Asashiba, H.
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ON ALGEBRAS OF SECOND LOCAL TYPE, III

Hideto ASASHIBA

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Throughout this paper, A denotes a (left and right) artinian ring with identity 1, J the Jacobson radical of A, and all modules are assumed to be (unitary) finitely generated one-sided A-modules unless otherwise stated. Further k denotes an algebraically closed field and by the word algebra we mean a finite dimensional algebra over k.

As defined in [I], we say that A is a ring of *right 2nd local type* in case the 2nd top $\mathrm{top}^2 M := M/MJ^2$ of every indecomposable right A-module M is indecomposable. In [II, sections 3 and 4], we have determined all possible structures of indecomposable projective right and left modules over an algebra of right 2nd local type; and proved that conversely this classification condition on indecomposable projectives is sufficient for left serial algebras to be of right 2nd local type. Here we will generalize these results. Namely in section 1, corresponding to the first result above, we give a complete classification of local left and right modules over an arbitrary artinian ring of right 2nd local type (Theorem I). And in section 2, we prove the same statement as the second result above is still true for left serial rings with selfduality (Theorem II). In the last section, as examples of algebras of right 2nd local type, we give all the sincere simply connected (representation-finite) algebras of right 2nd local type using the method outlined in Ringel [10], and then characterize such algebras in terms of their ordinary quivers which include interpretations of lemmas used in the proof of the classification theorem above (Theorem III). This characterization leads us to necessary and sufficient conditions for representation-finite algebras to be of right 2nd local type (Theorem IV), by making use of the covering technique developed in [5], [7] and [6].

Since the property to be of right 2nd local type is Morita invariant, we may and will assume throughout that A is a basic ring. We keep the notation and terminology used in [I] or [II].

1. Classification of local modules

We prove the following theorem.

Theorem I. *Assume that A is a ring of right 2nd local type. Then*
(R) *Every local right A-module P has one of the following structures:*
(R_1) *P is uniserial.*
(R_2) *P is of height 2 with the socle of length >2.*
(R_3) *P is non-uniserial of height 3 and colocal.*
(R_4) *PJ is a direct sum of a simple right module and a non-uniseri colocal right module of height 2.*
(R_5) *PJ is a direct sum of two non-zero uniserial right modules.*
(L) *Every local left A-module P has one of the following structures:*
(L_1) *P is uniserial.*
(L_2) *P is non-uniserial of height 2.*
(L_3) *P is non-uniserial of height 3 and colocal.*
(L_4) *P is non-uniserial of height 4 with J^2P a uniserial waist in P.*
If, in addition, A is a ring with selfduality, the case of (L_4) does not occur.

At present we have no example of a ring of right 2nd local type having a local left module of type (L_4). To prove the theorem we assume up to the end of this section that A is a ring of right 2nd local type. We quote the following as a special case of [II, Theorem 2.5 (1R)$\Rightarrow$(2R)].

Lemma 1.0. *Let L_1 and L_2 be local right modules and S_i a simple submodule of L_i for each i such that $S_1 \leq L_1J$ and $S_2 \leq L_2J^2$. Then for every isomorphism $\alpha: S_1 \to S_2$, α is extendable to a homomorphism $L_1 \to L_2$ or α^{-1} is extendable to a homomorphism $L_2 \to L_1$.*

Lemma 1.1. *Let P_A be a local module of height 3, L_A a local module of height 2 and S_A a simple submodule of LJ. Then every homomorphism $\alpha: S \to PJ^2$ is extendable to a homomorphism $L \to P$.*

Proof. We may assume that α is not zero, whence is a monomorphism. By Lemma 1.0, we have only to show that $\alpha^{-1}: \operatorname{Im}\alpha \to S$ is not extendable to any homomorphism $\varphi: P \to L$. But otherwise, $3 = \mathrm{h}(\operatorname{Coim}\varphi) = \mathrm{h}(\operatorname{Im}\varphi) \leq \mathrm{h}(L) = 2$, a contradiction.

Lemma 1.2. *Let P_A be a local module of height 3 and L_A a local submodule of PJ and put $\operatorname{soc} L = S_1 \oplus \cdots \oplus S_n$ with each S_i simple. Then there is a uniserial module L_i with $\operatorname{soc} L_i = S_i$ for each $i \in \{1, \cdots, n\}$ such that $L + L_1 + \cdots + L_{n-1} = L_1 \oplus L_2 \oplus \cdots L_{n-1} \oplus L_n$. In particular, L is contained in a sum of uniserial modules.*

Proof. By induction on n. If $n=1$, then the assertion holds for $L_n := L$. So assume $n>1$. Then $\mathrm{h}(L)=2$. Put $M := L/(S_2 \oplus \cdots \oplus S_n)$ and $S := \operatorname{soc} M$ ($\cong S_1$). Then by Lemma 1.1, the canonical injection $\alpha: S \to S_1 \leq PJ^2$ is extendable to a homomorphism $\varphi: M \to P$. Put $L_1 := \operatorname{Im}\varphi$. Then L_1 is uniserial of length 2 and $\operatorname{soc} L_1 = S_1$. Since $\varphi\pi: L \to L_1$ is an extension of the identity map of $S_1 = L \cap L_1$ (where $\pi: L \to M$ is the canonical projection), we have $L + L_1 = L_1 \oplus L'$ where L' is the image of the map $1_L - \varphi\pi: L \to L + L_1$ (whence L' is a local submodule of PJ). In addition, as easily verified we have $\operatorname{soc} L' = S_2 \oplus \cdots \oplus S_n$. Applying the induction hypothesis to L', we have uniserial modules $L_2, \cdots, L_n$ such that $\operatorname{soc} L_i = S_i$ for all $i \in \{2, \cdots, n\}$ and $L' + L_2 + \cdots + L_{n-1} = L_2 \oplus \cdots \oplus L_{n-1} \oplus L_n$. Hence $L + L_1 + \cdots + L_{n-1} = L_1 + L' + L_2 + \cdots + L_{n-1} = L_1 + (L_2 \oplus \cdots \oplus L_{n-1} \oplus L_n)$ and the last sum is direct since $\operatorname{soc} L_i = S_i$ for all $i \in \{1, \cdots, n\}$. //

Proposition 1.3. *If P_A is a local module of height* 3, *then PJ is expressed as an irredundant sum of uniserial modules.*

