivalently

$$(1.12) \quad \sum_{\mu} \Lambda^0_k \Gamma^0 = \sum \sum \Lambda^0_k \Theta_a(b)T^{p^k}.$$ 

This is a recursive formula to describe $\epsilon_k$.

A multiplicative operation $\Theta_a$ in $BP^*$ is called an automorphism of $BP^*$ if

$$\Theta_a(X, A): BP^*(X, A) \cong BP^*(X, A),$$

isomorphic for all finite CW-pair $(X, A)$. Clearly a multiplicative operation $\Theta_a$ is an automorphism of $BP^*$ iff it has an inverse. The set of all automorphisms of $BP^*$ forms a group, which will be denoted by $\text{Aut}(BP)$.

**Theorem 1.3.** $\text{Aut}(BP) = \text{Mult}(BP)$.

**Proof.** It is sufficient to prove that every multiplicative operation $\Theta_a$ has a right inverse.

Let $t = (t_1, t_2, \ldots)$ and $s = (s_1, s_2, \ldots)$ be sequences of indeterminates with $\dim t_k = \dim s_k = 2(1 - p^k)$. Put

$$(*)1 \quad \sum_{k \geq 0} u_k T^{p^k} = \sum_{k \geq 0} s_k T^{p^k},$$

where $s_0 = u_0 = 1$. Then over $BP^*(pt) [t, s]$ we have

$$\sum_{k \geq 0} u_k T^{p^k} \equiv T + u_1 T^p + u_2 T^{p^2} + \cdots \mod \tilde{I}^2,$$

and

$$\sum_{k \geq 0} s_k T^{p^k} \equiv T + (s_1 + t_1) T^p + (s_2 + t_2) T^{p^2} + \cdots \mod \tilde{I}^2,$$

where $\tilde{I} = (s, t)$, the ideal of $BP^*(pt) [s, t]$ generated by $s_1, s_2, \ldots, t_1, t_2, \ldots$. Thus we can put

$$(*)2 \quad u_k = t_k + s_k + P_k(t_1, \ldots, t_{k-1}, s_1, \ldots, s_{k-1}), \quad k \geq 1.$$ 

Here $P_k$ is a polynomial of $t_1, \ldots, t_{k-1}, s_1, \ldots, s_{k-1}$ with $\dim P_k = 2(1 - p^k)$ and
We want to find a right inverse of $\Theta_a$. Putting

\[ \Theta_a \circ \Theta_b = id \]

with undecided sequence $b = (b_1, b_2, \cdots)$, we shall decide the sequence $b$. By (1.12), (*1) and (*2), we get

\[ a_k + \Theta_a(b_k) + P_k(a_1, \cdots, a_{k-1}, \Theta_a(b_1), \cdots, \Theta_a(b_{k-1})) = 0 \]

for all $k \geq 1$. Since the coefficients of $P_k$ depend neither on $(a_1, a_2, \cdots)$ nor on $(\Theta_a(b_1), \Theta_a(b_2), \cdots)$ we may use (*4) as a recursive formula to obtain $\Theta_a(b_k)$, so we get $\Theta_a(b_k)$ as polynomials of $a_1, \cdots, a_k$ successively for $k \geq 1$. By Proposition 1.2 $\Theta_a(pt)$ is an isomorphism. Thus we get a sequence $(b_1, b_2, \cdots)$ so that it satisfies (*4). Thereby $\Theta_a$ is obtained to satisfy (*3). q.e.d.

2. Adams operations in $BP^*$. Let $\mathbb{Z}_p$ be the ring of integers localized at the prime $p$ and $\mathbb{Z}_p$ its completion, i.e., the ring of $p$-adic integers. As is well known the endomorphism

\[ [\alpha]_{BP} \in \text{End}(\mu_{BP}) \]

is defined for each $\alpha \in \mathbb{Z}_p$ so that

\[ [\alpha]_{BP}(T) = \alpha T + \text{higher terms.} \]

It is convenient for us to extend these endomorphisms $[\alpha]_{BP}$ to $\alpha \in \mathbb{Z}_p$. For this purpose we extend the grout ring $\mathbb{Z}_p$ of $BP^*$ to $\mathbb{Z}_p$ by tensoring, i.e., we consider $BP^*(\mathbb{Z}_p)$ whenever it is necessary to talk of $p$-adic integers.

