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Osaka University
On Sufficient Conditions for a Function to be Holomorphic in a Domain

By Zenjiro Kuramoto

§ 1

1. The problem under what condition it is sufficient for the continuous function \( f(z) = U(z) + iV(z) \) of a complex variable \( z = x + iy \) defined in a domain \( D \) of the \( z \)-plane to be holomorphic, has been studied from many points of view. In particular one is from the theory of a real function or the integral, and the other is from the properties of an analytic function in the neighbourhood of the regular point, for instance, the invariance of segment's ratio, of angles, etc. The latter is the starting point of Menchoff's study continued from 1923 to 1938.

In regarding this there may be enumerable algebraic singular points (i.e. branch point) at which the local properties in the neighbourhood will be lost to some extent, his allowance that there might be enumerable points at which the properties supposed as the conditions of his theorems, were not satisfied, renders to be more interesting in the case when \( f(z) \) is not univalent, because univalent and holomorphic function cannot have any branch points in its domain. The object of our study is to extend his theorems so as they may remain valid even when \( f(z) \) is not necessarily univalent, to shorten his proofs and generalize in some ways.

When \( \lim_{h \to 0} f(z_0 + h) - f(z_0) \) exists, we call \( f(z) \) is monogene at \( z = z_0 \). The necessary and sufficient conditions for \( f(z) \) to be monogene, is that \( f(z) \) is totally derivable and simultaneously satisfies the Cauchy-Riemann differential equations \( \frac{\partial U}{\partial x} = -\frac{\partial V}{\partial y}, \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} \) and the necessary and sufficient conditions for \( f(z) \) to be holomorphic in \( D \) is that \( f(z) \) is monogene at every point in \( D \). We see directly that the set in which \( f(z) \) is not regular forms a perfect set.

2. We denote the half lines issuing from \( z \) by \( \tau_i(z) ; i=1,2,3... \)

the angle made between \( \tau_i \) and \( \tau_j \) by \([\tau_i(z)^\wedge \tau_j(z)]\) and the amplitude of \( f(\xi) - f(z) \) by \( \operatorname{amp}[f(\xi) - f(z)] \). If the upper and lower limit \( \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} \) exist and when two extreme limits of \( \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} \) and \( \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z} \) are equal, we denote them by \( \tau_i \bar{A}(z) \), \( \tau_i \bar{B}(z) \) and \( \tau_i A(z) \), \( \tau_i B(z) \) respectively.

We say that \( f(z) \) satisfies the property \( K'' \), \( K^{**} \) and \( K^{***} \) at \( z = z_0 \), if the following conditions are satisfied respectively.

**PROPERTY \( K'' \)**

1° To \( z = z_0 \) three lines \( \tau_i(z_0) \) correspond such that \([\tau_i(z_0)^\wedge \tau_j(z_0)]\) \( \equiv 0 \) (mod \( \pi \))

2° \( \tau_i A(z) = \tau_j A(z) \) \( i, j = 1, 2, 3 \)

**PROPERTY \( K^{**} \)**

1° To \( z = z_0 \) two lines \( \tau_i(z) \) correspond such that \([\tau_i(z)^\wedge \tau_j(z)]\) \( \equiv 0 \) (mod \( \pi \))

2° \( \tau_i A(z) < +\infty \) and moreover two sequences \( q^1_i, q^2_i, q^3_i, ... \) on \( \tau_i(z) \) exist satisfying

\[
\lim_{\xi \to z} B(q^n_i) = \lim_{\xi \to z} B(q^n_j)
\]

**PROPERTY \( K^{***} \)**

1° To \( z = z_0 \) three lines \( \tau_i(z) \) correspond such that \([\tau_i(z)^\wedge \tau_j(z)]\) \( \equiv 0 \) (mod \( \pi \))

2° \( \tau_i A(z) < +\infty \) and moreover three sequences \( q^1_i, q^2_i, q^3_i, ... \) on \( \tau_i(z) \) exist satisfying

\[
\lim_{\xi \to \tau_i} A(q^n_i) = \lim_{\xi \to \tau_j} A(q^n_j) \quad i, j = 1, 2, 3, \text{ and } \operatorname{amp}
\]

3. **Condition \( S \)**. For a continuous function \( f(z) \) in \( D \), let \( \Omega \) be the image of \( D \) as \( z \) varies in \( D : \Omega = f(D) \). At every point of \( \Omega \), let \( s(w) : w \in \Omega \), be the number (finite or infinite) of times when \( w \) is covered by \( f(z) \). Then \( s(w) \) is measurable.

