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# ON REGULAR DIRICHLET SUBSPACES OF $H^1(I)$ AND ASSOCIATED LINEAR DIFFUSIONS

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**Abstract**

We will give a complete characterization of all regular Dirichlet subspaces of $H^1(I)$ for a finite open interval $I$ by a certain family of scale functions. Each associated diffusion will be constructed from a reflecting Brownian motion on a closed interval by a time change and a transformation of the state space.

## 1. Introduction

Throughout this paper let $I$ be a finite open interval $(a, b)$ or the real line $\mathbb{R}$. Denote by $L^2(I)$ the space of square integrable functions on $I$ and we let

$$H^1(I) = \{u \in L^2(I) \colon u \text{ is absolutely continuous and } u' \in L^2(I)\}.$$
$$\mathbf{D}(u, v) = \int_I u' \cdot v' \, dx \quad u, v \in H^1(I).$$

$(H^1(I), (1/2)\mathbf{D})$ can be considered as a regular local recurrent Dirichlet form on $L^2(\bar{I})$, where $\bar{I}$ denotes $[a, b]$ (resp. $\mathbb{R}$) for $I = (a, b)$ (resp. $I = \mathbb{R}$). The associated diffusion process on $\bar{I}$ is the reflecting Brownian motion (resp. the Brownian motion).

We call $(\mathcal{F}, \mathcal{E})$ a *Dirichlet subspace of* $(H^1(I), (1/2)\mathbf{D})$ if

$$\mathcal{F} \subset H^1(I), \ \ \mathcal{E}(u, v) = \frac{1}{2}\mathbf{D}(u, v), \quad u, v \in \mathcal{F}, \tag{1.1}$$

and $(\mathcal{F}, \mathcal{E})$ is a Dirichlet form on $L^2(I)$. It is called *regular on* $L^2(\bar{I})$ ($= L^2(I)$) if $\mathcal{F} \cap C_0(\bar{I})$ is dense both in $\mathcal{F}$ and $C_0(\bar{I})$, where $C_0(\bar{I})$ denotes the space of continuous functions on $\bar{I}$ with compact support. It is called *recurrent* if its extended Dirichlet space $\mathcal{F}_e$ contains the constant function 1. When $I$ is finite, any regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ is automatically recurrent.

In this paper, we shall prove that the Sobolev space $(H^1(I), (1/2)\mathbf{D})$ admits as its

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regular Dirichlet subspaces the following family of spaces $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})_{\mathtt{s}\in\mathbf{S}}$:

$$\mathcal{F}^{(\mathtt{s})} := \left\{ u \in L^2(I) \colon u \text{ is absolutely continuous} \right. \\ \left. \text{with respect to } d\mathtt{s}(x),\ \int_I \left(\frac{du}{d\mathtt{s}}(x)\right)^2 d\mathtt{s}(x) < \infty \right\} \tag{1.2}$$

$$\mathcal{E}^{(\mathtt{s})}(u, v) := \frac{1}{2}\int_I \frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}}\, d\mathtt{s}, \quad u, v \in \mathcal{F}^{(\mathtt{s})}, \tag{1.3}$$

for $\mathtt{s}$ belonging to the space of functions

$$\mathbf{S} = \{\mathtt{s} \colon \mathtt{s}(x) \text{ is absolutely continuous, strictly increasing in } x \in I \\ \text{and } \mathtt{s}'(x) = 0 \textit{ or } 1 \text{ for a.e. } x \in I,\ \ \mathtt{s}(\eta) = 0\}, \tag{1.4}$$

where $\eta$ denotes either $a$ or 0 according as $I$ is $(a, b)$ or $\mathbb{R}$.

We shall further consider the subfamily

$$\hat{\mathbf{S}} = \begin{cases} \mathbf{S} & \text{when } I = (a, b), \\ \{\mathtt{s} \in \mathbf{S} : \mathtt{s}(\pm\infty) = \pm\infty\} & \text{when } I = \mathbb{R}, \end{cases} \tag{1.5}$$

of $\mathbf{S}$ and prove that all recurrent regular Dirichlet subspaces of $(H^1(I), (1/2)\mathbf{D})$ are exhausted by the family of spaces $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})_{\mathtt{s}\in\hat{\mathbf{S}}}$.

For $\mathtt{s} \in \mathbf{S}$, we let $E_{\mathtt{s}} = \{x \in I \colon \mathtt{s}'(x) = 0\}$ and denote by $|\cdot|$ the Lebesgue measure. Denote by $\varphi$ the linear function $\varphi(x) = x,\ x \in I$. Clearly, $\varphi \in \mathcal{F}^{(\mathtt{s})}$ ($\mathcal{F}^{(\mathtt{s})}_{\mathrm{loc}}$ when $I = \mathbb{R}$) if and only if $|E_{\mathtt{s}}| = 0$, or equivalently, the inverse function of $\mathtt{s}$ is absolutely continuous. In this case, $\mathtt{s}(x)$ equals either $\varphi(x) - a$ or $\varphi(x)$ according as $I$ is $(a, b)$ or $\mathbb{R}$, and $\mathcal{F}^{(\mathtt{s})} = H^1(I)$ of course. A typical example of an element $\mathtt{s} \in \mathbf{S}$ for $I = (0, 2)$ with $|E_{\mathtt{s}}| > 0$ is provided by

$$\mathtt{s} := \mathtt{t}^{-1},\ \ \mathtt{t}(x) := c(x) + x, \quad x \in (0, 1), \tag{1.6}$$

where $c$ is the standard Cantor function on $(0, 1)$.

