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On Sufficient Conditions for a Function to be Holomorphic in a Domain

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§ 1

1. The problem under what condition it is sufficient for the continuous function $f(z)=U(z)+iV(z)$ of a complex variable $z=x+iy$ defined in a domain D of the z-plane to be holomorphic, has been studied from many points of view. In particular one is from the theory of a real function or the integral, and the other is from the properties of an analytic function in the neighbourhood of the regular point, for instance, the invariance of segment's ratio, of angles, etc. The latter is the starting point of Menchoff's study continued from 1923 to 1938.

In regarding this there may be enumerable algebraic singular points (i. e. branch point) at which the local properties in the neighbourhood will be lost to some extent, his allowance that there might be enumerable points at which the properties supposed as the conditions of his theorems, were not satisfied, renders to be more interesting in the case when $f(z)$ is not univalent, because univalent and holomorphic function cannot have any branch points in its domain. The object of our study is to extend his theorems so as they may remain valid even when $f(z)$ is not necessarily univalent, to shorten his proofs and generalize in some ways.

When $\lim\frac{f(z_0+h)-f(z_0)}{h}$ exists, we call $f(z)$ is monogene at $z=z_0$. The necessary and sufficient conditions for $f(z)$ to be monogene, is that $f(z)$ is totally derivable [1] and simultaneously satisfies the Cauchy-Riemann differential equations $\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}$, $\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}$ and the necessary and sufficient conditions for $f(z)$ to be holomorphic in D is that $f(z)$ is monogene at every point in D. We see directly that the set in which $f(z)$ is not regular forms a perfect set.

2. We denote the half lines issuing from z by $\tau_i(z)$; $i=1.2.3...$,

1) Pompeiu : Sur la continuité des fonctions de varibles complexes, Ann. Fac. Soc. Université Toulouse (2), pp. 262-315 (1905).

the angle made between τ_i and τ_j by $[\tau_i(z)\wedge\tau_j(z)]$ and the amplitude of $f(\xi)-f(z)$ by amp $[f(\xi)-f(z)]$. If the upper and lower limit $\overline{\lim_{\xi\to z}}\left|\frac{f(\xi)-f(z)}{\xi-z}\right|$ $\overline{\lim_{\xi\to z}}\frac{f(\xi)-f(z)}{\xi-z}$ exist and when two extreme limits of $\lim_{\xi\to z}\left|\frac{f(\xi)-f(z)}{\xi-z}\right|$ and $\lim_{\xi\to z}\frac{f(\xi)-f(z)}{\xi-z}$ are equal, we denote them by $\tau_i\bar{A}(z)$, $\tau_i\bar{B}(z)$ and $\tau_iA(z)$, $\tau_iB(z)$ respectively.

We say that $f(z)$ satisfies the property K'', $K''*$ and $K''**$ at $z=z_0$, if the folllowing conditions are satisfied respectively.

PROPERTY K''

1° To $z=z_0$ three lines $\tau_i(z_0)$ correspond such that $[\tau_i(z)\wedge\tau_j(z)]$ $\not\equiv 0 \pmod{\pi}$

2° $\tau_iA(z)=\tau_jA(z)\quad i.j.=1.\ 2.\ 3$

PROPERTY $K''*$

1° To $z=z_0$ two lines $\tau_i(z)$ correspond such that $[\tau_i(z)\wedge\tau_i(z)]\not\equiv 0$ $\pmod{\pi}$

2° $\tau_i\bar{A}(z)<+\infty$ and moreover two sequences $\overset{1}{q_i}\cdot\overset{2}{q_i}\cdot\overset{3}{q_i}\ldots$ on $\tau_i(z)$ exist satisfying

$$\lim_{\substack{n=\infty\\ \tau_i}} B(\overset{n}{q_i})=\lim_{\substack{n=\infty\\ \tau_j}} B(\overset{n}{q_j})$$

PROPERTY $K''**$

1° To $z=z_0$ three lines $\tau_i(z)$ correspond such that $[\tau_i(z)\wedge\tau_i(z)]\not\equiv 0$ $\pmod{\pi}$

2° $\tau_i\bar{A}(z)<+\infty$ and moreover three secquence $\overset{1}{q_i}\cdot\overset{2}{q_i}\cdot\overset{3}{q_i}\ldots$ on $\tau_i(z)$ exist satisfying $\lim_{n=\infty\tau_i} A(\overset{n}{q_i})=\lim_{n=\infty\tau_j} A(\overset{n}{q_j})$ $i,j=1.\ 2.\ 3$, and amp $[f(\overset{n}{q_i})-f(z)]\times[f(\overset{n}{q_j})-f(z)]>0$

3. Condition S. For a continuous function $f(z)$ in D, let Ω be the image of D as z varies in D: $\Omega=f(D)$. At every point of Ω, let $s(w)$: $w\in\Omega$, be the number (finite or infinite) of times when w is covered by $f(z)$. Then $s(w)$ is measurable.

Proof. Let a, b and c, d be the upper and lower bounds of x, y coordinates of $D: I^{(0)}=[a.\ b]$, $I'^{(0)}=[c.\ d]$. For each positive integer n, let us put $I_1^{(n)}=[a.\ a+(b-a)/2^n]$. $I_k^{(n)}=(a+(k-1)(b-a)/2^n\cdot a+k(b-a)/2^n]\ldots\ldots$ $I'^{(n)}_{k'}=(c+(k'-1)(d-c)/2^n\cdot c+k'(b-a)/2^n]$: $k.\ k'=1, 2, 3\ldots\ldots$

These define two subdivision $\mathfrak{F}^{(n)}$ and $\mathfrak{F}'^{(n)}$ of the intervals $I'^{(0)}$ and $I^{(0)}$ into 2^n subintervals, of which the first is closed and the other are half open on the left respectively. Let us denote the rectangle by

$R^{(n)}_{k,k'}$ of which the sides are $I^{(n)}_k$, and $I'^{(n)}_{k'}$, these $R^{(n)}_{k,k'}$ make up a subdivision of $R^{(0)}$ of which $I^{(0)}$ and $I'^{(0)}$ are sides, composed of 2^{2n} parts. For $k=1.2.3...2^n$, let $s^{(n)}_{k,k'}$ denote the characteristic function of the set $f(R^{(n)}_{k,k'})$ and let

$$s^{(n)}(w)=\sum_{k,k'} s_{k,k'}(w) : k, k'=1, 2. 2^2......2^n.$$

We see at once that the functions $s^{(n)}(w)$ constitute a non decreasing secquence which converges at each point of w to $s(w)$. Hence, the functions $s^{(n)}(w)$ being measurable, so is also the function $s(w)$, and $s(w)$ shows the number of times when w is covered by $f(z)$ in D.

We call, conditions S is satisfied in D if

$$\int_\Omega s(w)dU\cdot dV<+\infty : w=U+iV$$

Menchoff proved the following theorem [2]:

4. Theorem 1. *If $w=f(z)$ is a continuous function defined in D, if $f(z)$ is a topological and direct (i, e, sense preserving) transformation of the z-plane to the w-plane, and moreover K'' is satisfied at every point in D, except at most enumerable points, then $f(z)$ is holomorphic throughout in D.*

We shall prove the next modified theorem

Theorem 1'. *For the continuous function $w=f(z)$ defined in D (not necessarily topological or univalent), if the following conditions are satisfied,*

1° *$K''*$ (or $K''**$) is satisfied at every point except at most enumerable set,*

2° *Condition S is satisfied in D,*

then $f(z)$ is holomorphic in D.

In order to prove the theorem we proceed with some lemmas.

5. Lemma 1. *If $f(z)$ is the continuous function having two lines $\tau_i(z)$ on which $\lim {}_{\tau_i}\bar{A}(z)<+\infty$ at every point z except at most enumerable points, then $f(z)$ is almost everywhere totally derivable* [3]

To prove the lemma 1, we have only to show $\overline{\lim}\left|\frac{f(z+h)-f(z)}{h}\right|<\infty$ almost everywhere in D, by Stepanoff's Theorem [4].

If Lemma 1 were false, we can find a positive measure set E, in which $\overline{\lim\limits_{h\to 0}}\left|\frac{f(z+h)-f(z)}{h}\right|=\infty$, from which follows that $f(z)$ is not regular in E.

2) Menchoff: Sur les conditions monogènes, Bull. de Math. France pp, 141-182 (1928).

3) Menchoff: Sur les differentiales totales des fonctions univalents, Math. Ann. pp. 78-85 (1931).

4) Stepanoff: Ueber total Differentierbarkeit, Math. Ann. 70, pp. 318-320 (1925).

We denote the set of points of density of E, by E_1, we observe that mes $|E-E_1|=0$, and accordingly for any positive number ε, we can find a perfect set E_2, such as $E_1 \supseteqq E_2$, mes $|E_1-E_2|<\varepsilon$. We easily see that any portion of E_2 has positive measure.

We denote the set satisfying the following conditions by $G(P.N.n_1, n_2)$: where $P.N.n_1, n_2$ are all integers

1° $\left[\Delta \wedge \tau_i(z)-\frac{n_i}{NP}\right] \leqq \frac{1}{2NP}$: Δ is the fixed direction

2° $\frac{1}{P} \leqq \left[\frac{n_1}{NP} \wedge \frac{n_2}{NP}\right] < \pi - \frac{1}{P}$: $N \geqq 2$

3° $\left|\frac{f(\zeta)-+(z)}{\zeta-z}\right| \leqq P$: $0<|\zeta-z| \leqq \frac{1}{P}$: $\zeta \in \tau_i(z)$

4° dist (z, boundary of $D) \geqq \frac{1}{P}$

then $\quad E_2 \subseteqq \sum_{P.N.n_1,n_2} G(P.N, n_1, n_2)+H$

where H is the set in which $K''*$ (or $K''**$) is not satisfied which is enumerable at most.

