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MULTIPLICATIVE OPERATIONS IN *BP* COHOMOLOGY

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Introduction. In the present work we study multiplicative operations in BP cohomology. In § 1 we show that all multiplicative operations in BP^* are automorphisms (Theorem 1.3). Thus they from the group Aut (BP). In §2 we define Adams operations in BP^* by the formal group μ_{BP} of BP cohomology and study the basic proprties of them. These oprations are primarily defined for units in $\boldsymbol{Z}_{(p)}$ and then extended to p-adic units. Thereby we discuss BP^* by extending the ground ring $\boldsymbol{Z}_{(p)}$ to the ring of p-adic integers $\boldsymbol{Z}_p$. To achieve this extension simply by tensoring with $\boldsymbol{Z}p$ we restrict our cohomologies to the category of finite CW-complexes. Correspondingly we consider all multiplicative operations in $BP^*(\ \ )\otimes\boldsymbol{Z}_p$ whenever it becomes necessary to do so. Adams operations could be defined also for non-units, but we are not interested in such a case in this paper. In §3 we prove that the center of Aut (BP) consists of all Adams operations (Theorem 3.1).

We regard the lecture note [2] as our basic reference and use the results contained there rather freely.

1. Multiplicative operations in BP^*.

Let BP^* denote the Brown-Peterson cohomology for a specified prime p. By a *multiplicative* operation in BP^* we understand a stable, linear and degree-preserving cohomology operation

$$\Theta_a\colon BP^*(\ \ )\to BP^*(\ \ ) \tag{1.1}$$

which is multiplicative and $\Theta_a(1)=1$. The set of all multiplicative operations in BP^* forms a semi-group by composition, which will be denoted by Mult (BP).

With respect to the standard complex orientation of BP^* [1], [2], [7], we denote by $e^{BP}(L)$ the Euler class of a complex line bundle L and by μ_{BP} the associated formal group. Let $\Theta_a\in$ Mult (BP). Putting

$$\Theta_a(e^{BP}(L))=\sum_{i\geqq 0}\theta_i(e^{BP}(L))^i$$

for an arbitrary line bundle L, by naturality we obtain a well-determined power series

$$\theta_a(T) = \sum_{i\geq 0} \theta_i T^i, \quad \theta_i \in BP^{2-2i}(T).$$

By naturality $\theta_0=0$ and by stability $\theta_1=1$. In particular θ_a is invertible.

Put

$$\phi_a(T) = \theta_a^{-1}(T).$$

Then

$$(1.2) \qquad \Theta_a(pt)_* \mu_{BP} = \mu_a, \quad \mu_a = \mu_{BP}{}^{\phi_a}.$$

Recall that μ_{BP} is typical. Hence μ_a is a typical formal group and ϕ_a is a typical curve over μ_{BP}.

Conversely, given a typical curve ϕ_a over μ_{BP}, by the universality of BP^*, [2], Theorem 7.2, ϕ_a determines uniquely a multiplicative operation Θ_a in BP^* satisfying

$$(1.3) \qquad \Theta_a(e^{BP}(L)) = \phi_a^{-1}(e^{BP}(L)).$$

Thus, via (1.3) multiplicative operations Θ_a in BP^* correspond bijectively with typical curves ϕ_a over μ_{BP} such that

$$(1.4) \qquad \phi_a(T) \equiv T \bmod \deg 2 \quad \textit{and} \quad \dim \phi_a^{-1}(e^{BP}(L)) = 2$$

for complex line bundles L.

Recall that a typical curve ϕ_a satisfying (1.4) can be expressed uniquely as a Cauchy series

$$(1.5) \qquad \phi_a(T) = \sum_{k\geq 0}{}^{\mu} a_k T^{p^k}, \quad a_0 = 1, \quad a_k \in BP^{2(1-p^k)}(pt),$$

where $\mu=\mu_{BP}$ (cf., [2], [3]). Thus multiplicative operations Θ_a correspond bijectively with sequences

$$(1.6) \qquad a = (a_1, a_2, \cdots, a_n, \cdots), \ a_n \in BP^{2(1-p^n)}(pt),$$

via (1.3) and (1.5). The identity operation corresponds to the zero sequence $0=(0, 0, \cdots)$.

First we remark

Proposition 1.1. *Let Θ_a and Θ_b be multiplicative operations in BP^* such that*

$$\Theta_a(pt) = \Theta_b(pt).$$

Then $a=b$ as sequences (1.6). *Hence $\Theta_a=\Theta_b$.*

Proof. By (1.2) we see that

$$\mu_a = \mu_b.$$

Then, by the uniqueness of logarithm we see that

$$\log_{\mu_a} = \log_{\mu_b} ,$$

or

$$\log_{BP} \circ \phi_a = \log_{BP} \circ \phi_b .$$

thus $\phi_a = \phi_b$. q.e.d.

