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SAMPLE LYAPUNOV EXPONENT FOR A CLASS OF LINEAR MARKOVIAN SYSTEMS OVER $\mathbb{Z}^d$

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1. Introduction

Let $\mathbb{Z}^d$ be the d-dimensional cubic lattice space, and let $\{Y_i(t)\}_{i\in\mathbb{Z}^d}$ be independent copies of a one-dimensional Lévy process $Y(t)$ defined on a probability space $(\Omega, \mathcal{F}, P^Y)$. Regarding $\{Y_i(t)\}_{i\in\mathbb{Z}^d}$ as random noises, we consider the following linear stochastic partial differential equation (SPDE) over $\mathbb{Z}^d$;

$$(1.1) \qquad d\xi_i(t) = \kappa A\xi_i(t)dt + \xi_i(t-)dY_i(t) \quad (i \in \mathbb{Z}^d),$$

where $\kappa > 0$ is a constant and $A = \{a(i,j)\}_{i,j\in\mathbb{Z}^d}$ is an infinitesimal generator of a continuous time random walk on $\mathbb{Z}^d$, i.e.

$$(1.2) \quad a(0,i) \geq 0 \quad (i \neq 0), \quad \sum_{i\in\mathbb{Z}^d} a(0,i) = 0, \quad a(i,j) = a(0,j-i) \quad (i,j \in \mathbb{Z}^d)$$

and

$$A\xi_i = \sum_{j\in\mathbb{Z}^d} a(i,j)\xi_j.$$

Under a mild assumption on $Y(t)$ the equation (1.1) is well-posed and the solution defines a linear Markovian system in the sense of Liggett's book([7], Chap. IX).

When $\{Y_i(t)\}$ are independent copies of a standard Brownian motion, (1.1) is called *parabolic Anderson model* which has been extensively studied in [10], [8], [2], [3] from the view-point of intermittency. On the other hand when $Y_i(t) = -N_i(t)+t$ and $\{N_i(t)\}_{i\in\mathbb{Z}^d}$ are independent copies of a Poisson process with parameter one, (1.1) defines *a linear system with deterministic births and random deaths* introduced in [7], which was discussed from the view-point of ergodic problems. This process is a dual object of the survival probability of a random walker in a spatially and temporally fluctuating random environment, for which an asymptotic analysis was executed in [9]. The present form of the equation (1.1) was first treated in [1] where

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an asymptotic analysis of the moment Lyapunov exponents for solutions of (1.1) was discussed under a stronger moment condition on the Lévy measure.

In this paper we are concerned with asymptotic analysis of the sample Lyapunov exponent for the solutions of (1.1). Let $\psi(z)$ be the characteristic exponent of the Lévy process $Y(t)$,

$$(1.3)\quad \psi(z)=-\frac{\alpha^2}{2}z^2+\sqrt{-1}\beta z+\int_{\mathbb{R}\setminus\{0\}}\left(e^{\sqrt{-1}zu}-1-\sqrt{-1}zuI\left(|u|<\frac{1}{2}\right)\right)\rho(du),$$

where $I(A)$ stands for the indicator function of $A\subset\mathbb{R}$, α and β are real constants, and ρ is a Radon measure in $\mathbb{R}\setminus\{0\}$ satisfying

$$(1.4)\qquad \int_{\mathbb{R}\setminus\{0\}}\min\{u^2,1\}\rho(du)<\infty.$$

In order to formulate the sample Lyapunov exponent we restrict our consideration to the situation that (1.1) admits nonnegative solutions with finite mean, which is realized by the following condition.

Condition [A]

$$(1.5)\qquad \rho((-\infty,-1))=0,$$

and

$$(1.6)\qquad \int_{(1,\infty)}u\rho(du)<\infty.$$

Under the condition [A] there exists a unique nonnegative $L^1(\gamma)$ solution with $\xi_i(0)=1\quad(i\in\mathbb{Z}^d)$, which is denoted by $\xi^1(t)=\{\xi_i^1(t)\}$ (The definition of $L^1(\gamma)$ solutions are given later). We first establish that there exists a constant λ such that

$$(1.7)\qquad \lim_{t\to\infty}E^Y\left(\left|\frac{1}{t}\log\xi_i^1(t)-\lambda\right|\right)=0.$$

$\lambda=\lambda(\kappa A;Y)$ is called *sample Lyapunov exponent*in L^1-sense. In the section 3 we prove this result in a general setting and discuss some relations between the sample Lyapunov exponent and the almost sure Lyapunov exponent. In the section 4 we derive some inequalities on $\lambda(\kappa A;Y)$, from which it follows that $\lambda(\kappa A;Y)$ is continuous in $\kappa>0$ and Y in a suitable sense.

Our main concern of the present paper is to investigate an asymptotics of $\lambda(\kappa A;Y)$ as $\kappa\searrow 0$. Assume further that $Y(t)$ has zero mean, so that (1.3) turns to

$$(1.8)\qquad \psi(z)=-\frac{\alpha^2}{2}z^2+\int_{\mathbb{R}\setminus\{0\}}(e^{\sqrt{-1}zu}-1-\sqrt{-1}zu)\rho(du).$$

Let

$$\lambda_0(Y) = -\frac{\alpha^2}{2} + \int_{[-1,\infty)} (\log(1+u) - u)\rho(du), \tag{1.9}$$

which coincides with the sample Lyapunov exponent in the non-interacting case, i.e. $\kappa = 0$. Note that $-\infty \leq \lambda_0(Y) < 0$ in general, unless $Y(t) \equiv 0$.

In particular if we consider a two dimensional stochastic equation instead of (1.1), it is possible to obtain its precise asymptotics as $\kappa \searrow 0$ under some moment condition on the Lévy measure ρ at a neighbourhood of -1, which is discussed in the section 5. In the final section we discuss asymptotic estimates of $\lambda(\kappa A; Y)$ as $\kappa \searrow 0$ in two extremal cases involving a moment condition of ρ around -1, which combines with the inequalities on $\lambda(\kappa A; Y)$ to get a continuity result of $\lambda(\kappa A; Y)$ at $\kappa = 0$.

It is to be noted that the moment condition on ρ at a neighbourhood of -1 is significant for an asymptotics of $\lambda(\kappa A; Y)$ as $\kappa \searrow 0$. In fact it is shown that if $\rho(\{-1\}) > 0$, it holds $\lambda(\kappa A; Y) \approx \log \kappa$ for small $\kappa > 0$, which extends the previous result in [9]. On the other hand if $\rho = 0$ and $\alpha \neq 0$, i.e. $Y(t) = \alpha B(t)$, it is known that $\lambda(\kappa A; Y) - \lambda_0(Y) \approx 1/\log(1/\kappa)$ for small $\kappa > 0$, which was recently obtained by [3]. For a general $Y(t)$, under a moment condition on ρ at a neighbourhood of -1, we obtain the same lower bound estimate as in the Brownian case and a slightly weaker upper bound estimate for $\lambda(\kappa A; Y) - \lambda_0(Y)$ as $\kappa \searrow 0$ than the case.

2. Well-posedness of the SPDE and Feynman Kac formula

Let J be a countable set, $A = (a(i,j))_{i,j \in J}$ be a $J \times J$ real matrix satisfying that,

$$a(i,j) \geq 0 \quad (i \neq j), \sum_{j \in J} a(i,j) = 0, \quad \text{and} \quad \sup_{i \in J} |a(i,i)| < \infty, \tag{2.1}$$

and let $\{Y_i(t)\}_{i \in J}$ be independent copies of a one-dimensional Lévy process $Y(t)$ with the characteristic exponent $\psi(z)$ of (1.3). It is assumed that $\{Y_i(t)\}_{i \in J}$ are defined on a complete probability space $(\Omega, \mathcal{F}, P^Y)$ with a filtration $(\mathcal{F}_t)$ such that $\{Y_i(t)\}$ are $(\mathcal{F}_t)$-adapted and $\{Y_i(t+r) - Y_i(t); i \in J, r \geq 0\}$ are independent of $\mathcal{F}_t$ for each $t \geq 0$.

Let us consider the following linear stochastic equation:

$$\xi_i(t) - \xi_i(0) = \int_0^t \sum_{j \in J} a(i,j)\xi_j(s)ds + \int_0^{t+} \xi(s-)dY_i(s) \quad (i \in J). \tag{2.2}$$

To formulate a solution of (2.2) we first fix a positive and summable vector $\gamma =$

$\{\gamma_i\}_{i\in J}$ satisfying that for some constant $\Gamma > 0$

$$\sum_{i\in J}\gamma_i|a(i,j)| \le \Gamma\gamma_j \quad (j\in J), \tag{2.3}$$

and denote by $L^1(\gamma)$ the totality of vectors $\xi = \{\xi_i\}_{i\in J}$ such that

$$\|\xi\|_{L^1(\gamma)} = \sum_{i\in J}\gamma_i|\xi_i| < \infty.$$

$\xi(t) = (\xi_i(t))_{i\in J}$ is an $L^1(\gamma)$-solution of (2.2) if the following three conditions are satisfied;

(a) for each $i \in J$, $\xi_i(t)$ is an $(\mathcal{F}_t)$-predictable and right continuous process with left limit defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^Y)$,

(b) $\|\xi(t)\|_{L^1(\gamma)}$ is locally bounded in $t \ge 0$, P^Y-a.s.

(c) $\xi(t) = (\xi_i(t))_{i\in J}$ satisfies the equation (2.2).

The last term of (2.2) is the Itô's stochastic integral which is interpreted as follows. Consider the Lévy-Itô's decomposition of $Y_i(t)$,

$$Y_i(t) = \alpha B_i(t) + \beta t + \int_{[0,t]\times(|u|<1/2)} u\tilde{N}_i(dsdu) + \int_{[0,t]\times(|u|\ge 1/2)} uN_i(dsdu), \tag{2.4}$$

where $\{B_i(t)\}$ are independent copies of a standard Brownian motion, $\{N_i(dsdu)\}$ are independent copies of a Poisson random measure on $[0,\infty)\times\mathbb{R}\setminus\{0\}$ with intensity measure $ds\rho(du)$, and $\tilde{N}_i(dsdu) = N_i(dsdu) - ds\rho(du)$ $(i \in J)$. Then

$$\begin{aligned}\int_0^{t+}\xi_i(s-)dY_i(s) &= \alpha\int_0^t \xi_i(s)dB_i(s) + \beta\int_0^t \xi_i(s)ds \\ &\quad + \int_{[0,t]\times(|u|<1/2)} u\xi_i(s-)\tilde{N}_i(dsdu) \\ &\quad + \int_{[0,t]\times(|u|\ge 1/2)} u\xi_i(s-)N_i(dsdu).\end{aligned} \tag{2.5}$$

To guarantee the well-posedness of the equation (2.2) we impose the following condition.

$$\int_{(|u|>1)} |u|\rho(du) < \infty. \tag{2.6}$$

Theorem 2.1. *Assume* (2.6). *Let* $\xi(0) = \{\xi_i(0)\}$ *be an* $\mathcal{F}_0$*-measurable random vector satisfying* $\|\xi(0)\|_{L^1(\gamma)} < \infty$ P^Y*-a.s. Then the equation* (2.2) *has a pathwise-unique* $L^1(\gamma)$*-solution.*

To be selfcontained we give a proof of the theorem breafly, although it may not be novel except an $L^1(\gamma)$ argument, which is needed due to the weaker assumption (2.6) than in [1]. We may assume that $E^Y(\|\xi(0)\|_{L^1(\gamma)}) < \infty$.

1. Let $\{J_n\}$ be an increasing sequence of finite subsets of J with $\cup_{n\geq 1} J_n = J$. For $n \geq 1$ consider the following finite-dimensional equation.

$$(2.7)\quad \begin{aligned} \xi_i^{(n)}(t) &= \xi_i(0) + \int_0^t \sum_{j\in J} a(i,j)\xi_j^{(n)}(s)ds + \int_0^{t+} \xi_i^{(n)}(s-)dY_i(s) \quad (i \in J_n) \\ \xi_i^{(n)}(t) &= \xi_i(0) \quad (i \in J \setminus J_n). \end{aligned}$$

It is easy to see that (2.7) has a pathwise unique solution $\{\xi_i^{(n)}(t)\}$. Moreover letting $\eta_i^{(n)}(t) = \sup_{0\leq s\leq t} |\xi_i^{(n)}(s)|$, by the condition (2.6) one can see that

$$E^Y(\eta_i^{(n)}(t)) < \infty \quad (i \in J).$$

2. We claim that there is a constant $C > 0$ such that for every $n \geq 1$,

$$(2.8)\quad E^Y\left(\sup_{0\leq r\leq t} |\int_0^{r+} \xi_i^{(n)}(s-)dY_i(s)|\right) \leq \frac{1}{2}E^Y(\eta_i^{(n)}(t)) + \frac{C}{2}\int_0^t E^Y(\eta_i^{(n)}(s))ds.$$

To show this we apply the following maximal inequality for martingales.