By Baire's theorems we conclude that there is a portion Π [5] (we assume that mes $\Pi \neq 0$ without losing generality) defined by a certain open set D', and in Π a certain $G(P_0.N_0.\overset{0}{n_i})$ is dense, which will be denoted by G_0. In the case when $\Pi \cap G_0 \ni z$, $\tau_i(z)$ are defined already, and in the case when $\Pi \cap G_0 \not\ni z \in \Pi$ we define τ_i as the limit of $\tau_i(z_n)$ $z_n = z$: $z_n \in \Pi \cap G_0$. From the continuity of $f(z)$ we easily recognize that these $\tau_i(z)$: $z \in \Pi \cap D'$ satisfies all the conditions of Lemma 1.

Proof of the lemma 1. For a positive measure set Π, we know that the set of linearly density point of [6] Π with respect to a fixed direction, has the same measure as that of Π.

Now let us denote by X and Y axes the two half lines of the angles associated with the fixed directions $\frac{n_1^{(0)}}{NP}$, and $\frac{n_2^{(0)}}{NP}$, these axes intersect perpendicularily each other. If we denote by Π^* the set of points of linearly density of Π with respect to X, and Y directions simultaneously, then

$$\text{mes } |\Pi-\Pi^*|=0$$

By Egoroff's theorem for any small number ε and η we can find a positive measure set Π^{**} of Π^* and a positive number δ such that if

5) Saks: Theory of the Integral, p. 54 (1937).

6) Saks: loc. cit. p. 54.

l is the line containing a point Π^{**} at least parallel to X or Y axis and its length is smaller than δ then

$$\frac{\text{mes } l\bigwedge\Pi^{*}}{l}>1-\frac{\varepsilon}{2}, \quad \text{mes } |\Pi^{*}-\Pi^{**}|<\eta \ : \ \delta\leqq\frac{1}{P}$$

If z_0 is a point of Π^{**}, let us trace X, Y axes going through z_0 and denote by $V_s(z_0)$ the circular neighbourhood of z_0 with centre z_0 and diameter s, where

$$s\leqq\frac{\delta}{2(1-\varepsilon)^3 \ \text{cosec}\left(\frac{1}{2P}-\frac{1}{2NP}\right)\left\{1+\frac{1}{1-\varepsilon}\tan\left(\frac{1}{2P}+\frac{1}{2NP}\right)\cot\left(\frac{1}{2P}-\frac{1}{2NP}\right)\right\}} \quad (1)$$

Then $\left|\frac{f(z)-f(z_0)}{z-z_0}\right|\leqq MP : z\in\Pi\bigwedge V_s(z_0) : M$ depends only on P and N. Take p_1 so that $|z_0p_1|=s \text{ cosec}\left(\frac{1}{2P}-\frac{1}{2NP}\right)<\delta$ then there exists a point $p_1' : p_1'\in\Pi^{*}$, $|p_1'z_0|<\frac{1}{1-\varepsilon}|p_1z_0|$ on the left hand of p_1, and take $\tau_1(p_1')$ which intersects with Y axis at p_2 and denote by θ_1, the angle made X axis and $\tau_1(p_1')$. As $|z_2p|<\delta$, there exists a point $p_2'\in\Pi$ such as $|p_2'z_0|<\frac{|p_2z^0|}{1-\varepsilon}$ and trace $\tau_2(p_2')$ intersecting with X axists at p_3, we shall name as follow

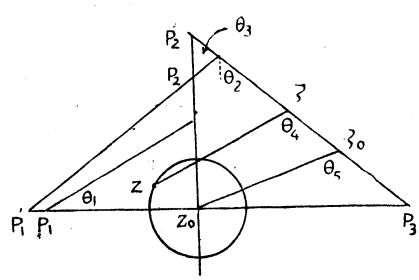

Fig. 1

$p_5=$ the intersecting point of $\tau_1(p_1')$ and $\tau_2(p_2')$

$\theta_2=$ angle $p_1'p_5p_3$

$\theta_3=$ angle $p_2'p_5p_2$

For $N\geqq2$ we have

$$0<\frac{1}{4P}<\frac{1}{2P}-\frac{1}{2NP}\leqq\theta_1\leqq\frac{1}{2P}+\frac{1}{2NP}<\frac{3}{4}<\frac{\pi}{2}$$

$$0<\frac{\pi}{2}-\frac{1}{2P}-\frac{1}{2NP}\leqq\theta_2\leqq\frac{\pi}{2}-\frac{1}{2P}+\frac{1}{2NP}<\frac{\pi}{2}$$

$$\frac{1}{P}-\frac{1}{NP}\leqq\theta_3\leqq\frac{1}{P}+\frac{1}{NP}$$

Let z be a point of Π lying on the periphery of $V_s(z_0)$, then $\tau_1(z)$ exists which has a point ζ with $\tau_2(p_2')$ in common and as $z_0\in\Pi^{**}\subseteqq\Pi$ there exists $\tau_1(z_0)$ which has a common point ζ_0 with $\tau_2(p_2')$.

Let $\theta_4=\text{angle } z\zeta p_3, \quad \theta_5=z_0\zeta_0p_3,$

then $\frac{\pi}{2}-\frac{1}{2P}-\frac{1}{NP}\leqq\theta_4\leqq\frac{\pi}{2}-\frac{1}{2P}+\frac{1}{NP}, \ \frac{\pi}{2}-\frac{1}{2P}-\frac{1}{NP}\leqq\theta_5\leqq\frac{\pi}{2}-\frac{1}{2P}+\frac{1}{NP}$

$$|z_0p_1|=s\operatorname{cosec}\left(\frac{1}{2P}-\frac{1}{2NP}\right)<\delta; \quad |p_1'z_0|\leqq|z_0p_1|\frac{1}{1-\varepsilon}; \quad p_1'\in\Pi^*$$

$$|z_2p_2'|\leqq|z_0p_2|(\tan\theta_1)\frac{1}{1-\varepsilon}; \quad |z_0p_3|=|z_0p_2'|\tan\theta_1; \quad |p_2p_2'|$$

$$=\frac{\varepsilon}{1+\varepsilon}|z_0p_2|=\frac{\varepsilon}{1+\varepsilon}|z_0p_1|\tan\theta_1\leqq\frac{\varepsilon}{1+\varepsilon}s\operatorname{cosec}\left(\frac{1}{2P}-\frac{1}{2NP}\right)\tan\left(\frac{1}{2P}+\frac{1}{2NP}\right)$$

$$|p_2'p_2|=\left|\frac{p_2p_2'\sin\theta_3}{\cos\theta_1}\right|<s: |z\zeta|=\frac{(s+z_0p_3)\sin\theta_4}{\sin\left(\frac{\pi}{2}-\theta_3\right)}<\frac{(s+z_0p_3)\sin\theta_4}{\cos\left(\frac{1}{P}+\frac{1}{NP}\right)}$$

$$|z\zeta_0|<\frac{z_0p_3\sin\theta_5}{\cos\theta_3} \quad \text{from (1).}$$

Thus

$$|z_0\zeta_0|, \ |\zeta_0p_2'|, \ |p_2'p_5|, \ |p_5p_1'|, \ |p_1'p_5|, \ |p_5p_2'|, \ |p_2'\zeta|, \ |\zeta, z|\leqq\delta$$

and all $<K_i|z-z_0|$ $i=1. 2...3$ and all $K_i<+\infty$ depend only on P and N.

In the same manner we proceed with $\tau_2(\bar{p}_1')$, etc, in the half plane under the X axis, and $\bar{p}_2'$, $\bar{p}_5$... etc. are denoted as in the former and p_2p_3 and $\overline{p_2p_3}$ intersect at p_4^*, then p_1, p_5, p_4^*, $\bar{p}_5$ and p_1' forms a quasi parallelogram $\square_s$.

Finally

$$|f(z)-f(z_0)|\leqq|f(z_0)-f(\zeta_0)|+|f(\zeta_0)-f(p_2')|+|f(p_2')-f(p_5)|$$
$$+|f(p_5)-f(p_1')|+|f(p_1')-f(p_5)|+|f(p_5)-f(p_2')|+|f(p_2')-f(\zeta)|$$
$$+|f(\zeta)-f(z)|\leqq M.P|z-z_0|,$$

where M depends only on P and N whenever $z\in V_s(z_0)\frown\Pi$.

In the case when $z\in\Pi$, we make s' so small that quasi parallelogram $\square_{s'}$ associated with s' and z_0 may be contained in $V_s(z_0)$ completely, then we have the same conclusion for any point of z' lying on the cercumference of $\square_{s'}$, that is

$$\left|\frac{f(z')-f(z)}{z'-z_0}\right|\leqq M'.P: \ z'\in\square_{s'}\text{'s periphery } \frown V_s\frown D': \ M'=M'(M.P)$$

If $z'\in\square_s\frown(V_s-\Pi)\frown D'$ $\frac{f(z')-f(z_0)}{z'-z_0}$ is regular

By the maximum principle of analytic functions

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \leqq MP : M'' = \max(M, M') : \text{ if } z \in V_s(z_0) \wedge \square_{s'}$$

Since ε and η any positive numbers, by Stepanoff's theorem $f(z)$ is totally derivable almost everywhere.

Remark. When $N \gtreqqless 1$ the proof is valid too with no essential alteration.

6. Lemma 2. *When at $z=z_0$, $f(z)$ is totally derivable and satisfies $K''**$, then $f(z)$ is monogene at $z=z_0$.*

$$f(z_1)-f(z_0)=(A_1+iA_2)(x_1-x_0)+(B_1+iB_2)(y_1-y_0)+\varepsilon(z_1)|z_1-z_0|=s(z_1)$$
$$+\varepsilon(z)|z_1-z_0| : \lim_{z_1 \to z_0} \varepsilon(z_1)=0 : z_i=x_i+y_i : i=1.0$$

$$\lim_{z_1 \to z_0} \left|\frac{f(z_1)-f(z_0)}{z_1-z_0}\right|$$
$$=\sqrt{(A_1+iA_2)\cos\theta_i+(B_1+iB_2)\sin\theta_i+2\sin\theta_i\cos\theta_i(A_1B_1+A_2B_2)}$$

A_1, A_2, B_1 and B_2 constants for θ_i; $i=1.2.3 \pmod{\pi}$

We easily have the relation $A_1=\pm B_2$, $A_2=\mp B_1$, but from the latter condition of $K''**$ we have $A_1=B_2$, $A_2=-B_1$, Therefore $f(z)$ is monogene, in the case of $K''*$ will be proved in the same manner.

