<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A criterion of exactness of the Clemens–Schmid sequences arising from semi-stable families of open curves</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Asakura, Masanori</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 40(4) P.977–P.980</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2003-12</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/6248">https://doi.org/10.18910/6248</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/6248</td>
</tr>
<tr>
<td><strong>Note</strong></td>
<td></td>
</tr>
</tbody>
</table>

Osaka University Knowledge Archive : OUKA
https://ir.library.osaka-u.ac.jp/repo/ouka/all/

Osaka University
A CRITERION OF EXACTNESS OF THE CLEMENS-SCHMID SEQUENCES ARISING FROM SEMI-STABLE FAMILIES OF OPEN CURVES

MASANORI ASAKURA

(Received April 4, 2002)

1. Introduction

Let \( \pi: X \to \Delta \) be a semistable family of projective algebraic varieties over the unit disk. In other words, \( \pi \) is flat and projective over \( \Delta \), smooth over \( \Delta^* = \Delta - \{0\} \), and the central fiber \( Y = \pi^{-1}(0) \) is a divisor with normal crossings without multiple components. We write \( X_t := \pi^{-1}(t) \) for \( t \in \Delta^* \) and \( X^* := X - Y \). There is the Wang exact sequence

\[
\cdots \to H^q(X^*, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{N} H^q(X_t, \mathbb{Q}) \to H^{q+1}(X^*, \mathbb{Q}) \to \cdots,
\]

and the localization exact sequence

\[
\cdots \to H^q_t(X, \mathbb{Q}) \to H^q(X, \mathbb{Q}) \to H^q(X^*, \mathbb{Q}) \to H^{q+1}_t(X, \mathbb{Q}) \to \cdots.
\]

Here \( N \) denotes the log monodromy around \( \Delta^* = \Delta - \{0\} \). Combining those sequences and the natural isomorphism \( H^q(X, \mathbb{Q}) \cong H^q(Y, \mathbb{Q}) \), we obtain a sequence

\[
\cdots \to H^{q-2}(X_t, \mathbb{Q}) \to H^q_t(X, \mathbb{Q}) \to H^q(Y, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{N} H^q(X_t, \mathbb{Q}) \to \cdots.
\]

This is called the Clemens-Schmid sequence. A theorem of Clemens and Schmid says that the sequence (1.3) is exact ([1, §3]). In particular, the first piece

\[
H^q(Y, \mathbb{Q}) \to H^q(X_t, \mathbb{Q}) \xrightarrow{N} H^q(X_t, \mathbb{Q})
\]

is called the local invariant cycle theorem ([3, (5.12)]).

In [2, (12.3.1)], S. Usui et al. proposed a problem whether the Clemens-Schmid sequence (1.3) or (1.4) is exact when we remove the assumption that \( \pi \) is proper. In this paper, we give a necessary and sufficient condition for that the sequence (1.4) is exact when \( \pi \) is a semistable family of open curves. In particular, we see that the Clemens-Schmid sequences of non-proper families are not exact in general.

The author would like to express his sincere gratitude to Professor Sampei Usui for stimulating discussions.
2. A Criterion and an example

In this section, we assume that $\pi$ is a family of curves. Let $D$ be a divisor on $X$ which is smooth and projective over $\Delta$, and such that $D + Y$ is a normal crossing divisor. We write $D_t = D \cap X_t$ for $t \in \Delta^*$ and $D^* = D \cap X^*$.

We discuss when the following sequence

$$(2.1) \quad H^1(Y - D \cap Y) \xrightarrow{i} H^1(X_t - D_t) \xrightarrow{N_0} H^1(X_t - D_t)$$

is exact, where $N_0$ denotes the log monodromy on $H^1(X_t - D_t)$. (We omit to write the coefficient $Q$.) The sequence (2.1) fits into the commutative diagram

$$\begin{array}{ccc}
H^0(D \cap Y) & \xrightarrow{\varpi} & H^0(D_t) \\
\uparrow r_0 & & \uparrow r_t \\
H^1(Y - D \cap Y) & \xrightarrow{i} & H^1(X_t - D_t) & \xrightarrow{N_0} & H^1(X_t - D_t) \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(X_t) & \xrightarrow{N} & H^1(X_t) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}$$

(2.2)

in which each vertical sequence is exact. The bottom sequence is exact by the local invariant cycle theorem ([3, (5.12)]). Therefore, the map $i$ is injective.

The following lemma is the key of our criterion.