Let $\Theta_a \in \mathrm{Mult}(BP)$. We have

$$\Theta_a(pt)_* \log_{BP}(T) = \log_{BP} \circ \phi_a(T)$$

over $BP^*(pt) \otimes \boldsymbol{Q}$. Putting

$$\log_{BP}(T) = \sum_{k \geqq 0} n_k T^{p^k}, \quad n_k = [CP_{p^k-1}]/p^k ,$$

expanding both sides of the above formula as power series of T and comparing coefficients of T^{p^k} we get

(1.7) $$€ \quad \sum_{j=0}^{k} n_j a_{k-j}^{p^j}, \quad k \geqq 0 .$$

This is a recursive formula to describe $\Theta_a(n_k)$, hence determines $\Theta_a(pt)$. We discuss another formula to describe $\Theta_a(pt)$.

Denote by $\boldsymbol{f}_p$ and $\boldsymbol{f}_p^a$ the Frobenius operators for the prime p on curves over μ_{BP} and μ_a respectively. Recall that, if we put

(1.8) $$(\boldsymbol{f}_p \gamma_0)(T) = \; ffv^* T^* >' \backslash \quad \mu = \mu_{BP}, \quad \gamma_0(T) = T ,$$

then $v_k \in BP^{2(1-p^k)}(pt)$ and the sequence $(v_1, v_2, \cdots, v_n, \cdots)$ forms a polynomial basis of $BP^*(pt)$, [2].

Since $\Theta_a(pt)_* \mu_{BP} = \mu_a$, we have

$$(\boldsymbol{f}_p^a \gamma_0)(T) = (\Theta_a(pt)_* \boldsymbol{f}_p \gamma_0)(T) = \sum_{k \geqq 1}{}^{\mu_a} \Theta_a(v_k) T^{p^{k-1}} .$$

Using the fact that $\phi_a \colon \mu_a \cong \mu_{BP}$, a strict isomorphism, we compute $(\phi_{a*} \boldsymbol{f}_p^a \gamma_0)(T)$ in two ways as follows:

$$\begin{aligned}
&(\phi_{a*} \boldsymbol{f}_p^a \gamma_0)(T) = (\boldsymbol{f}_p \phi_{a*} \gamma_0)(T) \\
&= (\boldsymbol{f}_p \phi_a)(T) = \sum_{k \geqq 0}{}^{\mu} \boldsymbol{f}_p(a_k T^{p^k}) \\
&= \quad (\boldsymbol{f}_p \gamma_0)(T) +^{\mu} \sum_{k \geqq 1}{}^{\mu} [p]_{BP}(a_k T^{p^{k-1}}) \\
&= \sum_{k \geqq 1}{}^{\mu} v_k T^{p^{k-1}} +^{\mu} \sum_{l \geqq 0} \sum_{k \geqq 1}{}^{\mu} w_l a_k{}^{p^l} T^{p^{k+l-1}}
\end{aligned}$$

by [2], Propositions 2.4, 2.5 and 2.9, on one hand, where

$$[p]_{BP}(T) = \sum_{l \geqq 0}{}^{\mu} w_l T^{p^l}, \quad w_0 \text{ - } p, \; w_k \in BP^{2(1-p^k)}(pt);$$

on the other hand

$$(\phi_{a\sharp}f_p^a\gamma_0)(T)=\phi_{a\sharp}\sum_{k\geqq 1}^{\mu_a}\Theta_a(v_k)T^{p^{k-1}}$$
$$=\sum_{k\geqq 1}^{\mu}\phi_a(\Theta_a(v_k)T^{p^{k-1}})=\sum_{k\geqq 1}\sum_{l\geqq 0}^{\mu}a_l\Theta_a(v_k)^{p^l}T^{p^{k+l-1}}.$$

Thus we obtain

(1.9)
$$\sum_{k\geqq 1}\sum_{l\geqq 0}^{\mu}a_l\Theta_a(v_k)^{p^l}T^{p^{k+l-1}}$$
$$=\sum_{k\geqq 1}^{\mu}v_kT^{p^{k-1}}+^{\mu}\sum_{l\geqq 0}\sum_{k\geqq 1}^{\mu}w_la_k^{p^l}T^{p^{k+l-1}}.$$

This is a recursive formula to describe $\Theta_a(v_k)$.