In this connection, we would like to mention that the second and the third authors have considered in [3] a slightly more general regular Dirichlet form than $(H^1(I), (1/2)\mathbf{D})$ for $I = (0, 1)$ and studied its regular Dirichlet subspace. Unfortunately, there is a flaw in the proof of Theorem 2 in [3]. As is corrected in [4], it should be replaced by the following weaker assertion for which the proof given in [3] works: *Let $\check{\mathcal{F}}$ be a subspace of $\mathcal{F}$ such that $(\check{\mathcal{F}}, \mathcal{E})$ is a regular Dirichlet space on $L^2(\overline{I}, \rho\, dx)$. Assume that a scale function $\mathtt{s}$ of the diffusion process on $\overline{I}$ associated with $(\check{\mathcal{F}}, \mathcal{E})$ admits an absolutely continuous inverse $\mathtt{t}$. Then $\check{\mathcal{F}} = \mathcal{F}$.*

The organization of the present paper is as follows. The next two sections are devoted to the proof of the above mentioned assertions. In particular, we shall show in §2 that any recurrent regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ has a scale function belonging to the class $\hat{\mathbf{S}}$.

In §3, we shall construct a recurrent diffusion process on $[a, b]$ (resp. $\mathbb{R}$) associated with the space $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on a closed interval (resp. the Brownian motion on $\mathbb{R}$) by a time change and a state space transformation. Since the infinitesimal generator of this diffusion is $(d/2dx)(d/d\mathtt{s})$ in Feller's canonical form, such a construction is well known in principle (cf. [6]), but we shall formulate it in relation to the transformations of Dirichlet forms in order to ensure the recurrence of the resulting diffusion and Dirichlet form.

In the last section, we shall state some useful descriptions of the space $\mathbf{S}$ and give examples of $\mathtt{s} \in \mathbf{S} \setminus \hat{\mathbf{S}}$ corresponding to transient regular Dirichlet subspaces of $(H^1(\mathbb{R}), (1/2)\mathbf{D})$.

## 2. Regular Dirichlet subspaces and scale functions

We recall (cf. [2, p.55]) that the extended Dirichlet space $H^1_e(I)$ of $H^1(I)$ is given by

$$H^1_e(I) = \{u : u \text{ is absolutely continuous on } I \text{ and } u' \in L^2(I)\}. \tag{2.1}$$

In particular, $1 \in H^1_e(I)$ and the Dirichlet form $(H^1(I), (1/2)\mathbf{D})$ is recurrent. $H^1_e(I)$ is continuously imbedded into $C(\overline{I})$ and in fact the following elementary inequality holds for any $x, y \in \overline{I}$:

$$|u(y) - u(x)|^2 \leq |y - x|\mathbf{D}(u, u), \quad u \in H^1_e(I). \tag{2.2}$$

When $I$ is finite, $H^1_e(I) = H^1(I)$.

Let $(\mathcal{F}, \mathcal{E})$ be a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$. Since $(\mathcal{F}, \mathcal{E})$ is strongly local, there exists a diffusion process $\mathbf{M} = (X_t, P_x)$ on $\overline{I}$ associated with it. Denote by $\sigma_y$ the hitting time of the one point set $\{y\}$, $y \in \overline{I}$, for $\mathbf{M}$. The next lemma about the existence of the scale function (a strictly increasing continuous function satisfying (2.3)) is well known for a more general one-dimensional diffusion process ([5]) but we give a self contained proof of it based on the inequality (2.2) in the present special situation.

**Lemma 2.1.** *There exists a strictly increasing function $\mathtt{s}$ on $\overline{I}$ uniquely up to a linear transformation such that*

$$P_x(\sigma_d < \sigma_c) = \frac{\mathtt{s}(x) - \mathtt{s}(c)}{\mathtt{s}(d) - \mathtt{s}(c)}, \quad c \leq x \leq d, \ c, d \in \overline{I}. \tag{2.3}$$

*$\mathtt{s}$ is absolutely continuous on $I$.*

Proof. Let $J$ be a connected open subset of $\overline{I}$. We denote by $\tau_J$ the leaving time from $J$ of the diffusion $\mathbf{M}$. We also consider the part $\mathbf{M}_J$ of $\mathbf{M}$ on $J$ the diffusion killed upon the leaving time $\tau_J$. $\mathbf{M}_J$ is then associated with the subspace $\mathcal{F}_J$ of $(\mathcal{F}, \mathcal{E})$ defined by

$$\mathcal{F}_J = \{u \in \mathcal{F} : u(x) = 0,\ \ x \in \overline{I} \setminus J\}.$$

(2.2) implies that each singleton of $J$ has a positive capacity with respect to the Dirichlet form $(\mathcal{F}_J, \mathcal{E})$. Consequently, the connectedness of the state space $J$ is a synonym for its quasi-connectedness for $(\mathcal{F}_J, \mathcal{E})$ and hence $(\mathcal{F}_J, \mathcal{E})$ is irreducible ([2, p.172]). This implies, by virtue of [2, Theorem 4.6.6], that

$$P_x(\sigma_y < \tau_J) > 0 \quad \forall x, y \in J. \tag{2.4}$$

For any $c, d \in \overline{I},\ -\infty < c < d < \infty$, we make the following choice of the intervals $J \subset \overline{I}$: when $\overline{I} = [a, b]$ (resp. $\overline{I} = \mathbb{R}$), we take $[a, d)$ and $(c, b]$ (resp. $(-\infty, d)$ and $(c, \infty)$). We then get from (2.4)

$$P_x(\sigma_c < \sigma_d) > 0,\ \ P_x(\sigma_d < \sigma_c) > 0, \quad \forall x \in (c, d).$$

We also note here that

$$P_c(\sigma_c < \sigma_d) = 1 \quad P_d(\sigma_d < \sigma_c) = 1 \tag{2.5}$$

because the positivity of the capacity of a point implies the $\mathbf{M}$-regularity of the point for itself.