Lemma 2.1. *For $0 < p \leq 1$, there is an absolute constant $C_p > 0$ such that for every $t > 0$,*

$$(2.9)\quad \begin{aligned} &E^Y\left(\sup_{0\leq r\leq t} |\alpha \int_0^r \xi_i^{(n)}(s)dB_i(s) + \int_{[0,r]\times(|u|<1/2)} u\xi_i^{(n)}(s-)\tilde{N}_i(dsdu)|^p\right) \\ &\leq C_p\left(\alpha^2 + \int_{(|u|<1/2)} |u|^2\rho(du)\right)^{p/2} E^Y\left(\left(\int_0^t \xi_i^{(n)}(s)^2 ds\right)^{p/2}\right) \end{aligned}$$

Proof. The maximal inequalities for continuous martingales are found in Theorem III.3.1 of [6], not all of which are valid for discontinuous martingales in general. However the inequality (2.9) can be proved for the discontinuous martingale without any change of the argument there. □

Thus by (2.9) we have a constant $C_1 > 0$ such that

$$E^Y\left(\sup_{0\leq r\leq t}\left|\alpha \int_0^r \xi_i^{(n)}(s)dB_i(s) + \int_{[0,r]\times(|u|<1/2)} u\xi_i^{(n)}(s-)\tilde{N}_i(dsdu)\right|\right)$$

$$\leq C_1\left(\alpha^2+\int_{(|u|<1/2)}|u|^2\rho(du)\right)^{1/2}E^Y\left(\sqrt{\int_0^t \xi_i^{(n)}(s)^2ds}\right)$$

$$\leq \frac{1}{2}E^Y(\eta_i^{(n)}(t))+\frac{1}{2}C_1^2\left(\alpha^2+\int_{(|u|<1)}|u|^2\rho(du)\right)\int_0^t E^Y(\eta_i^{(n)}(s))ds,$$

which yields (2.8) with

$$C=C_1^2\left(\alpha^2+\int_{(|u|<1/2)}|u|^2\rho(du)\right)+|\beta|+\int_{(|u|\geq 1/2)}|u|\rho(du).$$

3. Combining (2.7) and (2.8) we get

$$E^Y(\eta_i^{(n)}(t))\leq 2E^Y(|\xi_i(0)|)+\int_0^t 2\sum_{j\in J}|a(i,j)|E^Y(\eta_j^{(n)}(s))ds$$
$$+C\int_0^t E^Y(\eta_i^{(n)}(s))ds,$$

from which

$$E^Y(\eta_i^{(n)}(t))\leq 2\sum_{j\in J}(e^{tM})_{ij}E^Y(|\xi_j(0)|), \tag{2.10}$$

where

$$M_{ij}=2|a(i,j)|+C\delta_{ij}\quad (i,j\in J).$$

Particularly by (2.3) and (2.9) it holds that

$$E^Y(\|\eta^{(n)}(t)\|_{L^1(\gamma)})\leq e^{(2\Gamma+C)t}E^Y\left(\|\xi(0)\|_{L^1(\gamma)}\right). \tag{2.11}$$

4. For $n\geq m$, set

$$\eta_i^{(n,m)}(t)=\sup_{0\leq r\leq t}|\xi_i^{(n)}(r)-\xi_i^{(m)}(r)|.$$

Applying the same argument as in the step 2 we get

$$E^Y(\eta_i^{(n,m)}(t))\leq I(J_m)(i)\sum_{j\in J}M_{ij}\int_0^t E^Y(\eta_j^{(n,m)}(s))ds$$
$$+I(J_n\setminus J_m)(i)(E^Y(\eta_i^{(n)}(t))+E^Y(|\xi_i(0)|)),$$

from which it follows

$$E^Y(\|\eta^{(n,m)}(t)\|_{L^1(\gamma)})\leq te^{(2\Gamma+C)t}\sum_{j\in J_n\setminus J_m}\gamma_j(E^Y(\eta_j^{(n)}(t))+E^Y(|\xi_j(0)|)). \tag{2.12}$$

Moreover by (2.10) the r.h.s. of (2.12) vanishes as n and m tend to ∞. Accordingly there exists an $L^1(\gamma)$-valued stochastic process $\xi(t) = \{\xi_i(t)\}_{i\in J}$ such that for every $t > 0$

$$\lim_{n\to\infty} E^Y\left(\sup_{0\le r\le t} \|\xi^{(n)}(r) - \xi(r)\|_{L^1(\gamma)}\right) = 0.$$

Furthermore it is obvious that $\xi(t) = \{\xi_i(t)\}_{i\in J}$ is an $L^1(\gamma)$-solution of (2.2).

The proof of the pathwise uniqueness is routine, so it is omitted.

REMARK 2.1. The condition (2.6) is necessary for the solution of (2.2) to have finite first moment, that is

$$E^Y(|\xi_i(t)||\mathcal{F}_0) < \infty \quad (i \in J).$$

It would be rather delicate to discuss the well-posedness of (2.2) without the condition (2.6).

Next we discuss Feynman-Kac representation for the solution of (2.2). Let $(X(t), \mathbb{P}_i)$ be a continuous time Markov chain with state space J generated by the infinitesimal marix $A = \{a(i,j)\}$. We denote the associated transition matrix by $Q_t = (Q_t(i,j))$.

Theorem 2.2. *Under the same assumption as in Theorem* 2.1 *the solution* $\xi(t) = \{\xi_i(t)\}$ *is represented by*

$$\xi_i(t) = \mathbb{E}_i\left(\xi_{X(t)}(0) \exp \int_0^{t+} dZ_{X(t-s)}(s) : \int_0^{t+} N_{X(t-s)}(ds, \{-1\}) = 0\right), \tag{2.13}$$

where $\{Z_i(t)\}$ *are complex-valued independent Lévy processes given by*

$$\begin{aligned} Z_i(t) = \alpha B_i(t) &+ \int_{[0,t]\times(|u|<1/2)} \log(1+u)\tilde{N}_i(dsdu) \\ &+ \int_{[0,t]\times(|u|\ge 1/2, u\ne -1)} \log|1+u|N_i(dsdu) \\ &+ mt + \sqrt{-1}\pi N_i([0,t]\times(-\infty,-1)) \end{aligned} \tag{2.14}$$

with

$$m = \int_{(|u|<1/2)} (\log(1+u) - u)\rho(du) + \beta - \frac{\alpha^2}{2}, \tag{2.15}$$

$$\int_0^{t+} dZ_{X(t-s)}(s) = \sum_{j\in J} \int_0^{t+} I(X(t-s) = j)dZ_j(s),$$

and

$$\int_0^{t+} N_{X(t-s)}(ds,\{-1\}) = \sum_{j\in J}\int_0^{t+} I(X(t-s)=j)N_j(ds,\{-1\}).$$

Proof. First we note that the r.h.s. of (2.13) converges absolutely P^Y-a.s., because under P^Y, the distributions of $\int_0^{t+} dZ_{X(t-s)}(s)$ and $Z_i(t)$ coincide for given $t>0$ conditioned on $\{X(s)\}_{0\le s\le t}$, and

$$\begin{aligned} &E^Y(|\exp Z_i(t)|)\\ &= \exp\left(\frac{\alpha^2}{2} + \int_{(u\neq 0,-1)}\left(|1+u| - 1 - I\left(|u|<\frac{1}{2}\right)\log(1+u)\right)\rho(du) + m\right)t. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{i\in J}\gamma_i E^Y\left(\mathbb{E}_i\left(\left|\xi_{X(t)}(0)\exp\Big(\sum_{j\in J}\int_0^{t+} I(X(t-s)=j)dZ_j(s)\Big)\right|\right)\right)\\ &= \sum_{i\in J}\gamma_i\mathbb{E}_i(E^Y(|\xi_{X(t)}(0)|)E^Y(|\exp Z_i(t)|))\\ &= \sum_{i\in J}\gamma_i\sum_{j\in J}Q_t(i,j)E^Y(|\xi_j(0)|)E^Y(|\exp Z_i(t)|)\\ &\le e^{t\Gamma}E^Y(\|\xi(0)\|_{L^1(\gamma)})E^Y(|\exp Z_i(t)|) < \infty. \end{aligned}$$

The rest is essentially the same as the proof of Lemma 3.1 of [9]. For $p>0$, let

$$Y_i^{(p)}(t) = Y_i(t) + e^{-p}N_i([0,t]\times\{-1\}).$$

Using Itô's formula (cf. [6, Theorem II.5.1]) we have

$$\begin{aligned} (2.16)\qquad &\exp\left(\sum_{j\in J}\int_0^{t+} I(X(t-s)=j)(dZ_j(s) - pN_j(ds,\{-1\}))\right) - 1\\ &= \int_0^{t+}\left(\exp\sum_{l\in J}\int_0^{r-} I(X(t-s)=l)(dZ_l(s) - pN_l(ds,\{-1\}))\right)\\ &\qquad\times\sum_{j\in J}I(X(t-r)=j)dY_j^{(p)}(r). \end{aligned}$$

Set

$$\xi_i^{(p)}(t) = \mathbb{E}_i\left(\xi_{X(t)}(0)\exp\left(\sum_{j\in J}\int_0^{t+} I(X(t-s)=j)(dZ_j(s) - pN_j(ds,\{-1\}))\right)\right).$$

Using (2.16), Markov property of $(X(t), \mathbb{P}_i)$ and a stochastic Fubini theorem one can easily see

$$\begin{aligned}
&\xi_i^{(p)}(t) - \sum_{j\in J} Q_t(i,j)\xi_j(0) \\
&= \int_0^{t+} \mathbb{E}_i\left(\xi_{X(t)}(0)\exp\int_0^{r-}(dZ_{X(t-s)}(s) - pN_{X(t-s)}(ds,\{-1\}))dY^{(p)}_{X(t-s)}(r)\right) \\
&= \int_0^{t+} \sum_{j\in J} Q_{t-r}(i,j)\xi_j^{(p)}(r-)dY_j^{(p)}(r),
\end{aligned}$$

so letting $p \to \infty$, we obtain

$$\xi_i^{(\infty)}(t) = \mathbb{E}_i\left(\xi_{X(t)}(0)\exp\int_0^{t+} dZ_{X(t-s)}(s) : \int_0^{t+} N_{X(t-s)}(ds,\{-1\}) = 0\right),$$

and

$$\xi_i^{(\infty)}(t) - \sum_{j\in J} Q_t(i,j)\xi_j(0) = \int_0^{t+} \sum_{j\in J} Q_{t-r}(i,j)\xi_j^{(\infty)}(r-)dY_j(r). \tag{2.17}$$

It is easy to see that $\{\xi_i(t)\}$ itself satisfies the equation (2.17) and that the pathwise uniqueness holds for solutions of (2.17). Therefore $\xi_i(t) = \xi_i^{(\infty)}(t)$ holds for every $t \geq 0$, $i \in J$, P^Y-a.s, which completes the proof of Theorem 2.2. □

Corollary 2.1. *Suppose that the condition* [A] *is fulfilled. Then the equation* (2.2) *has a unique nonnegative $L^1(\gamma)$-solution, if*

$$\xi(0) = \{\xi_i(0)\} \in L^1(\gamma), \quad \text{and} \quad \xi_i(0) \geq 0 \quad \text{for all} \quad i \in J \quad P^Y\text{–}a.s.$$

The proof is immediate from Theorem 2.2 because $\{Z_i(t)\}_{i\in J}$ are real-valued processes due to the condition (1.4).