Let $I=\overline{BP}^*(pt)$, the kernel of the augmentation ε: $BP^*(pt)\to \boldsymbol{Z}_{(p)}$. By [2], §10, we see that

"the left hand side of (1.9)"

$$\equiv\sum_{k\geqq 1}^{\mu}\Theta_a(v_k)T^{p^{k-1}}\quad \bmod I^2$$
$$=\Theta_a(v_1)T+\Theta_a(v_2)T^p+\cdots\quad \bmod I^2,$$

and

"the right hand side of (1.9)"

$$\equiv\sum_{k\geqq 1}^{\mu}v_kT^{p^{k-1}}+^{\mu}\sum_{k\geqq 1}^{\mu}pa_kT^{p^{k-1}}\quad \bmod I^2$$
$$\equiv(v_1+pa_1)T+(v_2+pa_2)T^p+\cdots\quad \bmod I^2$$

Hence (1.9) implies

(1.10)
$$\Theta_a(v_k)=v_k+pa_k\quad \bmod I^2$$

for all $k\geqq 1$. In particular

$$\Theta_a(v_k)\equiv v_k\quad \bmod (p)+I^2$$

for $k\geqq 1$. This shows that $\{\Theta_a(v_k), k\geqq 1\}$ forms a polynomial basis of $BP^*(pt)$. Thus we obtain

Proposition 1.2. *For any* $\Theta_a\in$ Mult *(BP)*

$$\Theta_a(pt): BP^*(pt)\cong BP^*(pt), \textit{ an isomorphism.}$$

Let Θ_a and Θ_b be two multiplicative operations in BP^* with corresponding sequences $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$. Putting

$$\Theta_c=\Theta_a\circ\Theta_b,\quad c=(c_1, c_2, \cdots),$$

we shall discuss the sequence c. Put

$$\tilde{\phi}_b(T)=\Theta_a(pt)_*\phi_b(T)=\sum_{k\geqq 0}^{\mu_a}\Theta_a(b_k)T^{p^k}.$$

Then

$$\begin{aligned}\Theta_a(pt)_*\mu_b &= \Theta_a(pt)_*(\phi_b^{-1}\circ\mu\circ\phi_b\times\phi_b)\\ &= \tilde{\phi}_b^{-1}\circ\mu_a\circ\tilde{\phi}_b\times\tilde{\phi}_b = \mu_{BP}^{\phi_a\circ\tilde{\varphi}_b}\,.\end{aligned}$$

On the other hand

$$\Theta_a(pt)_*\mu_b = \Theta_a(pt)_*\Theta_b(pt)_*\mu_{BP} = \Theta_c(pt)_*\mu_{BP} = \mu_c\,.$$

Thus, likewise in the proof of Proposition 1.1, we have

$$\phi_c = \phi_a\circ\tilde{\phi}_b\,, \tag{1.11}$$

or equivalently

$$\begin{aligned}\sum_{k\geqq 0}^{\mu} c_k T^{p^k} &= \phi_{a*}\Big(\sum_{k\geqq 0}^{\mu}\Theta_a(b_k)T^{p^k}\Big)\\ &= \sum_{k\geqq 0}\sum_{l\geqq 0}^{\mu} a_l\Theta_a(b_k)^{p^l}T^{p^{k+l}}\,.\end{aligned} \tag{1.12}$$

This is a recursive formula to describe c_k.

A multiplicative operation Θ_a in BP^* is called an ***automorphism*** of BP^* if

$$\Theta_a(X,\ A)\colon BP^*(X,A)\cong BP^*(X,A),\quad \text{isomorphic}$$

for all finite CW-pair (X,A). Clearly a multiplicative operation Θ_a is an automorphism of BP^* iff it has an inverse. The set of all atutomorphisms of BP^* forms a group, which will be denoted by $\mathrm{Aut}(BP)$.

Theorem 1.3. $\mathrm{Aut}(BP)=\mathrm{Mult}(BP)$.

Proof. It is sufficient to prove that every multiplicative operation Θ_a has a right inverse.

Let $t=(r_1,t_2,\cdots)$ and $s=(s_1,s_2,\cdots)$ be sequences of indeterminates with $\dim t_k=\dim s_k=2(1-p^k)$. Put

$$\sum_{k\geqq 0}^{\mu} u_k T^{p^k} = \sum_{k\geqq 0}\sum_{l\geqq 0}^{\mu} t_l s_k^{p^l} T^{p^{k+l}}\,, \tag{*1}$$

where $s_0=t_0=u_0=1$. Then over $BP^*(pt)[t,s]$ we have

$$\sum_{k\geqq 0}^{\mu} u_k T^{p^k} \equiv T+u_1T^p+u_2T^{p^2}+\cdots \mod \tilde{I}^2\,,$$

and

$$\sum_{k\geqq 0}\sum_{l\geqq 0}^{\mu} t_l s_k^{p^l} T^{p^{k+l}} \equiv T+(s_1+t_1)T^p+(s_2+t_2)T^{p^2}+\cdots \quad \mod \tilde{I}^2\,,$$

where $\tilde{I}=(s,t)$, the ideal of $BP^*(pt)\ [s,\ t]$ generated by $s_1, s_2, \cdots, t_1, t_2, \cdots$. Thus we can put

$$u_k = t_k+s_k+P_k(t_1,\cdots,t_{k-1},s_1,\cdots,s_{k-1}),\quad k\geqq 1\,. \tag{*2}$$

Here P_k is a polynomial of $t_1,\ \cdots,\ t_{k-1}, s_1,\ \cdots, s_{k-1}$ with dim $P_k=2(1-p^k)$ and

$P_k \equiv 0 \bmod \hat{I}^2$.