7. Lemma 3. *A continuous function $f(x)$ is defined in the closed interval $[a \cdot b]$ and there is a closed set F. $[a \cdot b]-F=\sum I_i : I_i=(a_i \cdot b_i)$ are intervals contigus to F, with satisfiying the following couditions*

1° $$\left|\frac{f(z_i)-f(z_j)}{z_i-z_j}\right| \leqq M : \text{ if } z_i, z_j \in F$$

2° *$f'(x)$ exists almost everywhere and* $\sum \int_{I_n} |f'(x)| dx < +\infty$

3° *For each interval: $I_i=(a_i \cdot b_i)$, $f(x)$ is absolutely continuous*

then $\int_a^b f(x)dx = f(b)-f(a)$.

Let us denote the uppper and lower bound of F by a' and b' and

$$\bar{f}(x)=f(x)=f(x) \qquad \text{if } x \in F \text{ or } x<a' \text{ or } x>b'$$

$$\bar{f}(x)=\frac{\mu f(a_i)+\lambda f(c_i)}{\mu+\lambda} : x=\frac{\mu a_i+\lambda b_i}{\mu+\lambda}, \text{ if } x \bar{\in} F \text{ and } a'<x<b'$$

and $x \in I_i$

where λ, $\mu>0$

After elementary calculation we have

$$\left|\frac{\bar{f}(x_i)+\bar{f}(x_j)}{x_i-x_j}\right|\leq M \quad \text{if} \quad x_i,\ x_j \bar{\in} F,\ a'>x_i,\ x_j<b\ x_i\in,\ x_j\bar{\in} F$$

Consequently $\bar{f}(x)$ has the property N of Lusin, and from 2° $\bar{f}(x)$ is integrable. We denote the upper lower relative to F derivatives by $f'_F(x)$ or $\underline{\bar{f}}'_F(x)$ and when two are equal, by $\bar{f}'_F(x)$.

Then $\bar{f}'_F(x)=f'_F(x)=f'(x)$ almost everywhere in F, where

$$\underline{\lim}\, f'(x)\leq \underline{\lim}\, \underline{f}'_F(x)\leq \overline{\lim}\, \bar{\bar{f}}'_F(x)\leq \overline{\lim}\, \bar{f}'(x).$$

From 2° $\int_F |f'(x)-\bar{f}'(x)|dx+\sum\int_{I_i} |f'(x)-\bar{f}'(x)|dx=0,$ it follows

$$\bar{f}(b)-\bar{f}(a)=f(b)-f(a)=\int_a^b f'(x)dx$$

8. Proof of the theorem 1′.

We have only to show that $f(z)$ is holomorphic in D', for it follows that Π is empty set.

Let us take ξ, and η axies which are perpendecular to $\frac{n_1}{NP}$ and $\frac{n_2}{NP}$ directions respectively and denote by α and β the angles made between ξ and η and X axis, then we have

$$x-x_0=\xi\cos\alpha+\eta\cos\beta,\quad y-y_0=\xi\sin\alpha+\eta\sin\beta$$

$$\pi>\pi-\frac{1}{P}-\frac{1}{2NP}>[\tau_i(z_1)\wedge\tau_j(z_2)]>\frac{1}{P}-\frac{1}{NP}>0 \tag{2}$$

Take a so small parallelogram $\square_s$ in D' whose four sides are parallel ξ or η axis, of which the diameter is smaller than

$$\frac{1}{P}\sin\left(\frac{1}{P}-\frac{1}{NP}\right) \tag{3}$$

We shall prove that $f(z)$ is holomorphic in this prallelogram. If z_1, z_2 have the same ξ coordinates and both in $\Pi\cap D'$ then $\tau_2(z_1)$ and $\tau_1(z_2)$ exist which have a point z_3 in common. From (2) and (3)

$$|z_1-z_3|<\frac{1}{P},\quad |z_2-z_3|<\frac{1}{P},\quad |z_1-z_3|+|z_1-z_3|<M|z_1-z_2|:$$

$$M=M(P.N)$$

We see directly that $\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|\leq M.P$ in the same manner of Lemma 2, if $z\in\square\cap D'-\Pi$ then $f(z)$ is regular, therefore U, and V absolutely continious with respect to ξ. From condition S and change of variables,

$$\iint_{D'-II} |f'(z)|^2 d\xi d\eta = \iint_{D'-II} |f'(z)|^2 dxdy \leq \iint_{\Omega} s(w) dU \cdot dV < +\infty$$

By the theorem of Fubini

$$\int_{D'} |f'(z)| d\xi < +\infty \quad \text{for almost } \eta \text{ and } |f'(z)| \geq \left|\frac{\partial u}{\partial \xi}\right|, \ \geq \left|\frac{\partial v}{\partial \xi}\right|$$

by Lemma 3

$$\int_{\xi_1}^{\xi_2} \frac{\partial U}{\partial \xi} d\xi = U(\xi_2) - U(\xi_1), \quad \int_{\xi_1}^{\xi_2} \frac{\partial V}{\partial \xi} d\xi = V(\xi_2) - V(\xi_1) : \text{for almost } \eta$$

Similarly we have for η axis.

$$\int_{\eta_1}^{\eta_2} \frac{\partial U}{\partial \eta} d\eta = U(\eta_2) - U(\eta_1) \ : \ \int_{\eta_1}^{\eta_2} \frac{\partial V}{\partial \eta} d = V(\eta_2) - V(\eta_1) : \text{for almost } \xi$$

Denoting by C the circumference of $\square$

$$\int_D f(z)dz = \iint_{\square} (-U_\eta \cos\alpha + V_\eta \sin\alpha + U_\xi \cos\alpha - V_\xi \sin\beta) d\xi d\eta$$

$$+ i \iint_{\square} (-V_y \cos\alpha - U_\eta \sin\alpha + V_\xi \cos\beta + U_\xi \beta) d\xi d\eta$$

$$= \iint_{\square} (U_x - V_y) dxdy + i \iint_{\square} (U_y + V_x) dxdy = 0,$$

because $f(z)$ is monogene almost everywhere in D.

Finally we conclude that $f(z)$ is holomorphic in D', from which follows that $f(z)$ is holomorphic in D.

§ 2

9. In this paragraph we intend to enlarge the results in the preceedings, in the wide sense.

We denote by $f(z)=w$, a continuous function defined in a domain of the z-plane.

Proposition 1. *If $f(z)$ satisfies the following conditions.*

1° *$f(z)$ is contiuous and for almost y, $app_x\, U$* [7]*, app V_x and for almost x, app U_x, app V_y exist except at most enumerable set, relative x, and y axis respcetively.*

2°

$$\iint_D |app\, U_x| dxdy, \iint_D |app\, U_y| dxdy, \iint_D |app\, V_x| dxdy, \iint_D |app\, V_y| dxdy \quad < \infty$$

7) app means approximate derivate. Saks, p. 215. 300. 225.

3° *app* U_x=*app* V_y, *app* U_y=−*app* V_x *almost everywnere in* D, *then* $f(z)$ *is holomorphic in* D.

From Fubini's theorem for almost y $\int_{y=y} |\text{app}\, U_x| dx$ and 1°) follows

that $[U(x, y)]_{y=y_0}$ is function A. C. G.[8] We define $\bar{U}(x,y)=\int_{a_0}^{x}(\text{app}\, U_x(x,y))dx$,

then $U-\bar{U}$ is a function A. C. G, therefore $\bar{U}_x=\text{app}\,\bar{U}_x=\text{app}\,U_x$ almost everywhere with respect to x, so we have $U-\bar{U}=\text{const.}$, it follows that

$U(b)-U(a)=\bar{U}(b)-\bar{U}(a)$; $a>a_0$, after all we have $U(b)-U(a)=\int_a^b \text{app}\, U_x(x,y)dx$.

In the same way as in the proof of the theorem 1, for any square in D. $\int_C f(z)dz=\iint_{\square}(\text{app}\, U_x-\text{app}\, V_y)\, dxdy+i\iint_{\square}(\text{app}\, V_x+\text{app}\, U_y)\, dxdy=0$.

Proposition 2. *If* $f(z)$ *satisfies the following conditions*

1° *app* U_x, *app* U_y. *app* V_x *and app* V_y *exist except at most at enumerable point in* D, *and further* 2° *conditions* S *is satisfied, then* $f(z)$ *is holomorphic.*

Denote by $E(n_1, n_2)$ for *any given* ε_0 the set : n_i are integers.

$$\underset{z}{E}\left[\text{mes}\cdot\text{line}\ \underset{h}{E}\left[\left|\frac{f(z+h)-f(z)}{h}\right|\leqq n_1;\ 0<h<\frac{1}{n_1}\right]\geqq(1-\varepsilon_0)\frac{1}{n_1}:\quad h=\text{real}\right.$$

$$\left.\underset{z}{E}\left[\text{mes line}\ \underset{h}{E}\left[\left|\frac{f(z+ih)-f(z)}{h}\right|\leqq n_2;\ 0<h<\frac{1}{n_2}\right]\geqq(1-\varepsilon_0)\frac{1}{n_2}\right]\right].$$

If $f(z)$ is not holomorphic in D, we can find a portion Π defined by D' in which $E(n_1^0, n_2^0)$ is dense, and by taking limit, Π is contained in the closure of a certain $E(n_1^0, n_2^0)$ completely. We term this operation B.

If Π is defined by D' from condition 1°) app U_x, app U_y, app V_x, and app V_y exist, therefore, they are $\leqq\text{Max}(n_1^0, n_2^0)$ in absolute value. $f(z)$ is regular, if $z\in D'-\Pi$.

From proposition 1 we conclude that $f(x)$ is holomorphic in D.