Let $A = BP^*(pt) \otimes \mathbb{Z}_p$. Let $F$ and $G$ be formal groups over $A$. Let

\[ c : \text{Hom}_A(F, G) \to A \]

be the homomorphism sending $f$ to $a_1$ when $f(T) = a_1 T + \text{higher terms}$. Since $A$ is an integral domain of characteristic zero, $c$ is injective as is well known (cf., [4], [5]).

Since $A$ is a direct sum of copies of $\mathbb{Z}_p$ (corresponding to each monomials of $v_i$'s) we give a direct limit topology to $A$. (Each direct summand is given the topology of $\mathbb{Z}_p$). Then, using the argument of Lubin [5], Lemma 2.1.1, we see that $c$ is an isomorphism onto a closed subgroup of $A$.

In case $F = G = \mu_{BP}$,

\[ \text{Im } c \supseteq \mathbb{Z}_p, \]

because $c([\alpha]_{BP}) = \alpha$ for $\alpha \in \mathbb{Z}_p$. Hence

\[ \text{Im } c \supseteq \mathbb{Z}_p = \mathbb{Z}_p. \]
Since \( c \) is injective, for each \( \alpha \in \mathbb{Z} \) there exists a unique
\[
[\alpha]_{BP} \in \text{End}_A(\mu_{BP})
\]
such that \( c([\alpha]_{BP}) = \alpha \). Thus the definition of \([\alpha]_{BP}\) is extended to \( \mathbb{Z}_p \).

Since \( c: \text{End}_A(\mu_{BP}) \rightarrow A \) is a ring homomorphism, for any \( p \)-adic integers \( a \) and \( \beta \) we have the following relations:

\[
\begin{align*}
(2.1) & \quad [\alpha]_{BP}(T) = \alpha T + \text{higher terms}, \\
(2.2) & \quad [\alpha]_{BP} + [\beta]_{BP} = [\alpha + \beta]_{BP}, \quad \mu = \mu_{BP}, \\
(2.3) & \quad [\alpha]_{BP} \circ [\beta]_{BP} = [\alpha \beta]_{BP}.
\end{align*}
\]

Let \( \alpha \in \mathbb{Z}_{(\varphi)} \) (or \( \in \mathbb{Z}_p \)) be a unit. Put
\[
\Psi_{\alpha}(T) = [\alpha^{-1}]_{BP}(\alpha T).
\]

Since
\[
(f_{\psi_{\alpha}})(T) = f_{\psi([\alpha^{-1}]_{BP}(\alpha T))} = [\alpha^{-1}]_{BP}([\alpha^e]f_{\psi_{\alpha}}(T)) = 0
\]
for every \( q > 1 \) such that \( (\varphi, q) = 1 \) by [2], Propositions 2.3 and 2.9, where \( \gamma_{\varphi}(T) = T \), we see that \( \psi_{\alpha} \) is a typical curve over \( \mu_{BP} \). Moreover \( \psi_{\alpha} \) satisfies (1.4) as is easily seen. Thus there corresponds a multiplicative operation in \( BP^* \) to \( \psi_{\alpha} \).

We denote this multiplicative operation by \( \Psi^* \) and call Adams operations in \( BP^* \).

REMARK 1. Even for non-units \( a \) Adams operations can be defined on the same way as above. But these operations are defined in \( BP^* \otimes \mathbb{Q} \) or \( BP^* \otimes \mathbb{Q}_p \). And these cohomology theories are essentially ordinary cohomologies (corresponding to generalized Eilenberg-MacLane spectra), so we are not interested in these operations in the present work.