Proof. Clear from Lemma 1.2. //

Lemma 1.4. *Let L be a right module. Then the following are equivalent*:
(1) *L is uniserial.*
(2) $\operatorname{top}^2 L$ *is uniserial.*
(3) $\operatorname{soc}^2 L$ *is uniserial.*

Proof. The implications (1)$\Rightarrow$(2) and (1)$\Rightarrow$(3) are clear.
(2)$\Rightarrow$(1). Assume that $\operatorname{top}^2 L$ is uniserial but L is not. Let t be the smallest natural number such that $|LJ^t/LJ^{t+1}| > 1$. Then by assumption $t > 1$. Put $P := LJ^{t-2}/LJ^{t+1}$. Then P is a local module of height 3 and PJ is also local. Hence PJ is uniserial by Proposition 1.3. This implies $1 = |PJ^2| = |LJ^t/LJ^{t+1}| > 1$, a contradiction.
(3)$\Rightarrow$(1). Assume that $\operatorname{soc}^2 L$ is uniserial but L is not. Let t be the smallest natural number such that $|\operatorname{soc}^{t+1} L/\operatorname{soc}^t L| > 1$. Then by assumption $t > 1$. Put $P := \operatorname{soc}^{t+1} L/\operatorname{soc}^{t-2} L$. Then P is of height 3 and $\operatorname{soc}^2 P = \operatorname{soc}^t L/\operatorname{soc}^{t-2} L$ is uniserial. Again by assumption $|\operatorname{top} P| > 1$. Hence P is expressed as an irredundant sum of at least two local modules L_i. Put $M := L_1 + L_2$. Then $\operatorname{soc}^2 M = \operatorname{soc}^2 P$ and $L_1 J = \operatorname{soc}^2 M = L_2 J$. Hence, in particular, L_1 and L_2 are uniserial of length 3. By Lemma 1.1 $1_{\operatorname{soc} M}$ is extendable to a homomorphism $\varphi: L_1 \to L_2$. Putting $L' := (1_{L_1} - \varphi)(L_1)$ $(\neq 0)$, we have $M = L' + L_2$ and $|L' \cap L_2| < |L_1 \cap L_2| = |\operatorname{soc}^2 M| = 2$. Since M is indecomposable, $L' \cap L_2 \neq 0$ whence $L' \cap L_2 = \operatorname{soc} M$. Then $1 = |\operatorname{soc}^2 M/\operatorname{soc} M| = |\operatorname{soc}(M/\operatorname{soc} M)| = |\operatorname{soc}((L' + L_2)/(L' \cap L_2))| = 2$, a contradiction. //

Lemma 1.5 ([2]). *Let L be a non-zero proper submodule of a right module M. Then L is a waist in M iff L/LJ is a waist in M/LJ.* //

Corollary 1.6. *Let L be a right module of height* > 2 *such that* $\operatorname{top}^2 L$ *is colocal. Then L is uniserial.*

Proof. Since $\mathrm{top}^2 L=L/LJ^2$ is colocal, we have $LJ/LJ^2=\mathrm{soc}(L/LJ^2)$ is a waist in L/LJ^2. Hence by Lemma 1.5, LJ is a waist in L. We first show that LJ is uniserial. Let x be any element in $L\backslash LJ$. Then $xe\notin LJ$ for some $e\in\mathrm{pi}(A)$. Putting $N:=xeA$, N is a local submodule of L with $N\nleq LJ$. By the above, $LJ<N$. Then since $NJ\leq LJ<N$ and N is local, we have $NJ=LJ$ and $NJ/NJ^2=LJ/LJ^2$ is simple. Thus $\mathrm{top}^2 N$ is uniserial. Hence by Lemma 1.4, N is uniserial. So LJ is uniserial since $LJ<N$. Now put $h:=\mathrm{h}(L)$. Since $LJ=\mathrm{soc}^{h-1}L$ is uniserial and $h-1>1$, $\mathrm{soc}^2 L$ is uniserial, whence so is L by Lemma 1.4. //

Lemma 1.7. *Let L be a local right module. If LJ^2 is simple, then* $|\mathrm{soc}\,L|\leq 2$; *more precisely L is colocal or LJ is a direct sum of a simple right module and a colocal right module of height* 2.

Proof. Since $LJ^3=0$ and A/J^3 is also a ring of right 2nd local type, we may assume that $J^3=0$. Consider the following exact sequence

$$(1)\qquad 0\to I\hookrightarrow eA\to L\to 0$$

where $e\in\mathrm{pi}(A)$ and eA is the projective cover of L. Put $T:=eJ^2\cap I$, $M:=eA/T$ and $S:=MJ^2$. Then M is local and $S=eJ^2/T\cong LJ^2$ is simple.

We first show that $|\mathrm{soc}\,M|\leq 2$. Assume $|\mathrm{soc}\,M|\geq 3$. Then there are simple right modules S_1 and S_2 such that $S_1\oplus S_2\oplus S\leq\mathrm{soc}\,M$. Define an isomorphism α by the following commutative diagram

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d, "\wr"'] & S \arrow[d, "\wr"] \\ (S\oplus S_1)/S_1 \arrow[r, dashed, "\alpha"'] & (S\oplus S_2)/S_2 \end{tikzcd}$$

where the vertical maps are the canonical isomorphisms. Then it follows from $(S\oplus S_i)/S_i=(M/S_i)J^2$ for each i that α is extendable to a homomorphism $\beta\colon M/S_1\to M/S_2$ or α^{-1} is extendable to a homomorphism $\beta\colon M/S_2\to M/S_1$. In either case β is an isomorphism. In fact, $(M/S_i)J^2\nleq\mathrm{Ker}\,\beta$ implies $\mathrm{h}(\mathrm{Im}\,\beta)=3$, thus β is an epimorphism, whence is an isomorphism since $|M/S_1|=|M/S_2|$. Therefore we may suppose that α is extendable to some $\beta\colon M/S_1\to M/S_2$. Let S_i' be the unique submodule of eA containing T such that $S_i'/T=S_i$ for each i. Then $M/S_i\cong eA/S_i'$ for each i. The homomorphism β induces an isomorphism $\beta'\colon eA/S_1'\to eA/S_2'$ which is clearly given by the left multiplication by an element t in $eAe\backslash eJe$. Thus $tS_1'=S_2'$. Taking $S=eJ^2/T$ into account it follows from $S_i\cap S=0$ that $S_i'\cap eJ^2=T$ for each i. Hence $tT=t(S_1'\cap eJ^2)=tS_1'\cap teJ^2=S_2'\cap eJ^2=T$. Thus the left multiplication by t induces an automorphism γ of M. As easily checked γ is a lift of β, i.e. the diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & M \\ \downarrow & & \downarrow \\ M/S_1 & \xrightarrow[\beta]{} & M/S_2 \end{array}$$

is commutative where the vertical maps are the canonical epimorphisms. Then since $\gamma(S)=\gamma(MJ^2)=MJ^2=S$, the diagram above induces the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\gamma|S} & S \\ \downarrow & & \downarrow \\ (S\oplus S_1)/S_1 & \xrightarrow[\alpha]{} & (S\oplus S_2)/S_2\,. \end{array}$$