3. Sample Lyapunov exponent

We first establish an existence theorem of sample Lyapunov exponent in L^1-sense for a class of stochastic partial differential equations in a general setting. Let G be a topological abelian group with a Haar measure m. Suppose that we are given a system of random kernels $\{p^\omega(s,y;t,x); 0 \leq s < t, x, y \in G\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_s^t)_{0\leq s\leq t}$ that satisfies the following conditions.

[B-1] $p^\omega(s,y;t,x)$ is a jointly measurable non-negative random field in $(s,y;t,x)$ satisfying that for $0 \leq s < r < t$ and $y, x \in G$,

$$p^\omega(s,y;t,x) = \int_G m(dz)p^\omega(s,y;r,z)p^\omega(r,z;t,x).$$

[B-2] (independent increments)

(i) for any $0 \le s < t$, $\{p^\omega(s,y;t,x); x, y \in G\}$ are $\mathcal{F}_s^t$ measurable and

(ii) for any $0 \le t_1 < t_2 < \cdots < t_n$, $\{\mathcal{F}_0^{t_1}, \mathcal{F}_{t_1}^{t_2}, \cdots, \mathcal{F}_{t_{n-1}}^{t_n}\}$ are independent.

[B-3] (homogenuity) For any $h > 0, z \in G$, the probability laws of$\{p^\omega(s,y;t,x); 0 \le s < t, x, y \in G\}$ and $\{p^\omega(s+h, y+z; t+h, x+z); 0 \le s < t, x, y \in G\}$ coincide.

[B-4] Let $u_1^\omega(t,x) = \int_G m(dy)p^\omega(0,y;t,x)$. There exists a constant $\alpha > 0$ such that for every $t > 0$,

$$Eu_1^\omega(t,x) + Eu_1^\omega(t,x)^{-\alpha} < \infty.$$

Then we obtain

Theorem 3.1. *Under the conditions* [B-1]–[B-4] *there exists a constant* λ *such that*

$$\lim_{t\to\infty} E\left(\frac{1}{t}\log u_1^\omega(t,x)\right) = \sup_{t>0} E\left(\frac{1}{t}\log u_1^\omega(t,x)\right) = \lambda. \tag{3.1}$$

Moreover for every $p > 0$,

$$\lim_{t\to\infty} E\left|\frac{1}{t}\log u_1^\omega(t,x) - \lambda\right|^p = 0. \tag{3.2}$$

We call λ the sample Lyapunov exponent of $\{p^\omega(s,y;t,x); 0 \le s < t, x, y \in G\}$.

Corollary 3.1. *Suppose that* $u^\omega(0,x)$ *is an* $\mathcal{F}_0^0$*-measurable nonnegative random field satisfying that*

$$E(u^\omega(0,x)) + E(|\log u^\omega(0,x)|) \text{ is bounded in } x \in G, \tag{3.3}$$

and set

$$u^\omega(t,x) = \int_G m(dy)u^\omega(0,y)p^\omega(0,y;t,x). \tag{3.4}$$

Then

$$\lim_{t\to\infty} E\left|\frac{1}{t}\log u^\omega(t,x) - \lambda\right| = 0. \tag{3.5}$$

The proof of Theorem 3.1 is rather standard in the situation that the moment Lyapunov exponent $\gamma(p) = \lim_{t\to\infty}(1/t)\log E(u(t,x)^p)$ is well-posed in a neighbourhood of $p = 0$, for which the condition [B-4] is an essential requirement. In fact, it will be shown that λ of (3.2) coincides with the left derivative of $\gamma(p)$ at $p=0$.

We may assume $0<\alpha\le 1$ in [B-4]. For $-\alpha<p\le 1$, let

$$M_p(t)=Eu_1^\omega(t,x)^p.$$

Noting that $M_p(t)$ is constant in $x\in G$ by [B-3], and set

$$\gamma(p,t)=\frac{1}{t}\log M_p(t).$$

Lemma 3.1.

(i) $M_1(t)=e^{ct}\quad (t\ge 0)$ *for some real c.*

(ii) *For* $-\alpha\le p\le 0$,

$$\gamma(p)=\lim_{t\to\infty}\frac{1}{t}\log M_p(t)=\inf_{t>0}\frac{1}{t}\log M_p(t)\quad \textit{exists,} \tag{3.6}$$

and

$$\gamma(p)\ge cp\quad \textit{for}\quad p\in[-\alpha,0]. \tag{3.7}$$

(iii) *For* $0\le p\le 1$,

$$\gamma(p)=\lim_{t\to\infty}\frac{1}{t}\log M_p(t)=\sup_{t>0}\frac{1}{t}\log M_p(t)\quad \textit{exists,} \tag{3.8}$$

and

$$\gamma(p)\le cp\quad \textit{for}\quad p\in[0,1]. \tag{3.9}$$

(iv) $\gamma(p)$ *is a convex function of* $p\in[-\alpha,1]$ *with* $\gamma(0)=0$, *so that the left and right derivatives* $\gamma'(p-)$ *and* $\gamma'(p+)$ *exist for* $-\alpha<p<1$. *In particular* $\gamma(p)/p$ *is non-decreasing in* $p\in[-\alpha,0)\cup(0,1]$, *and*

$$\lim_{p\nearrow 0}\frac{\gamma(p)}{p}=\gamma'(0-)\quad \textit{and}\quad \lim_{p\searrow 0}\frac{\gamma(p)}{p}=\gamma'(0+). \tag{3.10}$$

Proof. First we note that by [B-1] to [B-4]

$$M_1(t+s)=M_1(t)M_1(s)\quad (t,s\ge 0),$$

which yields (i). Next by [B-1]

$$u_1^\omega(t+s,x)=\left(\int_G m(dy)u_1^\omega(s,y)\hat{p}^\omega(s,y;s+t,x)\right)\int_G m(dz)p^\omega(s,z;t+s,x),$$

where

$$(3.11) \qquad \hat{p}^{\omega}(s,y;s+t,x) = p^{\omega}(s,y;s+t,x)\left(\int_G m(dz)p^{\omega}(s,z;t+s,x)\right)^{-1}.$$

Using this, [B-2]–[B-4], and Jensen's inequality with a convex function x^p $(p<0)$, we see the subadditivity of $\log M_p(t)$, i.e.

$$M_p(t+s) \le M_p(t)M_p(s) \quad (s>0, t>0),$$

and

$$M_p(t) \ge M_{|p|}(t)^{-1} \ge M_1(t)^p = e^{cpt}.$$

Thus we get

$$\lim_{t\to\infty}\frac{1}{t}\log M_p(t) = \inf_{t>0}\frac{1}{t}\log M_p(t) \ge cp \quad (-\alpha \le p \le 0),$$

which yields (3.6) and (3.7) and a similar argument yields (iii). (iv) is elementary, so omitted. □

The following lemma follows immediately from Lemma 3.1 by a standard large deviation argument to use the exponential Chebyshev inequality and (3.10).

Lemma 3.2.
(i) *For any* $\epsilon>0$ *there exist* $c(\epsilon)>0$ *and* $t(\epsilon)>0$ *such that for* $t>t(\epsilon)$

$$(3.12) \qquad P(u_1^{\omega}(t,x) \le e^{t(\gamma'(0-)-\epsilon)}) \le e^{-c(\epsilon)t},$$

and

$$(3.13) \qquad P(u_1^{\omega}(t,x) \ge e^{t(\gamma'(0+)+\epsilon)}) \le e^{-c(\epsilon)t}.$$

(ii) *For every* $p>0$ *and* $\epsilon>0$

$$(3.14) \qquad \lim_{t\to\infty} E(|\log u_1^{\omega}(t,x)|^p; u_1^{\omega}(t,x) \le e^{t(\gamma'(0-)-\epsilon)}) = 0.$$

(iii) *For every* $p>0$

$$(3.15) \qquad E\left(\left|\frac{1}{t}\log u_1^{\omega}(t,x)\right|^p\right) \quad \textit{is bounded in } t>0.$$

REMARK 3.1. From (3.12), (3.13) and the Borel-Cantelli lemma it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log u_1^\omega(n,x) \le \gamma'(0+) \quad P\text{–}a.s. \tag{3.16}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log u_1^\omega(n,x) \ge \gamma'(0-) \quad P\text{–}a.s. \tag{3.17}$$

Accordingly, if $\gamma(p)$ is differentiable at $p = 0$, it holds

$$\lim_{n\to\infty} \frac{1}{n} \log u_1^\omega(n,x) = \gamma'(0) \quad P\text{–}a.s. \tag{3.18}$$

However it seems extremely difficult to verify the differentiability of $\gamma(p)$. On the other hand it is obvious that

$$\liminf_{t\to\infty} \frac{1}{t} \log u_1^\omega(t,x) \le \liminf_{n\to\infty} \frac{1}{n} \log u_1^\omega(n,x) \quad P^Y\text{–}a.s. \tag{3.19}$$

It should be noted that the equality in (3.19) does not hold in general, which will be discussed later, (see Theorem 3.3).

The following lemma is a key point for the proof of Theorem 3.1.

Lemma 3.3.

$$\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) = \gamma'(0-). \tag{3.20}$$

Proof. Since $\log M_p(t)$ is subadditive in $t > 0$ if $p < 0$,

$$t\gamma(p,t) = E(\log u_1^\omega(t,x)) = \lim_{p\nearrow 0} \frac{\log M_p(t)}{p}$$

is a superadditive function of $t > 0$, which implies

$$\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) = \sup_{t>0} \frac{1}{t} E(\log u_1^\omega(t,x)) \le \sup_{t>0} \frac{1}{t} \log E(u_1^\omega(t,x)) \le c. \tag{3.21}$$

Noting that $\gamma(p,t)$ is convex in p, so that $\gamma(p,t)/p$ is nondecreasing as $p \nearrow 0$, we see that for $-\alpha < p < 0$,

$$\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) = \lim_{t\to\infty} \lim_{p\nearrow 0} \frac{\gamma(p,t)}{p} \ge \frac{\gamma(p)}{p},$$

which yields

$$\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) \geq \gamma'(0-). \tag{3.22}$$

On the other hand by (3.6) and (3.21)

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) &= \sup_{t>0} \lim_{p\nearrow 0} \frac{\gamma(p,t)}{p} \\ &\leq \lim_{p\nearrow 0} \sup_{t>0} \frac{\gamma(p,t)}{p} = \lim_{p\nearrow 0} \frac{\inf_{t>0}\gamma(p,t)}{p} = \lim_{p\nearrow 0} \frac{\gamma(p)}{p},\end{aligned}$$

which yields

$$\lim_{t\to\infty} \frac{1}{t} E(\log u_1^\omega(t,x)) \leq \gamma'(0-). \tag{3.23}$$

Hence (3.20) follows from (3.22) and (3.23). □

Proof of Theorem 3.1. Let $X(t) = (1/t)\log u_1^\omega(t,x) - \gamma'(0-)$, then by Lemma 3.3

$$\lim_{t\to\infty} E(X(t)) = 0. \tag{3.24}$$

Moreover by Lemma 3.2 we have

$$\lim_{t\to\infty} E(X(t) : X(t) \leq -\epsilon) = 0 \quad \text{for every } \epsilon > 0. \tag{3.25}$$

(3.24) and (3.25) imply

$$\lim_{t\to\infty} E(|X(t)|) = 0.$$

Moreover combining this with (3.15) we see that for any $p > 0$,

$$\lim_{t\to\infty} E(|X(t)|^p) = 0,$$

which completes the proof of Theorem 3.1. □

Proof of Corollary 3.1. Note that by (3.11) and Jensen's inequality,

$$\begin{aligned}\log u^\omega(t,x) - \log u_1^\omega(t,x) &= \log\left(\int_G m(dy) u^\omega(0,y)\hat{p}^\omega(0,y;t,x)\right) \\ &\geq -\int_G m(dy)|\log u^\omega(0,y)|\hat{p}^\omega(0,y;t,x).\end{aligned}$$

Using $\log x \leq x - 1$ $(x > 0)$, we have

$$|\log u^\omega(t,x) - \log u_1^\omega(t,x)| \leq \int_G m(dy)(|\log u^\omega(0,y)| + u^\omega(0,y) - 1)\hat{p}^\omega(0,y;t,x).$$