REMARK 2. Adams operations in complex cobordism are defined by Novikov [6]. When we regard \( BP^* \) as a direct summand of \( U^* \otimes \mathbb{Q}_p \), our Adams operations will be the restrictions of Novikov's Adams operations to \( BP^* \).

Let \( a \) be a unit of \( \mathbb{Z}_{(\varphi)} \) (or of \( \mathbb{Z}_p \)). Since
\[
\Psi_{\alpha}(\alpha^i[\alpha]_{BP}(T)) = [\alpha^{-1}]_{BP} \circ [\alpha]_{BP}(T) = T,
\]
we see that
\[
(2.4) \quad \Psi^*(e_{BP}(L)) = \alpha^{-i}[\alpha]_{BP}(e_{BP}(L))
\]
for any complex line bundle \( L \).

Since \( \Psi^*(pt) \cap \mu_{BP} = \mu_{BP}^\vee \) we see that
\[ \Psi^a(pt) \log_{BP} = \log_{BP} \circ \psi_a. \]

Here
\[ (\log_{BP} \circ \psi_a)(T) = \log_{BP}[\alpha^{-1}]_{BP}(\alpha T) \]
\[ = \alpha^{-1} \cdot \log_{BP}(\alpha T) = \sum_{k \geq 0} \alpha^{\beta_k - 1} n_k T^{\beta_k}. \]
Thus
\[ \sum_{l \geq 0} \Psi(n_k) T^{\beta_l} = \sum_{l \geq 0} \alpha^{\beta_k - 1} n_k T^{\beta_k}, \]
or
\[ (2.5) \quad \Psi fo) = \alpha^{\beta_k - 1} n_k, \quad k \geq 1, \]
after extending \( \Psi^a(pt) \) to \( \Psi^a(pt) \otimes 1_Q \).

**Proposition 2.1.** \( \Psi^a(pt) BP^{-2s}(pt) = \alpha^s id. \)

Proof. \( (n_1, n_2, \cdots) \) is a polynomial basis of \( BP^*(pt) \otimes Q \). Since \( \Psi^a \) is linear and multiplicative, for every polynomials \( x_s \) of \( n_k \)'s with \( \dim x_s = -2s \) by (2.5) we see easily that
\[ \Psi^a(x_s) = \alpha^s x_s. \quad \text{q.e.d.} \]

**Corollary 2.2.** *If we put*
\[ \mu_{BP}(X, Y) = \sum_{i,j} a_{ij} X_i Y_j, \]
\[ \text{then} \]
\[ \mu_{BP}^\beta(X, Y) = \sum_{i,j} \alpha^{i+j-1} a_{ij} X_i Y_j. \]

Next we prove

**Proposition 2.3.** \( \Psi^a \Psi^\beta = \Psi^{a \beta} \Psi^a. \)

Proof. Put
\[ [\alpha]_{BP}(T) = \sum_{s \geq 0} \alpha_s T^s \tau^{-1} \tau^{s+1}, \quad \alpha_s \in BP^{-2s \tau^{-1} \tau^{s}}(pt). \]

For any complex line bundle \( L \) we have
\[ \Psi^\beta(\Psi^a(e_{BP}(L))) = \Psi^\beta(\alpha^{-1}[\alpha]_{BP}(e_{BP}(L))) \]
\[ = \alpha^{-1} \Psi^\beta(\sum_{s \geq 0} a_s(e_{BP}(L))^{s \tau^{-1} \tau^{s+1}}) \]
\[ = \alpha^{-1} \sum_{s \geq 0} \beta^{s \tau^{-1} \tau^{s}} a_s(\Psi^\beta(e_{BP}(L)))^{s \tau^{-1} \tau^{s+1}} \quad \text{by Proposition 1} \]
\[ = \alpha^{-1} \beta^{-1} \sum_{s \geq 0} a_s(\beta \Psi^\beta(e_{BP}(L)))^{s \tau^{-1} \tau^{s+1}} \]
\[ = \alpha^{-1} \beta^{-1} \cdot [\alpha]_{BP}(e_{BP}(L))_{\beta} \quad \text{by (2.4)} \]
\[ = (\alpha \beta)^{-1} [\alpha \beta]_{BP}(e_{BP}(L)) \quad \text{by (2.3)} \]
\[ = \Psi^{a \beta}(e_{BP}(L)). \]
Therefore, by the universality of $BP^*$, [2], Theorem 7.2, we concludes the Proposition.