Thus $\gamma|S=1_S$. This means that $t-e\in eJe$, whence $t=e+j$ for some $j\in eJe$. Then $S_2'=tS_1'=(e+j)S_1'\leq eS_1'+jS_1'\leq S_1'+eJ^2$ since $jS_1'\leq jeJ\leq eJ^2$. Thus $S_2\leq S_1\oplus S$, a contradiction. Accordingly we must have $|\operatorname{soc} M|\leq 2$.

Since $T\leq I$, (1) induces an exact sequence

$$0\to K\hookrightarrow M\to L\to 0\,.$$

$LJ^2\neq 0$ implies $MJ^2\not\leq K$. Hence $K\cap MJ^2=0$ since MJ^2 is simple. By [II, Lemma 3.2.2] $MJ=K\oplus H$ for some $H\leq MJ$. Hence $|\operatorname{soc} L|=|\operatorname{soc} LJ|=|\operatorname{soc}(MJ/K)|=|\operatorname{soc} H|\leq|\operatorname{soc} MJ|=|\operatorname{soc} M|\leq 2$. This completes the proof of the first part of the statement. Now suppose that L is not colocal. Then by the above $|\operatorname{soc} L|=2$. Since LJ^2 is simple, $\operatorname{soc} L=X\oplus LJ^2$ for some simple $X\leq L$. In particular $X\cap LJ^2=0$. It follows from $X\leq LJ$ that $LJ=X\oplus Y$ for some $Y\leq LJ$ by [II, Lemma 3.2.2]. Since $LJ^2=YJ$ is simple, $\mathrm{h}(Y)=2$. Further Y is colocal since $2=|\operatorname{soc} L|=|\operatorname{soc} LJ|=|X|+|\operatorname{soc} Y|$. Hence L is a direct sum of a simple X and a colocal Y of height 2. //

Proposition 1.8. *If P_A is a local module, then $|PJ^2/PJ^3|\leq 2$.*

Proof. We may assume that $\mathrm{h}(P)=3$. Suppose that $|PJ^2|\geq 3$. Then by Proposition 1.3 we have an irredundant sum $PJ=\sum_{i=1}^{n} L_i$ of PJ for some uniserial modules L_i. So $PJ^2=\sum_{i=1}^{n} L_iJ$ where L_iJ is simple or zero, whence we may assume that $PJ^2=\bigoplus_{i=1}^{m} L_iJ$ for some $m\geq 3$. Put $L:=P/(\bigoplus_{i\neq 1} L_iJ)$ and $\pi\colon P\to L$ to be the canonical epimorphism. Then L is local and $LJ^2\cong L_1J$ is simple. Hence by Lemma 1.7 $|\operatorname{soc} L|\leq 2$. But by construction $3=|\pi L_1J\oplus \pi L_2\oplus\pi L_3|\leq|\operatorname{soc} L|$, a contradiction. //

Proposition 1.9. *Let P_A be a local module. If $|PJ^2/PJ^3|=1$, then P has* *one of the following structures*:

(1) *P is colocal of height* 3.

(2) *PJ is a direct sum of a simple right module and a colocal right module of height* 2.

(3) *P is uniserial.*

(4) *PJ is a direct sum of a simple module and a nonzero uniserial module.*

Proof. If $h(P)=3$, then by Lemma 1.7 P is of type (1) or (2). So we may assume that $h(P)>3$. By Lemma 1.7 P/PJ^3 is of type (1) or (2).

(a) In case P/PJ^3 is of type (1). Since $\mathrm{top}^2(PJ)=PJ/PJ^3$ is colocal and $h(PJ)\geq 3$, PJ is uniserial by Corollary 1.6. Thus P is of type (3).

(b) In case P/PJ^3 is of type (2). Put $PJ/PJ^3=X\oplus Y$ where X is simple and Y is colocal of height 2. Then since $\mathrm{top}^2(PJ)=PJ/PJ^3$ is decomposable, so is PJ. Let $PJ=X'\oplus Y'$ where X' and Y' are non-zero. Then $X\oplus Y\cong \mathrm{top}^2 X'\oplus \mathrm{top}^2 Y'$. By Krull-Schmidt's theorem we may assume that $X\cong \mathrm{top}^2 X'$ and $Y\cong \mathrm{top}^2 Y'$. Since X is simple, so is X' by Nakayama's lemma. Thus $h(Y')\geq 3$ since $0\neq PJ^3=Y'J^2$. Then since $\mathrm{top}^2 Y'\cong Y$ is colocal, Y' is uniserial by Corollary 1.6. Hence P is of type (4). //

Lemma 1.10. *Let P_A be a local module of height* 3. *If $|PJ^2|=2$, then*

(1) *Every colocal submodule of PJ is uniserial*; *and*

(2) $\mathrm{soc}\,P=PJ^2$ (*i.e. PJ has no simple direct summand*).

Proof. (1) Let C be a colocal submodule of PJ. If $h(C)=1$, then the assertion is trivial. So we may assume that $h(C)=2$. Since $CJ=\mathrm{soc}\,C$ is simple and $|PJ^2|=2$, $PJ^2=CJ\oplus S$ for some simple $S\leq PJ^2$. Put $L:=P/CJ$. Then L is local and $LJ^2\cong S$ is simple. Hence by Lemma 1.7 $|\mathrm{soc}\,L|\leq 2$. Then since $C/CJ\cap LJ^2=0$, $C/CJ\oplus LJ^2\leq \mathrm{soc}\,L$, whence $|C/CJ|=1$. Thus C is uniserial.