On the other hand, for the finite open interval $J = (c, d) \subset I$, the space $(\mathcal{F}_J, \mathcal{E})$ admits a 0-order potential operator $G^0$ by virtue of (2.2) again: for any $f \in L^2(J)$,

$$G^0 f \in \mathcal{F}_J,\ \ \mathcal{E}_J(G^0 f, v) = \int_J f v\, dx, \quad \forall v \in \mathcal{F}_J.$$

Therefore

$$E_x(\sigma_c \wedge \sigma_d) = G^0 1_J(x) < \infty, \quad x \in (c, d),$$

and

$$P_x(\sigma_c < \sigma_d) + P_x(\sigma_d < \sigma_c) = 1, \quad x \in (c, d).$$

In particular, the function $p_{c,d}(x) = P_x(\sigma_d < \sigma_c),\ x \in \overline{I}$, is not only strictly positive but also strictly increasing in $x \in (c, d)$ because the sample path continuity and the strong Markov property of $\mathbf{M}$ implies

$$p_{c,d}(x) = p_{c,y}(x) p_{c,d}(y) < p_{c,d}(y), \quad c < x < y < d. \tag{2.6}$$

In the same way, we have, for $c' \leq c < d \leq d'$, $c', d' \in \overline{I}$, that

$$\begin{aligned} p_{c',d'}(x) &= p_{c,d}(x)p_{c',d'}(d) + (1 - p_{c,d}(x))p_{c',d'}(c) \\ &= (p_{c',d'}(d) - p_{c',d'}(c))p_{c,d}(x) + p_{c',d'}(c) \quad c \leq x \leq d. \end{aligned} \tag{2.7}$$

When $I = (a, b)$, we let

$$\mathsf{s}(x) = p_{a,b}(x) \quad x \in \overline{I}.$$

Then $\mathsf{s}$ is strictly increasing and its property (2.3) follows from (2.7) with $c' = a$, $d' = b$. When $I = \mathbb{R}$, we put, for any $c < d$ such that $c \leq x \leq d$ and $c < 0$, $1 < d$,

$$\mathsf{s}(x) = \alpha p_{c,d}(x) + \beta,$$

and determines constants $\alpha$, $\beta$ by

$$\mathsf{s}(0) = 0, \quad \mathsf{s}(1) = 1.$$

Then, $\mathsf{s}(x)$ is independent of such a choice of $(c, d)$ because, for any interval $(c', d') \supset (c, d)$, $p_{c,d}$ is a linear function of $p_{c',d'}$ on $[c, d]$ in view of (2.4). Further $\mathsf{s}$ satisfies (2.3) because $p_{c,d}(c) = 0$, $p_{c,d}(d) = 1$.

Finally, in order to show the absolute continuity of $\mathsf{s}$, we take any finite interval $(c, d) \subset I$. It suffices to prove that the function $p(x) = p_{c,d}(x)$, $x \in I$, is absolutely continuous since $\mathsf{s}$ is a linear function of $p$ on $(c, d)$.

When $I = (a, b)$, $p(x)$ is known to be the 0-order equilibrium potential of $\{d\}$ with respect to the Dirichlet space

$$\mathcal{F}_{(c,b]} = \{u \in \mathcal{F} : u(x) = 0, \ \forall x \leq c\},$$

and $p(x)$ is characterized by

$$p \in \mathcal{F}_{(c,b]}, \ \ p(d) = 1, \ \ \mathcal{E}(p, v) \geq 0, \quad \forall v \in \mathcal{F}_{(c,b]}, \ \ v(d) \geq 0. \tag{2.8}$$

In particular, $p$ is absolutely continuous.

When $I = \mathbb{R}$, we consider the space

$$\mathcal{F}_{(c,\infty)} = \{u \in \mathcal{F} : u(x) = 0, \ \forall x \leq c\}.$$

By virtue of (2.2), we see that the Dirichlet space $(\mathcal{F}_{(c,\infty)}, \mathcal{E})$ is transient and the function $p(x)$ is the associated 0-order equilibrium potential of $\{d\}$ characterized by

$$p \in \mathcal{F}_{(c,\infty),e}, \quad p(d) = 1, \quad \mathcal{E}(p, v) \geq 0, \quad \forall v \in \mathcal{F}_{(c,\infty),e}, \ \ v(d) \geq 0, \tag{2.9}$$

where $\mathcal{F}_{(c,\infty),e} (\subset H^1_e(\mathbb{R}))$ is the extended Dirichlet space of $\mathcal{F}_{(c,\infty)}$. Hence $p$ is absolutely continuous. $\square$

We call the function $\mathtt{s}$ in Lemma 2.1 the *scale function* associated with the regular Dirichlet subspace $(\mathcal{F}, \mathcal{E})$ of $(H^1(I), (1/2)\mathbf{D})$.

We continue to consider a finite open interval $J = (c, d) \subset I$ and the corresponding function $p(x) = p_{c,d}(x)$ as in the proof of Lemma 2.1. By virtue of (2.2), the space $(\mathcal{F}_J, \mathcal{E})$ admits the reproducing kernel $g^0(x, y),\ x, y \in J$ characterized by

$$g^0(\cdot, y) \in \mathcal{F}_J,\ \ \mathcal{E}(g^0(\cdot, y), v) = v(y), \quad \forall v \in \mathcal{F}_J.$$

**Lemma 2.2.** *There exists a constant $C > 0$, such that, for any $x, y \in J$,*

$$g^0(x, y) = \begin{cases} Cp(x)(1 - p(y)), & x \le y; \\ C(1 - p(x))p(y), & x \ge y. \end{cases}$$

Proof. We consider the function

$$p^0_y(x) := P_x(\sigma_c \wedge \sigma_d > \sigma_y), \quad x, y \in J, \tag{2.10}$$

$p^0_y(\,\cdot\,)$ is the 0-order equilibrium potential of $\{y\}$ with respect to $(\mathcal{F}_J, \mathcal{E})$ characterized by

$$p^0_y \in \mathcal{F}_J,\ \ p^0_y(y) = 1,\ \ \mathcal{E}(p^0_y, v) \ge 0, \quad \forall v \in \mathcal{F}_J,\ \ v(y) \ge 0. \tag{2.11}$$

The above two characterizations lead us to

$$p^0_y(x) = \frac{g^0(x, y)}{g^0(y, y)} \quad x, y \in J.$$

On the other hand, we have $p^0_y(x) = p_{c,y}(x),\ \ c < x \le y$, and we get from (2.7)

$$p^0_y(x) = \begin{cases} \dfrac{p(x)}{p(y)}, & x \le y \\[2ex] \dfrac{1 - p(x)}{1 - p(y)}, & x \ge y, \end{cases}$$

for $x, y \in J$. The desired expression of $g^0(x, y)$ follows from the above two identities. □