Let $a$ and $\beta$ be $p$-adic units. By Propositions 1.1 and 2.1 we see that

\begin{equation}
\Psi^a = \Psi^\beta \quad \text{iff} \quad \alpha^{p^{-1}} = \beta^{p^{-1}}.
\end{equation}

Let $U(\mathbb{Z}_p)$ be the multiplicative group of $p$-adic units and $U_i(\mathbb{Z}_p)$ be its subgroup consisting of $p$-adic integers $a$ such that

$$\alpha \equiv 1 \mod p.$$ 

As is well known

$$U_i(\mathbb{Z}_p) = \{\alpha^{p^{-1}}; \alpha \in U(\mathbb{Z}_p)\}.$$ 

By Proposition 2.3 all Adams operations (for $p$-adic units) form a multiplicative subgroup of $\text{Aut}(BP)$. We denote this subgroup by $\text{Ad}(BP)$. Then, (2.6) implies that

**Proposition 2.4.** $\text{Ad}(BP) \cong U_i(\mathbb{Z}_p)$.

And also

**Proposition 2.5.** $\Psi^\lambda = 1$ \iff $\lambda^{p^{-1}} = 1$.

Next we discuss the relations of Adams operations with Quillen operations (of Landweber-Novikov type). We recall the definition of Quillen operations, [2], [7]. Let $t=(t_1, t_2, \ldots)$ be a sequence of indeterminates such that $\dim t_k = 2(1-p^k)$ and

$$\phi_t(T) = \sum_{\sum e_k t_k^p, \ t_0 = 1} \phi_t,$$

a typical curve over $\mu_{BP}$ by extending the ground ring of $\mu_{BP}$ to $BP^*(pt)/t$.

Then

$$r_t: BP^*(\ ) \to BP^*(\ )[t]$$

is the multiplicative operation such that

$$r_t(e^{BP}(L)) = \phi_t^{-1}(e^{BP}(L))$$

for any complex line bundle $L$. Putting

$$r_t(x) = \sum_{E} r_E(x)t^E, \ x \in BP^*(X, A),$$

where $E=(e_1, e_2, \cdots)$ runs over all sequences of non-negative integers such that all $e_k$ but a finite are zero, we get linear stable operations

$$r_E: BP^*(\ ) \to BP^{*+\|E|}(\ )$$
of degree $2|E|$, where $|E| = \sum_{i} e_i (p^i - 1)$.

Now for a $p$-adic unit $\alpha$ we have

$$\phi_i \circ r_i (e^{BP}(L)) = r_i (\psi_\alpha^{-1}(e^{BP}(L)))$$

$$= (r_i(pt) \ast \psi_\alpha)^{-1}(r_i(e^{BP}(L)))$$

$$= (\phi_i \circ r_i (pt) \ast \psi_\alpha)^{-1}(e^{BP}(L)).$$

And

$$r_i (pt) \ast \psi_\alpha(T) = r_i (pt) \ast ([\alpha^{-1}]_{BP}(\alpha T)) = [\alpha^{-1}]_{BP}(\alpha T),$$

where $\mu' = \mu_{BP}$. Thus

$$\phi_i \circ r_i (pt) \ast \psi_\alpha(T_p) = \phi_i ([\alpha^{-1}]_{BP}(\alpha T))$$

$$= [\alpha^{-1}]_{BP}(\phi_i(\alpha T)) = [\alpha^{-1}]_{BP}(\sum_{k \geq 0} \alpha^{p^k} t_k T^{p^k}).$$