(2) Assume that $\mathrm{soc}\,P\neq PJ^2$. Then P has a simple direct summand S. By Proposition 1.3 there are uniserial modules L_i such that $PJ=\sum_{i=1}^{n} L_i$ is an irredundant sum. Since $|PJ^2|=2$, we may assume that $PJ^2=L_1J\oplus L_2J$. Put $L:=P/L_2J$ and $\pi: P\to L$ to be the canonical epimorphism. Then L is local and $LJ^2\cong L_1J$ is simple. So by Lemma 1.7 $|\mathrm{soc}\,L|\leq 2$. But $3\leq|\pi L_1J\oplus\pi L_2\oplus \pi S|\leq|\mathrm{soc}\,L|$, a contradiction. //

Lemma 1.11. *Let P_A be a local module of height* 3 *with $|PJ^2|=2$ and $PJ=\sum_{i=1}^{n} L_i$ be an irredundant sum of uniserial modules. Then $\mathrm{soc}\,L_i\neq \mathrm{soc}\,L_j$ for each $i\neq j$ in $\{1, \cdots, n\}$.*

Proof. Assume $\mathrm{soc}\,L_i=\mathrm{soc}\,L_j=:S$. If the identity map 1_S of S is extendable to a homomorphism $\varphi: L_i \to L_j$, then $L_i + L_j = L \oplus L_j$ where $L := (1_{L_i} - \varphi)(L_i)$ is simple. So since $PJ = \sum_{t \neq i,j} L_t + L + L_j$ is an irredundant sum of local modules, PJ has a simple direct summand L. This contradicts Lemma 1.10 (2). Hence 1_S is not extendable to any homomorphism $L_i \to L_j$. Therefore $L_i + L_j$ is colocal by [12, Lemma 1.2 (2)]. Hence $L_i + L_j$ is uniserial by Lemma 1.10 (1). Thus $L_i = L_j$, i.e. $i = j$. //

Proposition 1.12. *Let P_A be a local module. If $|PJ^2/PJ^3| = 2$, then PJ is a direct sum of two non-zero uniserial modules.*

Proof. First we prove the assertion in the case of $\mathrm{h}(P) = 3$. By Proposition 1.3 $PJ = \sum_{i=1}^{n} L_i$ is an irredundant sum of some uniserial modules L_i. Put $S_i := L_i J$ for each i. By Lemma 1.10 (2) $|L_i| = \mathrm{h}(L_i) = 2$ for each i. Since $|PJ^2| = 2$, we may assume that $\mathrm{soc}\, PJ = PJ^2 = S_1 \oplus S_2$. We show that $n = 2$. Assume $n \geq 3$. Define $\alpha_i: S_3 \to S_i$ by $\alpha_i := \pi_i|_{S_3}$ where $\pi_i: S_1 \oplus S_2 \to S_i$ is the canonical projection for each $i = 1, 2$. Then $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. For, otherwise, say $\alpha_1 = 0$, we would have $S_3 = S_2$ contradicting Lemma 1.11. By Lemma 1.1 α_1 is extendable to a monomorphism $\varphi: L_3 \to P$. If $\varphi(L_3) + L_1$ is colocal, then $\varphi(L_3) = L_1$ by Lemma 1.10 (1). In this case we put $\lambda := 1_{L_1}$. If $\varphi(L_3) + L_1$ is not colocal, then 1_{S_1} is extendable to an isomorphism $\varphi(L_3) \to L_1$. In this case we put λ to be this isomorphism. In either case we have a commutative diagram

$$\begin{array}{ccccc} L_3 & \xrightarrow{\varphi} & \varphi(L_3) & \xrightarrow{\lambda} & L_1 \\ \cup & & \cup & & \cup \\ S_3 & \xrightarrow[\alpha_1]{} & S_1 & = & S_1 . \end{array}$$

Then $L_1 + L_3 = L_1 + L_3'$ where $L_3' := (1_{L_3} - \lambda\varphi)(L_3)$. Since $(1_{L_3} - \lambda\varphi)|_{S_3} = 1_{S_3} - \alpha_1 = \alpha_2 \neq 0$, $1_{L_3} - \lambda\varphi: L_3 \to L_3'$ is an isomorphism. Hence $\mathrm{soc}\, L_3' = (1_{L_3} - \lambda\varphi)(\mathrm{soc}\, L_3) = \alpha_2 S_3 = S_2 = \mathrm{soc}\, L_2$. But the sum $PJ = \sum_{i \neq 3} L_i + L_3'$ is also an irredundant sum of uniserial modules. This contradicts Lemma 1.11. As a consequence $n \leq 2$. If $n = 1$, then $|PJ^2| = 1$, a contradiction. Hence $n = 2$. Thus $PJ = L_1 \oplus L_2$.

In the general case, since $\mathrm{h}(P/PJ^3) = 3$, we have $\mathrm{top}^2 PJ = PJ/PJ^3 = L_1 \oplus L_2$ for some uniserial modules L_i of length 2 by the special case above. Then PJ is decomposable. Put $PJ = X_1 \oplus X_2$ with $X_i \neq 0$ for each i. Then $L_1 \oplus L_2 = \mathrm{top}^2 X_1 \oplus \mathrm{top}^2 X_2$. By Krull-Schmidt's theorem each $\mathrm{top}^2 X_i$ is uniserial. Hence each X_i is uniserial by Lemma 1.4. //

The following statement holds without the assumption that A is a ring of right 2nd local type.