**Lemma 2.3.** *Any function in $\mathcal{F}$ is absolutely continuous with respect to $d\mathtt{s}$.*

Proof. For any finite interval $J = (c, d) \subset I$, let $G^0$ be the 0-order potential operator associated with $(\mathcal{F}_J, \mathcal{E})$ as was considered in the proof of Lemma 2.1. Then it follows from Lemma 2.2 that, for $f \in L^2(J)$, $x \in J$,

$$G^0 f(x) = \int_J g^0(x, y) f(y)\, dy$$

$$= C(1-p(x))\int_c^x p(y)f(y)\,dy + Cp(x)\int_x^d (1-p(y))f(y)\,dy$$
$$= Cp(x)\int_c^d (1-p(y))f(y)\,dy - C\int_c^x\int_c^y f(z)\,dz\,dp(y),$$

which means that $G^0 f$ is absolutely continuous with respect to $p$, namely, it can be expressed as $\int_c^x \varphi(y)p'(y)\,dy$ by some function $\varphi \in L^1(J;dp)$.

Since $G^0(L^2(J))$ is dense in $\mathcal{F}_J$, there exist, for any $u \in \mathcal{F}_J$, $f_n \in L^2(J)$ such that $u_n = G^0 f_n = \int_0^\cdot \varphi_n(y)p'(y)\,dy$ is $\mathcal{E}$ convergent to $u$. Hence, for any $B \subset J$ on which $p'(x) = 0$ a.e.,

$$\int_B u'(x)^2\,dx = \int_B (u'(x) - \varphi_n(x)p'(x))^2\,dx \le \mathcal{E}(u-u_n, u-u_n) \to 0, \quad n \to \infty,$$

which implies $u'(x) = 0$ a.e. on $B$, namely, $u$ is absolutely continuous with respect to $dp$.

Finally, any $u \in \mathcal{F}$ can be expressed as

$$u(x) = u(c) + (u(d)-u(c))p(x) + [(u(x)-u(c)) - (u(d)-u(c))p(x)], \quad x \in J, \tag{2.12}$$

the last term being a member of $\mathcal{F}_J$. Therefore $u$ is absolutely continuous on $J$ with respect to $dp$ and hence with respect to $d\mathtt{s}$. □

Suppose that $(\mathcal{F},\mathcal{E})$ is recurrent. Then, by [2, Theorem 4.6.6], the property (2.4) for $J = \overline{I}$ is strengthened to

$$P_x(\sigma_y < \infty) = 1 \quad \forall x, y \in \overline{I}. \tag{2.13}$$

Note that, when $I = (a,b)$, $(\mathcal{F},\mathcal{E})$ is automatically recurrent because, owing to the regularity, $\mathcal{F}$ contains a continuous function $v$ greater than 1 on $[a,b]$ and hence the constant function $1 \wedge v$ as well.

For the scale function $\mathtt{s}$ associated with $(\mathcal{F},\mathcal{E})$, we let

$$E_{\mathtt{s}} = \left\{ x \in I : \limsup_{h\to 0} \frac{\mathtt{s}(x+h)-\mathtt{s}(x)}{h} = 0 \right\}. \tag{2.14}$$

**Lemma 2.4.** *Suppose $(\mathcal{F},\mathcal{E})$ is recurrent.*
(i) *$\mathtt{s}'$ is constant a.e. on $I \setminus E_{\mathtt{s}}$.*
(ii) *$\mathtt{s}(\pm\infty) = \pm\infty$.*

Proof. (i) Again we fix an arbitrary interval $(c,d) \subset I$ and denote by $p(x)$, $x \in I$, the function $p_{c,d}(x)$ in the proof of Lemma 2.1. We know that $p$ is absolutely continuous on $I$, strictly increasing on $(c,d)$ and $p((c,d)) = (0,1)$. Denote by $q$ the inverse function of $p|_{(c,d)}$.

We have then

$$|p(A)| = \int_A p'(x)\,dx, \quad \text{for any Borel set } A \subset (c,d). \tag{2.15}$$

This is clear for a disjoint union of finite number of subintervals of $(c,d)$, and the monotone class lemma (cf. [1]) then applies.

We next let

$$E = \left\{ x \in (c,d) \colon \limsup_{h\to 0} \frac{p(x+h)-p(x)}{h} = 0 \right\}$$

and $F = p(E)$. Then $|F| = 0$ by (2.15). (2.15) further means that, if $A \subset (c,d)\setminus E$ and $|A| > 0$, then $|p(A)| > 0$. Hence $q$ is absolutely continuous on $(0,1)\setminus F$.

On the other hand, for any $\varphi \in C_0^{(1)}((0,1))$, $\varphi(p) \in \mathcal{F}_{(c,b]}$ (resp. $\mathcal{F}_{(c,\infty),e}$) when $I = (a,b)$ (resp. $\mathbb{R}$.) Further $\varphi(p(x)) = 0$, $x \geq d$, $x \in I$, because $p(x) = 1$, $x \geq d$, $x \in I$, on account of (2.13) and (2.5) . Hence, in view of (2.8) and (2.9),

$$\int_0^1 p'(q(x))\varphi'(x)\,dx = \int_c^d p'(x)\varphi'(p(x))p'(x)\,dx = 2\mathcal{E}(p,\varphi(p)) = 0.$$

It follows that $p'(q(x))$ is constant a.e. on $(0,1)$. Therefore $p'$ is constant a.e. on $(c,d)\setminus E$. Since $\mathsf{s}$ is a linear function of $p$ on $(c,d)$, $\mathsf{s}'$ is constant a.e. on $(c,d)\setminus E_{\mathsf{s}}$ as was to be proved.

(ii) Since the recurrence assumption implies the conservativeness of the process $\mathbf{M}$ ([2]), it is easy to see that

$$P_x\left(\lim_{y\to\pm\infty} \sigma_y = \infty\right) = 1 \quad x \in \overline{I},$$

and we can get $\mathsf{s}(-\infty) = -\infty$ by noting (2.11) and letting $c \to -\infty$ in (2.3). Similarly we get $\mathsf{s}(\infty) = \infty$. □

We are now in a position to state a main theorem of this paper. Let $\mathbf{S}$ be the class of functions $\mathsf{s}$ defined by (1.4) and $\hat{\mathbf{S}}$ be its subclass defined by (1.5). For $\mathsf{s} \in \mathbf{S}$, we introduce the space $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ by (1.2) and (1.3).