10. Proposition 3. *If* $f(x)$ *is a continuous function defined in a closed interval* $[a, b]$, *and if there is a closed set* $F\subseteqq[a, b]$, $I_i=(a_i, b_i)$ *denoting the intervals contigus satisfying the following conditions.*

1°) $\int_{I_i} f'(z)\, dx=f(b_i)-f(a_i)$ *for each interval and* $\sum_i \int_{I_i}|f'(x)|dx<\infty$

8) see 7).

2°) *$f'_F(x)$ exists except at most at mumerable set and* $\int_F |f'_F(x)|dx < \infty$,

then $f(b)-f(a)=\sum_i \int_{I_i} f'(x)dx + \int_F f'_F(x)\,dx$.

Proof. If $x \in F$ and x is isolated from F, $f_F(x)$ loses its meaning, but the set where x is isolated, is at most enumerable, therefore $f'_F(x)$ has finite value everywhere in F except at most enumerable set in F, we define a function such as

$$\bar{f}(x)=f(x)\ ; \quad \text{if } x \in F$$

$$\bar{f}(x)=\frac{\lambda f(a_i)+\mu f(b_i)}{\lambda+\mu} \ : \text{if } x \in I_i=(a_i,\ b_i) \quad x=\frac{\lambda a_i+\mu b_i}{\lambda+\mu} \quad \lambda.\ \mu>0.$$

When $|f'_F(x)|<K$; $|K|<\infty$, there exists a secquence x_i converging to x, $x_i \in F$ and there is number δ exists so that

$$\text{if } x_i \in (x \pm \delta) \frown F$$

a) In the case when x_i, $x \in F$ $|x-x_i|<\delta$ follows $K-\varepsilon<\frac{f(x_i)-f(x)}{x_i-x}$ $<K+\varepsilon$

b) In the case when $x \in F$, and $x_i \bar{\in} F$

b,1) $F \ni x_i > x_l =$ lower bound of $(x-\delta) \frown F$

b,2) $F \bar{\in} x_i < x_u =$ upper bound of $(x+\delta) \frown F$, there exists a $I'_i = (a'_i,\ b_i) \in x_i$

from this it is clear $\left|\frac{\bar{f}(x_i)-\bar{f}(x)}{x_i-x}\right|<K+\varepsilon$.

2) If $x \bar{\in} F$ $|f(x)| \leq M$ (because $f(x)$ is continuous in closed interval, there exists an interval $I_i=(a_i,\ b_i) \ni x_i$, x_j therefore for $x_i.\ x_j$

$$\left|\frac{f(x_i)-f(x_j)}{x_i-x_i}\right|=\left|\frac{f(b_i)-f(a_i)}{b_i-a_i}\right| \leq \frac{2M}{b_i-a_i}<\infty, \quad M=\max\ |f(x)|;\ x \in [a,\ b].$$

Finally all $\bar{f}(x)$ has finite Dini's derivatives everywhere except at most enumerable set, from 2°) $\bar{f}(x)$ is an absolutely continuous function, on the other hand $\bar{f}'_F(x)=\bar{f}'(x)=f'_F(x)$ almost everwhere in F, then

$$f(b)-f(a)=\sum_i \int_{I_i} \bar{f}'(x)\ dx+\int_F \bar{f}'(x)\ dx=\sum \int_{I_i} f'(x)\ dx+\int_F f'_F(x)\ dx\ .$$

11. Theoreme 2. *$f(z)$ is a continuous*[9] *function in D, and D is*

9) Kametani: On conditions for a function to be regular, Jap. Journ. of Math. 17, pp. 337-345 (1941).

expressed in the form $D=\sum_i E_i+H$, *where* H *is an enumerable set, and satisfies the following conditions.*

1°) *For each* $E_j \ni z$ *two lines (fixed direction) denoted by* τ_i *issuing from* z, *correspond, and for* $z' \in E_j \cap \tau_i$ *and* $|z'-z|<\delta(z)$

$$\tau_1 B_{E_j}=\lim_{\substack{\zeta\to z\\ \zeta\in E_j\cap\tau_1}} \frac{f(\zeta)-f(z)}{\zeta-z},\quad \tau_2 B_{E_j}=\lim_{\substack{\zeta\to z\\ \zeta\in E_j\cap\tau_2}} \frac{f(\zeta)-f(z)}{\zeta-z}$$

exist except at most enumerable set in E_j, *and when two* $\tau_i B$ *exist,* $\tau_1 B_{E_j}=\tau_2 B_{E_j}$ *almost everywhere in* E_j, *and* S *is satisfied, then* $f(z)$ *is holomorpic in* D. (Of course on $\tau_i(z)\cap E_j$, when z is isolated from $\tau_i(x)\cap E_j$, relative derivative loses its meaning)

Generality will not be lost by assuming that the two fixed directions are that of x and y axis. H_j denotes the set of E_j where (1°) is not satisfied.

Then $$D=\sum_j E_j+H_j+H\ .$$

Denote by E_{sp} the set E_z satisfying the following conditions

1°) $E_z\left[\left|\frac{f(z+h)-f(z)}{h}\right|<P\right]$ if $z,\ z+h\in E_s$: $0<h<\frac{1}{P}$:

$h=$ real or imaginary

2°) dist (z, boundary of D) $\geqq\frac{1}{P}$

$$E_s=\sum_p E_{sp},\ D=\sum_{sp} E_{sp}+H_s+H\ .$$

If $f(z)$ is not holomorphic in D, by operation B we can find a portion Π defined by D' in which a certain E_{sp} is dense, we conclude by taking limit of $\tau_i(z_n)$: $z_n\in E_{sp}$, $\lim\limits_n z_n=z$. For any $z\in D'\cap\Pi$ 1°) and 2°) is satisfied,

$$f(z) \text{ is regular : if } z\in D'-\Pi\ ,$$

$$\left|\frac{\partial f}{\partial x}\right|_\Pi,\ \left|\frac{\partial f}{\partial y}\right|_\Pi\leqq P\ :\ \text{if } z\in\Pi\ .$$

By using Fubini's theorem about S condition $\iint_{D'-\Pi}|f'(z)|dxdy<\infty$ and proposition 3, we conclude that for almost all y

$$U(x_2,\ y)-U(x_1,\ y)=\left[\int_{x_1}^{x_2} U'_{x_{D'-\Pi}}(x,\ y)\,dx\right]_{y=y}+\int_\Pi U'_x(x,\ y)_\Pi dx,\ \text{etc.}$$

and further ${}_{\Pi}\dfrac{\partial f}{\partial x}={}_{\Pi}\dfrac{\partial f}{\partial y}$, etc. almost everywhere in Π. Finally we have

$$\int_C f(z)\,dz = 0$$

12. Proposition 4. *$w=f(z)$ is approximately monogene except at most enumerable set and condition S is satisfied in D, then $f(z)$ is holomorphic in D.*

If $f(z)$ is not holomophic in D, we can find by B operation a portion Π defined by D', there exist a certain ε_0 and r_0 and M_0 not depending on $z \in \Pi$.

$$f(z) \text{ is regular, if } z \in D'-\Pi$$

$$1^\circ)\quad \text{mes}\left|E\left[\left|\frac{f(z+he^{i\theta})-f(z)}{h}-A\right|<\varepsilon_0\right]\right|>(1-\varepsilon_0)h_2^0\pi : 0\leq\theta<2\pi :$$

$$\text{where } |A|=M_0,\quad h<h_0<r_0 : \text{ if } z\in\Pi .$$

We have only to show that $f(z)$ is holomorphic for any small square in D' for this purpose, we take a square with its diametre smaller than $<\frac{r_0}{2}$, then for z_1, $z_2 \in \Pi$ we find a cercle $C(z_1)$ and, $C(z_2)$ their diametre $|z_1-z_2|$, in which

(1°) is satisfied and mes $|C(z_1)\frown C(z_2)|>\frac{\pi}{3}\ |z_1-z_2|^2 \leq (1-\varepsilon_0)|z_1-z_2|^2\pi$ therefore there exists at least a point $z_3 \in C(z_1)\frown C(z_2)$

$$\left|\frac{f(z_1)-f(z_3)}{z_1-z_3}\right|\leq P,\quad \left|\frac{f(z_2)-f(z_3)}{z_2-z_3}\right|\leq P$$

and so $$\left|\frac{f(z_1)-f(z_2)}{z_1-z_2}\right|\leq 2MP : \text{ if } z_1,\ z_2\in\Pi\ ;\ M=M(A,\ \varepsilon_0)$$

On the other hand $f(z)$ is approximately monogene

$$f(z_2)-f(z_1)=(A_1+iA_2)(x_2-x_1)+(B_1+iB_2)(y_2-y_1)+\varepsilon(z_2)|z_2-z_1| :$$

$$\lim_{z_2\to z_1}\varepsilon(z_2)=0$$

(approximately totally derivable)

but directions are fixed

$$f(z_2)-f(z_1)=(A_1+iA_2)(x_2-x_1)+(B_1+iB_2)(y_2-y_1)+\varepsilon(z_2)|z_2-z_1|$$

then we have $(A_1+iA_2)=\text{app}\, f_x$, $(B_1+iB_2)=\text{app}\, f_y$ almost everywhere in Π. Finally from the theorem 2, $f(z)$ is holomorphic in D.

§ 3

We give the simplest proof under a little change of the conditions of the theorem 1.

We denote by $l(z)$ straight line passing through z and denote

$$\lim_{\substack{\zeta\to z\\ \zeta\in l_i(z)}} \frac{f(\zeta)-f(z)}{\zeta-z}, \quad \overline{\lim_{\substack{\zeta\to z\\ \zeta\in l_i(z)}}} \left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|$$

by $_{l_i}B(z)$ or $\overline{\lim}\, _{l_i}\bar{A}(z)$

13. Theoreme 3 (Menchoff)[10]. *If $f(z)$ is a continuous function with the following conditions.*

1°) *To every point except at most at enumerable points, correspond two lines passing through z, $[l_1 \wedge l_2] \not\equiv 0 \pmod{\pi}$.*

2°) $$B_{l_1} = B_{l_2} .$$

Then $f(z)$ is holomorphic in D.