Lemma 1.13. *Let e be in* $\mathrm{pi}(A)$ *such that* $eJ=U\oplus V$ *for some uniserial right modules U and V which are of length* 2, *and f and g be in* $\mathrm{pi}(A)$ *with* $\mathrm{top}\, U\cong fA/fJ$ *and* $\mathrm{soc}\, U\cong gA/gJ$. *Assume that Af and Ag are nniserial left modules· Then every local submodule L of eJ with* $\mathrm{top}\, L\cong fA/fJ$ *is uniserial.*

Proof. If $\mathrm{h}(L)=1$, then the statement is trivial. So we may assume that $\mathrm{h}(L)=2$. Assume that L is not uniserial. Then $LJ=eJ^2$. Put $L=u_1A$ and $U=u_2A$ where $u_i\in eJf\setminus eJ^2f$ for each i. Since $L\cap U=\mathrm{soc}\, U\cong gA/gJ$, we have $0\neq u_1a_1=u_2a_2$ for some $a_i\in fJg\setminus fJ^2g$. Since Ag is uniserial, Jg/J^2g is simple, whence there is some $t\in fAf\setminus fJf$ such that $a_2\in a_1t+J^2g$. Thus $u_1a_1=u_2a_2=u_2ta_1$ since $u_2J^2g\leq eJ^3=0$. Hence $(u_1-u_2t)a_1=0$. Consider the homomorphism $\alpha\colon Af\to Ag$ defined by the right multiplication by a_1. Since Af is uniserial and $\alpha(Jf)\ni u_1a_1\neq 0$, $\mathrm{Ker}\,\alpha\leq J^2f$. As a consequence $u_1-u_2t\in J^2f$. Thus $L=u_1A\leq u_2tA+eJ^2fA\leq U+eJ^2=U\oplus VJ$. Comparing the composition lengths, we see that $L=U\oplus VJ$ which contradicts that L is local. //

REMARK. In the above it is necessary that Af is uniserial as the following example says. Let A^{op} be an algebra defined by the bounden quiver

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & 2 \arrow[ur, "\gamma"] \arrow[dr, "\delta"'] & \\ & & 4 \end{tikzcd} \quad ;\ \delta\alpha=\gamma\beta=0\,.$$

Then $e_1J=\alpha A\oplus\beta A$ with αA and βA uniserial, and $\mathrm{top}\,\alpha A\cong e_2A/e_2J$ and $\mathrm{soc}\,\alpha A\cong e_3A/e_3J$. But e_1J has a non-uniserial local module $L:=(\alpha+\beta)A$ with $\mathrm{top}\, L\cong e_2A/e_2J$. In this case Ae_3 is uniserial but Ae_2 is not.

Proof of Theorem I. The statement (R) immediately follows from Propositions 1.8, 1.9 and 1.12. We show the statement (L). By [II, Theorem 2.5 (1R)⇒(3-iL)] J^2e is a uniserial waist in Ae if $J^2e\neq 0$ for each $e\in\mathrm{pi}(A)$. Hence we have only to show that every local left module P of height 5 is uniserial. Further since A/J^5 is also a ring of right 2nd local type, we may assume that $J^5=0$ and $P=Ah$ for some $h\in\mathrm{pi}(A)$. Let $Jh/J^2h\cong\bigoplus_{i=1}^{n}Ag_i/Jg_i$, $J^2h/J^3h\cong Af/Jf$ and $J^3h/J^4h\cong Ae/Je$ where e, f and g_i are in $\mathrm{pi}(A)$. Then all Ag_i and Af are uniserial by the argument used in the proof of [I, Proposition 2.4.1]. It is enough to show that $n=1$. Suppose that $n>1$. Then there are some $x_i\in g_iJh\setminus g_iJ^2h$ for $i=1$, 2 such that the sum $Y:=Ax_1+Ax_2$ is irredundant and $Ax_1\cap Ax_2=Jx_1=Jx_2=J^2h=JY$. Hence we have $y_1x_1=y_2x_2$ and $Ayx_1=J^2h$ for some $y_i\in fJg_i\setminus fJ^2g_i$. Further there is some $z\in eJf\setminus eJ^2f$ such that $Azy_1x_1=J^3h$. Consider a homomorphism

$$\alpha := (\cdot zy_1, \cdot zy_2)\colon Ae/Je \to Ag_1/J^3g_1 \oplus Ag_2/J^3g_2$$

where for each $t \in A$, $\cdot t$ is the map defined by the right multiplication by t. By the argument similar to that used in the proof of the implication (3)⇒(1) in Lemma 1.4, it follows from the structure of Y/J^3Y that α is infusible. Therefore by [I, Corollary 1.4.2] the map

$$\alpha^* := (zy_1\cdot, zy_2\cdot)\colon g_1A/g_1J \oplus g_2A/g_2J \to eJ^2/eJ^3$$

is a monomorphism where for each $t \in A$, $t\cdot$ is the map defined by the left multiplication by t. Thus $[zy_1]A \oplus [zy_2]A \leq eJ^2/eJ^3$ where $[zy_i] := zy_i + eJ^3$ for each i. By the statement (R), we have $eJ/eJ^3 = U_1 \oplus U_2$ for some uniserial modules U_i of length 2. Hence $eJ^2/eJ^3 = [zy_1]A \oplus [zy_2]A \leq [z]A$ where $[z] := z + eJ^3$. On the other hand, since $eJf/eJ^2f \neq 0$, we have $fA/fJ \cong \text{top } U_i$ for some i, say $i = 1$. Further since $\text{soc } U_1$ is a direct summand of $[zy_1]A \oplus [zy_2]A$, $\text{soc } U_1 \cong g_jA/g_jJ$ for some j. Hence by Lemma 1.13, $[z]A$ must be a uniserial module, a contradiction. This completes the proof of (L). Finally assume that A is a ring with selfduality. Then by Lemma 1.4, the following holds.

(a) Let L be a left module. If $\text{soc}^2 L$ is uniserial, then so is L.

As easily seen this implies that every local left module of height≥ 4 is uniserial. Hence in this case (L_4) does not occur. //

2. Left serial rings with selfduality

Theorem II. *Assume that A is a left serial ring with selfduality. Then the following are equivalent*:

(1) *A is of right 2nd local type.*

(2) *For every $e \in \text{pi}(A)$, if $\text{h}(eA) > 2$, then eJ is a direct sum of two uniserial modules.*

(3) *Every indecomposable right module of height>2 is local.*

Proof. The proof is quite similar to that of [II, Theorem 4.1]. The implication (3)⇒(1) is trivial, and (1)⇒(2) follows from Theorem I and the selfduality of A. By the selfduality of A the condition (3) is equivalent to the following.

(3′) *Every indecomposable left module of height>2 is colocal.*

The implication (2)⇒(3′) also follows as in [II], with the following (necessary) changes and additions.

First the definition of a simple left module being of V-type is changed.

DEFINITION. Let S be a simple left module. Then S is said to be of *V-type* in case $\text{E}(S)/S$ is a direct sum of two uniserial left modules and eJ is a direct sum of two uniserial right modules where $E(-)$ denotes the injective hull of $(-)$ and e is an element in $\mathrm{pi}(A)$ such that $Ae/Je \cong S$.