Hence (3.3), [B-2] and [B-3] imply that

$$E(|\log u^\omega(t,x) - \log u_1^\omega(t,x)|) \quad \text{is bounded in } t > 0,$$

thus (3.5) follows from Theorem 3.1. □

EXAMPLE 3.1. Let $G = \mathbb{R}^d$, m be the Lebesgue measure on $\mathbb{R}^d$, and $\{W(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ be a continuous centered Gaussian field defined on a probability space $(\Omega, \mathcal{F}, P^W)$ with filtration $\{\mathcal{F}_t\}$ satisfying that for each $t \geq 0$, $W(t,\cdot)$ is $\mathcal{F}_t$-adapted and that

$$E^W(W(s,x)W(t,y)) = t \wedge sC(x-y) \quad (s,t \geq 0, x,y \in \mathbb{R}^d), \tag{3.26}$$

where $C(x)$ is assumed to be a bounded smooth function on $\mathbb{R}^d$. Then $W(s,x)$ has a continuous modification. Let us consider the following stochastic partial differential equation (SPDE):

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + u(t,x)\frac{\partial^2 W(t,x)}{\partial t \partial x}. \tag{3.27}$$

It is easy to see that the SPDE (3.27) admits a fundamental solution $\{p^\omega(s,y;t,x); s,t \geq 0, x,y \in \mathbb{R}^d\}$ which satisfies the conditions [B-1]–[B-3]. In this case [B-4] can be verified as follows. For an $\mathbb{R}^d$-valued continuous function $f(t)$ one can define a stochastic integral $\int_0^t dW(s,f(s))$ that is a centered Gaussian process satisfying that

$$E^W\left(\int_0^t dW(s,f(s)) \int_0^t dW(s,g(s))\right) = \int_0^t C(f(s)-g(s))ds.$$

Using this stochastic integral we have Feynman-Kac representation for the solution of (3.27) as follows.

$$u_1^\omega(t,x) = \mathbb{E}_x\left(\exp \int_0^t dW(s,B(t-s))\right) \exp -\frac{C(0)t}{2}, \tag{3.28}$$

where $(B(t), \mathbb{P}_x)$ is a d-dimensional Brownian motion. Obviously $E^W(u_1^\omega(t,x)) = 1$. On the other hand, noting that

$$(u_1^\omega(t,x))^{-1} \leq \mathbb{E}_x\left(\exp \int_0^t -dW(s,B(t-s))\right) \exp \frac{C(0)t}{2},$$

we get

$$E^W((u_1^\omega(t,x))^{-1}) \le \exp C(0)t,$$

which verifies the condition [B-4]. Accordingly by Theorem 3.1 the sample Lyapunov exponent is well-defined for the SPDE (3.27).

Next we apply Theorem 3.1 to show existence of the sample Lyapunov exponent for the solutions of the stochastic equation (1.1).

Now let $J = \mathbb{Z}^d$, and let us consider the stochastic equation (2.2) with $A = \{a(i,j)\}$ satisfying (1.2).

Theorem 3.2. *Suppose that* (1.2) *and the condition* [A] *are fulfilled, and let* $\xi^1(t) = \{\xi_i^1(t)\}_{i\in\mathbb{Z}^d}$ *be the* $L^1(\gamma)$*-solution of the equation* (1.1) *with* $\xi_i(0) = 1$ *for all* $i \in \mathbb{Z}^d$. *Then* $\{\xi_i^1(t)\}_{i\in\mathbb{Z}^d}$ *are non-negative and there exists a constant* λ *such that for every* $p > 0$,

$$\lim_{t\to\infty} E^Y\left(\left|\frac{1}{t}\log\xi_i^1(t) - \lambda\right|^p\right) = 0. \tag{3.29}$$

The nonnegativity of $\{\xi_i^1(t)\}_{i\in\mathbb{Z}^d}$ follows from Corollary 2.1. Let $\{p^\omega(s,i;t,j);\ 0 \le s \le t, i,j \in \mathbb{Z}^d\}$ be the fundamental solution of (2.2). Choosing $\mathcal{F}_s^t$ as the σ-field generated by $\{Y_i(r) - Y_i(s); s \le r \le t, i \in \mathbb{Z}^d\}$ and taking $\mathcal{F}_0^0$ the σ-field independent of $\{\mathcal{F}_s^t; 0 \le s \le t < \infty\}$, it is easy to verify the conditions [B-1]–[B-3]. In order to apply Theorem 3.1 it suffices to verify the condition [B-4], which will be reduced to the following lemma.

Lemma 3.4. *Let* $\{\eta_i(t)\}_{i\in\mathbb{Z}^d}$ *be the solution of the following stochastic equation,*

$$\eta_i(t) - 1 = \int_0^t \sum_{j\in\mathbb{Z}^d} a(i,j)\eta_j(s)ds - \int_0^{t+} \eta_i(s-)N_i\left(ds, \left[-1, -\frac{1}{2}\right)\right) \quad (i \in \mathbb{Z}^d). \tag{3.30}$$

Then

$$\eta_i(t) = \mathbb{P}_i\left(\int_0^{t+} N_{X(t-s)}\left(ds, \left[-1, -\frac{1}{2}\right)\right) = 0\right), \tag{3.31}$$

and moreover for every $0 < \alpha < 1$

$$E^Y(\eta_i(t)^{-\alpha}) < \infty \quad \textit{for every } t > 0, i \in \mathbb{Z}^d. \tag{3.32}$$

Proof. (3.31) follows from Theorem 2.2, and (3.32) follows from a combination of Lemma 2.3 and 2.4 in [9]. □

Proof of Theorem 3.2. By Theorem 2.2 we have

$$(3.33)\ \xi_i^1(t) = \mathbb{E}_i\left(\exp\int_0^{t+} dZ_{X(t-s)}(s) : \int_0^{t+} N_{X(t-s)}(ds,\{-1\}) = 0\right)$$
$$\geq \mathbb{E}_i\left(\exp\int_0^{t+} dZ'_{X(t-s)}(s) : \int_0^{t+} N_{X(t-s)}\left(ds,\left[-1,-\frac{1}{2}\right)\right) = 0\right),$$

where

$$(3.34)\qquad Z_i(t) = \alpha B_i(t) + \int_{[0,t]\times(|u|<1/2)} \log(1+u)\tilde{N}_i(ds,du)$$
$$+ \int_{[0,t]\times(-1,-1/2]\cup[1/2,\infty)} \log(1+u)N_i(ds,du) + mt$$

where m is of (2.15), and

$$(3.35)\qquad Z'_i(t) = Z_i(t) - \int_{[0,t]\times(-1,-1/2]} \log(1+u)N_i(ds,du).$$

Define a random variable U and an event A by

$$U = \exp\int_0^{t+} dZ'_{X(t-s)}(s),\quad A = \left[\int_0^{t+} N_{X(t-s)}\left(ds,\left[-1,-\frac{1}{2}\right)\right) = 0\right].$$

Then it is easy to see that for every $p > 1$ and $t > 0$

$$E^Y(\mathbb{E}_i(U^{-p})) = E^Y(\exp -pZ'_0(t)) < \infty.$$

Also using Jensen's inequality and Hölder's inequality we see that for $0 < \alpha < 1$ and p, $q > 1$ with $1/p + 1/q = 1$,

$$\begin{aligned} E^Y(\mathbb{E}_i(U : A)^{-\alpha}) &\leq E^Y(\mathbb{E}_i(U^{-\alpha} : A)\mathbb{P}_i(A)^{-1-\alpha}) \\ &\leq E^Y(\mathbb{E}_i(U^{-\alpha p})^{1/p}\mathbb{P}_i(A)^{1/q-1-\alpha}) \\ &\leq E^Y(\mathbb{E}_i(U^{-\alpha p}))^{1/p}E^Y(\mathbb{P}_i(A)^{1-q(1+\alpha)})^{1/q}. \end{aligned}$$

Hence choosing $q > 1$ close to 1, by (3.32) we have

$$E^Y\left(\mathbb{E}_i(U : A)^{-\alpha}\right) < \infty.$$

Combining this with (3.33) we obtain

$$E^Y(\xi_i^1(t)^{-\alpha}) < \infty \quad \text{for} \quad t > 0, i \in \mathbb{Z}^d.$$

Thus the condition [B-4] is verified and the proof of Theorem 3.2 is completed. □

By virtue of Theorem 3.1 the sample Lyapunov exponent in L^1-sense is well-defined for a class of SPDEs satisfying the conditions [B-1]–[B-4]. However the almost sure sample Lyapunov exponent is not well-defined in general. Concerning this problem we have the following result. Recall $\gamma(p)$ is the moment Lyapunov exponent of $\xi_i^1(t)$, which is well-defined by Lemma 3.1.

Theorem 3.3. *In the situation of Theorem* 3.2 *assume that*

$$\rho(\{-1\}) = 0 \quad \text{and} \quad \int_{(-1,-1/2)} |\log(1+u)|\rho(du) < \infty. \tag{3.36}$$

Then

$$\liminf_{t\to\infty} \frac{1}{t} \log \xi_i^1(t) = \gamma'(0-) = \lambda(A;Y) \quad P^Y\text{-}a.s., \tag{3.37}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log \xi_i^1(t) \le \gamma'(0+) \quad P^Y\text{-}a.s. \tag{3.38}$$

To the contrary if either $\rho(\{-1\}) > 0$ *or*

$$\int_{(-1,-1/2)} |\log(1+u)|\rho(du) = \infty, \tag{3.39}$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \log \xi_i^1(t) = -\infty \quad P^Y\text{-}a.s. \tag{3.40}$$

Proof. 1. $\rho(\{-1\}) = 0$ implies that $\xi_i^1(t) > 0$ holds for every $t \ge 0$, so that by Itô's formula

$$\log \xi_i^1(t) - \log \xi_i^1(s) = \int_s^t \frac{A\xi_i^1(r)}{\xi_i^1(r)} dr + Z_i(t) - Z_i(s) \quad (s \le t, i \in \mathbb{Z}^d), \tag{3.41}$$

where $Z_i(t)$ is of (3.34). From this it follows

$$\inf_{n\le t\le n+1} \log \xi_i^1(t) - \log \xi_i^1(n) \ge a(0,0) + \inf_{n\le t\le n+1} Z_i(t) - Z_i(n), \tag{3.42}$$

and

$$\log \xi_i^1(n+1) - \sup_{n\le t\le n+1} \log \xi_i^1(t) \ge a(0,0) + Z_i(n+1) - \sup_{n\le t\le n+1} Z_i(t). \tag{3.43}$$

Since (3.36) implies that $\{\sup_{n\le t\le n+1}|Z_i(t)-Z_i(n)|\}$ are i.i.d. random variables with finite mean, we have

(3.44) $$\lim_{n\to\infty}\frac{1}{n}\sup_{n\le t\le n+1}|Z_i(t)-Z_i(n)|=0\quad P^Y\text{-}a.s.$$

thus a combination of (3.42)–(3.44) with (3.16)–(3.17) implies (3.38) and

(3.45) $$\liminf_{n\to\infty}\frac{1}{t}\log\xi_i^1(t)\ge\gamma'(0-),\quad P^Y\text{-}a.s.$$

Therefore (3.37) follows from (3.45) and Theorem 3.2.

2. Next assume (3.39) since it is trivial in the case $\rho(\{-1\})>0$. It is easy to see that

(3.46) $$\limsup_{t\to\infty}\frac{1}{t}\log\xi_i^1(t)<\infty\quad P^Y\text{-}a.s.$$

Let $i\in\mathbb{Z}^d$ be fixed, and let $\{(\tau_i^n,U_i^n)\}$ be the sequence of random variables such that

$$N_i(ds,du)|_{[0,\infty)\times(-1,-1/2)}=\sum_{n=1}^{\infty}\delta_{(\tau_i^n,U_i^n)}(ds,du),$$

where $\delta_{(s,u)}$ stands for the unit mass at (s,u). Note that $\{U_i^n\}_{n=1,2,\ldots}$ is an i.i.d. sequence with distribution $c\rho|_{(-1,-1/2)}$ and it holds that

(3.47) $$\lim_{n\to\infty}\frac{\tau_i^n}{n}=c,$$

where $c=\rho((-1,-1/2))^{-1}$. Moreover (3.39) implies that

(3.48) $$\limsup_{n\to\infty}\frac{1}{n}|\log(1+U_i^n)|=\infty\quad P^Y\text{-}a.s.$$

Since

$$\xi_i^1(\tau_i^n)=(1+U_i^n)\xi_i^1(\tau_i^n-),$$

by (3.47) and (3.48) we have

$$\begin{aligned}2\limsup_{t\to\infty}\frac{1}{t}|\log\xi_i^1(t)|&\ge\limsup_{n\to\infty}\frac{1}{\tau_i^n}|\log\xi_i^1(\tau_i^n)-\log\xi_i^1(\tau_i^n-)|\\&=\lim_{n\to\infty}\frac{1}{\tau_i^n}|\log(1+U_i^n)|\\&=\infty\quad P^Y\text{-}a.s.\end{aligned}$$

Thus (3.40) follows from this and (3.46). □

4. Inequalities for the sample Lyapunov exponent

By Theorem 3.2 the sample Lyapunov exponent is well-defined for the SPDE (1.1) over $\mathbb{Z}^d$ under the condition [A], which is denoted by $\lambda(\kappa A; Y)$.