We want to find a right inverse of Θ_a. Putting

$$(*3) \qquad \Theta_a \circ \Theta_b = id$$

with undecided sequence $b=(b_1, b_2, \cdots)$, we shall decide the sequence b. By (1.12), (*1) and (*2), we get

$$(*4) \qquad a_k + \Theta_a(b_k) + P_k(a_1, \cdots, a_{k-1}, \Theta_a(b_1), \cdots, \Theta_a(b_{k-1})) = 0$$

for all $k \geqq 1$. Since the coefficients of P_k depend neither on $(a_1, a_2, \cdots)$ nor on $(\Theta_a(b_1), \Theta_a(b_2), \cdots)$ we may use (*4) as a recursive formula to obtain $\Theta_a(b_k)$, so we get $\Theta_a(b_k)$ as polynomials of $a_1, \cdots, a_k$ successively for $k \geqq 1$. By Proposition 1.2 $\Theta_a(pt)$ is an isomorphism. Thus we get a sequence $(b_1, b_2, \cdots)$ so that it satisfies (*4). Thereby Θ_b is obtained to satisfy (*3). q.e.d.

2. Adams operations in BP^*.

Let $\boldsymbol{Z}_{(p)}$ be the ring of integers localized at the prime p and $\boldsymbol{Z}_p$ its completion, i.e., the ring of p-adic integers. As is well known the endomorphism

$$[\alpha]_{BP} \in \operatorname{End}(\mu_{BP})$$

is defined for each $\alpha \in \boldsymbol{Z}_{(p)}$ so that

$$[\alpha]_{BP}(T) = \alpha T + \text{ higher terms.}$$

It is convenient for us to extend these endomorphisms $[\alpha]_{BP}$ to $\alpha \in \boldsymbol{Z}_p$. For this purpose we extend the groud ring $\boldsymbol{Z}_{(p)}$ of BP^* to $\boldsymbol{Z}_p$ by tensoring, i.e., we consider $BP^*(\ \ ) \otimes \boldsymbol{Z}_p$ whenever it is necessary to takl of p-adic integers.

Let $A = BP^*(pt) \otimes \boldsymbol{Z}_p$. Let F and G be formal groups over A. Let

$$c \colon \operatorname{Hom}_A(F, G) \to A$$

be the homomorphism sending f to a_1 when $f(T) = a_1 T +$ higher terms. Since A is an integral domain of characteristic zero, c is injective as is well known (cf., [4], [5]).

Since A is a direct sum of copies of Zp (corresponding to each monomials of v_k's) we give a direct limit topology to A. (Each direct summand is given the topology of Zp). Then, using the argument of Lubin [5], Lemma 2.1,1, we see that c is an isomorphism onto a closed subgroup of A.

In case $F = G = \mu_{BP}$,

$$\operatorname{Im} c \supset \boldsymbol{Z}_{(p)},$$

because $c([\alpha]_{BP}) = \alpha$ for $\alpha \in \boldsymbol{Z}_{(p)}$. Hence

$$\operatorname{Im} c \supset \bar{\boldsymbol{Z}}_{(p)} = Zp.$$

Since c is injective, for each $\alpha \in \boldsymbol{Z}_p$ there exists a unique

$$[\alpha]_{BP} \in \mathrm{End}_A(\mu_{BP})$$

such that $c([\alpha]_{BP})=\alpha$. Thus the definition of $[\alpha]_{BP}$ is extended to Zp.

