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Osaka University
A CRITERION OF EXACTNESS OF THE CLEMENS-SCHMID SEQUENCES ARISING FROM SEMI-STABLE FAMILIES OF OPEN CURVES

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1. Introduction

Let $\pi: X \to \Delta$ be a semistable family of projective algebraic varieties over the unit disk. In other words, $\pi$ is flat and projective over $\Delta$, smooth over $\Delta^* = \Delta - \{0\}$, and the central fiber $Y = \pi^{-1}(0)$ is a divisor with normal crossings without multiple components. We write $X_t := \pi^{-1}(t)$ for $t \in \Delta^*$ and $X^* := X - Y$. There is the Wang exact sequence

\[
\cdots \to H^q(X^*, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{\Delta} H^q(X_t, \mathbb{Q}) \to H^{q+1}(X^*, \mathbb{Q}) \to \cdots,
\]

and the localization exact sequence

\[
\cdots \to H^q(Y, \mathbb{Q}) \to H^q(X, \mathbb{Q}) \to H^q(X^*, \mathbb{Q}) \to H^{q+1}(X, \mathbb{Q}) \to \cdots.
\]

Here $N$ denotes the log monodromy around $\Delta^* = \Delta - \{0\}$. Combining those sequences and the natural isomorphism $H^q(X, \mathbb{Q}) \simeq H^q(Y, \mathbb{Q})$, we obtain a sequence

\[
\cdots \to H^q(Y, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \to H^q(Y, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{N} H^q(X_t, \mathbb{Q}) \to \cdots.
\]

This is called the Clemens-Schmid sequence. A theorem of Clemens and Schmid says that the sequence (1.3) is exact ([1, §3]). In particular, the first piece

\[
H^q(Y, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{N} H^q(X_t, \mathbb{Q})
\]

is called the local invariant cycle theorem ([3, (5.12)]).

In [2, (12.3.1)], S. Usui et al. proposed a problem whether the Clemens-Schmid sequence (1.3) or (1.4) is exact when we remove the assumption that $\pi$ is proper. In this paper, we give a necessary and sufficient condition for that the sequence (1.4) is exact when $\pi$ is a semistable family of open curves. In particular, we see that the Clemens-Schmid sequences of non-proper families are not exact in general.

The author would like to express his sincere gratitude to Professor Sampei Usui for stimulating discussions.
2. A Criterion and an example

In this section, we assume that $\pi$ is a family of curves. Let $D$ be a divisor on $X$ which is smooth and projective over $\Delta$, and such that $D + Y$ is a normal crossing divisor. We write $D_t = D \cap X_t$ for $t \in \Delta^*$ and $D^* = D \cap X^*$.

We discuss when the following sequence

$$(2.1) \quad H^1(Y - D \cap Y) \xrightarrow{i} H^1(X_t - D_t) \xrightarrow{N_0} H^1(X_t - D_t)$$

is exact, where $N_0$ denotes the log monodromy on $H^1(X_t - D_t)$. (We omit to write the coefficient $Q$.) The sequence (2.1) fits into the commutative diagram

$\begin{array}{ccc}
H^0(D \cap Y) & \xrightarrow{\cong} & H^0(D_t) \\
\uparrow r_0 & & \uparrow r_t \\
H^1(Y - D \cap Y) & \xrightarrow{i} & H^1(X_t - D_t) \\
\uparrow & & \uparrow \downarrow N_0 & \downarrow \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(X_t) \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0,
\end{array}$

(2.2)

in which each vertical sequence is exact. The bottom sequence is exact by the local invariant cycle theorem ([3, (5.12)]). Therefore, the map $i$ is injective.

The following lemma is the key of our criterion.

**Lemma 2.1.** The image of $N$ coincides with the image of $N_0$.

Proof. Image $N \subset \text{Image } N_0$ is trivial. We show the converse. The cohomology groups $H^1(X_t, Q)$ and $H^1(X_t - D_t, Q)$ carry the limit mixed Hodge structures due to Steenbrink and Zucker ([3], [4]). The weight filtration $W_*$ on $H^1(X_t, Q)$ is the weight monodromy filtration, which is explicitly given as follows:

$W_2H^1(X_t) = H^1(X_t), \quad W_1H^1(X_t) = \text{ker } N,$

$W_0H^1(X_t) = \text{Image } N, \quad W_{-1}H^1(X_t) = 0.$

We put the weight filtration on $H^0(D_t)$ by $\text{Gr}^W_1H^0(D_t) = H^0(D_t)$. Then the weight filtration $W_*$ on $H^1(X_t - D_t)$ (called the relative monodromy filtration) is explicitly given by

$W_2H^1(X_t - D_t) = H^1(X_t - D_t), \quad W_1H^1(X_t - D_t) = W_1H^1(X_t),$  

$W_0H^1(X_t - D_t) = W_0H^1(X_t), \quad W_{-1}H^1(X_t - D_t) = 0,$

$W_{-2}H^1(X_t - D_t) = 0.$
The map $N_0$ is a morphism of the mixed Hodge structure of type $(-1, -1)$ \((4, 3.13\text{.iii})\). Therefore $\text{Image } N_0$ is contained in $W_0 H^1(X_t - D_t, \mathbb{Q}(0)) = W_0 H^1(X_t)$. However, this is the image of $N$. Thus we have $\text{Image } N_0 \subset \text{Image } N$.

Since the map $i$ is injective, the sequence (2.1) is exact if and only if

\[(2.3) \quad h^1(Y - D \cap Y) = \dim H^1(Y - D \cap Y) = \dim \ker N_0.\]

By Lemma 2.1, we have

\[
\dim \ker N_0 = h^1(X_t - D_t) - \dim \text{Image } N
= h^1(X_t - D_t) - h^1(X_t) + h^1(Y)
= \dim \text{Image } r_t + h^1(Y).
\]

On the other hand,

\[h^1(Y - D \cap Y) = \dim \text{Image } r_0 + h^1(Y).\]

Therefore, (2.3) holds if and only if

\[(2.4) \quad \dim \text{Image } r_0 = \dim \text{Image } r_t.\]

Let $n$ be the number of the irreducible components of $Y$, and $e$ the number of the components $Y_i$ of $Y$ such that $D \cap Y_i \neq \emptyset$. Then we have

\[
\dim \text{Image } r_t = \dim \ker (H^0(D_t) \to H^2(X_t))
= \deg D_t - 1,
\]

and

\[
\dim \text{Image } r_0 = \dim \ker (H^0(D \cap Y) \to H^2(Y))
= h^0(D \cap Y) - h^2(Y) + h^2(Y - D \cap Y)
= \deg D_t - n + (n - e)
= \deg D_t - e.
\]

Thus we have the following criterion:

**Theorem 2.2.** The Clemens-Schmid sequence (2.1) is exact if and only if $e = 1$, that is, the support of $D \cap Y$ is contained in one irreducible component of $Y$.

**Example 2.3.** We denote by $(x, y, z)$ homogeneous coordinates of the projective plane $\mathbb{P}_C^2$. Let $X$ be a complex submanifold of $\mathbb{P}^2(\mathbb{C}) \times \Delta$ defined by an equation

\[zy^2 = zx^2 + t(x^3 - z^3),\]
and $\pi: X \to \Delta$ the projection. The morphism $\pi$ is smooth and proper over $\Delta^\ast$, and the central fiber $Y = \pi^{-1}(0)$ is a reduced divisor with normal crossing. Let $Y_1$, $Y_2$ and $Y_3$ be the irreducible components of $Y$ given by

\begin{align*}
Y_1 : & z = 0, \ t = 0, \\
Y_2 : & y + x = 0, \ t = 0, \\
Y_3 : & y - x = 0, \ t = 0
\end{align*}

respectively. Let $D = D_1 + D_2 + D_3$ be flat sections of $\pi$ given by

\begin{align*}
D_1 & = \{(0, 1, 0)\} \times \Delta, \\
D_2 & = \{(1, -1, \omega)\} \times \Delta, \\
D_3 & = \{(1, 1, \omega')\} \times \Delta,
\end{align*}

where $\omega$ and $\omega'$ are 3rd roots of unity. The divisor $D + Y$ is normal crossing.

The divisor $D_1$ (resp. $D_2$, $D_3$) meets only with $Y_1$ (resp. $Y_2$, $Y_3$). Thus, the Clemens-Schmid sequence (2.1) is not exact by Theorem 2.2.

A family $\pi: X \times X' \to \Delta$ with $X' \to \Delta$ a constant family of a projective nonsingular variety gives a non-exact Clemens-Schmid sequence (1.4) for all $q \geq 1$.

References