**Lemma 2.1.** The image of $N$ coincides with the image of $N_0$.

**Proof.** Image $N \subset \text{Image } N_0$ is trivial. We show the converse. The cohomology groups $H^1(X_t, Q)$ and $H^1(X_t - D_t, Q)$ carry the limit mixed Hodge structures due to Steenbrink and Zucker ([3], [4]). The weight filtration $W_\bullet$ on $H^1(X_t, Q)$ is the weight monodromy filtration, which is explicitly given as follows:

$$W_2H^1(X_t) = H^1(X_t), \quad W_1H^1(X_t) = \ker N,$$

$$W_0H^1(X_t) = \text{Image } N, \quad W_{-1}H^1(X_t) = 0.$$

We put the weight filtration on $H^0(D_t)$ by $\text{Gr}_2^W H^0(D_t) = H^0(D_t)$. Then the weight filtration $W_\bullet$ on $H^1(X_t - D_t)$ (called the relative monodromy filtration) is explicitly given by

$$W_2H^1(X_t - D_t) = H^1(X_t - D_t), \quad W_1H^1(X_t - D_t) = W_1H^1(X_t),$$

$$W_0H^1(X_t - D_t) = W_0H^1(X_t), \quad W_{-1}H^1(X_t - D_t) = 0,$$
The map $N_0$ is a morphism of the mixed Hodge structure of type $(-1, -1)$ ([4, (3.13).iii]). Therefore $\text{Image} N_0$ is contained in $W_0 H^1(X - D_t, \mathbb{Q}(0)) = W_0 H^1(X_t)$. However, this is the image of $N$. Thus we have $\text{Image} N_0 \subset \text{Image} N$. 

Since the map $i$ is injective, the sequence (2.1) is exact if and only if

\[(2.3) \quad h^1(Y - D \cap Y) = \dim H^1(Y - D \cap Y) = \dim \ker N_0.\]

By Lemma 2.1, we have

\[
\dim \ker N_0 = h^1(X_t - D_t) - \dim \text{Image} N \\
= h^1(X_t - D_t) - h^1(X_t) + h^1(Y) \\
= \dim \text{Image} r_t + h^1(Y).
\]

On the other hand,

\[
h^1(Y - D \cap Y) = \dim \text{Image} r_0 + h^1(Y).
\]

Therefore, (2.3) holds if and only if

\[(2.4) \quad \dim \text{Image} r_0 = \dim \text{Image} r_t.\]

Let $n$ be the number of the irreducible components of $Y$, and $e$ the number of the components $Y_i$ of $Y$ such that $D \cap Y_i \neq \emptyset$. Then we have

\[
\dim \text{Image} r_t = \dim \ker (H^0(D_t) \to H^2(X_t)) \\
= \deg D_t - 1,
\]

and

\[
\dim \text{Image} r_0 = \dim \ker (H^0(D \cap Y) \to H^2(Y)) \\
= h^0(D \cap Y) - h^2(Y) + h^2(Y - D \cap Y) \\
= \deg D_t - n + (n - e) \\
= \deg D_t - e.
\]

Thus we have the following criterion:

**Theorem 2.2.** The Clemens-Schmid sequence (2.1) is exact if and only if $e = 1$, that is, the support of $D \cap Y$ is contained in one irreducible component of $Y$.

**Example 2.3.** We denote by $(x, y, z)$ homogeneous coordinates of the projective plane $\mathbb{P}^2_C$. Let $X$ be a complex submanifold of $\mathbb{P}^2(C) \times \Delta$ defined by an equation

\[zy^2 = zx^2 + t(x^3 - z^3),\]
and $\pi: X \to \Delta$ the projection. The morphism $\pi$ is smooth and proper over $\Delta^*$, and the central fiber $Y = \pi^{-1}(0)$ is a reduced divisor with normal crossing. Let $Y_1$, $Y_2$ and $Y_3$ be the irreducible components of $Y$ given by

\begin{align*}
Y_1 : & \quad z = 0, \quad t = 0, \\
Y_2 : & \quad y + x = 0, \quad t = 0, \\
Y_3 : & \quad y - x = 0, \quad t = 0
\end{align*}

respectively. Let $D = D_1 + D_2 + D_3$ be flat sections of $\pi$ given by

\begin{align*}
D_1 &= \{(0, 1, 0)\} \times \Delta, \\
D_2 &= \{(1, -1, \omega)\} \times \Delta, \\
D_3 &= \{(1, 1, \omega')\} \times \Delta,
\end{align*}

where $\omega$ and $\omega'$ are 3rd roots of unity. The divisor $D + Y$ is normal crossing.

The divisor $D_1$ (resp. $D_2$, $D_3$) meets only with $Y_1$ (resp. $Y_2$, $Y_3$). Thus, the Clemens-Schmid sequence (2.1) is not exact by Theorem 2.2.

A family $\pi: X \times X' \to \Delta$ with $X' \to \Delta$ a constant family of a projective nonsingular variety gives a non-exact Clemens-Schmid sequence (1.4) for all $q \geq 1$.

References


Graduate School of Mathematics
Kyushu University
Hakozaki Higashi-ku, Hukuoka 812-8581
JAPAN

e-mail: asakura@math.kyushu-u.ac.jp