**Theorem 2.1.** (i) *For any $\mathsf{s} \in \mathbf{S}$, the space $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ is a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$. The scale function associated with $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ equals $\mathsf{s}$ up to a linear transform.*

(ii) *Let $(\mathcal{F}, \mathcal{E})$ be a regular recurrent Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$ and $\mathsf{s}$ be the associated scale function. Then, by making a linear modification of $\mathsf{s}$ if necessary, $\mathsf{s}$ belongs to the class $\hat{\mathbf{S}}$ and*

$$\mathcal{F} = \mathcal{F}^{(\mathsf{s})}, \ \ \mathcal{E}(u,v) = \mathcal{E}^{(\mathsf{s})}(u,v), \quad u,v \in \mathcal{F}.$$

REMARK. The converse to (ii) (the recurrence of the space $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ for $\mathsf{s} \in \hat{\mathbf{S}}$) will be shown in the next section.

Proof. (i) Suppose $\mathsf{s} \in \mathbf{S}$ and $u, v \in \mathcal{F}^{(\mathsf{s})}$. Then $u, v$ are absolutely continuous with respect to $dx$ and

$$\begin{aligned} \frac{1}{2}\int_I \frac{du}{dx}\frac{dv}{dx}\,dx &= \frac{1}{2}\int_I \frac{du}{d\mathsf{s}}\frac{dv}{d\mathsf{s}}\mathsf{s}'(x)^2\,dx \\ &= \frac{1}{2}\int_I \frac{du}{d\mathsf{s}}\frac{dv}{d\mathsf{s}}\,d\mathsf{s}, \quad u, v \in \mathcal{F}^{(\mathsf{s})}. \end{aligned} \tag{2.16}$$

Hence $\mathcal{F}^{(\mathsf{s})} \subset H^1(I)$ and $\mathcal{E}^{(\mathsf{s})}(u, v) = (1/2)\mathbf{D}(u, v)$, $u, v \in \mathcal{F}^{(\mathsf{s})}$.

Since $u(d) - u(c) = \int_c^d (du/d\mathsf{s})\,d\mathsf{s}$, we see that

$$|u(d) - u(c)|^2 \le 2|d - c|\mathcal{E}^{(\mathsf{s})}(u, u) \quad (c, d) \subset I,\ u \in \mathcal{F}^{(\mathsf{s})},$$

and any $\mathcal{E}_1^{(\mathsf{s})}$-Cauchy sequence is uniformly convergent on any compact interval of $I$. Hence $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ is a closed symmetric form on $L^2(I)$. Clearly it is Markovian.

The regularity is also verifiable. Indeed, when $I$ is a finite interval, $\mathcal{F}^{(\mathsf{s})}$ contains $\mathsf{s}$ and constant functions and hence an algebra generated by them, which separates points of $\overline{I}$. Consequently $\mathcal{F}^{(\mathsf{s})}$ is dense in $C(\overline{I})$ by the Weierstrass theorem. Since the above inequality implies that $\mathcal{F}^{(\mathsf{s})} \subset C(\overline{I})$, we see that $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ is a regular Dirichlet form on $L^2(\overline{I})$.

When $I = \mathbb{R}$, we consider the space

$$\mathcal{C} = \{\varphi(\mathsf{s}) : \varphi \in C_0^1(\mathbb{R})\}.$$

Then $\mathcal{C} \subset \mathcal{F}^{(\mathsf{s})}$. Since $\mathcal{C}$ is an algebra separating points of $I$, it is dense in $C_0(\mathbb{R})$. Suppose $u \in \mathcal{F}^{(\mathsf{s})}$ is $\mathcal{E}_1$-orthogonal to $\mathcal{C}$: $\mathcal{E}_1(u, v) = 0\ \forall v \in \mathcal{C}$. Then $u$ is a solution of the equation

$$\frac{1}{2}\frac{d}{dx}\frac{du}{d\mathsf{s}} = u.$$

It is known that the solutions of this equation form a 2-dimensional vector space spanned by a positive increasing function $u^{(1)}$ and a positive decreaing function $u^{(2)}$ ([6]). Obviously, neither $u^{(1)}$ nor $u^{(2)}$ is in $L^2(\mathbb{R})$ and $u$ must vanish. Hence $\mathcal{C}$ is dense in $\mathcal{F}^{(\mathsf{s})}$. Therefore $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ is a regular Dirichlet subspace of $(H^1(I), (1/2)\mathbf{D})$.

In order to prove the second assertion in (i), we consider any finite interval $J = (c, d) \subset \mathbb{R}$, take any $d_1 \in J$ and put

$$r(x) = \left(\frac{\mathsf{s}(x) - \mathsf{s}(c)}{\mathsf{s}(d_1) - \mathsf{s}(c)}\right)^+ \wedge \left(\frac{\mathsf{s}(d) - \mathsf{s}(x)}{\mathsf{s}(d) - \mathsf{s}(d_1)}\right)^+, \quad x \in \mathbb{R}.$$

We readily see that $r \in \mathcal{F}_J^{(\mathsf{s})}$, $r(d_1) = 1$ and, for any $v \in \mathcal{F}_J^{(\mathsf{s})}$,

$$\begin{aligned}\mathcal{E}^{(\mathsf{s})}(r, v) &= \frac{1}{2(\mathsf{s}(d_1) - \mathsf{s}(c))} \int_c^{d_1} \frac{dv}{d\mathsf{s}} d\mathsf{s} - \frac{1}{2(\mathsf{s}(d) - \mathsf{s}(d_1))} \int_{d_1}^{d} \frac{dv}{d\mathsf{s}} d\mathsf{s} \\ &= \frac{1}{2(\mathsf{s}(d_1) - \mathsf{s}(c))} v(d_1) + \frac{1}{2(\mathsf{s}(d) - \mathsf{s}(d_1))} v(d_1).\end{aligned}$$