Recall that

$$-\infty \leq \lambda_0(Y) = -\frac{\alpha^2}{2} + \int_{[-1,\infty)} (\log(1+u) - u)\rho(du) \leq 0,$$

then $\lambda_0(Y) > -\infty$ holds if and only if

$$\rho(\{-1\}) = 0 \quad \text{and} \quad \int_{(-1,-1/2)} |\log(1+u)|\rho(du) < \infty. \tag{4.1}$$

Let $Y'(t)$ be another Lévy process with the characteristic exponent

$$\psi'(z) = -\frac{(\alpha')^2}{2}z^2 + \int_{(-1,\infty)} (e^{\sqrt{-1}zu} - 1 - \sqrt{-1}zu)\rho'(du). \tag{4.2}$$

Theorem 4.1. *Assume that*

$$|\alpha| \leq |\alpha'| \quad \text{and} \quad \rho \leq \rho'. \tag{4.3}$$

Then the following inequalities hold.

$$\lambda(A; Y') \leq \lambda(A; Y) \leq 0. \tag{4.4}$$

$$\frac{1}{\kappa}\lambda(\kappa A, Y) \leq \frac{1}{\kappa'}\lambda(\kappa' A, Y) \quad \text{if} \quad 0 < \kappa < \kappa'. \tag{4.5}$$

$$0 \leq \lambda(A; Y) - \lambda_0(Y) \leq \lambda(A; Y') - \lambda_0(Y') \tag{4.6}$$

whenever ρ and ρ' satisfy (4.1).

Corollary 4.1. *Suppose that ρ and ρ' satisfy* (4.1). *Then $\lambda(\kappa A; Y)$ is continuous in (κ, Y) in the following sense.*

$$\left|\frac{1}{\kappa}\lambda(\kappa A, Y) - \frac{1}{\kappa'}\lambda(\kappa' A, Y)\right| \leq \left|\frac{1}{\kappa} - \frac{1}{\kappa'}\right| |\lambda_0(Y)|. \tag{4.7}$$

$$|\lambda(\kappa A, Y) - \lambda(\kappa A, Y')| \leq \frac{|\alpha^2 - (\alpha')^2|}{2} + \int_{(-1,\infty)} |\log(1+u) - u||\rho - \rho'|_{var}(du), \tag{4.8}$$

where $|\rho - \rho'|_{var}$ stands for the total variational measure of $\rho - \rho'$.

For the proof of Theorem 4.1 we apply the following comparison lemma which is a modification of the comparison theorem of [4].

Let $\mathbb{F}$ be the totality of bounded C^2-functions defined on $[0,\infty)^J$ depending on finitely many components and having first and second bounded derivatives such that $D_iD_jf \geq 0$ $(i,j \in \mathbb{Z}^d)$.

Lemma 4.1. *Assume the condition* (4.3). *Then for the solutions* $\xi(t)$ *and* $\xi'(t)$ *of the equation* (2.2) *associated with* $Y(t)$ *and* $Y'(t)$ *and* $\xi(0) = \xi'(0) \geq 0$, *it holds that*

$$E^Y(f(\xi(t))) \leq E^{Y'}(f(\xi'(t))) \quad \textit{for} \quad f \in \mathbb{F} \quad \textit{and} \quad t > 0. \tag{4.9}$$

Proof. It is sufficient to prove (4.9) in a simpler situation that J is a finite set and ρ is compactly supported in $(-1,0) \cup (0,\infty)$ because general case can be reduced to this case by a standard approximation procedure. Let T_t and T'_t be the transition semi-group of the Markov processes $\xi(t)$ and $\xi'(t)$, and denote by L and L' their infinitesimal generators respectively. In this case it is easy to see the following perturbation formula,

$$T_tf - T'_tf = \int_0^t T'_{t-s}(L-L')T_sf\,ds. \tag{4.10}$$

Note that if $f \in \mathbb{F}$,

$$\begin{aligned}(L'-L)f(\boldsymbol{x}) &= \frac{(\alpha')^2-\alpha^2}{2}\sum_{i\in J} D_i^2 f(\boldsymbol{x}) \\ &\quad + \sum_{i\in J}\int_{(-1,\infty)} (f(\pi_i^u\boldsymbol{x}) - f(\boldsymbol{x}) - x_iuD_if(\boldsymbol{x}))(\rho'-\rho)(du) \\ &\geq 0,\end{aligned} \tag{4.11}$$

where

$$(\pi_i^u\boldsymbol{x})_i = (1+u)x_i \quad \text{and} \quad (\pi_i^u\boldsymbol{x})_j = x_j \quad (j \neq i).$$

Because the second term of the r.h.s. of (4.11) is

$$\sum_{i\in J} x_i\int_{(-1,\infty)}\left(\int_0^u (D_if(\pi_i^v\boldsymbol{x}) - D_if(\boldsymbol{x}))dv\right)(\rho'-\rho)(du) \geq 0 \text{ for } f \in \mathbb{F}.$$

On the other hand it holds that

$$T_tf \in \mathbb{F} \quad \text{if} \quad f \in \mathbb{F}, \tag{4.12}$$

since by making use of the fundamental solution $p^{\omega}(s,i;t,j)$ of the equation (2.2) $T_t f$ can be represented by

$$T_t f(\boldsymbol{x}) = E^Y\left(f\left(\left\{\sum_{j\in J} x_j p^{\omega}(0,j;t,i)\right\}\right)\right),$$

which yields (4.12) by straightforward differentiations. Therefore (4.9) follows from (4.10)–(4.12). □

Proof of Theorem 4.1. Let $\xi_i(0) = \xi_i'(0) = 1 \quad (i \in \mathbb{Z}^d)$. By Lemma 4.1 we have

$$0 \geq E^Y(\log \xi_i(t)) \geq E^{Y'}(\log \xi_i'(t)),$$

which yields (4.4). Note that by a time rescaling

$$\frac{1}{\kappa}\lambda(\kappa A, Y) = \frac{1}{\kappa'}\lambda\left(\kappa' A, Y\left(\frac{\kappa'}{\kappa}\cdot\right)\right). \tag{4.13}$$

Hence a combinaiton of (4.4) and (4.13) yields (4.5). To show (4.6) note that by Theorem 2.2

$$\xi_i(t) = \mathbb{E}_i\left(\exp\int_0^t dZ_{X(t-s)}(s)\right)\exp\lambda_0(Y)t,$$

where

$$Z_i(t) = \alpha B_i(t) + \int_{[0,t]\times(-1,\infty)} \log(1+u)\tilde{N}_i(ds,du).$$

By the assumption (4.3) we may assume that the probability laws of$(\{Y_i'(\cdot)\}, P^{Y'})$ and $(\{Y_i(\cdot)+\widehat{Y}_i(\cdot)\}, P^Y \times P^{\widehat{Y}})$ coincide, where $(\widehat{Y}_i(t), P^{\widehat{Y}})$ are independent copies of a Lévy process $\widehat{Y}(t)$ associated with $(\hat{\alpha} = \sqrt{(\alpha')^2-\alpha^2}, \hat{\rho} = \rho' - \rho)$. Then

$$\xi_i'(t) = \mathbb{E}_i\left(\exp\int_0^t dZ_{X(t-s)}(s)\exp\int_0^t d\widehat{Z}_{X(t-s)}(s)\right)\exp\lambda_0(Y')t,$$

where $\{\widehat{Z}_i(t)\}$ are independent copies of a Lévy process $\{\widehat{Z}(t)\}$ satisfying $E^{\widehat{Y}}(\widehat{Z}(t)) = 0$ corresponding to $\widehat{Y}(t)$. Using Jensen's inequality we see

$$
\begin{aligned}
&\lambda(A;Y') - \lambda_0(Y') - \lambda(A;Y) + \lambda_0(Y) \\
&= \lim_{t\to\infty} \frac{1}{t} E^Y \times E^{\widehat{Y}} \left(\log \left(\mathbb{E}_i \left(\exp \int_0^t dZ_{X(t-s)}(s) \exp \int_0^t d\widehat{Z}_{X(t-s)}(s) \right) \right.\right. \\
&\qquad\qquad \left.\left. \times \mathbb{E}_i \left(\exp \int_0^t dZ_{X(t-s)}(s) \right)^{-1} \right) \right) \\
&\le \lim_{t\to\infty} \frac{1}{t} E^Y \left(\log \left(\mathbb{E}_i \left(\exp \int_0^t dZ_{X(t-s)}(s) E^{\widehat{Y}} \left(\int_0^t d\widehat{Z}_{X(t-s)}(s) \right) \right) \right.\right. \\
&\qquad\qquad \left.\left. \times \mathbb{E}_i \left(\exp \int_0^t dZ_{X(t-s)}(s) \right)^{-1} \right) \right) \\
&= 0,
\end{aligned}
$$

which yields (4.6), since

$$
E^{\widehat{Y}} \left(\int_0^t d\widehat{Z}_{X(t-s)}(s) \right) = E^{\widehat{Y}}(Z_i(t)) = 0. \qquad \square
$$

Proof of Corollary 4.1. Using the scaling property (4.13) and (4.6) we have

$$
\begin{aligned}
\left| \frac{1}{\kappa} \lambda(\kappa A, Y) - \frac{1}{\kappa'} \lambda(\kappa' A, Y) \right| &= \frac{1}{\kappa'} \left| \lambda(\kappa' A, Y) - \lambda \left(\kappa' A, Y \left(\frac{\kappa'}{\kappa} \cdot \right) \right) \right| \\
&\le \frac{1}{\kappa'} \left| \lambda_0(Y) - \lambda_0 \left(Y \left(\frac{\kappa'}{\kappa} \cdot \right) \right) \right| \\
&= \left| \frac{1}{\kappa} - \frac{1}{\kappa'} \right| |\lambda_0(Y)|,
\end{aligned}
$$

which yields (4.7). (4.8) follows immediately from (4.6). $\square$

Next we discuss a comparison of the sample Lyapunov exponents between finite systems and infinite system of (2.2). Let $\Lambda_n = (-n, n]^d \cap \mathbb{Z}^d$, and let $A^{(n)} = \{a^{(n)}(i,j)\}$ be a $\Lambda_n \times \Lambda_n$ matrix induced by $A = (a(i,j))$, i.e.

$$
a^{(n)}(i,j) = \sum_{k-j \in 2n\mathbb{Z}^d} a(i,k) \quad (i, j \in \Lambda_n).
$$

We denote by $\lambda^{(n)}(A;Y)$ the sample Lyapunov exponent of (2.2) with $J = \Lambda_n$ and $A^{(n)} = (a^{(n)}(i,j))$. Furthermore we denote by $\lambda^{(2)}(\kappa;Y)$ the sample Lyapunov exponent of the following two dimensional stochastic equation,

$$
\xi_1(t) - \xi_1(0) = \kappa \int_0^t (\xi_2(s) - \xi_1(s)) ds + \int_0^{t+} \xi_1(s-) dY_1(s) \tag{4.14}
$$

$$\xi_2(t) - \xi_2(0) = \kappa \int_0^t (\xi_1(s) - \xi_2(s))ds + \int_0^{t+} \xi_2(s-)dY_2(s),$$

where

$$\kappa = \sum_{j \in \mathbb{Z}^d, j:\text{odd}} a(0, j) \tag{4.15}$$

and $\{Y_1(t), Y_2(t)\}$ are independent copies of $Y(t)$.