Since c: $\mathrm{End}_A(\mu_{BP}) \to A$ is a ring homomorphism, for any p-adic integers a and β we have the following relations:

$$[\alpha]_{BP}(T) = \alpha T + \text{higher terms}, \tag{2.1}$$

$$[\alpha]_{BP} +^{\mu} [\beta]_{BP} = [\alpha+\beta]_{BP}, \quad \mu = \mu_{BP}, \tag{2.2}$$

$$[\alpha]_{BP} \circ [\beta]_{BP} = [\alpha\beta]_{BP}. \tag{2.3}$$

Let $\alpha \in \boldsymbol{Z}_{(p)}$ (or $\in \boldsymbol{Z}_p$) be a unit. Put

$$\psi_\alpha(T) = [\alpha^{-1}]_{BP}(\alpha T).$$

Since

$$(\boldsymbol{f}_q \psi_\alpha)(T) = \boldsymbol{f}_q([\alpha^{-1}]_{BP}(\alpha T)) = [\alpha^{-1}]_{BP}([\alpha^q]\boldsymbol{f}_q \gamma_0(T)) = 0$$

for every $q>1$ such that $(p, q)=1$ by [2], Propositions 2.3 and 2.9, where $\gamma_0(T) = T$, we see that ψ_α is a typical curve over μ_{BP}. Moreover ψ_α satisfies (1.4) as is easily seen. Thus there corresponds a multiplicative operation in BP^* to ψ_α. We denote this multiplicative operation by Ψ^α and call *Adams operations* in BP^*.

REMARK 1. Even for non-units a Adams operations can be defined on the same way as above. But these operations are defined in $BP^*(\quad) \otimes \boldsymbol{Q}$ or $BP^*(\quad) \otimes \boldsymbol{Q}_p$ And these cohomology theories are essentially ordinary cohomologies (corresponding to generalized Eilenberg-MacLane spectra), so we are not interested in these operations in the present work.

REMARK 2. Adams operations in complex cobordism are defined by Novikov [6]. When we regard BP^* as a direct summand of $U^*(\quad)_{(p)}$, our Adams operations will be the restrictions of Novikov's Adams operations to BP^*.

Let a be a unit of $\boldsymbol{Z}_{(p)}$ (or of Zp). Since

$$\Psi_\alpha(\alpha^{-1}[\alpha]_{BP}(T)) = [\alpha^{-1}]_{BP} \circ [\alpha]_{BP}(T) = T,$$

we see that

$$\Psi^\alpha(e^{BP}(L)) = \alpha^{-1}[\alpha]_{BP}(e^{BP}(L)) \tag{2.4}$$

for any complex line bundle L.

Since $\Psi^\alpha(pt)_* \mu_{BP} = \mu_{BP}{}^{\psi_\alpha}$, we see that

$$\Psi^{\alpha}(pt)_{*}\log_{BP}=\log_{BP}\circ\psi_{\alpha}\,.$$

Here

$$(\log_{BP}\circ\psi_{\alpha})(T)=\log_{BP}[\alpha^{-1}]_{BP}(\alpha T)$$
$$=\alpha^{-1}\cdot\log_{BP}(\alpha T)=\sum_{k\geqq 0}\alpha^{p^k-1}n_kT^{p^k}\,.$$

Thus $$\sum_{\text{fe≡gO}}\Psi\ (n_k)T^{p^k}=\sum_{\text{fc;>0}}\alpha^{p^k-1}n_kT^{p^k}\,,$$

or

$$\Psi\ \text{fo})=\alpha^{p^k-1}n_k,\quad k\geqq 1\,, \tag{2.5}$$

after extending $\Psi^{\alpha}(pt)$ to $\Psi^{\alpha}(pt)\otimes 1_{Q}$.

Proposition 2.1. $\Psi^{\alpha}(pt)|BP^{-2s}(pt)=\alpha^{s}id.$

Proof. $(n_1, n_2, \cdots)$ is a polynomial basis of $BP^*(pt)\otimes\boldsymbol{Q}$. Since Ψ^{α} is linear and multiplicative, for every polynomials x_s of n_k's with dim $x_s=-2s$ by (2.5) we see easily that

$$\Psi^{\alpha}(x_s)=\alpha^{s}x_s\,.\qquad \text{q.e.d.}$$

Corollary 2.2. *If we put*

$$\mu_{BP}(X,\ Y)=\sum_{i,j}a_{ij}X^iY^j\,,$$

them

$$\mu_{BP}{}^{\psi_{\alpha}}(X,\ Y)=\sum_{i,j}\alpha^{i+j-1}a_{ij}X^iY^j\,.$$

Next we prove

Proposition 2.3. $\Psi^{\alpha}\Psi^{\beta}=\Psi^{\alpha\beta}=\Psi^{\beta}\Psi^{\alpha}$.