Hence $r$ satisfies the condition (2.11) for $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ and $r(x)$ coincides with the function $p^0_{d_1}(x)$ defined by (2.10) on $J$ for the diffusion $(X_t, P_x)$ associated with $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ and in particular

$$r(x) = P_x(\sigma_{d_1} < \sigma_c) \quad x \in (c, d_1).$$

Since

$$r(x) = \frac{\mathsf{s}(x) - \mathsf{s}(c)}{\mathsf{s}(d_1) - \mathsf{s}(c)} \quad c < x < d_1,$$

we have shown that $\mathsf{s}$ is a scale function for the space $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$.
(ii) The scale function $\mathsf{s}$ associated with a given regular recurrent Dirichlet subspace $(\mathcal{F}, \mathcal{E})$ of $(H^1(I), (1/2)\mathbf{D})$ belongs to $\hat{\mathbf{S}}$ (after an appropriate linear transform) by virtue of Lemma 2.1 and Lemma 2.4. We further see from Lemma 2.3 and identity (2.16) for $u, v \in \mathcal{F}$ that $\mathcal{F} \subset \mathcal{F}^{(\mathsf{s})}$ and $\mathcal{E}(u, v) = \mathcal{E}^{(\mathsf{s})}(u, v)$, $u, v \in \mathcal{F}$.

Take an interval $J = (c, d) \subset I$. Consider any function $u \in \mathcal{F}^{(\mathsf{s})}$ with $u(x) = 0$ for $x \notin J$ and assume that $u$ is $\mathcal{E}^{(\mathsf{s})}$-orthogonal to the space $\mathcal{F}_J$:

$$\mathcal{E}^{(\mathsf{s})}(u, v) = 0, \quad \forall v \in \mathcal{F}_J.$$

By the function $p = p_{c,d}$ for $(\mathcal{F}, \mathcal{E})$ as in the proof of Lemma 2.1, we may write

$$\mathsf{s}(x) = c_0 p(x) + c_1, \;\; u(x) = \int_c^x \varphi(\xi)\, dp(\xi), \quad c \le x \le d.$$

Choosing as $v$ the Green function $g^{0,y}(x) = g^0(x, y) \in \mathcal{F}_J$ of Lemma 2.2 for each fixed $y \in J$, we are led to

$$\begin{aligned}\mathcal{E}^{(\mathsf{s})}(u, g^{0,y}) &= \int_0^d \frac{du}{ds} \frac{dg^{0,y}}{ds} ds \\ &= Cc_0^{-1} \int_c^y \varphi(x)(1 - p(y))\, dp(x) - Cc_0^{-1} \int_y^d \varphi(x) p(y)\, dp(x) \\ &= Cc_0^{-1} \int_c^y \varphi(x)\, dp(x) - Cc_0^{-1} p(y) \int_c^d \varphi(x)\, dp(x) = Cc_0^{-1} u(y),\end{aligned}$$

and $u = 0$. Hence any function in $\mathcal{F}^{(\mathsf{s})}$ with compact support belongs to the space $\mathcal{F}$. Since we have seen in (i) that $(\mathcal{F}^{(\mathsf{s})}, \mathcal{E}^{(\mathsf{s})})$ is regular, we have the desired inclusion $\mathcal{F}^{(\mathsf{s})} \subset \mathcal{F}$. $\square$

## 3. Constructions by time change and state space transform

If the scale function of the diffusion associated with the regular Dirichlet subspace is $\mathtt{s}$, then we know intuitively that, after a state space transformation $\mathtt{s}\colon I \to \mathtt{s}(I)$, the diffusion becomes another diffusion with hitting distributions identical with that of Brownian motion and that this new diffusion differs from the Brownian motion by a time change. This suggests a way of constructing the original diffusion and Dirichlet subspace from the Brownian motion and Sobolev space.

In this section, we construct a recurrent diffusion process $\tilde{X}$ associated with the Dirichlet form (1.2), (1.3) on $L^2(I)$ for $\mathtt{s} \in \hat{\mathbf{S}}$ from the reflecting Brownian motion on $\mathtt{s}(\overline{I})$ when $I$ is finite and the Brownian motion on $\mathbb{R}$ when $I = \mathbb{R}$ by a time change and a transformation of the state space. We also notice that $\tilde{X}$ is the one-dimensional diffusion on $\overline{I}$ with infinitesimal generator $(1/2)(d/dx)(d/d\mathtt{s})$ in Feller's sense ([5]).

We prepare a lemma.

**Lemma 3.1.** *Let $(E, m)$ be a $\sigma$-finite measure space, $X = (X_t, P_x)$ be an $m$-symmetric Markov process on $E$ and $(\mathcal{F}, \mathcal{E})$ be the associated Dirichlet space on $L^2(E; m)$. Let $\gamma$ be a one-to-one measurable transformation from $E$ onto a space $\tilde{E}$ and $\tilde{m}$ be the image measure; $\tilde{m}(B) = m(\gamma^{-1}(B))$. We put*

$$\tilde{X}_t = \gamma(X_t), \quad \tilde{P}_x = P_{\gamma^{-1}x}, \quad x \in \tilde{E}.$$

*Then $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E}$ and the associated Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ on $L^2(\tilde{E}, \tilde{m})$ satisfies*

$$\begin{aligned} \tilde{\mathcal{F}} &= \{u \in L^2(\tilde{E}; \tilde{m})\colon u \circ \gamma \in \mathcal{F}\} \\ \tilde{\mathcal{E}}(u, v) &= \mathcal{E}(u \circ \gamma, v \circ \gamma) \quad u, v \in \tilde{\mathcal{F}}. \end{aligned}$$

Proof. It was proved in [1, p. 325] that $\tilde{X}$ is a Markov process on $\tilde{E}$ with transition function

$$\tilde{p}_t f(y) = p_t(f \circ \gamma)(\gamma^{-1}(y)) \quad y \in \tilde{E}, \ \ f \in \mathcal{B}^+,$$

where $p_t$ is the transition function of $X$.