Let

$$\sigma_\alpha: Z_{(p)}[t] \to Z_{(p)}[t]$$

be an algebra homomorphism such that

$$\sigma_\alpha(t_k) = \alpha^{p^{k-1}} t_k, \quad k \geq 1,$$

and define an operation

$$\overline{\Psi}^*: BP^*(\ )[i] \to BP^*(\ )[i]$$

by $\overline{\Psi}^* = \Psi^* \otimes \sigma_\alpha$. Then

$$\overline{\Psi}^* (\phi_i \circ r_i (pt) \ast \psi_\alpha(T)) = \overline{\Psi}^* \ast (\phi_i^{-1}(e^{BP}(L)))$$

$$= (\overline{\Psi}^*(pt) \ast \phi_i)^{-1}(\overline{\Psi}^*(e^{BP}(L)))$$

$$= (\overline{\Psi}^*(pt) \ast \phi_i)(e^{BP}(L)).$$

Remark that

$$\overline{\Psi}^*(pt) \ast \mu_{BP} = \mu_{BP} \phi_i.$$

Thus

$$\overline{\Psi}^*(pt) \ast \mu_{BP} = \mu_{BP} \phi_i.$$

$$\overline{\Psi}^*(pt) \ast \mu_{BP} = \mu_{BP} \phi_i.$$
Thus by (2.8) and (2.10) we see that
\[ \phi_t \circ r_t(pt) \circ \psi_s = \psi_s \circ \bar{\psi}^s(pt) \circ \phi_t, \]
then, by (2.7) (2.9) and the universality of \( BP^* \) we obtain

**Proposition 2.6.** For any unit of \( \mathbb{Z}_p \) there holds the commutativity
\[ r_t \circ \psi_s = \bar{\psi}^s \circ r_t. \]

**Corollary 2.7.** Let \( E = (e_1, e_2, \cdots) \) be a sequence of non negative integers of which all but a finite terms are zero. There holds the commutativity
\[ r_E \circ \psi_s = \alpha^{1_E} \psi_s \circ r_E. \]

**Corollary 2.8.** For any linear stable cohomology operation
\[ \Xi_s: BP^*(\ ) \to BP^{s+2}(\ ) \]
of degree 2s there holds the commutativity
\[ \Xi_s \circ \psi_s = \alpha^s \psi_s \circ \Xi_s. \]

Remark that every stable cohomology operation in \( BP^* \) can be expressed as linear combinations of Quillen operations \( r_E \) over \( BP^*(pt) \). Then Corollary 2.8 follows from Proposition 2.1 and Corollary 2.7.

**Corollary 2.9.** Adams operations in \( BP^* \) commute with all multiplicative operations.

REMARK. Properties of Adams operations in complex cobordism which correspond to Propositions 2.1, 2.2, 2.3, 2.7 and 2.8 are obtained in Novikov [7] by different arguments.

### 3. The center of \( \text{Aut}(BP) \).

For any \( b \in BP^{2(1-p^k)}(pt) \) we define a sequence
\[ (b, k) = (0, \cdots, 0, b, 0, \cdots) \]
with \( b \) as the \( k \)-th term and with all other terms zero. By (1.9) we obtain
\[ \sum_{i=1}^s \psi_i T^{s-1} + \sum_{i=1}^s b \cdot \Theta_{(b, k)}(\psi_i) T^{p^{s-1}} + \sum_{i=1}^{s-1} \psi_i T^{p^{s-1}} + \sum_{i=1}^{s-1} b \cdot \Theta_{(b, k)}(\psi_i) T^{p^{s-1}}. \]

