Asakura, M.
Osaka J. Math.
40 (2003), 977–980

A CRITERION OF EXACTNESS OF THE CLEMENS-SCHMID SEQUENCES ARISING FROM SEMI-STABLE FAMILIES OF OPEN CURVES

MASANORI ASAKURA

(Received April 4, 2002)

1. Introduction

Let $\pi\colon X \to \Delta$ be a semistable family of projective algebraic varieties over the unit disk. In other words, π is flat and projective over Δ, smooth over $\Delta^* = \Delta - \{0\}$, and the central fiber $Y = \pi^{-1}(0)$ is a divisor with normal crossings without multiple components. We write $X_t := \pi^{-1}(t)$ for $t \in \Delta^*$ and $X^* := X - Y$. There is the Wang exact sequence

$$\cdots \to H^q(X^*, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to H^{q+1}(X^*, \mathbf{Q}) \to \cdots, \tag{1.1}$$

and the localization exact sequence

$$\cdots \to H^q_Y(X, \mathbf{Q}) \to H^q(X, \mathbf{Q}) \to H^q(X^*, \mathbf{Q}) \to H^{q+1}_Y(X, \mathbf{Q}) \to \cdots. \tag{1.2}$$

Here N denotes the log monodromy around $\Delta^* = \Delta - \{0\}$. Combining those sequences and the natural isomorphism $H^q(X, \mathbf{Q}) \simeq H^q(Y, \mathbf{Q})$, we obtain a sequence

$$\cdots \to H^{q-2}(X_t, \mathbf{Q}) \to H^q_Y(X, \mathbf{Q}) \to H^q(Y, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to \cdots. \tag{1.3}$$

This is called the *Clemens-Schmid sequence*. A theorem of Clemens and Schmid says that the sequence (1.3) is exact ([1, §3]). In particular, the first piece

$$H^q(Y, \mathbf{Q}) \longrightarrow H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \tag{1.4}$$

is called the *local invariant cycle theorem* ([3, (5.12)]).

In [2, (12.3.1)], S. Usui et al. proposed a problem whether the Clemens-Schmid sequence (1.3) or (1.4) is exact when we remove the assumption that π is proper. In this paper, we give a necessary and sufficient condition for that the sequence (1.4) is exact when π is a semistable family of open curves. In particular, we see that the Clemens-Schmid sequences of non-proper families are not exact in general.

The author would like to express his sincere gratitude to Professor Sampei Usui for stimulating discussions.

2. A Criterion and an example

In this section, we assume that π is a family of curves. Let D be a divisor on X which is smooth and projective over Δ, and such that $D+Y$ is a normal crossing divisor. We write $D_t = D \cap X_t$ for $t \in \Delta^*$ and $D^* = D \cap X^*$.

We discuss when the following sequence

$$H^1(Y - D \cap Y) \xrightarrow{i} H^1(X_t - D_t) \xrightarrow{N_0} H^1(X_t - D_t) \tag{2.1}$$

is exact, where N_0 denotes the log monodromy on $H^1(X_t - D_t)$. (We omit to write the coefficient $\mathbf{Q}$.) The sequence (2.1) fits into the commutative diagram

$$\begin{tikzcd}
 & H^0(D \cap Y) \arrow[r, "\simeq"] & H^0(D_t) & H^0(D_t) \\
 & H^1(Y - D \cap Y) \arrow[r, "i"] \arrow[u, "r_0"] & H^1(X_t - D_t) \arrow[r, "N_0"] \arrow[u, "r_t"'] & H^1(X_t - D_t) \arrow[u] \\
0 \arrow[r] & H^1(Y) \arrow[r] \arrow[u] & H^1(X_t) \arrow[r, "N"] \arrow[u] & H^1(X_t) \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0, \arrow[u]
\end{tikzcd} \tag{2.2}$$

in which each vertical sequence is exact. The bottom sequence is exact by the local invariant cycle theorem ([3, (5.12)]). Therefore, the map i is injective.

The following lemma is the key of our criterion.

Lemma 2.1. *The image of N coincides with the image of N_0.*

Proof. Image $N \subset$ Image N_0 is trivial. We show the converse. The cohomology groups $H^1(X_t, \mathbf{Q})$ and $H^1(X_t - D_t, \mathbf{Q})$ carry the limit mixed Hodge structures due to Steenbrink and Zucker ([3], [4]). The weight filtration $W_\bullet$ on $H^1(X_t, \mathbf{Q})$ is the weight monodromy filtration, which is explicitly given as follows:

$$W_2 H^1(X_t) = H^1(X_t), \quad W_1 H^1(X_t) = \ker N,$$
$$W_0 H^1(X_t) = \mathrm{Image}\, N, \quad W_{-1} H^1(X_t) = 0.$$