Or more generally $\overline{\lim}\, _{l_1}A$, $\overline{\lim}\, _{l_2}A < \infty$ and two sequences on them

$$\lim_n B(q_1^n) = \lim_n B(q_2^n) .$$

We prove this theorem as an application of following Pompeiu's theorem. A complex function $f(z)$, continuous in an open set D, is regular in D, if it is monogene at almost all the point D and if further $\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}\right| < \infty$ at each point except at most enumerable set.

Proof. It is not regular in D we can find as in the case of theorem 1, the portion Π defined by D' and followingly conditioned.

1°) $f(z)$ is regular, if $z \in D'-\Pi$

2°) l_1^0, l_2^0 are fixed direction $[l_i^0 \wedge l_i] \leqq \frac{1}{2NP}$; $N \geqq 2$

3°) $\frac{1}{P}-\frac{1}{NP} < [l_1^0 \wedge l_2^0] < \pi - \frac{1}{P}+\frac{1}{NP}$

dist (z, boundary of D) $\geqq \frac{1}{P}$

4°) $\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right| \leqq P$ if $\zeta \in l_i(z)$, $0 \lneqq |\zeta - z| \leqq \frac{1}{P}$: $i=1.2$

If we associate a sector $S(z)$ (fixed direction and fixed opening angle) to each point z of the plane set Π, of which z is the vertex of the sector $S(z)$. It is clear that the set of z which is isolated from $S(z) \cap \Pi$ is at most enumerable.

Let R be a subset of Π, which is isolated from Π in any one of four sectors, then R is at most numerable.

14 Lemma 1. *Let us denote by $V_s(z)$ the circular neighbourhood of z with the centre at z and the radius s.*

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10) Menchoff: Sur les conditions de Cauchy-Riemann, Fund. Math. (1935). pp. 59-97

$$s<\frac{1}{2P}\times\sin\left(\frac{1}{P}-\frac{1}{NP}\right),$$

then

$$\overline{\lim}\left|\frac{f(z'')-f(z)}{z''-z}\right| \textit{ is bounded if } z''\in\Pi\wedge V_s(z)\ :\ z\in\Pi-R\ .$$

Proof. We take a point $z''\in\Pi\wedge V_s(z)$, then exist two $l_1(z'')$ and $l_2(z'')$, which intersect with $l_2(z)$ and $l_1(z)$ at points p_1 and p_2, and denote the angle $\theta_3=$ angle $z'p_2z$, $\theta_2=$ angle p_2zz'' then

$$0\leq\theta_2\leq\frac{1}{P}+\frac{1}{PN}:$$

$$0<\frac{1}{P}-\frac{1}{NP}<\theta_3<\pi-\frac{1}{P}+\frac{1}{NP}$$

$$N\geqq 1$$

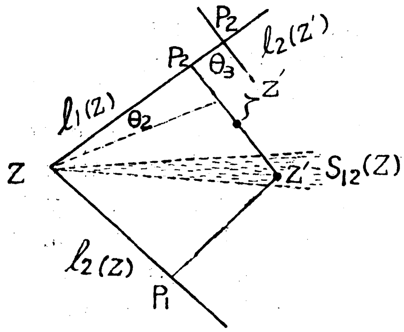

Fig. 2

accordingly

$$|z''-p_2|+|p_2-z|<\frac{|z'-z|(\sin(\theta_2+\theta_3)+\sin\theta_3)}{\sin\theta_3}<K_i|z''-z|<\frac{1}{P}$$

$$K_i=K_i(P.N)$$

We directly see that $\left|\frac{f(z'')-f(z)}{z''-z}\right|\leq P.M$ if $z''\in V_s(z)\wedge\Pi$ in the same way as in Theorem 1, where M depends only on P and N.

From that z is not contained in R, there exists z' such as

$$|z'-z|<s,\ z'\in S_{ij}(z)\wedge\Pi$$

and two lines $l_i(z')$ exist which intersect $l_j(z)$ at p_1, and p_2 where $S_{ij}(z)$ is a sector of which vertex is z and its half line is the half line $l_i^0(z)$ and $l_j^0(z)$ and its opening angle sufficiently small given number ε_0.

Then, diametre of $(zp_2z'p_1)<\frac{1}{P}$, therefore $\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|\leq PM$: if ζ lies on the circumference of $(zp_2z'p_1)$, which can be proved as usual.

Finally $\left|\frac{f(z')-f(z)}{z'-z}\right|\leq PM$: if $z'\in V_s(z)\wedge(zp_2z'p_1)$.

In the long run we conclude that

$$\overline{\lim}\left|\frac{f(z')-f(z)}{z'-z}\right|\leq PM: \text{ if } z\in\Pi\wedge D'.$$

When z is contained in $D'-\Pi$, $f(z)$ is regular, so $\overline{\lim\limits_{h\to 0}}\left|\frac{f(z+h)-f(z)}{h}\right|<\infty$

at every point except at most enumerable set, from condition 2°, $f(z)$ must be monogene almost everywhere in D'. By the theorem of Pompeiu $f(z)$ is holomorphic ni D',

Remark. It is clear that this method is applicable when K'' is under the condition that three lines issuing from z never lie on the same side of any line passing through z.

When four lines issuing from z, we can this apply without any satisfied further condition. It is important not that $\lim B(z)$ exist, but that $\lim A<\infty$.

15. We know what effect the number of $\tau_i(z)$ of which $\lim \tau_i A <\infty$ has on the condition of regularity.

1) two lines, condition S. $B_1=B_2$

2) three lines condition S. $A_1=A_2=A_3$

3) two lines passing through or four lines issing from z. $B_1=B_2$

4) two lines (fixed direction) relative or approximate derivateve conditions S.

§ 4

16. Invariance of angles. The properties studied in the preceeding paragraphes are quantative relations between the behaviours of z and w in the sense of segment's ratio or its extended meaning. Nevertheless on the contrary this property is not direct relation between them but it only tells us the indirectly, in the other word, it means the connection of quantatives (angles) defined by pairs $(z.y)$ and $(U.V)$.

Property K'

With $z=z_0$ three half lines $\tau_i(z): i=1.2.3$ issuing from z_0 are associated and any Jordan curve J terminating in z_0 with one of $\tau_i(z_0)$ as its tangent, has its image $f(J)$ with a half line $T_i(w_0): w_0=f(z_0)$ issuing from w_0 as its tangent in the w-plane.

$$[\tau_i(z)^\wedge\tau_j(z)]=[T_i(w)^\wedge T_j(w)]\neq 0 \pmod{\pi} \quad i.j.=1.2.3$$

Menchoff proved the following theorem [11].

Theorem 4. *If $w=f(z)$ is univalent and continuous function defined in a domain of the z-plane and if it has K' at every point except at most enumerable point, then $f(z)$ is holomorphic in D.*

For the purpose to make this theorem remain valid, in the case when $f(z)$ is not univalent, we take a little changed property K'^* as it follows.

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11) Menchoff: Sur les représentations qui conservent les angles, Math. Ann. 109, p. 101-159 (1934).

Property K'^*

With $z=z_0$ three lines $l_i(z): i=1.2.3$ passing through z are associated having its image $f(l_i(z))$ in the w-plane which has a tangent T_i in the neighbourhood of w and at $w=f(z)$

$$[l_i \wedge l_j] = [T_i \wedge T_j] \neq 0 \pmod{\pi} \quad i,\ j=1.2.3$$

17. Theorem 4'. *If $w=f(z)$ is a continuous function which has K'^* at every points except at most enumerable points, and further if condition S is satisfied in D, then $f(z)$ is holomorphic in D.*

Let us denote by $T_i(w)$ the tangent of $f(l_i(z))$ at $w: i=1.2.3$ and by $G(P.N.\ n_1,\ n_2,\ n_3)$ the set conditioned followingly.

$$1^\circ \quad -\frac{1}{2NP}<\left[l_i(z)\wedge\frac{n_i}{2NP}\right]<\frac{1}{2NP} : \ -\frac{1}{2NP}<\left[T_i\wedge\frac{n_i}{2NP}\right]<\frac{1}{2NP}$$

$$2^\circ \quad \frac{1}{P}<\left[l_i(z)\wedge l_j(z)\right]<\pi-\frac{1}{P} : \ \frac{1}{P}<\left[T_i\wedge T_j\right]<\pi-\frac{1}{P}$$

$$3^\circ \quad [l_i\wedge l_j] = [T_i\wedge T_j]$$

$$4^\circ \quad -\frac{1}{2NP}<\left[T_i(w)\wedge T_i(w_0)\right]<\frac{1}{2NP} : \text{ if } |z-z_0|<\frac{1}{P}$$

$$5^\circ \quad \text{dist } (z.\ \text{boundary of } D)\geqq\frac{1}{P} \quad N\geqq 4\ .$$

Then $\qquad D=\sum G(P.N.\ n_1,\ n_2,\ n_3)+H$

where $P.N\ n_1,\ n_2,\ n_3$ are all integers, and H is enumerable set.

If $f(z)$ where not holomorphic in D, we can find the portion Π defined by a certain open set D', and in Π a certain $G(P^0.\ N^0\ \overset{0}{n_1},\ \overset{0}{n_2},\ \overset{0}{n_3})$ is dense. In the case when $z\in G_0\frown\Pi$, $l_i(z)$ are defined already, in the case when $z\overline{\in} G_0\frown\Pi$, we can define $l_i(z)$ by the limit of $l_i(z_n)$: $\lim z_n=z: z_n\in G_0\frown\Pi$, then by the continuity conditions 1°......5° are satisfied

$$1^\circ \quad -\frac{1}{2NP}<\left[\Delta_i\wedge l_i(z)\right]<\frac{1}{2NP}; \ -\frac{1}{2NP}\left[\overline{\Delta}_i\wedge T_i(w)\right]<\frac{1}{2NP}$$

$$2^\circ \quad \frac{1}{P}-\frac{1}{NP}\leqq\left[\Delta_i\wedge\Delta_j\right]\leqq\pi-\left(\frac{1}{P}-\frac{1}{NP}\right); \ \frac{1}{P}-\frac{1}{NP}\leqq\left[\overline{\Delta}_i\wedge\overline{\Delta}_j\right]\leqq\pi -\left(\frac{1}{P}-\frac{1}{NP}\right)$$

$$4^\circ \quad -\frac{1}{2NP}\leqq\left[T_i(w)\wedge T_i(w_0)\right]\leqq\frac{1}{2NP} : \text{ if } |z-z_0|\leqq\frac{1}{P}$$

where Δ_i, and $\overline{\Delta_i}$ are all fixed directions in the z or w-plane respectively.