The $\tilde{m}$-symmetry of $\tilde{p}_t$ and the above relation of the Dirichlet spaces follow from

$$\begin{aligned} \int_{\tilde{E}} \tilde{p}_t f \cdot g d\tilde{m} &= \int_{\tilde{E}} p_t(f \circ \gamma)(\gamma^{-1}(y))(g \circ \gamma)(\gamma^{-1}(y))\, dm(\gamma^{-1}y) \\ &= \int_E p_t(f \circ \gamma)(g \circ \gamma)\, dm, \end{aligned}$$

and

$$\frac{1}{t} \int_{\tilde{E}} (f - \tilde{p}_t f) \cdot g\, d\tilde{m} = \frac{1}{t} \int_E (f \circ \gamma - p_t(f \circ \gamma)) g \circ \gamma\, dm.$$

That completes the proof. □

The process $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)_{x\in\tilde{E}}$ in the above lemma is called the process obtained from $X = (X_t, P_x)_{x\in E}$ by the transformation $\gamma$ of the state space from $E$ to $\tilde{E}$.

Take any $\mathtt{s}$ from the class $\hat{\mathtt{S}}$ defined by (1.5) and let $\mathtt{t}$ be its inverse function. Clearly

$$J = \mathtt{s}(I) = \begin{cases} (0, b-a-|E_{\mathtt{s}}|), & I = (a,b), \\ \mathbb{R}, & I = \mathbb{R}. \end{cases}$$

Let $(B_t, P_x)_{x\in\bar{J}}$ be the reflecting Brownian motion on $\bar{J}$ when $I = (a,b)$ and the Brownian motion on $\mathbb{R}$ when $I = \mathbb{R}$. It is associated with the regular local recurrent Dirichlet form $(\mathcal{H}^1(J), (1/2)\mathbf{D})$ on $L^2(\bar{J})$. The transition function of $(B_t, P_x)$ is absolutely continuous with respect to $dx$. Each one point set has a positive 1-capacity with respect to this Dirichlet form. Hence the quasi-support of a positive Radon measure on $\bar{J}$ coincides with its topological support.

Let $A_t$ be the PCAF (positive continuous additive functional) in the strict sense $(B_t, P_x)$ with Revuz measure $d\mathtt{t}$. Since the support of $d\mathtt{t}$ is $\bar{J}$, the fine support of $A_t$ is also $\bar{J}$ and $A_t$ is strictly increasing in $t$ a.s. Let $\tau_t$ be the inverse of $A_t$ and denote by $X$ the time change of $B_t$ by $(\tau_t)$:

$$X_t = B_{\tau_t}. \tag{3.1}$$

**Theorem 3.1.** (i) *Let*

$$\tilde{X}_t = \mathtt{t}\left(B_{\tau_t}\right), \; t \geq 0, \quad \tilde{P}_x = P_{\mathtt{s}(x)}, \; x \in \bar{I}. \tag{3.2}$$

*Then $(\tilde{X}_t, \tilde{P}_x)_{x\in\bar{I}}$ is a diffusion process on $\bar{I}$ associated with the regular Dirichlet subspace $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ on $L^2(\bar{I})$ of $(H^1(I), (1/2)\mathbf{D})$.*
(ii) $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ *is recurrent.*

Proof. (i) By virtue of (6.2.22) in [2], the time changed process $(X_t, P_x)_{x\in\bar{J}}$ is $d\mathtt{t}$-symmetric and its Dirichlet space $(\mathcal{F}^J, \mathcal{E}^J)$ on $L^2(\bar{J}; d\mathtt{t})$ is given by

$$\mathcal{F}^J = H^1_e(J) \cap L^2(\bar{J}, d\mathtt{t}), \;\; \mathcal{E}^J(u,v) = \frac{1}{2}\mathbf{D}(u,v), \quad u, v \in \mathcal{F}^J,$$

for the extended Dirichlet space $H^1_e(J)$ defined by (2.1) for $H^1(J)$.

Since $\tilde{X} = (\tilde{X}_t, \tilde{P}_x)$ is obtained from the time changed process $(X_t, P_x)$ of (3.1) by means of the transformation $\mathtt{t}$ of the state space from $\bar{J}$ onto $\bar{I}$, we see by Lemma 3.1 that $\tilde{X}$ is symmetric with respect to the image measure by $\mathtt{t}$ of $d\mathtt{t}$, which is obviously the Lebesgue measure $dx$ on $\bar{I}$, and the associated Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ on $L^2(\bar{I}) =$

$L^2(I)$ is given by

$$\tilde{\mathcal{F}} = \{u \in L^2(I) : u \circ \mathtt{t} \in \mathcal{F}_J\} = \{u \in L^2(I) : u \circ \mathtt{t} \in H_e^1(J)\}, \tag{3.3}$$

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2}\mathbf{D}(u \circ \mathtt{t}, v \circ \mathtt{t}), \quad u, v \in \tilde{\mathcal{F}}. \tag{3.4}$$

We claim that

$$\tilde{\mathcal{F}} = \mathcal{F}^{(\mathtt{s})}, \ \ \tilde{\mathcal{E}}(u, v) = \mathcal{E}^{(\mathtt{s})}(u, v), \quad u, v \in \tilde{\mathcal{F}}. \tag{3.5}$$