Then the following lemma (quite similar to [II, Lemma 4.3]) holds.

Lemma 2.1. *Assume that A is a left serial ring with selfduality which satisfies the condition* (2) *of Theorem II. Let S be a simple left module. If $S \cong \mathrm{soc}\, L$ for some left module L of height>2, then S is of V-type.*

Proof. Let D be a selfduality of A. Since $\mathrm{h}(\mathrm{E}(S)) \geq \mathrm{h}(L) > 2$, $D(\mathrm{E}(S))J$ is a direct sum of two uniserial modules by the condition (2) of Theorem II, whence so is $\mathrm{E}(S)/S \cong D(D(\mathrm{E}(S))J)$. Let e be in $\mathrm{pi}(A)$ such that $S \cong Ae/Je$. There is some local submodule M of L such that $\mathrm{h}(M) > 2$. Since A is left serial, $M \cong Af/J^h f$ for some f in $\mathrm{pi}(A)$ where $h := \mathrm{h}(M) > 2$. Since $Ae/Je \cong S = \mathrm{soc}\, L = \mathrm{soc}\, M \cong J^{h-1}f/J^h f$, $eJ^{h-1}f \neq 0$. Thus $eJ^{h-1} \neq 0$, i.e. $\mathrm{h}(eA) \geq h > 2$. Hence eJ is a direct sum of two uniserial right modules by the condition (2) of Theorem II. //

Lemma 2.2. *Let L be a quasi-projective uniserial left module of length* 2 *such that* $\mathrm{soc}L$ *is of V-type. Then* $\dim_{D_2(L)} D_1(L) \leq 2$ ([II, 4.2]).

Proof. Assume that $\dim_{D_2(L)} D_1(L) \geq 3$. Then there exist α_2 and α_3 in $\mathrm{D}_1(L)$ such that $\{1_S, \alpha_2, \alpha_3\}$ is a linearly independent set in ${}_{D_2(L)}\mathrm{D}_1(L)$. Put $S := \mathrm{soc}\, L$ and $\alpha := \begin{pmatrix} \alpha_2 & -1_S & 0 \\ \alpha_3 & 0 & -1_S \end{pmatrix} : S^{(2)} \to L^{(3)}$. Since L is quasi-projective, $M := \mathrm{Coker}\, \alpha$ is a colocal left module of height 2 and $|M/S| = 3$ by the argument used in [12, Lemma 5.3] (or the dual version of [1, Theorem]). This contradicts that S is of V-type, since $M \leq \mathrm{E}(S)$. //

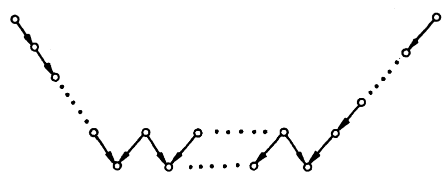
Completion of the proof of Theorem II. The above lemma enables us to prove [II, Lemma 4.8] in the selfdual artinian case by the argument used in [12, Lemma 3.7]. Now the proof of the implication (2)$\Rightarrow$(3′) of Theorem II proceeds along the same way of that of [II, Theorem 4.1 (2)$\Rightarrow$(3)′]. //

3. Examples: representation-finite algebras of right 2nd local type

In this section we give necessary and sufficient conditions for representation-finite algebras to be of right 2nd local type, using the method outlined in Ringel [10]. This gives many examples of algebras of right 2nd type, and helps further considerations. Throughout this section, we assume that our algebra A is (basic) representation-finite and, in addition, (without loss of generality) is connected. Following Ringel [11], we say that a left A-module M is *sincere* in case $eM \neq 0$ for every e in $\mathrm{pi}(A)$, and that an algebra is *sincere* if it has a sincere indecomposable left module. First we give the result for sincere, simply connected (in the sense of Bongartz-Gabriel [5]) algebras, and then reduce the general case to this special one. We identify algebras with their isomorphism classes and with their bounden quivers below.

EXAMPLE 3.1. (1) Consider the following list $\mathcal{L}$ of sincere simply connected algebras:

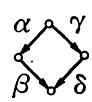
(A_n) Quiver algebras of (Dynkin) type A_n with the following orientation.



(D_n) Quiver algebras of type D_n of height 2 ($n \geq 4$).

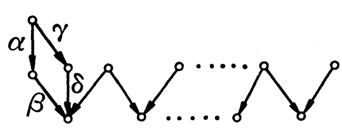
(E_n) Quiver algebras of type E_n of height 2 ($n = 6, 7, 8$).

(X_4) A bounden quiver algebra



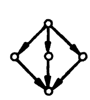
; $\beta\alpha = \delta\gamma$.

(X_n) Bounden quiver algebras ($n \geq 5$)



; $\beta\alpha = \delta\gamma$.

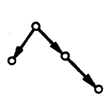
(Y_5) An algebra defined by the quiver

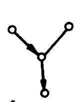


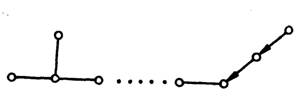
with full commutativity relations.

Then it is easy to see that all algebras in $\mathcal{L}$ are of right 2nd local type.

(2) The quiver algebras in the following list $\mathcal{S}$ are not of right 2nd local type:

(A_4)  .

(D_4)  .

(D_n) ($n \geq 5$)  .

where each edge represents an arrow with an arbitrary orientation.

The two lists in Example 3.1 determines the right 2nd local type for sincere simply connected algebras as the following theorem says.

Theorem III. *Assume that A is a sincere simply connected algebra. Then the following are equivalent:*

(1) *A is of right 2nd local type.*

(2) *$e_E A e_E \notin \mathcal{S}$ for any $E \subseteq \mathrm{pi}(A)$.*

(3) *$A \in \mathcal{L}$.*

where we put $e_E := \sum_{e \in E} e$.