Theorem 4.2.

$$\lambda^{(2)}(\kappa; Y) \leq \lambda^{(n)}(A; Y) \leq \lambda(A; Y). \tag{4.16}$$

For the proof of the theorem we apply another comparison theorem. Let $\{J_n\}$ be a partition of J, $\{Y_n(t)\}$ be independent Lévy processes associated with the characteristic exponent $\psi(z)$ of (1.1), and set

$$\widehat{Y}_i(t) = Y_n(t) \quad \text{if} \quad i \in J_n.$$

Let us consider an equation similar to (2.2);

$$\eta_i(t) - \eta_i(0) = \int_0^t \sum_{j \in J} a(i, j)\eta_j(s)ds + \int_0^{t+} \eta_i(s-)d\widehat{Y}_i(s) \quad (i \in J), \tag{4.17}$$

where $\eta(0) = \{\eta_i(0)\}$ is assumed to be non-negative and $E(\|\eta(0)\|_{L^1(\gamma)}) < \infty$. By the same method as Theorem 2.1 one can show that (4.17) has the pathwise unique $L^1(\gamma)$-solution $\eta(t) = \{\eta_i(t)\}$.

Now we compare it with the solution of (2.2).

Lemma 4.2. *Let $f \in \mathbb{F}$. If $\xi(0) = \eta(0)$, then for every $t > 0$,*

$$E^Y(f(\xi(t))) \leq E^{\widehat{Y}}(f(\eta(t))). \tag{4.18}$$

Proof. It is essentially the same as Lemma 2.2 of [9] where a special case is treated, so we omit it. □

Proof of Theorem 4.2. Let J_1 and J_2 be the set of all odd points and the set of all even points in Λ_n respectively. For $i \in J_p$, we set $\widehat{Y}_i(t) = Y_p(t)$ $(p = 1, 2)$. For the solution $(\xi_1^{(2)}(t), \xi_2^{(2)}(t))$ of (4.14) with $\xi_1(0) = \xi_2(0) = 1$, we set $\eta_i(t) = \xi_p^{(2)}(t)$ for $i \in J_p$ $(p = 1, 2)$. Then $\{\eta_i(t)\}_{i \in \Lambda_n}$ is a solution of (4.17) with $J = \Lambda_n$ and $A^{(n)}$.

Furthermore let $\xi^{(n)}(t) = \{\xi_i^{(n)}(t)\}_{i\in\Lambda_n}$ be the solution of the equation (2.2) with $\xi_i(0) = 1$ $(i \in \Lambda_n)$. Then by Lemma 4.2 we have

$$E^Y(\log \xi_1^{(2)}(t)) \le E^Y(\log \xi_1^{(n)}(t))$$

which yields the first inequality of (4.16). The second inequality of (4.16) can be proved in a similar way. □

5. Two dimensional case

Recall the two dimensional stochastic equation

$$\begin{aligned}\xi_1(t) - \xi_1(0) &= \kappa \int_0^t (\xi_2(s) - \xi_1(s))ds + \int_0^{t+} \xi_1(s-)dY_1(s) \\ \xi_2(t) - \xi_2(0) &= \kappa \int_0^t (\xi_1(s) - \xi_2(s))ds + \int_0^{t+} \xi_2(s-)dY_2(s),\end{aligned} \tag{4.14}$$

and denote by $\lambda^{(2)}(\kappa; Y)$ its sample Lyapunov exponent. In this section we investigate an asymptotics of $\lambda^{(2)}(\kappa; Y)$ as $\kappa \searrow 0$.

Theorem 5.1. *Assume that* (1.8), *the condition* [A] *and the following* (5.1) *are fulfilled.*

$$\rho(\{-1\}) = 0 \quad and \quad \int_{(-1,\infty)} (\log(1+u))^2 \rho(du) < \infty. \tag{5.1}$$

Then

$$\lambda^{(2)}(\kappa; Y) - \lambda_0(Y) \sim \frac{c}{\log(1/\kappa)} \quad as \quad \kappa \searrow 0. \tag{5.2}$$

where

$$c = \frac{1}{2}\left(\alpha^2 + \int_{(-1,\infty)} (\log(1+u))^2 \rho(du)\right). \tag{5.3}$$

Here $\alpha(\kappa) \sim \beta(\kappa)$ *as* $\kappa \searrow 0$ *means* $\lim_{\kappa\searrow 0} \alpha(\kappa)/\beta(\kappa) = 1$.

Let $\hat{\rho}$ be a Radon measure on $\mathbb{R} \setminus \{0\}$ defined by the following relation: for every continuous function $f \ge 0$,

$$\int_{\mathbb{R}\setminus\{0\}} \hat{\rho}(du) f(u) = \int_{(-1,\infty)} \rho(du)(f(\log(1+u)) + f(-\log(1+u))). \tag{5.4}$$

Note that the condition (5.1) is equivalent to

$$\int_{\mathbb{R}\setminus\{0\}} \hat{\rho}(du) u^2 < \infty. \tag{5.5}$$

Let $\widehat{Z}(t)$ be a Lévy process with the characteristic exponent

$$\hat{\psi}(z) = -\alpha^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{\sqrt{-1}zu} - 1)\hat{\rho}(du), \tag{5.6}$$

and consider the following stochastic equation

$$\zeta(t) - \zeta(0) = \int_0^t \kappa(e^{-\zeta(s)} - e^{\zeta(s)})ds + \widehat{Z}(t). \tag{5.7}$$

The equation (5.7) has the unique solution, which defines a one-dimensional Markov process $(\zeta(t), P_x)$ with the infinitesimal generator

$$Lf(x) = \kappa(e^{-x} - e^{x})f'(x) + \alpha^2 f''(x) + \int_{\mathbb{R}\setminus\{0\}} (f(x+y) - f(x))\hat{\rho}(dy). \tag{5.8}$$

It is easy to see that $(\zeta(t), P_x)$ has the unique stationary probability measure $\mu^{(\kappa)}$ which is characterized by

$$\int_{\mathbb{R}} \mu^{(\kappa)}(dx)Lf(x) = 0 \quad \text{for} \quad f \in C_c^2(\mathbb{R}), \tag{5.9}$$

where $C_c^1(\mathbb{R})$ stands for the set of all C^1-functions defined on $\mathbb{R}$ with compact support.

Note that $\mu^{(\kappa)}$ is indeed a symmetric measure on $\mathbb{R}$, and that (5.9) is valid for $f(x) = x^2$ by the condition (5.1) so that

$$\begin{aligned} 2\kappa \int_{\mathbb{R}} \mu^{(\kappa)}(dx)e^{x}x &= \frac{1}{2}\int_{\mathbb{R}} \hat{\rho}(dx)x^2 + \alpha^2 \\ &= \int_{(-1,\infty)} \rho(du)(\log(1+u))^2 + \alpha^2. \end{aligned} \tag{5.10}$$

$\lambda^{(2)}(\kappa; Y)$ is given in terms of $\mu^{(\kappa)}$ as follows.

Lemma 5.1.

$$\lambda^{(2)}(\kappa; Y) - \lambda_0(Y) = \kappa \int_{\mathbb{R}} \mu^{(\kappa)}(dx)e^{x} - \kappa. \tag{5.11}$$

Proof. Let $(\xi_1(t), \xi_2(t))$ be the solution of (4.14) under an assumption that the distribution of $\log \xi_1(0)$ coincides with $\mu^{(\kappa)}$ satisfying (5.9) and $\xi_2(0) = 1$. Then by Corollary 3.1 we have

$$\lambda^{(2)}(\kappa; Y) = \lim_{t\to\infty} \frac{1}{t} E^Y(\log \xi_1(t)). \tag{5.12}$$

Let

$$\zeta(t) = \log \frac{\xi_1(t)}{\xi_2(t)}. \tag{5.13}$$

From (4.14) it follows

$$\begin{aligned}\log \xi_1(t) - \log \xi_1(0) = \kappa \int_0^t (e^{-\zeta(s)} - 1)ds + \alpha B_1(t) + \lambda_0(Y)t \\ + \int_{[0,t]\times(-1,\infty)} \log(1+u)\tilde{N}_1(ds, du),\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\zeta(t) - \zeta(0) = \int_0^t \kappa(e^{-\zeta(s)} - e^{\zeta(s)})ds + \alpha B_1(t) - \alpha B_2(t) \\ + \int_{[0,t]\times(-1,\infty)} \log(1+u)(N_1(ds,du) - N_2(ds,du)),\end{aligned} \tag{5.15}$$

which is equivalent to (5.7). Note further that $\zeta(t)$ is stationary, so that it follows from (5.14) that

$$E^Y(\log \xi_1(t)) - E^Y(\log \xi_1(0)) = \kappa t \left(\int_{\mathbb{R}} \mu^{(\kappa)}(dx) e^{-x} - 1 \right) + \lambda_0(Y)t.$$

Thus (5.11) follows from this and (5.12). □

Proof of Theorem 5.1. First we give the upper bound. By (5.10)

$$\kappa \int_{\mathbb{R}} \mu^{(\kappa)}(dx) e^x x = c, \tag{5.16}$$

where c is given in (5.3).
Set

$$M(\kappa) = \int_{\mathbb{R}} \mu^{(\kappa)}(dx) e^x. \tag{5.17}$$

Since $g(x) = x \log x$ is convex, using Jensen's inequality we get

$$g(M(\kappa)) \le \frac{c}{\kappa}.$$

Hence denoting by g^{-1} the inverse function of $g(x) = x \log x$ in $(1, \infty)$, we have that for small $\kappa > 0$,

$$M(\kappa) \le g^{-1}\left(\frac{c}{\kappa}\right) \sim \frac{c}{\kappa}\left(\log \frac{c}{\kappa}\right)^{-1},$$

which yields

$$\limsup_{\kappa\searrow 0} \frac{\kappa M(\kappa)}{(1/\log(1/\kappa))} \leq c. \tag{5.18}$$

Thus we obtain the upper bound.

To get the lower bound we first assume

$$c_1 = \int_{\mathbb{R}} \hat{\rho}(dx)(e^x - 1) = \int_{(-1,\infty)} \rho(du)\frac{u^2}{1+u} < \infty. \tag{5.19}$$

Then (5.9) is valid for $f(x) = e^x$ and

$$\kappa\left(1 - \int_{\mathbb{R}} \mu^{(\kappa)}(dx)e^{2x}\right) + (c_1 + \alpha^2)M(\kappa) = 0. \tag{5.20}$$

Using Jensen's inequality and (5.16) we have

$$\begin{aligned}\int_{\mathbb{R}} \mu^{(\kappa)}(dx)e^{2x} &\geq M(\kappa)\exp(M(\kappa)^{-1}\int_{\mathbb{R}} \mu^{(\kappa)}(dx)e^x x) \\ &= M(\kappa)\exp\frac{c}{\kappa M(\kappa)}.\end{aligned} \tag{5.21}$$

Setting

$$h_\kappa(x) = xe^{\frac{c}{x}} - \frac{c_2}{\kappa}x \quad \text{with} \quad c_2 = c_1 + \alpha^2,$$

by (5.20) and (5.21) we get

$$h_\kappa(\kappa M(\kappa)) \leq \kappa. \tag{5.22}$$

Since $h_\kappa(x)$ is decreasing in $(0, c)$ and $\kappa M(\kappa)$ vanishes as $\kappa \searrow 0$ by (5.18), setting $x(\kappa) = h_\kappa^{-1}(\kappa)$ we see

$$\kappa M(\kappa) \geq x(\kappa) \quad \text{for small } \kappa > 0. \tag{5.23}$$

Note that $h_\kappa(x(\kappa)) = \kappa$ implies that $x(\kappa) \to 0$ as $\kappa \searrow 0$ and

$$\frac{c}{x(\kappa)} = \log\left(\frac{\kappa}{x(\kappa)} + \frac{c_2}{\kappa}\right) \leq \log\frac{c}{x(\kappa)} + \log\frac{c_2}{\kappa} \quad \text{for small } \kappa > 0,$$

from which and (5.23) it follows

$$\liminf_{\kappa\searrow 0} \frac{\lambda^{(2)}(\kappa; Y) - \lambda_0(Y)}{\log(1/\kappa)} \geq \liminf_{\kappa\searrow 0} \frac{\kappa M(\kappa)}{\log(1/\kappa)} \geq \liminf_{\kappa\searrow 0} \frac{x(\kappa)}{\log(1/\kappa)} \geq c. \tag{5.24}$$