18. Lemma 1. *$f(z)$ is totally derivable almost everywhere in* Π.

Let us denote by $\Delta_{1\cdot 2}$ the half line of the angle made by Δ_1, and Δ_2 (Δ_i are fixed directions) which is named X axis the other axis perpendecular to this axis will be named Y axis, and denote by $l(y)$ the line passing through y and parallel to X axis. In the same way the half line of $\bar{\Delta}_1$ and $\bar{\Delta}_2$ and the other will be named U. and V axis, (this is possible by rotation of the coordinates).

Remark 1.

If $|z_i - z_k| < \min\left(\frac{1}{P}\cos\left(X\wedge\Delta_i\right).\ \frac{1}{P}\cos\left([X\wedge\Delta_2]\right)\right.$ and $z_i,\ z_k \in l(y)\frown\Pi$

then $\quad \tan\left(\Delta_2 - \frac{1}{NP}\right) \leq \frac{V(z_i)-V(z_n)}{U(z_i)-U(z_k)} \leq \tan\left(\Delta_1 + \frac{1}{NP}\right)$.

Proof. If it were not so, there is at least one point where the branches of $l_1(z_k)$ anc $l_2(z_i)$ intersects. But their images f (branch of $l_1(z_k)$) and f (branch of $l_2(z_i)$) do not intersect, this is impossible.

This follows clearly that $\underset{y=y}{U(x.\ y)}$ is monoton increasing function of x : $x \in l(y)\frown\Pi$, accordingly if $x \in l(y)\frown\Pi$, then $U(x.y)$ and $V(x.y)$ are functions of bounded variation on $l(y)\frown\Pi$. But on the other hand from condition $S \iint_{D'-\Pi} |f'(z)|^2\, dxdy < \infty$. We see directly that $U(x.y)$ and $V(x.y)$ are bounded variation on $l(y)\frown D'$ for almost y., consequently ($w = U + iV$) is a rectifiable curve for almonst y as a function of x.

19. Remark 2. U and V are bounded variation, therefore they are derivable with respect to x almost everywhere in $l(y)\frown D'$, and from remark 1

$$\left|\frac{\frac{\partial V}{\partial x}}{\frac{\partial U}{\partial x}}\right| \leq M\left(< \tan\Delta_i \pm \frac{1}{NP}\right) \quad i=1.2$$

almot everywhere in $l(y)\frown\Pi$.

Let us denote by E_k the set satisfying the following condition on $l(y)\frown\Pi$ and denote by $E(y)$ the set $l(y)\frown\Pi$

$$\tan\frac{K-1}{NP} < \frac{\frac{\partial V}{\partial x}}{\frac{\partial U}{\partial x}} \leq \tan\frac{K}{NP} : \text{ if } z \in E_k : K < NP : N \geq 2$$

$$\text{mes } E(y) = |l(y)\frown\Pi| = \sum \text{mes } E_k.$$

To prove the total derivability, we have only to show that

$$\overline{\lim_{h\to 0}}\left|\frac{f(z+h)-f(z)}{h}\right|<\infty$$

almost everywhere in Π, we assume that $\overline{\lim\limits_{h\to 0}}\left|\frac{f(z+h)-f(z)}{h}\right|=\infty$ in a positive measure set Π^0 : $\Pi^0\subseteqq\Pi$.

$$\text{mes}\,|\,\Pi\wedge\sum_v E(y)\,|=\sum_v\sum_k E_k=\text{mes}\,\Pi>\text{mes}\,\Pi^0\geqq d>0\ ,$$

therefore there is a certain m such as at least

$$\text{mes}\ |\,\Pi^0\wedge\sum_v E_m\,|>\frac{d}{NP}\ .$$

If $\quad z\in\Pi^0\wedge\sum\limits_v E_m$, then $\quad \tan\dfrac{m-1}{NP}<\dfrac{\dfrac{\partial V}{\partial x}}{\dfrac{\partial U}{\partial x}}\leqq\tan\dfrac{m}{NP}\ .$

By Egoroff's theorem for any positive number ε. There exist δ and a closed subset Π' such as

$$\text{mes}\ |\,\Pi^0-\Pi'\,|<\varepsilon$$

If $z\in\Pi'$ and h is real number and $|h|<\delta$ then

$$-\frac{1}{NP}\leqq\left[\frac{m}{NP}\ \wedge\frac{V(z+h)-V(z)}{U(z+h)-U(z)}\right]\leqq\frac{1}{NP}\ .$$

From Π', we take a set Π^2 which is linearly density with respect to any line $l(y)$ and by Egoroff's theorem we can find a subset Π^2 of Π' such as

If lengh of $l(y)<\delta$, then $\quad\dfrac{\text{mes}\ |\,l(y)\wedge\Pi'\,|}{\text{mes}\ l(y)}>1-\dfrac{\varepsilon}{2}\ .$

We denote by θ_1 and θ_2 the angles which is made Δ_1 and Δ_2 with X-axis we can assume that $-\dfrac{2}{NP}>\left(\theta_1-\dfrac{m}{NP}\right)>\dfrac{2}{NP}$, $\dfrac{\pi}{2}-\dfrac{2}{NP}>\left(\dfrac{m}{NP}-\theta_2\right)>\dfrac{2}{NP}$ by choosing adequate Δ_1, Δ_2 among Δ_1, Δ_2, Δ_3, and now let δ be smaller than

$$\min\left(\frac{1}{1-\varepsilon}\ \frac{1}{P}\sin\left(\theta_1-\frac{1}{NP}\right),\quad\frac{1}{1-\varepsilon}\ \frac{1}{P}\sin\left(\theta_2+\frac{1}{NP}\right)\right)$$

20. Remark 3. Maximal and minimal quasi parallelogram in the z-plane with centre z and radius h.

If $z\in\Pi^2$, then $\overline{\lim}\left|\dfrac{f(z+h)-f(z)}{h}\right|=\infty$. Therefore there exists $z_n=z+h$ such as $\left|\dfrac{f(z+h)-f(z)}{h}\right|\geqq M$ for any large number M.

We write the circle with centre at z and radius

$$h \begin{cases} < \dfrac{1}{2P} \sin\left(\theta_1 + \dfrac{1}{2NP}\right) \\ < \dfrac{1}{2P} \sin\left(\theta_2 - \dfrac{1}{2NP}\right). \end{cases}$$

We can find α_1 and α_1' on $l(y) \frown \Pi$ satisfying the following conditions

$$\max \begin{Bmatrix} h \sec\left(\theta_2 + \dfrac{1}{2NP}\right) \\ h \sec\left(\theta_1 - \dfrac{1}{2NP}\right) \end{Bmatrix} \leqq \begin{Bmatrix} |z-\alpha_1| \\ |z-\alpha_1'| \end{Bmatrix} \leqq \max \begin{Bmatrix} h \sec\left(\theta_2 + \dfrac{1}{2NP}\right)\dfrac{1}{1-\varepsilon} \\ h \sec\left(\theta_1 - \dfrac{1}{2NP}\right)\dfrac{1}{1-\varepsilon} \end{Bmatrix}.$$

From α_1 and α_1' we trace $l_1(\alpha_1)$ and $l_2(\alpha_1)$ and $l_1(\alpha_1')$ and $l_2(\alpha_1')$. These lines forms a quasi parallelogram. This will be called maximal quasi parallelogram $\underset{\max z}{\square}$ with centre at z and radius h.

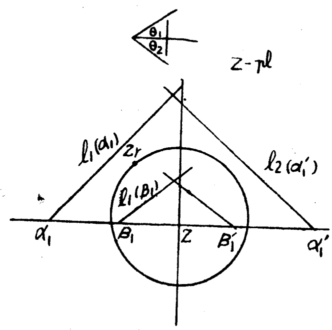

Fig. 3

Next we can find β_1 and β_1' on $l(y) \frown \Pi$ satisfying following conditions

$$\text{mini} \begin{Bmatrix} h \sec\left(\theta_2 - \dfrac{1}{2NP}\right) \\ h \sec\left(\theta_1 + \dfrac{1}{2NP}\right) \end{Bmatrix} \leqq \begin{Bmatrix} |z-\beta_1| \\ |z-\beta_1'| \end{Bmatrix} \leqq \text{mini} \begin{Bmatrix} \dfrac{1}{1-\varepsilon}\, h \cos\left(\theta_2 - \dfrac{1}{2NP}\right) \\ \dfrac{1}{1-\varepsilon}\, h \cos\left(\theta_1 + \dfrac{1}{2NP}\right) \end{Bmatrix}$$

and we trace $l_i(\beta)$ in the same manner as in the preceding, we call this quasi minimal perallelogram $\underset{\text{mini } z}{\square}$ with centre z radius h.

Evidently
$$z_n \in \underset{\max z}{\square} - \underset{\text{mini } z}{\square}$$

$$\frac{\text{dia} \underset{\max}{\square}}{\text{dia} \underset{\text{mini}}{\square}} \leqq K_1, \quad \frac{\text{area} \underset{\max}{\square}}{\text{area} \underset{\text{mini}}{\square}} \leqq K_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

$$K_1,\ K_2 = K_i(P.N)$$

21. Remark 4. Outer minimal, and outest parallelogram in the w-plane and their property.

In general, let us denote the image of p by $\bar{p}$ in the w-plane.