Proof of (3)$\Rightarrow$(1) and (1)$\Rightarrow$(2). Example 3.1 (1) says the implication (3)$\Rightarrow$(1). By Example 3.1 (2), the implication (1)$\Rightarrow$(2) follows from the next statement for an arbitrary artinian ring A. //

Proposition 3.2. *If A is a ring of right 2nd local type, then so is eAe for every idempotent e in A.*

Proof. Obviously we may assume that $e \neq 0$. Put $B := eAe$ and denote by T and by H the functors $- \otimes_B eA: \mathrm{mod}\, B \to \mathrm{mod}\, A$ and $\mathrm{Hom}_A(eA, -): \mathrm{mod}\, A \to \mathrm{mod}\, B$, respectively where $\mathrm{mod}\, C$ denotes the category of finitely generated right C-modules for $C = A$ and B. Then T is a left adjoint of H and $HT \cong 1_{\mathrm{mod}\, B}$, the identity functor of mod B. We first claim

(i) *T preserves indecomposables.*

In fact, let M be an indecomposable module in $\mathrm{mod}\, B$ and assume that TM is decomposable, say $TM = X \oplus Y$ with $X, Y \neq 0$. Then $M \cong HTM = HX \oplus HY$. Since M is indecomposable, we may assume that $HX = 0$. But then $0 \neq \mathrm{Hom}_A(TM, X) \cong \mathrm{Hom}_B(M, HX) = 0$, a contradiction.

Next putting $j := eJe$, direct calculation shows

(ii) *Let M be in $\mathrm{mod}\, B$ and $\alpha: Mj^2 \to M$ the inclusion map. Then $\alpha_*(T(Mj^2)) \leq (TM)J^2$ where $\alpha_* := T(\alpha): T(Mj^2) \to T(M)$.*

Now since T is right exact, $T(\mathrm{top}^2 M) \cong TM/\alpha_*(T(Mj^2))$. Assume that B is not of right 2nd local type, thus $\mathrm{top}^2 M = X \oplus Y$ with $X, Y \neq 0$ for some indecomposable M in $\mathrm{mod}\, B$. Then by (i), TM is indecomposable in $\mathrm{mod}\, A$ but $\mathrm{top}^2(TM)$ is decomposable as follows.

$$\begin{aligned}\mathrm{top}^2(TM) = TM/(TM)J^2 &\cong \frac{TM/\alpha_*(T(Mj^2))}{(TM)J^2/\alpha_*(T(Mj^2))} = \frac{TM/\alpha_*(T(Mj^2))}{[TM/\alpha_*(T(Mj^2))]J^2} \\ &\cong \frac{T(\mathrm{top}^2 M)}{[T(\mathrm{top}^2 M)]J^2} \cong \mathrm{top}^2(TX) \oplus \mathrm{top}^2(TY). \quad //\end{aligned}$$

We denote by $n(A)$ the number of isomorphism classes of simple right A-modules, i.e. $n(A)$ is equal to the cardinality of $\mathrm{pi}(A)$. The following result in [11] makes it possible to prove the implication (2)⇒(3) in Theorem III by induction on $n(A)$. We put $D:=\mathrm{Hom}_k(-, k)$ to be the usual selfduality of A.

Proposition 3.3 ([11, Proposition 6.5 (2)]). *Assume that A is a sincere simply connected algebra with $n(A)>1$. Then A is a one-point extension $\begin{pmatrix} B & R \\ 0 & k \end{pmatrix}$ or a one-point coextension $\begin{pmatrix} B & 0 \\ DR & k \end{pmatrix}$ of a sincere simply connected algebra B by an indecomposable left B-module R such that $\mathrm{Hom}_B(R, N)\neq 0$ or $\mathrm{Hom}_B(N, R)\neq 0$ for some sincere indecomposable left B-module N, respectively.*

Proof of (2)⇒(3) of Theorem III. First note that this assertion is easily verified for quiver algebras and for bounden quiver algebras whose relation ideal is generated by a single commutativity relation. The proof proceeds by induction on $n(A)$. If $n(A)=1$, then $A=k$, and so there is nothing to show. Assume $n(A)>1$. Then by Proposition 3.3 A is a one-point extension or a one-point coextension of a sincere simply connected algebra B by an indecomposable left B-module R such that $\mathrm{Hom}_B(R, N)\neq 0$ or $\mathrm{Hom}_B(N, R)\neq 0$ for some sincere indecomposable left B-module N, respectively. Assume that A satisfies the condition (2) of Theorem III. Then so does B since $B\cong eAe$ where $e:=\begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix}$. By the induction hypothesis it follows from $n(B)=n(A)-1$ that B is in $\mathcal{L}$. Hence it suffices to check that the algebras obtained by the above type of one-point extension and one-point coextension from some B in $\mathcal{L}$ are again in $\mathcal{L}$ if they are representation-finite and satisfy the condition (2) of Theorem III. In doing so, as noted above, we can remove the quiver algebras and the bounden quiver algebras whose relation ideal is generated by a single commutativity relation. Further the condition that A is representation-finite requires that (i) $|\mathrm{top}\, R|\leq 3$ (resp. $|\mathrm{soc}\, R|\leq 3$) in the one-point extension (resp. coextension) case, and (ii) $\dim R(a)\leq 1$ for the vertex a of (D_n) or (E_n) in $\mathcal{L}$ which has three neighbours. In particular, the second condition yields that $\dim R(a)\leq 1$ for all vertices a of (D_n) or (E_n) in $\mathcal{L}$. This makes the verification easy, and details are left to the reader. Above all, since the condition $\mathrm{Hom}_B(R, N)\neq 0$ (resp. $\mathrm{Hom}_B(N, R)\neq 0$) in the extension (resp. coextension) case guarantees that the obtained algebras are sincere, we can ignore nonsincere algebras from the start. //

REMARK. (a) In Theorem III, the condition (2) is equivalent to the following condition which is easier to check.

(2′) *$e_E A e_E \notin S$ for any set of vertices $E \subseteq \mathrm{pi}(A)$ of any connected subquiver of the ordinary quiver of A.*
For, the verification of (2)⇒(3) is, in fact, done by that of (2′)⇒(3).

(b) In case $n(A)>13$, the implication (2)⇒(3) of Theorem III is easily verified by Bongartz' theorem [4, Klassifikationssatz] (see also [11, Theorem 6.3 (1)]).