By (3.3), $u \in \tilde{\mathcal{F}}$ if and only if $u \in L^2(I)$ and there exists a function $\phi \in L^2(J)$ such that

$$u(\mathtt{t}(x)) = \int_0^x \phi(y)\, dy + C, \quad x \in J,$$

for some constant $C$. In this case,

$$u(x) = \int_0^{\mathtt{s}(x)} \phi(y)\, dy + C = \int_a^x \phi(\mathtt{s}(y))\, d\mathtt{s}(y) + C, \ \ x \in I$$

and

$$\frac{1}{2}\int_I \left(\frac{du}{d\mathtt{s}}\right)^2 d\mathtt{s} = \frac{1}{2}\int_I \phi(\mathtt{s}(x))^2\, d\mathtt{s}(x) = \frac{1}{2}\int_J \phi(x)^2\, dx,$$

and hence $\tilde{\mathcal{F}} \subset \mathcal{F}^{(\mathtt{s})}$ and $\tilde{\mathcal{E}} = \mathcal{E}^{(\mathtt{s})}$ on $\tilde{\mathcal{F}} \times \tilde{\mathcal{F}}$. Converse inclusion can be shown in the same way.
(ii) We have only to show this for $I = \mathbb{R}$. By virtue of [2, (6.2.23)], the extended Dirichlet space of $(\mathcal{F}^{\mathbb{R}}, \mathcal{E}^{\mathbb{R}})$ coincides with $(H_e^1(\mathbb{R}), (1/2)\mathbf{D})$ and hence contains constant functions. Since the Dirichlet space $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ is obtained by (3.3) and (3.4), its extended Dirichlet space also contains constant functions. □

From the proof, it also follows that ,for $\mathtt{s} \in \hat{\mathbf{S}}$, $u \in \mathcal{F}^{(\mathtt{s})}$ if and only if $u \circ \mathtt{t} \in H^1(J)$. Equivalently $\mathcal{F}^{(\mathtt{s})} = \{u \circ \mathtt{s} : u \in H^1(J)\}$.

## 4. Some descriptions of the class S

We can give more tractable descriptions of the class **S** of scale functions defined by (1.4).

Let **T** be the totality of function $\mathtt{t}$ defined on some open interval $J \subset \mathbb{R}$ expressed as

$$\mathtt{t}(x) = c(x) + x, \quad x \in J, \tag{4.1}$$

for a non-decreasing singular continuous function $c(x)$ on $J$.

Let $\mathbf{E}$ be the totality of measurable subset $E$ of $I$ satisfying that, for any $x, y \in I$, $x < y$, $|(I \setminus E) \cap (x, y)| > 0$, i.e., the complement of $E$ has a positive measure on any non-empty open subinterval. Two sets in $\mathbf{E}$ are regarded to be equivalent if they differ by a zero-measure set.

The following theorem illustrates the structure of $\mathbf{S}$ and shows that any regular recurrent Dirichlet subspace of $(H^1(I), \mathbf{D})$ may be obtained in the same way as done in the example in §1.

**Theorem 4.1.** *Let $\mathtt{s}$ be a strictly increasing function on $I$.*
(1) *$\mathtt{s} \in \mathbf{S}$ if and only if its inverse function belongs to $\mathbf{T}$.*
(2) *$\mathtt{s} \in \mathbf{S}$ if and only if there exists a set $E \in \mathbf{E}$ such that*

$$\mathtt{s}(x) = \int_\eta^x 1_{E^c}(y)\, dy, \quad x \in I, \tag{4.2}$$

*where $\eta$ denotes $a$ when $I = (a, b)$ and $0$ when $I = \mathbb{R}$. The set $E$ is uniquely determined by $\mathtt{s}$ up to the equivalence.*

Proof. (1) For $\mathtt{s} \in \mathbf{S}$, we let $\mathtt{t}(x) = \mathtt{s}^{-1}(x)$, $x \in J = \mathtt{s}(I)$. In view of the first part of the proof of Lemma 2.4 (i), we see that $\mathtt{t}'(x) = 1$ a.e. $x \in J$, and accordingly

$$\mathtt{t}(x) = c(x) + x, \quad x \in J,$$

for some nondecreasing singular continuous function $c(x)$. Hence $\mathtt{t} \in \mathbf{T}$.

Conversely if $\mathtt{t} \in \mathbf{T}$, then $\mathtt{t}(x) = c(x) + x$ is a strictly increasing continuous function with $\mathtt{t}' = 1$ a.e. on $J$. Further, for any $x, y \in J$, $x < y$, $(y - x) \le \mathtt{t}(y) - \mathtt{t}(x)$. It follows that $\mathtt{s}(x) = \mathtt{t}^{-1}(x)$, $x \in I = \mathtt{t}(J)$, is absolutely continuous. Clearly $\mathtt{s}$ is differentiable at $\mathtt{t}(x)$ if and only if $\mathtt{t}$ has a non-zero derivative at $x \in J$ and hence

$$\mathtt{s}'(\mathtt{t}(x)) = \frac{1}{\mathtt{t}'(x)} = 1, \quad \text{a.e. } x \in J,$$

which implies that $\mathtt{s}' = 1$ a.e. on $I \setminus E_{\mathtt{s}}$ in the same way as in the second part of the proof Lemma 2.4 (i).

As for (2), for any $\mathtt{s} \in \mathbf{S}$, $E_{\mathtt{s}} \in \mathbf{E}$ and conversely for $E \in \mathbf{E}$, it is easy to check that $\mathtt{s} \in \mathbf{S}$ as defined in (4.2). □

By this theorem, we can readily conceive functions in $\mathbf{S} \setminus \hat{\mathbf{S}}$ when $I = \mathbb{R}$. For example, for any non-decreasing singular continuous function $c(x)$ on $\mathbb{R}$ with $c(\pm\infty) = \pm\infty$, we put

$$\mathtt{t}(x) = c\left(\frac{x}{1 - |x|}\right) + x, \quad x \in (-1, 1) \tag{4.3}$$

and let $\mathtt{s}$ be the inverse function of $\mathtt{t}$.

Another example is provided by

$$\mathtt{s}(x) = \int_0^x 1_G(y)\,dy, \quad x \in \mathbb{R}, \quad \text{for } G = \bigcup_{r_n \in Q} \left( r_n - \frac{1}{2^{n+1}}, r_n + \frac{1}{2^{n+1}} \right), \tag{4.4}$$

where $Q = \{r_n\}$ is the set of all rational numbers.

In both cases, $\mathtt{s}(-\infty)$ and $\mathtt{s}(\infty)$ are finite and the corresponding spaces $(\mathcal{F}^{(\mathtt{s})}, \mathcal{E}^{(\mathtt{s})})$ are transient Dirichlet subspaces of $(H^1(\mathbb{R}), (1/2)\mathbf{D})$ by Theorem 2.1.

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