In particular
\[ \sum_{i=1}^s \Theta_{(b, k)}(\psi_i) T^{p^{s-1}} \equiv \sum_{i=1}^s \psi_i T^{p^{s-1}} + \sum_{i=1}^{s-1} b \cdot \Theta_{(b, k)}(\psi_i) T^{p^{s-1}} \text{ mod } \deg p^{s-1} + 1. \]
Recursively on $/ , 1 \leq l < k$, and deleting the same terms successively we see that

\begin{equation}
\Theta_{(b, k)}(v_l) = v_l, \quad 1 \leq l < k,
\end{equation}

and

\begin{equation}
\Theta_{(b, k)}(v_k) = v_k + pb.
\end{equation}

These imply that

\begin{equation}
\Theta_{(b, k)}(x) = x \quad \text{for any } x \in BP^{-2s}(pt), s < p^k - 1,
\end{equation}

and

\begin{equation}
\Theta_{(b, k)}(y) = y + pcb \quad \text{for } y \in BP^{2(1-p^k)}(pt)
\end{equation}

when $y = cv_k \mod \text{decomposables}, c \in \mathbb{Z}_p$.

Let $\Theta_a$ be in the center of $\text{Aut}(BP)$. Then

\begin{equation}
\Theta_{(v_k, k)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k, k)}
\end{equation}

for all $k \geq 1$. And by (1.12) we have

\begin{equation}
\sum_{l \leq b} \mu_a \Theta_{(v_k, k)}(a_l) T^{p_l} + \sum_{l \geq b} \mu_a v_k \Theta_{(v_k, k)}(a_l) T^{p_l + i} = \sum \mu_a a_l T^{p_l} + \sum \mu_a v_k \Theta_a a_l T^{p_l + i}.
\end{equation}

In particular

\begin{equation}
\Theta_{(v_k, k)}(a_k) T^{p_k} + \mu_a T^{p_k} + \mu_a v_k T^{p_k} = a_k + \Theta_a v_k T^{p_k} \mod \text{deg } p^k + 1.
\end{equation}

Thus

\begin{equation}
\Theta_{(v_k, k)}(a_k) + v_k = a_k + \Theta_a(v_k).
\end{equation}

Put

\begin{equation}
a_k \equiv \lambda_k v_k \mod \text{decomposables}, \lambda_k \in \mathbb{Z}_p.
\end{equation}

Then by (3.4) and (3.5) we obtain

\begin{equation}
\Theta_{a}(v_k) = (1 + p \lambda_k) v_k, \quad k \geq 1.
\end{equation}

Next, putting

\begin{equation}
v_{k}^{'} = v_k + v_{l}^{(p^k - 1)/p - 1)}
\end{equation}

for $k > 1$, by commutativity

\begin{equation}
\Theta_{(v_k, k)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k, k)}^{(p^k - 1)/p - 1)}
\end{equation}
and by the same argument as (3.5) we obtain

\[(3.8) \quad \Theta_{\Theta, k}(a_k) + h_k' = a_k + \Theta_a(h_k').\]

Applying (3.4) and (3.7) to (3.8) we obtain

\[(1 + p\lambda_k)\phi((\phi^{p^{-1}}/\phi^{-1}) = ((1 + p\lambda_1)\phi((\phi^{p^{-1}}/\phi^{-1})\]

thus

\[(3.9) \quad 1 + p\lambda_k = (1 + p\lambda_1)\phi^{p^{-1}}/\phi^{-1}.\]

Let \(\lambda\) be a \(p\)-adic unit such that

\[\lambda^{p^{-1}} = 1 + p\lambda_1.\]

Then (3.9) implies that

\[(3.10) \quad 1 + p\lambda_k = \lambda^{p^{-1}}\]

for all \(k \geq 1\). Thus, by (3.7), (3.10) and Proposition 2.1 we see that

\[\Theta_a BP^*(pt) = \Psi^\lambda BP^*(pt).\]

Then by Proposition 1.1

\[\Theta_a = \Psi^\lambda.\]

In other words every multiplicative operation which is in the center of \(\text{Aut}(BP)\)

is a suitable Adams operation. Let \(Z(\text{Aut}(BP))\) denote the center of \(\text{Aut}(BP)\).

The above result and Corollary 2.9 imply

**Theorem 3.1.** \(\text{Ad}(BP) = Z(\text{Aut}(BP)).\)

**Corollary 3.2.** \(Z(\text{Aut}(BP)) = U_1(Z_p).\)

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**References**


80 (1964), 464–484.