We go on to the general case. This is done as usual by the covering technique developed in [5], [7] and [6]. Put Γ to be the Auslander-Reiten quiver of A and $\tilde{\Gamma}$ the universal cover of Γ (see [5]). By $\mathrm{mod}\,A$ and by $\mathrm{ind}\,A$ we denote the category of finite dimensional right A-modules and the full subcategory of $\mathrm{mod}\,A$ consisting of the chosen representatives of isomorphism classes of indecomposable right A-modules where projective indecomposables are canonical ones, respectively. Then as well known, there exists a well-behaved ([9]), covering functor $E: k(\tilde{\Gamma})\to \mathrm{ind}\,A$ where $k(\tilde{\Gamma})$ is the mesh category of $\tilde{\Gamma}$. Let $\tilde{A}$ be the full subcategory of $k(\tilde{\Gamma})$ consisting of projective vertices of $\tilde{\Gamma}$. Then it is well known that $k(\tilde{\Gamma})\cong \mathrm{ind}\,\tilde{A}$ by which we identify these categories. We have a commutative diagram

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & \mathrm{ind}\,\tilde{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle E} \\ A & \longrightarrow & \mathrm{ind}\,A \end{array}$$

where the horizontal functors are Yoneda embeddings, and E and F are covering functors. Since $\mathrm{ind}\,A$ is locally bounded and $\mathrm{ind}\,\tilde{A}$ is basic, $\mathrm{ind}\,\tilde{A}$ is locally bounded by [7, Proposition 1.2 a)], i.e. $\tilde{A}$ is locally representation-finite. Hence by [7, Proposition 2.7], the push down $F_\lambda: \mathrm{mod}\,\tilde{A}\to \mathrm{mod}\,A$ preserves Auslander-Reiten sequences and $E\cong F_\lambda|_{\mathrm{ind}\,\tilde{A}}$. In particular, for every $M\in \mathrm{mod}\,\tilde{A}$, M is indecomposable iff so is $F_\lambda(M)\in \mathrm{mod}\,A$.

Proposition 3.4. *A is of right 2nd local type iff so is $\tilde{A}$.*

Proof. Note that for every $M\in \mathrm{ind}\,A$, $M=F_\lambda(\tilde{M})$ for some $\tilde{M}\in \mathrm{ind}\,\tilde{A}$ by [7, Proposition 1.2 b)]. Now let M be in $\mathrm{ind}\,A$ and $\tilde{M}$ in $\mathrm{ind}\,\tilde{A}$ such that $M=F_\lambda(\tilde{M})$. Since F_λ is an exact functor and preserves the radical by [5, Proposition 3.2], $F_\lambda(\mathrm{top}^2\tilde{M})\cong \mathrm{top}^2 M$. Hence by the above $\mathrm{top}^2\tilde{M}$ is indecomposable iff so is $\mathrm{top}^2 M$. //

Note that by [6, Remark 3.3 (a)] $\tilde{A}$ is directly constructed from A. Since A is representation-finite, there exists a connected subquiver Q of $Q_{\tilde{A}}$ with the set Q_0 of vertices finite such that for every $M\in \mathrm{ind}\,A$, there exists an $\tilde{M}\in \mathrm{ind}\,\tilde{A}$ such that $F_\lambda(\tilde{M})=M$ and $\mathrm{sp}\,M\subseteq Q_0$, where $\mathrm{sp}\ \tilde{M}$ is the *support* of $\tilde{M}$, i.e. the set of vertices v of $Q_{\tilde{A}}$ such that $\tilde{M}(v)\neq 0$. By $\bar{A}$ we denote the factor category of $\tilde{A}$ by the ideal generated by all the vertices v of $Q_{\tilde{A}}$ such that

$v \notin Q_0$. Since $\bar{A}$ has only a finite number of objects, $\bar{A}$ can be regarded as a representation-finite algebra. Further the Auslander-Reiten quiver $\bar{\Gamma}$ of $\bar{A}$ has no oriented cycle since so does $\tilde{\Gamma}$.

Theorem IV. *Assume that A is a (basic, connected) representation-finite algebra (of finite dimension over k). Then the following are equivalent.*

(1) *A is of right 2nd local type.*

(2) *$e_E\bar{A}e_E \notin \mathcal{S}$ for every $E \subseteq \mathrm{pi}(\bar{A})$.*

(3) *$\mathrm{Sp}(\bar{M}) \in \mathcal{L}$ for every indecomposable right $\bar{A}$-module $\bar{M}$.*

In the above, $\mathrm{Sp}(\bar{M})$ denotes the *support algebra* of $\bar{M}$, i.e. $\mathrm{Sp}(\bar{M}) := e_S\bar{A}e_S$ where $S := \mathrm{sp}(\bar{M})$.

Proof. (1)$\Rightarrow$(2). Assume that A is of right 2nd local type. Then so is $\tilde{A}$ by Proposition 3.4, and so is $\bar{A}$ since $\bar{A}$ is a factor of $\tilde{A}$. Hence $e_E\bar{A}e_E \notin \mathcal{S}$ for every $E \subseteq \mathrm{pi}(\bar{A})$ by Proposition 3.2.

(2)$\Rightarrow$(3). Assume that (2) holds, and let $\bar{M} \in \mathrm{ind}\,\bar{A}$. Then $e_E\,\mathrm{Sp}(\bar{M})e_E \notin \mathcal{S}$ for every $E \subseteq \mathrm{sp}(\bar{M})$, and $\mathrm{Sp}(\bar{M})$ is a sincere representation-finite algebra. Further the Auslander-Reiten quiver of $\mathrm{Sp}(\bar{M})$ has no oriented cycle since so does $\bar{\Gamma}$. Hence $\mathrm{Sp}(\bar{M})$ is a sincere tilted algebra by [8, Corollary 8.3]. Thus $\mathrm{Sp}(\bar{M})$ is a sincere simply connected algebra by [3, Theorem 3.1]. Hence by Theorem III, $\mathrm{Sp}(\bar{M}) \in \mathcal{L}$.

(3)$\Rightarrow$(1). Let $M \in \mathrm{ind}\,A$. Then $M = F_\lambda(\bar{M})$ for some $\bar{M} \in \mathrm{ind}\,\bar{A}$. Clearly we can regard $\bar{M} \in \mathrm{ind}\,\mathrm{Sp}(\bar{M})$. Since $\mathrm{Sp}(\bar{M}) \in \mathcal{L}$ is of right 2nd local type, $\mathrm{top}^2\bar{M}$ is indecomposable, whence so is $\mathrm{top}^2 M$. //

REMARK. The list $\mathcal{S}$ is closely related to Lemmas 1.4 and 1.7 and to the conditions (R) and (L) in Theorem I. In fact, it is possible to describe the condition (2) in Theorems III and IV by certain conditions on local A-modules.

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Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558, Japan