Proof. Put

$$[\alpha]_{BP}(T)=\sum_{s\geqq 0}\alpha_sT^{(p-1)s+1},\ \alpha_s\in BP^{-2(p-1)s}(pt)\,.$$

For any complex line bundle L we have

$$\begin{aligned}
&\Psi^{\beta}(\Psi^{\alpha}(e^{BP}(L))=\Psi^{\beta}(\alpha^{-1}[\alpha]_{BP}(e^{BP}(L)))\\
&=\alpha^{-1}\cdot\Psi^{\beta}(\sum_{s\geqq 0}\alpha_s(e^{BP}(L))^{(p-1)s+1})\\
&=\alpha^{-1}\sum_{s\geqq 0}\beta^{(p-1)s}\alpha_s(\Psi^{\beta}(e^{BP}(L)))^{(p-1)s+1}\qquad \text{by Proposition 2.1}\\
&=\alpha^{-1}\beta^{-1}\sum_{s\geqq 0}\alpha_s(\beta\Psi^{\beta}(e^{BP}(L)))^{(p-1)s+1}\\
&=\alpha^{-1}\beta^{-1}\cdot[\alpha]_{BP}([\beta]_{BP}(e^{BP}(L)))\qquad \text{by (2.4)}\\
&=(\alpha\beta)^{-1}[\alpha\beta]_{BP}(e^{BP}(L))\qquad \text{by (2.3)}\\
&=\Psi^{\alpha\beta}(e^{BP}(L))\,.
\end{aligned}$$

Therefore, by the universality of BP^*, [2], Theorem 7.2, we concludes the Proposition.

Let α and β be p-adic units. By Propositions 1.1 and 2.1 we see that

$$\text{(2.6)} \qquad \Psi^{\alpha} = \Psi^{\beta} \quad \text{iff} \quad \alpha^{p-1} = \beta^{p-1} .$$

Let $\boldsymbol{U}(\boldsymbol{Z}_p)$ be the multiplicative group of p-adic units and $\boldsymbol{U}_1(\boldsymbol{Z}_p)$ be its subgroup consisting of p-adic integers α such that

$$\alpha \equiv 1 \mod p .$$

As is well known

$$\boldsymbol{U}_1(\boldsymbol{Z}_p) = \{\alpha^{p-1};\ \alpha \in \boldsymbol{U}(\boldsymbol{Z}_p)\} .$$

By Proposition 2.3 all Adams operations (for p-adic units) form a multiplicative subgroup of $\mathrm{Aut}(BP)$. We denote this subgroup by $\mathrm{Ad}(BP)$. Then, (2.6) implies that

Proposition 2.4. $\mathrm{Ad}(BP) \cong \boldsymbol{U}_1(\boldsymbol{Z}_p)$.

And also

Proposition 2.5. $\Psi^{\lambda} = 1$ *iff* $\lambda^{p-1} = 1$.

Next we discuss the relations of Adams operations with Quillen operations (of Landweber-Novikov type). We recall the definition of Quillen operations, [2], [7]. Let $t = (t_1, t_2, \cdots)$ be a sequence of indeterminates such that $\dim t_k = 2(1-p^k)$ and

$$\phi_t(T) = \sum_{k \geq 0}^{\mu} t_k T^{p^k}, \quad t_0 = 1 ,$$

a typical curve over μ_{BP} by extending the ground ring of μ_{BP} to $BP^*(pt)[t]$. Then

$$r_t \colon BP^*(\ \ ) \to BP^*(\ \ )\,[t]$$

is the multiplicative operation such that

$$r_t(e^{BP}(L)) = \phi_t^{-1}(e^{BP}(L))$$

for any complex line bundle L. Putting

$$r_t(x) = \sum_E r_E(x) t^E, \quad x \in BP^*(X, A) ,$$

where $E = (e_1, e_2, \cdots)$ runs over all sequences of non-negative integers such that all e_k but a finite are zero, we get linear stable operations

$$r_E \colon BP^*(\ \ ) \to BP^{*+2|E|}(\ \ )$$

of degree $2|E|$, where $|E|=\sum_i e_i(p^i-1)$.

Now for a p-adic unit α we have

$$(2.7)\qquad \begin{aligned} r_t\circ\Psi\ (e^{BP}(L)) &= r_t(\psi_\alpha^{-1}(e^{BP}(L))) \\ &= (r_t(pt)_*\psi_\alpha)^{-1}(r_t(e^{BP}(L))) \\ &= (\phi_t\circ r_t(pt)_*\psi_\alpha)^{-1}(e^{BP}(L)) . \end{aligned}$$

And

$$(r_t(pt)_*\psi_\alpha)\,(T) = r_t(pt)_*([\alpha^{-1}]_{BP}(\alpha T)) = [\alpha^{-1}]_{\mu'}(\alpha T)\,,$$

where $\mu'=\mu_{BP}{}^{\phi_t}$ Thus

$$(2.8)\qquad \begin{aligned} (\phi_t\circ r_t(pt)_*\psi_\alpha)(T) &= \phi_{t\sharp}([\alpha^{-1}]_{\mu'}(\alpha T)) \\ &= [\alpha^{-1}]_{BP}(\phi_t(\alpha T)) = [\alpha^{-1}]_{BP}(\textstyle\sum_{k\geqq 0}^{\mu}\alpha^{p^k}t_kT^{p^k}) . \end{aligned}$$