We put the weight filtration on $H^0(D_t)$ by $\mathrm{Gr}_2^W H^0(D_t) = H^0(D_t)$. Then the weight filtration $W_\bullet$ on $H^1(X_t - D_t)$ (called the relative monodromy filtration) is explicitly given by

$$W_2 H^1(X_t - D_t) = H^1(X_t - D_t), \quad W_1 H^1(X_t - D_t) = W_1 H^1(X_t),$$
$$W_0 H^1(X_t - D_t) = W_0 H^1(X_t), \quad W_{-1} H^1(X_t - D_t) = 0.$$

The map N_0 is a morphism of the mixed Hodge structure of type $(-1,-1)$ ([4, (3.13).iii)]). Therefore $\operatorname{Image} N_0$ is contained in $W_0H^1(X_t - D_t, \mathbf{Q}(0)) = W_0H^1(X_t)$. However, this is the image of N. Thus we have $\operatorname{Image} N_0 \subset \operatorname{Image} N$. $\square$

Since the map i is injective, the sequence (2.1) is exact if and only if

$$h^1(Y - D \cap Y) = \dim H^1(Y - D \cap Y) = \dim\ker N_0. \tag{2.3}$$

By Lemma 2.1, we have

$$\begin{aligned}\dim\ker N_0 &= h^1(X_t - D_t) - \dim\operatorname{Image} N\\ &= h^1(X_t - D_t) - h^1(X_t) + h^1(Y)\\ &= \dim\operatorname{Image} r_t + h^1(Y).\end{aligned}$$

On the other hand,

$$h^1(Y - D \cap Y) = \dim\operatorname{Image} r_0 + h^1(Y).$$

Therefore, (2.3) holds if and only if

$$\dim\operatorname{Image} r_0 = \dim\operatorname{Image} r_t. \tag{2.4}$$

Let n be the number of the irreducible components of Y, and e the number of the components Y_i of Y such that $D \cap Y_i \neq \emptyset$. Then we have

$$\begin{aligned}\dim\operatorname{Image} r_t &= \dim\ker\bigl(H^0(D_t) \to H^2(X_t)\bigr)\\ &= \deg D_t - 1,\end{aligned}$$

and

$$\begin{aligned}\dim\operatorname{Image} r_0 &= \dim\ker\bigl(H^0(D \cap Y) \to H^2(Y)\bigr)\\ &= h^0(D \cap Y) - h^2(Y) + h^2(Y - D \cap Y)\\ &= \deg D_t - n + (n - e)\\ &= \deg D_t - e.\end{aligned}$$

Thus we have the following criterion:

Theorem 2.2. *The Clemens-Schmid sequence* (2.1) *is exact if and only if* $e = 1$, *that is, the support of* $D \cap Y$ *is contained in one irreducible component of* Y.

EXAMPLE 2.3. We denote by (x, y, z) homogeneous coordinates of the projective plane $\mathbf{P}^2_{\mathbf{C}}$. Let X be a complex submanifold of $\mathbf{P}^2(\mathbf{C}) \times \Delta$ defined by a equation

$$zy^2 = zx^2 + t(x^3 - z^3),$$

and $\pi : X \to \Delta$ the projection. The morphism π is smooth and proper over Δ^*, and the central fiber $Y = \pi^{-1}(0)$ is a reduced divisor with normal crossing. Let Y_1, Y_2 and Y_3 be the irreducible components of Y given by

$$\begin{aligned} Y_1 &: z = 0, \quad t = 0, \\ Y_2 &: y + x = 0, \quad t = 0, \\ Y_3 &: y - x = 0, \quad t = 0 \end{aligned}$$

respectively. Let $D = D_1 + D_2 + D_3$ be flat sections of π given by

$$\begin{aligned} D_1 &= \{(0, 1, 0)\} \times \Delta, \\ D_2 &= \{(1, -1, \omega)\} \times \Delta, \\ D_3 &= \{(1, 1, \omega')\} \times \Delta, \end{aligned}$$

where ω and ω' are 3rd roots of unity. The divisor $D + Y$ is normal crossing.

The divisor D_1 (resp. D_2, D_3) meets only with Y_1 (resp. Y_2, Y_3). Thus, the Clemens-Schmid sequence (2.1) is not exact by Theorem 2.2.

A family $\pi : X \times X' \to \Delta$ with $X' \to \Delta$ a constant family of a projective nonsingular variety gives a non-exact Clemens-Schmid sequence (1.4) for all $q \geq 1$.

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Graduate School of Mathematics
Kyushu University
Hakozaki Higashi-ku, Hukuoka 812-8581
JAPAN
e-mail: asakura@math.kyushu-u.ac.jp