Thus we have shown the lower bound under the assumption (5.19). However it is easy to remove (5.19) by virtue of Theorem 4.1. In fact let $Y^{(n)}(t)$ be a Lévy process with the characteristic exponent $\psi(z)$ of (1.8) with (α, ρ_n) where ρ_n is the restriction of ρ on $[-1+1/n, n]$. Then by (5.24)

$$\liminf_{\kappa\searrow 0} \frac{\lambda^{(2)}(\kappa; Y^{(n)}) - \lambda_0(Y^{(n)})}{\log(1/\kappa)} \geq \frac{1}{2}\left(\alpha^2 + \int_{[-1+1/n,n]} \rho(du)(\log(1+u))^2\right)$$
$$\text{for any } n \geq 1,$$

so by Theorem 4.1 we get

$$\liminf_{\kappa\searrow 0} \frac{\lambda^{(2)}(\kappa; Y) - \lambda_0(Y)}{\log(1/\kappa)} \geq \lim_{n\to\infty} \liminf_{\kappa\searrow 0} \frac{\lambda^{(2)}(\kappa; Y^{(n)}) - \lambda_0(Y^{(n)})}{\log(1/\kappa)} = c,$$

which yields the lower estimate of (5.2). □

6. Asymptotical estimates of $\lambda(\kappa A; Y)$ as $\kappa \searrow 0$

In this section we investigate asymptotical estimates of the sample Lyapunov exponent $\lambda(\kappa A; Y)$ as $\kappa \searrow 0$ for the SPDE (1.1) over $\mathbb{Z}^d$ in two extremal cases depending upon singularity of the Lévy measure ρ at the neighbourhood of -1.

Let us begin with a non-singular case that satisfies (4.1). Recall that the solution $\xi^1(t) = \{\xi_i^1(t)\}$ with the initial condition $\xi_i(0) = 1$ $(i \in \mathbb{Z}^d)$ is represented by the following Feynman-Kac formula;

$$\xi_i^1(t) = \mathbb{E}_i\left(\exp\int_0^t dZ_{X(t-s)}(s)\right)\exp\lambda_0(Y)t, \tag{6.1}$$

where $(X(t), \mathbb{P}_i)$ denotes a continuous time random walk generated by κA,

$$Z_i(t) = \alpha B_i(t) + \int_{(0,t]\times(-1,\infty)} \log(1+u)\tilde{N}_i(ds, du) \quad (i \in \mathbb{Z}^d), \tag{6.2}$$

and by (4.1),

$$-\infty < \lambda_0(Y) = -\frac{\alpha^2}{2} + \int_{(-1,\infty)} (\log(1+u) - u)\rho(du) < 0. \tag{6.3}$$

To investigate asymptotics of $\lambda(\kappa A; Y)$ as $\kappa \searrow 0$ we use the following expression,

$$\lambda(\kappa A; Y) - \lambda_0(Y) = \lim_{t\to\infty} \frac{1}{t}\log\mathbb{E}_i\left(\exp\int_0^t dZ_{X(s)}(s)\right) \quad \text{in } L^1(P^Y), \tag{6.4}$$

since for fixed $t > 0$, $\{Z_i(s) : 0 \leq s \leq t, i \in \mathbb{Z}^d\}$ and $\{Z_i(t) - Z_i(t-s) : 0 \leq s \leq t, i \in \mathbb{Z}^d\}$ have the same probability law under P^Y, which inherits that

$\mathbb{E}_i(\exp\int_0^t dZ_{X(t-s)}(s))$ and $\mathbb{E}_i(\exp\int_0^t dZ_{X(s)}(s))$ have the same distribution under P^Y.

For technical reason we impose the following stronger condition than (4.1).

Theorem 6.1. *Assume further that* $A = (a(i,j))$ *is of finite range, i.e. for some* $R > 0$,

$$a(0,j) = 0 \quad \textit{for} \quad j \in \mathbb{Z}^d \quad \textit{with} \quad |j| > R,$$

and that

$$\int_{(-1,-1/2)} \frac{1}{1+u}\rho(du) < \infty. \tag{6.5}$$

Then there exist constants $c_1 > 0$, $c_2 > 0$ *and* $\kappa_0 > 0$ *such that*

$$c_1\frac{1}{\log(1/\kappa)} \le \lambda(\kappa A; Y) - \lambda_0(Y) \le c_2\frac{\log(\log(1/\kappa))}{\log(1/\kappa)} \quad (0 < \kappa < \kappa_0). \tag{6.6}$$

Next we consider the following extremely singular case,

$$\rho(\{-1\}) > 0. \tag{6.7}$$

Theorem 6.2. *Assume* (6.7) *in addition to the situation of Theorem* 3.2. *Then it holds*

$$\lambda(\kappa A; Y) \approx \log\kappa \quad \textit{as} \quad \kappa \searrow 0. \tag{6.8}$$

For the proof of Theorem 6.1 we adopt the method of [8] where is exploited an approximation of the continuous time random walk on $\mathbb{Z}^d$ by a discrete time stochastic process. However unlike the case of Brownian motions in [8], [3], Lévy process $Y(t)$ lacks sufficient moment conditions which makes arguments more complicated.

In what follows we normalize $a(0,0) = -1$. First we mention the following lemma from [8].

Lemma 6.1 ((Lemma 3.2 of [8])). *Let* $(X(t), \mathbb{P}_i)$ *be a continuous time random walk on* $\mathbb{Z}^d$ *generated by* κA, *and let* Π_t *be the number of jump times of* $X(\cdot)$ *up to time* t. *Then* (Π_t) *is a Poisson process with parameter* κ. *Moreover there is a discrete time* $\mathbb{Z}^d$*-valued stochastic process* $(\widehat{X}(n))$ *satisfying that*

$$|\widehat{X}(n) - \widehat{X}(n-1)| \le R \quad (n = 1, 2, \ldots), \tag{6.9}$$

$$\int_0^n I(X(s) \ne \widehat{X}[s])ds \le 2\Pi_n, \tag{6.10}$$

$$\widehat{\Pi}_n \equiv \sharp\{1 \le m \le n | \widehat{X}(m) \ne \widehat{X}(m-1)\} \le \Pi_n \quad \textit{and} \quad [\widehat{\Pi}_n = 0] = [\Pi_n = 0]. \tag{6.11}$$

Lemma 6.2. *Conditioned on the sample paths $X(\cdot)$ and $\widehat{X}(\cdot)$, it holds that*

$$E^Y\left(\exp\int_0^t dZ_{X(s)}(s)\right) = e^{|\lambda_0(Y)|t} \quad \mathbb{P}_i\text{–}a.s., \tag{6.12}$$

and

$$E^Y\left(\exp\int_0^t d(Z_{X(s)} - Z_{\widehat{X}[s]})(s)\right) \le e^{c\Pi_n} \quad \mathbb{P}_i\text{–}a.s., \tag{6.13}$$

where

$$c = 2\alpha^2 + 2\int_{(-1,\infty)} \frac{u^2}{1+u}\rho(du).$$

Proof. Since $\int_0^t dZ_{X(s)}(s)$ is equivalent to the Lévy process $Z_i(t)$ of (6.3) for $\mathbb{P}_i$-a.s. fixed $X(\cdot)$, (6.12) is obvious.

Note that the l.h.s. of (6.13) is

$$\begin{aligned}
&E^Y\left(\exp\int_0^t \sum_{j\in\mathbb{Z}^d}\left(I(X(s)=j) - I(\widehat{X}[s]=j)\right)dZ_j(s)\right)\\
&= \exp\left(\sum_{j\in\mathbb{Z}^d}\int_0^t I(X(s)=j\ne\widehat{X}[s])ds\left(\frac{\alpha^2}{2}+\int_{(-1,\infty)}(u-\log(1+u))\rho(du)\right)\right.\\
&\quad+\sum_{j\in\mathbb{Z}^d}\int_0^t I(X(s)\ne j=\widehat{X}[s])ds\\
&\qquad\left.\times\left(\frac{\alpha^2}{2}+\int_{(-1,\infty)}\left(\frac{1}{1+u}-1+\log(1+u)\right)\rho(du)\right)\right)\\
&= \exp\left(\frac{c}{2}\int_0^t I(X(s)\ne\widehat{X}[s])ds\right),
\end{aligned}$$

hence this and (6.10) yield (6.13). □

Lemma 6.3. *Assume the condition* (5.1). *Then there exists a constant $C_1 > 0$ such that*

$$P^Y(Z(t)\ge\lambda t) \le \exp -C_1 t\lambda^2 \quad \textit{for} \quad 0<\lambda\le 1 \quad \textit{and} \quad t>0. \tag{6.14}$$

Proof. Note that

$$M(a) \equiv \log E^Y(\exp aZ(1)) = \frac{\alpha^2 a^2}{2} + \int_{(-1,\infty)}((1+u)^a - 1 - a\log(1+u))\rho(du)$$

is a $C^2((0,1])$-function, and by (5.1)

$$M''(0+) = \alpha^2 + \int_{(-1,\infty)} (\log(1+u))^2 \rho(du) < \infty. \tag{6.15}$$

Setting

$$L(\lambda) = \sup_{0\le z\le 1} (z\lambda - M(z)) \quad (\lambda \ge 0),$$

by the exponential Chebyshev's inequality we get

$$P^Y(Z(t) > \lambda t) \le \exp -tL(\lambda). \tag{6.16}$$

So it suffices to show that

$$C_1 \equiv \inf_{0<\lambda\le 1} \frac{L(\lambda)}{\lambda^2} > 0. \tag{6.17}$$

But as easily seen, $L(0) = 0$, $L(\lambda) > 0$ $(\lambda > 0)$ and

$$\lim_{\lambda\searrow 0} \frac{L(\lambda)}{\lambda^2} = \frac{1}{2M''(0+)},$$

which yields (6.17). □

Proof of Theorem 6.1. Let $0 < \epsilon < 1$ be fixed.

1. Note that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}_0 \left(\exp \int_0^n dZ_{X(s)}(s) : \Pi_n = 0 \right) \\
&= \lim_{n\to\infty} \frac{1}{n} (Z_0(n) + \log \mathbb{P}_0(\Pi_n = 0)) \\
&= -\kappa.
\end{aligned} \tag{6.18}$$

2. Using Fubini's theorem and (6.12) we have

$$\begin{aligned}
E^Y\left(\mathbb{E}_0\left(\exp\int_0^n dZ_{X(s)}(s) : \Pi_n > \epsilon n\right)\right) &= e^{|\lambda_0(Y)|n}\mathbb{P}_0(\Pi_n > \epsilon n) \\
&\le \mathbb{E}_0\left(\exp\left(\frac{|\lambda_0(Y)|}{\epsilon}\Pi_n\right)\right) \\
&= \exp\left(\kappa n\left(\exp\left(\frac{|\lambda_0(Y)|}{\epsilon}\right) - 1\right)\right),
\end{aligned}$$

so that by Chebyshev's inequality for every $\delta > 0$,

$$\begin{aligned}
&P^Y\left(\frac{1}{n}\log\mathbb{E}_0\left(\exp\int_0^n dZ_{X(s)}(s) : \Pi_n > \epsilon n\right) > (\kappa+\delta)\left(\exp\left(\frac{|\lambda_0(Y)|}{\epsilon}\right) - 1\right)\right) \\
&\le \exp\left(-\delta n\left(\exp\left(\frac{|\lambda_0(Y)|}{\epsilon}\right) - 1\right)\right),
\end{aligned}$$

which is summable in $n \geq 1$. Hence by the Borel-Cantelli lemma it holds that that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}_0 \left(\exp \int_0^n dZ_{X(s)}(s) : \Pi_n > \epsilon n \right) \leq \kappa \left(\exp \left(\frac{|\lambda_0(Y)|}{\epsilon} \right) - 1 \right) \quad P^Y\text{-}a.s. \tag{6.19}$$