From $\overline{\alpha}_1$ and $\overline{\alpha}_1'$ we trace lines $L_i(\overline{\alpha}_1)$ and $L_i(\overline{\alpha}_1')$ etc,

direction $L_1(\overline{\alpha}_1)=\overline{\Delta}_1+\frac{1}{2NP}$ direction $L_2(\overline{\alpha}_1)=\overline{\Delta}_2-\frac{1}{2NP}$

direction $L_1(\overline{\alpha}_1')=\pi-\overline{\Delta}_1-\frac{1}{2NP}$ direction $L_2(\overline{\alpha}_1')=\pi-\overline{\Delta}_2+\frac{1}{2NP}$

These L_i form a parallelogram named outest $\underset{O\ W}{\square}$. From $\overline{\alpha}_1$ and $\overline{\alpha}_1'$ we trace lines L_i so that

direction $L_1(\overline{\alpha}_1)=\overline{\Delta}_1-\frac{1}{2NP}$

direction $L_2(\overline{\alpha})=\overline{\Delta}_2+\frac{1}{2NP}$

direction $L_1(\overline{\alpha}_1')=\pi-\overline{\Delta}_1+\frac{1}{2NP}$

dirdction $L_2(\overline{\alpha}')=\pi-\overline{\Delta}_2-\frac{1}{2NP}$.

This is named outer minimal parallelogram $\underset{O\ \mathrm{mini}\ W}{\square}$.

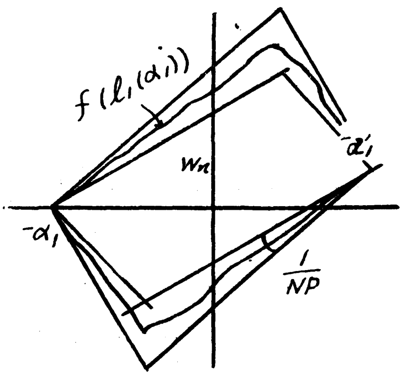

Fig. 4

As $\alpha_1\cdot\alpha_1'\in\Pi$. So $\tan\frac{m-1}{NP}\leqq\frac{\frac{\partial V}{\partial x}}{\frac{\partial U}{\partial x}}\leqq\tan\frac{m}{NP}$ $\overline{\beta}_1,\ \overline{\beta}_1'\in\underset{O\ W}{\square}$.

From $\overline{\alpha}_1$, $\overline{\alpha}_1'$ we make imges $f(l_i(\alpha_1))$, etc this forms a quasi parallelogram with four curves $\square\ (\overline{\alpha}_1,\ \overline{\alpha}_1')$.

It is evident $\underset{O\ \mathrm{mini}\ W}{\square}\leqq\square\ (\overline{\alpha}_1,\ \overline{\alpha}_1')\leqq\underset{O\ W}{\square}$.

and from $\frac{\pi}{2}-\frac{2}{NP}>\left(\theta_1-\frac{m}{NP}\right)>\frac{2}{NP}$, $\frac{\pi}{2}-\frac{2}{NP}>\left(\frac{m}{NP}-\theta_2\right)>\frac{2}{NP}$, then $\frac{1}{NP}\left[L_i\wedge\overline{\overline{\alpha}_1\,\overline{\alpha}_1'}\right]$, it follows that area $\underset{C\ W}{\square}$, and area $\underset{O\ \mathrm{mini}\ W}{\square}=\underline{M}\,|\overline{\alpha},\overline{\alpha}_1'|^2$: where

$$0<m^{**}(N.P)\leqq m^{*}(N.P.m)\leqq\underline{M}\leqq\underline{M}^{*}(N.P.m)\leqq M^{**}(N.P)>+\infty$$

If $z_n\in\Pi$ then $w_n\in\underset{O\ \mathrm{mini}\ W}{\square}$ this is proved easily as in remark 1.

22. Remark 5.

$$\frac{\text{area } \square(\bar{\alpha}, \bar{\alpha}_1')}{\text{area } \underset{o\ w}{\square}} > K_3$$

Case 1 $z_n \in \Pi \cap \underset{\max z}{\square} - \underset{\text{mini } z}{\square}$, then $f(z_n) \subset \square(\bar{\alpha}_1, \bar{\alpha}_1')$

because to z_n $f(l_i(z_n))$ correspond which must intersect the peripherie of $\square(\bar{\alpha}_1, \bar{\alpha}_1')$ to outer side

Case 2 $z_n \bar{\in} \Pi \cap \underset{\max z}{\square} - \underset{\text{mini } z}{\square}$

$\frac{f(z_n)-f(z)}{z_n-z}$ is regular, therefore the maximum of this absolute value is attaiend at the point p of Π or the peripherie of $\underset{\max z}{\square} - \underset{\text{mini } z}{\square}$ therefore from case 1 or 2. There exists a point ζ_0 on the peripherie of $\underset{\max z}{\square} - \underset{\text{mini } z}{\square}$) such as $|f(\zeta_0)-f(z)| \geq M|\zeta_0-z|$., but in the z-plane $|\zeta_0-z| > K_3 h$, or at a point of Π (this is case 1), accordingly in $\underset{\max z}{\square}$, there are two point z, ζ_0 such as $f(\zeta_0)$ and$f$ $(z) \in \underset{o\ w}{\square}$, $|f(\zeta_0)-f(z)| > Mk_3 h$, this follows that

$$\text{area of } \underset{o\ w}{\square} > K_4 h^2 M \ : K_3,\ K_4 : K_i = K(P.N)$$

23. Remark 6. *If two maximal quasi parallelogram has no point in common in the z-plane, then corresponding two minmal outer parallelogram has no poinit in common in the w-plane,*

Case 1 $\square_j$ lies on one side of $l_i(a_n)$. Let such l_i be $l_1(a_i)$ then A_jB_j C_jD_j lies on one side of A_iB_i, if it were not so d_j. A_iB_i opposite side, then a_jd_j intersect with $l_i(a_i)$ or $l_i(b_i)$, but A_jD_j or C_jD_j cannot intersects with A_iB_i or C_iD_i on its extension. This is impossible. Where $A_i = f(a_i)$ etc.

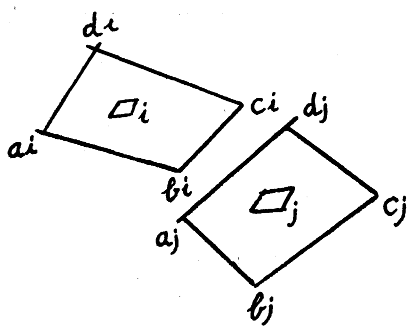

Fig. 5

Case 2 (not case 1) in this case, we can prove in the same way in 1 using the continuity of angle. Let $\square_j$ be not contained in the angle $a_id_ic_i$, then D_j lies $C_iD_iB_iD_i$ same side, therefore D_i is not contained in the angle $A_iD_iB_i$, therefore minimal $\underset{o\ \text{mini}\ w}{\square}$ never overlappe.

But $\dfrac{\text{area} \underset{O\ W}{\square}}{\text{area} \underset{O \text{ mini } W}{\square}} < K_5$ and from 4 area $\underset{\text{max } z}{\square} > K_6 h^2$

finally $$\frac{\text{area} \underset{O \text{ mini } W}{\square}}{\text{area} \underset{\text{max } z}{\square}} \geq K_7 M.$$

By Vitali's covering theorem, we can find a secquence of $\underset{\text{max } z}{\square}$ not overlapping each other and

$$\sum^{n} \text{mes} \underset{\text{max } z}{\square} > \frac{d_0}{2}$$

mes $f(D) >$ mes $f(D') > \sum^{n}$ mes $\underset{O \text{ mini } W}{\square} > K_7 M \frac{d_0}{2}$, but $M \to \infty$, this is a contradiction.

24. Lemma 2. *If $f(z)$ is totally derivable at $z=z_0$ and satisfies K'^* then $f(z)$ is monogene at $z=z_0$*

$$f(z)-f(z_0)=(A_1+iA_2)(x-x_0)+(B_1+iB_2)(y-y_0)+\varepsilon(z)|z-z_0|$$

$$\lim_{z \to z_0} \varepsilon(z)=0$$

$$\tan \Theta = \frac{A_2(x-x_0)+B_2(y-y_0)}{A_1(x-x_0)+B_1(y-y_0)} = \frac{A_2+B_2 \tan \theta_i}{A_1+B_1 \tan \theta_i}$$

$$\tan \Theta - \tan \theta = \text{const for } \theta_i \quad i=1.2.3.$$

Then we easily have $A_1 = B_2$, $A_2 = -B_1$.

25. Lemma 3. *$f(z)$ has property N on $l(y)$ for almost all y.*

If it were not so there exists a positive measure set G on Δ_2 such that, for any $y \in G_1$, $[\underset{y=y}{f(z)}]$ are rectifiable and on which $f(z)$ has not N, this fact follows that there exists a set $q(y)$ for line mes $|q(y)|=0$ but $f(q(y))$ has line measure >0. By Lusin's theorem there exists a such a perfect set as mes line $q(y)$ of which any portion q of it line mes $f(q(y))>0$, of course $q(y) \subset \Pi$ for $D'-\Pi \ni z$, $f(z)$ is regular accordingly absolutely continuous.

If $z_k, z_k' \in l \cap \Pi$ and $|z_k - z_k'| < \frac{1}{2P}$ then $[\overline{w_k, w_k'} \wedge \overline{\Delta_1}]$ is contained in $\left[\theta_1 + \frac{1}{2NP} \wedge \theta_2 - \frac{1}{2NP}\right]$. Let us denote the set of y such as

$$G_m = \underset{y}{E}\left[\text{lin mes } f\left(\underset{z \in l(y)}{E}\right)\frac{m}{NP} \leq \left[X\text{-axis} \wedge \tan^{-1}\frac{\frac{\partial V}{\partial x}}{\frac{\partial U}{\partial x}}\right] < \frac{m+1}{NP}\right) > \frac{\lambda}{NP}\right]:$$

$$\text{mes } f(q(y)) > \lambda.$$

Then there exists at least a set such as outer mes $G^m > \mu > 0$, which is denoted by G_0.