Let

$$\sigma_\alpha\colon \boldsymbol{Z}_{(p)}[t] \to \boldsymbol{Z}_{(p)}[t]$$

be an algebra homomorphism such that

$$\sigma_\alpha(t_k) = \alpha^{p^k-1}t_k,\quad k\geqq 1\,,$$

and define an operation

$$\bar{\Psi}^\alpha\colon BP^*(\quad)\,[t] \to BP^*(\quad)\,[t]$$

by $\bar{\Psi}^\alpha=\Psi^\alpha\otimes\sigma_\alpha$. Then

$$(2.9)\qquad \begin{aligned} (\bar{\Psi}^\alpha\circ r_t)\,(e^{BP}(L)) &= \bar{\Psi}^\alpha(\phi_t^{-1}(e^{BP}(L)) \\ &= (\bar{\Psi}^\alpha(pt)_*\phi_t)^{-1}(\bar{\Psi}^\alpha(e^{BP}(L))) \\ &= (\Psi_\alpha\circ\bar{\Psi}^\alpha(pt)_*\phi_t)^{-1}(e^{BP}(L)) . \end{aligned}$$

Remark that

$$\bar{\Psi}^\alpha(pt)_*\mu_{BP} = \mu_{BP}{}^{\psi_\alpha} .$$

Thus

$$(\bar{\Psi}^\alpha(pt)_*\phi_t)\,(T) = \sum_{k\geqq 0}^{\mu''}\alpha^{p^k-1}t_kT^{p^k}\,,$$

where $\mu''=\mu_{BP}{}^{\psi_\alpha}$. And

$$(2.10)\qquad \begin{aligned} (\psi_\alpha\circ\bar{\Psi}^\alpha(pt))_*\phi_t)\,(T) &= \psi_{\alpha\sharp}(\textstyle\sum_{k\geqq 0}^{\mu''}\alpha^{p^k-1}t_kT^{p^k}) \\ &= \textstyle\sum_{k\geqq 0}^{\mu}\psi_\alpha(\alpha^{p^k-1}t_kT^{p^k}) \\ &= \textstyle\sum_{k\geqq 0}^{\mu}[\alpha^{-1}]_{BP}(\alpha^{p^k}t_kT^{p^k}) \\ &= [\alpha^{-1}]_{BP}(\textstyle\sum_{k\geqq 0}^{\mu}\alpha^{p^k}t_kT^{p^k}) . \end{aligned}$$

Thus by (2.8) and (2.10) we see that

$$\phi_t \circ r_t(pt)_* \psi_\alpha = \psi_\alpha \circ \bar{\Psi}^\alpha(pt)_* \phi_t ,$$

then, by (2.7) (2.9) and the universality of BP^* we obtain

Proposition 2.6. *For any unit of $\boldsymbol{Z}_p$ there holds the commutativity*

$$r_t \circ \Psi^\alpha = \bar{\Psi}^\alpha \circ r_t .$$

Corollary 2.7. *Let $E=(e_1, e_2, \cdots)$ be a sequence of non negative integers of which all but a finite terms are zero. There holds the commutativity*

$$r_E \circ \Psi^\alpha = \alpha^{|E|} \Psi^\alpha \circ r_E .$$

Corollary 2.8. *For any linear stable cohomology operation*

$$\Xi_s \colon BP^*(\ \ ) \to BP^{*+2s}(\ \ )$$

of degree 2s there holds the commutativity

$$\Xi_s \circ \Psi^\alpha = \alpha^s \Psi^\alpha \circ \Xi_s .$$

Remark that every stable cohomology operation in BP^* can be expressed as linear combinations of Quillen operations r_E over $BP^*(pt)$. Then Corollary 2.8 follows from Proposition 2.1 and Corollary 2.7.

Corollary 2.9. *Adams operations in BP^* commute with all multiplicative operations.*

REMARK. Properties of Adams operations in complex cobordism which correspond to Propositions 2.1, 2.2, 2.3, 2.7 and 2.8 are obtained in Novikov [7] by different arguments.