3. Let W_k^n is the totality of $\mathbb{Z}^d$-valued paths $w = (w(m))_{0\leq m\leq n}$ such that $|w(m)-w(m-1)| \leq R$ $(1 \leq m \leq n)$ and that $\sharp\{1 \leq m \leq n : w(m) \neq w(m-1)\} = k$. Note that for a function $\delta(r)$ defined on $[0, \epsilon]$, if

$$\int_0^n dZ_{w[s]}(ds) \leq n\delta\left(\frac{k}{n}\right) \quad \text{for every } 1 \leq k \leq \epsilon n \quad \text{and } w \in W_k^n,$$

then

$$\begin{aligned}
&\mathbb{E}_0 \left(\exp \int_0^n dZ_{X(s)}(s) : 1 \leq \Pi_n \leq \epsilon n \right) \\
&= \sum_{k=1}^{[\epsilon n]} \sum_{w\in W_k^n} \exp \int_0^n dZ_{w[s]}(s) \\
&\quad \times \mathbb{E}_0 \left(\exp \int_0^n d(Z_{X(s)} - Z_{\widehat{X}[s]})(s) : 1 \leq \Pi_n \leq \epsilon n, \widehat{\Pi}_n = k, \widehat{X}(\cdot) = w \right) \\
&\leq \mathbb{E}_0 \left(\exp \left(n\delta\left(\frac{\widehat{\Pi}_n}{n}\right) + \int_0^n d(Z_{X(s)} - Z_{\widehat{X}[s]}(s)) \right) : 1 \leq \Pi_n \leq \epsilon n \right).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&P^Y \left(\mathbb{E}_0 \left(\exp \int_0^n dZ_{X(s)}(s) : 1 \leq \Pi_n \leq \epsilon n \right) \geq \mathbb{E}_0 \left(\exp 2n\delta\left(\frac{\Pi_n}{n}\right) : \Pi_n \leq \epsilon n \right) \right) \\
&\leq \sum_{k=1}^{[\epsilon n]} \sum_{w\in W_k^n} P^Y \left(\int_0^n dZ_{w[s]}(s) > n\delta\left(\frac{k}{n}\right) \right) + \\
&\quad P^Y \left(\mathbb{E}_0 \left(\exp \left(n\delta\left(\frac{\widehat{\Pi}_n}{n}\right) + \int_0^n d(Z_{X(s)} - Z_{\widehat{X}[s]})(s) \right) : 1 \leq \Pi_n \leq \epsilon n \right) \right. \\
&\qquad \left. > \mathbb{E}_0 \left(\exp 2n\delta\left(\frac{\Pi_n}{n}\right) : \Pi_n \leq \epsilon n \right) \right) \\
&\leq \sum_{k=1}^{[\epsilon n]} {}_nC_k R^{2dk} \exp \left(-C_1 n\delta\left(\frac{k}{n}\right)^2 \right) \\
&\quad + \left(\mathbb{E}_0 \left(\exp \left(2n\delta\left(\frac{\Pi_n}{n}\right) : \Pi_n \leq \epsilon n \right) \right) \right)^{-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}_0\left(\exp\left(n\delta\left(\frac{\widehat{\Pi}_n}{n}\right)+c\Pi_n\right) : 1 \le \Pi_n \le \epsilon n\right)\\
&= J_1(n) + J_2(n) \quad (\text{say,})
\end{aligned}
$$

where at the last inequality we used Lemma 6.3 together with Chebyshev's inequality. Using Stirling formula we get

$$
J_1(n) \le \text{const.} \sum_{k=1}^{[\epsilon n]} k^{-1/2} \exp\left(-k\log\frac{k}{n} + 2dk\log R - C_1 n\delta\left(\frac{k}{n}\right)^2\right).
$$

So, letting

$$
C_1\delta(r)^2 = 3r\log\frac{1}{r} + 2dr\log R, \tag{6.20}
$$

we get

$$
J_1(n) \le \text{const.} \sum_{k=1}^{[\epsilon n]} k^{-1/2} \exp 2k\log\frac{k}{n},
$$

which yields that $\{J_1(n)\}$ is summable.

Next choose a small $\epsilon > 0$ so that $\delta(r)$ of (6.20) is increasing in $r \in [0,\epsilon]$, then

$$
J_2(n) \le \exp(n \sup_{0\le r\le\epsilon} \{cr - \delta(r)\}).
$$

Accordingly, for small ϵ, $\{J_2(n)\}$ also is summable, thus the Borel-Cantelli lemma implies that P^Y-almost surely

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_0\left(\exp\int_0^n dZ_{X(s)}(s) : 1 \le \Pi_n \le \epsilon n\right)\\
&\le \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}_0\left(\exp 2n\delta\left(\frac{\Pi_n}{n}\right) : \Pi_n \le \epsilon n\right)\\
&= \sup_{0\le r\le\epsilon}(2\delta(r) - I(r))\\
&\le \text{const.} \sup_{0\le r\le\epsilon}\left(\sqrt{r\log\frac{1}{r}} - r\log\frac{1}{\kappa}\right)\\
&\approx \frac{\log(\log(1/\kappa))}{\log(1/\kappa)} \quad \text{as} \quad \kappa\searrow 0,
\end{aligned}
$$

which completes the proof of the upper bound of (6.6). On the other hand the lower bound of (6.6) follows from Theorem 4.2 and Theorem 5.1 since the condition (5.1) is satisfied by (6.5). Therefore the proof of Theorem 6.1 is complete. □

Proof of Theorem 6.2. The proof is reduced to the case $\alpha = 0$, and $\rho = \delta_{\{-1\}}$, for which (6.8) was proved in [9], so that by virtue of Theorem 1.4 of [9], for $Y'(t) = -N((0,t],\{-1\}) + \rho(\{-1\})t$

$$\lambda(\kappa A; Y') \approx \log \kappa \quad \text{as} \quad \kappa \searrow 0. \tag{6.21}$$

Furthermore, by the comparison result of Theorem 4.1 we obtain the upper estimate of (6.8).

On the other hand by Theorem 2.2

$$\xi_i^1(t) = \mathbb{E}_i\left(\exp\int_0^t dZ_{X(t-s)}(s) : \int_0^t N_{X(t-s)}(ds, \{-1\}) = 0\right), \tag{6.22}$$

where

$$\begin{aligned} Z_i(t) = \alpha B_i(t) &+ \int_{[0,t]\times(|u|<1/2)} \log(1+u)\tilde{N}_i(ds,du) \\ &+ \int_{[0,t]\times(-1,-1/2]\cup[1/2,\infty)} \log(1+u)N_i(ds,du) + \beta t, \end{aligned}$$

with

$$\beta = \int_{[-1,\infty)}\left(I\left(|u| < \frac{1}{2}\right)\log(1+u) - u\right)\rho(du).$$

Let

$$Z_i'(t) = \alpha B_i(t) + \int_{[0,t]\times(|u|<1/2)} \log(1+u)\tilde{N}_i(ds,du).$$

By the proof of Theorem 6.1

$$\lim_{t\to\infty}\frac{1}{t}E^Y\left(\log\mathbb{E}_i\left(\exp - \int_0^t dZ'_{X(t-s)}(s)\right)\right) \leq \text{const.}\frac{\log(\log(1/\kappa))}{\log(1/\kappa)} \quad \text{for small} \quad \kappa > 0. \tag{6.23}$$

By the previous result in [9] we know that

$$\begin{aligned} &\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_i\left(\int_0^t N_{X(t-s)}\left(ds, \left(|u| \geq \frac{1}{2}\right)\right) = 0\right) \\ &\approx \log\kappa \quad \text{for small} \quad \kappa > 0. \end{aligned} \tag{6.24}$$

Noting that by the Schwarz inequality

$$\mathbb{P}_i\left(\int_0^t N_{X(t-s)}\left(ds, \left(|u| \geq \frac{1}{2}\right)\right) = 0\right)$$

$$\leq \mathbb{E}_i\left(\exp\int_0^t dZ'_{X(t-s)}(s) : \int_0^t N_{X(t-s)}\left(ds,\left(|u|\geq\frac{1}{2}\right)\right)=0\right)^{1/2}$$
$$\times\,\mathbb{E}_i\left(\exp -\int_0^t dZ'_{X(t-s)}(s)\right)^{1/2}$$
$$\leq e^{\beta t}\mathbb{E}_i\left(\exp\int_0^t dZ_{X(t-s)}(s) : \int_0^t N_{X(t-s)}(ds,\{-1\})=0\right)^{1/2}$$
$$\times\,\mathbb{E}_i\left(\exp -\int_0^t dZ'_{X(t-s)}(s)\right)^{1/2},$$

by (6.22)–(6.24) we have a constant $C>0$ satisfying that

$$\lambda(\kappa A;Y)\geq C\log\kappa \quad \text{for small} \quad \kappa>0,$$

which combines with (6.23) to complete the proof of Theorem 6.2. □

A combination of Theorem 6.1 and Theorem 4.1 yield a continuity result of $\lambda(\kappa A;Y)$ as $\kappa\searrow 0$ as follows.

Theorem 6.3. *Assume that $A=(a(i,j))$ is of finite range.*
(i) *If* (5.1) *is fulfilled, then* $\lim_{\kappa\searrow 0}\lambda(\kappa A;Y)=\lambda_0(Y)>-\infty$.
(ii) *While if* (5.1) *is violated, then* $\lambda_0(Y)=-\infty$ *and* $\lim_{\kappa\searrow 0}\lambda(\kappa A;Y)=-\infty$.

Proof. Let $Y^{(n)}(t)$ be a Lévy process with the characteristic exponent (1.7) with $\rho_n=\rho|_{(-1+1/n,\infty)}$ in place of ρ. Then Theorem 6.1 is applicable for $Y^{(n)}(t)$ and it holds

$$\lim_{\kappa\searrow 0}\lambda(\kappa A;Y^{(n)})=\lambda_0(Y^{(n)}). \tag{6.25}$$

Also by Theorem 4.1,

$$\lambda_0(Y)\leq\lambda(\kappa A;Y)\leq\lambda(\kappa A;Y^{(n)}). \tag{6.26}$$

Since $\lim_{n\to\infty}\lambda_0(Y^{(n)})=\lambda_0(Y)\geq-\infty$, (6.25) and (6.26) yield (i) and (ii). □

REMARK 6.1. Assume the condition [A] together with $\sum_{j\in\mathbb{Z}^d}|j|^2a(0,j)<\infty$, and that $Y(t)$ has zero mean.
(i) If $d=1$ or 2, for every $\kappa>0$ it holds that

$$\lim_{t\to\infty}\xi_i(t)=0 \quad \text{in probability} \quad (i\in\mathbb{Z}^d) \tag{6.27}$$

for every nonnegative solution of of (1.1) $\xi(t)=\{\xi_i(t)\}$ satisfying $\sup_{i\in\mathbb{Z}^d}E^Y(\xi_i(0))<\infty$.

(ii) If $d \geq 3$, there is a constant $\kappa_0 > 0$ such that (6.27) holds for $0 < \kappa < \kappa_0$.

Proof. (i) can be proved by the same method as Theorem 4.5 in [7], Chap. IX. For (ii) apply Theorem 6.3 that asserts that $\lambda(\kappa A; Y) < 0$ for small $\kappa > 0$. Moreover Corollary 3.1 implies (6.9). □

REMARK 6.2. Let $d \geq 3$, and let $(\Omega, \mathcal{F}, P_\xi, \xi(t) = \{\xi_i(t)\})$ be a linear Markovian system associated with (1.1). Remark 6.1 implies that if $\kappa > 0$ is small, any $\mathbb{Z}^d$-shift invariant stationary distribution μ satisfying $E^\mu(\xi_i) < \infty$ coincides with δ_0 (the point mass at $\xi_i = 0$ $(i \in \mathbb{Z}^d)$). On the other hand if

$$\int_{(u>1)} u^2 \rho(du) < \infty, \tag{6.28}$$

for a large $\kappa > 0$, $(\Omega, \mathcal{F}, P_\xi, \xi(t) = (\xi_i(t)))$ has non-trivial stationary distributions, that can be proved by making use of standard second moment computations as in [7], Chap. IX. However if (6.28) is violated, it holds $E_1(\xi_i(t)^2) = \infty$ for $t > 0$, so that the second moment arguments are not applied. In this case it is an open problem how to show existence of non-trivial stationary distributions for a large $\kappa > 0$.

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