For any $y \in G_0$, let us devide $l(y)$ in equal length segments δ_i: $\delta_i \frown \delta_j = 0$ and denote by z_k. z'_k the ends of $\delta_k \frown \Pi$ and construct the parallelogram $\square$ formed by $l_i(z_k)$, $l_i(z'_k)$ $i=1.2..$ From mes $q(y)=0$ follows $\sum^p$ length $\delta_k < \frac{\lambda}{A}$ for any large number A, and if $w_k = f(z_k)$,

$w'_k = f(z_k)$, $|w_k - w'_k| = \lambda_k$ are denoted then $\sum^p |w_k - w'_k| > \frac{\lambda}{2NP}$ for sufficiently large p, and

$$\frac{1}{NP} < [\overline{w_k - w'_k} \wedge T_m] < \pi - \frac{1}{NP} : \text{direction } T_m = \frac{m}{NP} .$$

From the construction of $f(\square)$, we see that $f(\square)$ is a quasi parallelogram in the w-plane which has outer minimal parallelogram in in its interior. These minimal parallelograms $\underset{\text{mini } W}{\square}$ never overlapp, when corresponding maxima $\underset{\max z}{\square}$ have no common point in the z-plane (see Lemma 1).

$$\text{area of min } \underset{\text{mini } W}{\square} > C\lambda_k^2 \quad (C \text{ depends only on } P \text{ and } N)$$

$$\sum^p \lambda_k > \frac{\lambda}{2NP} \text{ for sufficiently large } p .$$

$\sum'$ and $\sum''$ means the summation over k satisfying (1) or (2)

$$\lambda_k \geqq \frac{A}{4NP} \delta_k(y) \tag{1}$$

$$\lambda_k < \frac{A}{4NP} \delta_k(y) \tag{2}$$

$$\frac{\lambda}{2NP} < \sum = \sum' + \sum'' \quad \text{then} \quad \sum' > \frac{\lambda}{4NP}$$

$$\text{area of minimal parallelogram } \underset{\text{mini } W}{\square} > \lambda_k^2 C > \left(\frac{A}{4NP}\delta_k\right)^2 C$$

$$\sum^p \text{area of } \underset{\text{mini } W}{\square} > \sum' > \sum' \left(\frac{A}{4NP}\delta_k\right)^2 C > C\left(\frac{A}{4NP}\right)^2 \delta_k^2 > \frac{CA}{16NP}\delta_k \lambda$$

We denote by $s(K)$ the projection of parallelogram of which the diagonal is $\overline{z_k, z'_k} = s_k$ on Δ_2.

We can find a secquence of intervals I_i has no common point each other on Δ_2

$$\sum I_i > \frac{\mu}{8} \text{ follows } \sum^{m} s_k > \frac{\mu}{8} \qquad \text{for large number } m.$$

This operation will be used for each I_i then we have

$$\sum^{m}\sum^{n} \text{ area of } \underset{\text{mini } W}{\square} > C\sum{}' \frac{A}{64NP}\delta\lambda > \frac{CA\lambda\mu}{128NP}.$$

This is a contradiction for $A\to\infty$ and mes $|f(D)| < +\infty$, the same fact occurs for another $l(x)$, accordingly we can conclude

$$U(x_1)-U(x_2)=\int_{x_1}^{x_2}\frac{\partial U}{\partial x}\,dx \qquad \text{for almost all } y, \text{ etc.,}$$

then we can prove that $\int_C f(z)dz=0$ in the same manner as used in Theorem 1.

§ 5

25. When the topological property of a regular function is characterized, this is called an inner transformation satisfying the following two fundamental conditions.

1° Light transformation: for any $w\in f(D)$, $f^{-1}(w)$ is totally disconnected, then $f(z)$ is called a light transformation.

2° Open transformrtion: any open set is transformed into an open set.

Property K'^s. If at $z=z$, $f(z)$ satisfies K' and further in the neibourhood of z, any Jordan curve issuing from z contained in the sector $S_{ij}(z)$ formed τ_i and τ_j, has its image in the w-plane in the corresponding sector $\overline{S}_{ij}$ which is not whole direction, then we call that $f(z)$ has K'^s at $z=z_0$.

In regarding that $f(\tau_i(z))$ has a tengent at $w: w=f(z)$, there exists such r_0; if $|\zeta-z|<r_0 : \zeta\in\tau_i(z)$ then $f(\zeta)\neq f(z)$. We define $\overline{\overline{S}}_{ij}=2\pi-\overline{S}_{ij}$ and $\overline{T}_{ij}$ is the half liene of T_i and T_j.

We denote by $G(N.P, n_1, n_2, n_3)$ the set satsfying the following conditions

1° $$\frac{1}{2NP}<\left[\tau_i\wedge\frac{n_i}{2NP}\right]<\pi-\frac{1}{2NP}$$

2° $$\frac{1}{P}<\left[\tau_i\wedge\tau_j\right]<\pi-\frac{1}{P}$$

3° $$[T_i\wedge T_j]=[\tau_i\wedge\tau_j]$$

4° $\left[T_i \wedge f(\zeta)-f(z)\right] < \frac{1}{NP}$ if $|\zeta - z| \leq \frac{1}{P}$: $\zeta \in \tau_i(z)$

5° $f(\zeta) \in \overline{\overline{S}}_{ij}$: if $|\zeta - z| \leq \frac{1}{P}$: $\zeta \in S_{ij}$

26. Menchoff proved the following theorem [12]

Theorem 5. *If $f(z)$ is topological and direct in D, and* $\lim_{\zeta \to z} \arg \frac{f(\zeta)-f(z)}{\zeta - z}$ *exists at every point except at most enumerable points, then $f(z)$ is holomorphic in D.*

Theorem 5′. *If $f(z)$ is continuons (not necessarily univalent) and K'' is satisfied at every point except at most enumerable points, then $f(z)$ is holomorphic in D.*

Lemma 1. *$f(z)$ is a light transformation in D.*

If $f(z)$ were not so, there exists at least such a point of w as $f^{-1}(w)$ is a continum being clearly closed. A continum is a perfect set, then there exists a portion Π of the continum in which a certain G_0 is dense, therefore there is secquence of points converging to p, and then there is also the subsecquence of points converging to p in certaine sector $S(p)$ with the opning angle smaller than $\frac{1}{2NP}$ and the vertex is p. If we denote by $q_{i,i+1}$ the intersectiong point of $\tau_1(p_i)$ and $\tau_2(p_{i+1})$, then there exists at least a pair of p_i, p_{i+1} in $S(p)$ satisfying conditions

1° $p_i,\ p_{i+1} \in S(p)$

2° dist $|q_{i,i+1} p_i| < \frac{1}{P}$; dist $|q_{i,i+1},\ p_{i+1}| < \frac{1}{P}$

3° $f(p_i) \neq f(q_{i,i+1})$, $f(p_{i+1}) \neq f(q_{i,i+1})$

4° If length of $\tau_1(p_i)$, $\tau_2(p_{i+1}) < \frac{1}{P}$, then $f(\tau_1(p_i)) \subset \overline{S}_1(f(p_i))$

$f(\tau_2(p_{i+1})) \in \overline{S}_2(f(p))$, where the opening angle of $\overline{S}_i$ is $\frac{1}{NP}$

and the half line of $\overline{S}_i$ is $T_i(w)$: $w=f(p)$ respectively.

But from $f(p_i)=f(p_{i+1})=f(p)$: $f(q_{i,i+1}) \subset \overline{S}_1 \wedge \overline{S}_2 = f(p)$.

$f(q_{i,i+1}) = f(p_{i+1}) = f(p_i) = f(p)$

12) Menchoff : Sur la représentation conforme des domaines plans, Math. Ann. 95, p. 642 (1926).

This is a contradiction.

If $f(z)$ is not holomorphic in D, we can find a portion Π defined by D' which is completely contained in the closure of certain G_0.

Lemma 2. *$f(z)$ is an open transformation in D'.*

If $z \in D'-\Pi$, $f(z)$ is regular, therefore if $f(z)$ were not an open transformation, then there exists such a point $p \in \Pi \frown D'$ and an open set G as $p \in$ interior of G, and $f(p) \in$ boundary of $f(G)$.

We take a neighbourhood $V(p)$ of p: dia $V(p) < \frac{1}{P}$: $V(p) \Subset G \subset D'$. Since $f(z)$ is a light transformation $f^{-1}f(p)$ is closed and disconnected. We take 3 points a. b. c. on $\tau_i(p) \frown V(p) \frown$ complement of $f^{-1}f(p)$; $i=1.2.3$ and connect by the ${}_aC_b$ a and b in $V(p) \frown \bar{S}_{ij}(p) \frown$ complement of $f^{-1}f(p)$, and so on about b, c and c, a in $\bar{S}_{ji}(p)$, $\bar{S}_{jk}(p)$ respectively to make a closed curve C, then it is clear that

dist $(f(C),\ f(p)) \geqq \delta_0 > 0$,

the order of $f(C)$ with respect to $f(p)$ is 1.

Hence $f(p) \in$ boundary of $f(G)$, then there exists another point q and another neighbourhood $V'(f(p))$; dia $V'(f(p)) \leqq \frac{\delta_0}{2}$, $V'(f(p)) \ni q$: $f(G) \not\ni q$; dist $(f(p),q) = \varepsilon < \frac{\delta_0}{4}$, then

the order of $f(C)$ with respect to q is 1.

In $V(p)$ we deform continuously C into C' ; so that dia $f(C') < \frac{\varepsilon}{4}$ and enclosing p, then

the order of $f(C')$ with respect to q is 0.

This shows that q is covered by the schar of images of curves from C to C' in this deforming process, which contradicts that $q \bar{\in} f(G)$.

As $f(z)$ is an inner transformation in D', therefore it is locally univalent and topological, consequently theorem 4 is applicable locally except enumerable points (branch point), finally $f(z)$ is holomorphic in D.

Remark. Theorem 5 is clearly contained in Theorem 5′ therefore the condition of univalency of Menchoff's theorem is surplus.

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