3. The center of Aut(BP).

For any $b \in BP^{2(1-p^k)}(pt)$ we define a sequence

$$(b, k) = (0, \cdots, 0, b, 0, \cdots)$$

with b as the k-th term and with all other terms zero. By (1.9) we obtain

$$\begin{aligned}&\sum_{l\geqq 1}{}^{\mu}\Theta_{(b,k)}(v_l)T^{p^{l-1}} +^{\mu}\sum_{l\geqq 1}{}^{\mu} b\cdot\Theta_{(b,k)}(v_l)^{p^k} T^{p^{k+l-1}} \\ &= \sum_{l\geqq 1}{}^{\mu} v_l T^{p^{l-1}} +^{\mu}\sum_{l\geqq 0}{}^{\mu} w_l b^{p^l} T^{p^{k+l-1}} .\end{aligned}$$

In particular

$$\sum_{l=1}^{k}{}^{\mu}\Theta_{(b,k)}(v_l)T^{p^{l-1}} \equiv \sum_{l=1}^{k}{}^{\mu} v_l T^{p^{l-1}} +^{\mu} pbT^{p^{k-1}} \bmod \deg p^{k-1}+1 .$$

Recursively on l, $1 \leqq l < k$, and deleting the same terms successively we see that

$$\Theta_{(b,k)}(v_l) = v_l, \quad 1 \leqq l < k, \tag{3.1}$$

and

$$\Theta_{(b,k)}(v_k) = v_k + pb. \tag{3.2}$$

These imply that

$$\Theta_{(b,k)}(x) = x \qquad \text{for any } x \in BP^{-2s}(pt), s < p^k - 1, \tag{3.3}$$

and

$$\Theta_{(b,k)}(y) = y + pcb \qquad \text{for } y \in BP^{2(1-p^k)}(pt) \tag{3.4}$$

when $y = cv_k$ mod decomposables, $c \in \boldsymbol{Z}_p$.

Let Θ_a be in the center of $\mathrm{Aut}(BP)$. Then

$$\Theta_{(v_k,k)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k,k)}$$

for all $k \geqq 1$. And by (1.12) we have

$$\begin{aligned} &\sum_{l \geqq 0}{}^{\mu} \Theta_{(v_k,k)}(a_l) T^{p^l} +^{\mu} \sum_{l \geqq 0}{}^{\mu} v_k \cdot \Theta_{(v_k,k)}(a_l)^{p^k} T^{p^{k+l}} \\ = &\sum_{l \geqq 0}{}^{\mu} a_l T^{p^l} +^{\mu} \sum_{l \geqq 0}{}^{\mu} a_l \cdot \Theta_a(v_k)^{p^l} T^{p^{k+l}}. \end{aligned}$$

In particular

$$\begin{aligned} &\Theta_{(v_k,k)}(a_k) T^{p^k} +^{\mu} v_k T^{p^k} \\ \equiv &a_k T^{p^k} +^{\mu} \Theta_a(v_k) T^{p^k} \qquad \text{mod deg } p^k + 1. \end{aligned}$$

Thus

$$\Theta_{(v_k,k)}(a_k) + v_k = a_k + \Theta_a(v_k). \tag{3.5}$$

Put

$$a_k \equiv \lambda_k v_k \qquad \text{mod decomposables, } \lambda_k \in \boldsymbol{Z}_p. \tag{3.6}$$

Then by (3.4) and (3.5) we obtain

$$\Theta_a(v_k) = (1 + p\lambda_k) v_k, \quad k \geqq 1. \tag{3.7}$$

Next, putting

$$v_k' = v_k + v_1^{(p^k-1)/(p-1)}$$

for $k > 1$, by commutativity

$$\Theta_{(v_k',k)} \circ \Theta_a = \Theta_a \circ \Theta_{(v_k',k)}$$

and by the same argument as (3.5) we obtain

$$\Theta_{(v_k',k)}(a_k)+v_k' = a_k+\Theta_a(v_k') . \tag{3.8}$$

Applying (3.4) and (3.7) to (3.8) we obtain

$$(1+p\lambda_k)v_1^{(p^k-1)/(p-1)} = ((1+p\lambda_1)v_1)^{(p^k-1)/(p-1)} .$$

thus

$$1+p\lambda_k = (1+p\lambda_1)^{(p^k-1)/(p-1)} . \tag{3.9}$$

Let λ be a p-adic unit such that

$$\lambda^{p-1} = 1+p\lambda_1 .$$

Then (3.9) implies that

$$1+p\lambda_k = \lambda^{p^k-1} \tag{3.10}$$

for all $k \geqq 1$. Thus, by (3.7), (3.10) and Proposition 2.1 we see that

$$\Theta_a \mid BP^*(pt) = \Psi^\lambda \mid BP^*(pt).$$

Then by Proposition 1.1

$$\Theta_a = \Psi^\lambda .$$

In other words every multiplicative operation which is in the center of Aut(BP) is a suitable Adams operation. Let Z(Aut(BP)) denote the center of Aut(BP). The above result and Corollary 2.9 imply

Theorem 3.1. Ad(BP)$=Z$(Aut(BP)).

Corollary 3.2. Z(Aut(BP))$\cong U_1(\boldsymbol{Z}_p)$.

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