Proof. Let \( a, b \) and \( c, d \) be the upper and lower bounds of \( x, y \) coordinates of \( D : \Gamma^0 = [a, b], \hat{\Gamma}^0 = [c, d] \). For each positive integer \( n \), let us put \( \hat{I}_i^{(n)} = [a, a + (b-a)/2^n] \), \( \hat{I}_j^{(n)} = (a + (k-1)(b-a)/2^n, a + k(b-a)/2^n] \), \( \hat{I}_i^{(n)} = (c + (k'-1)(d-c)/2^n, c + k'(b-a)/2^n] : k, k' = 1, 2, 3 .... \)

These define two subdivision \( \hat{\Sigma}^{(n)} \) and \( \hat{\Sigma}^{(m)} \) of the intervals \( \hat{\Gamma}^{(n)} \) and \( \hat{\Gamma}^{(m)} \) into \( 2^n \) subintervals, of which the first is closed and the other are half open on the left respectively. Let us denote the rectangle by
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For $k=1, 2, 3, \ldots, 2^n$, let $s_{k, k'}(w)$ denote the characteristic function of the set $f(R_{k, k'}^{(n)})$ and let

$$s^{(n)}(w) = \sum_{k, k'} s_{k, k'}(w) : k, k' = 1, 2, 2^2, \ldots, 2^n.$$ 

We see at once that the functions $s^{(n)}(w)$ constitute a non-decreasing sequence which converges at each point of $w$ to $s(w)$. Hence, the functions $s^{(n)}(w)$ being measurable, so is also the function $s(w)$, and $s(w)$ shows the number of times when $w$ is covered by $f(z)$ in $D$.

Menchoff proved the following theorem $^2$:

4. Theorem 1. If $w=f(z)$ is a continuous function defined in $D$, if $f(z)$ is a topological and direct (i.e., sense preserving) transformation of the $z$-plane to the $w$-plane, and moreover $K''$ is satisfied at every point in $D$, except at most enumerable points, then $f(z)$ is holomorphic throughout in $D$.

We shall prove the next modified theorem

Theorem 1'. For the continuous function $w=f(z)$ defined in $D$ (not necessarily topological or univalent), if the following conditions are satisfied,

1° $K''$ (or $K''^*$) is satisfied at every point except at most enumerable set,

2° Condition $S$ is satisfied in $D$,

then $f(z)$ is holomorphic in $D$.

In order to prove the theorem we proceed with some lemmas.

5. Lemma 1. If $f(z)$ is the continuous function having two lines $\tau_i(z)$ on which $\lim \tau_i \overline{A}(z) < \infty$ at every point $z$ except at most enumerable points, then $f(z)$ is almost everywhere totally derivable $^3$.

To prove the lemma 1, we have only to show $\lim_{h \to 0} \frac{|f(z+h) - f(z)|}{h} = \infty$, from which follows that $f(z)$ is not regular in $E$.


We denote the set of points of density of $E$, by $E_1$, we observe that $\text{mes} \{ E - E_1 \} = 0$, and accordingly for any positive number $\varepsilon$, we can find a perfect set $E_2$, such as $E_1 \supseteq E_2$, $\text{mes} |E_1 - E_2| < \varepsilon$. We easily see that any portion of $E_2$ has positive measure.

We denote the set satisfying the following conditions by $G(P. N. n_1, n_2)$: where $P. N. n_1, n_2$ are all integers

1° \[ \left[ \Delta \wedge \tau_i (z) \right] \wedge \frac{N_1}{N} \leq \frac{1}{2N^2} : \Delta \text{ is the fixed direction} \]

2° \[ \frac{1}{P} \leq \left[ \frac{n_1 \wedge n_2}{NP} \right] \wedge \frac{1}{P} : N \geq 2 \]

3° \[ \frac{f(z)}{\xi - z} \leq \frac{1}{P} : 0 < |\xi - z| \leq \frac{1}{P} : \xi \in \tau_i (z) \]

4° \[ \text{dist} (z, \text{boundary of } D) \geq \frac{1}{P} \]

then \[ E_2 \subseteq \sum_{P, N, n_1, n_2} G(P. N, n_1, n_2) + H \]

where $H$ is the set in which $K^{**}$ (or $K^{***}$) is not satisfied which is enumerable at most.

By Baire's theorems we conclude that there is a portion $\Pi$ \(^5\) (we assume that $\text{mes} \Pi = 0$ without losing generality) defined by a certain open set $D'$, and in $\Pi$ a certain $G(P, N, n_1, n_2)$ is dense, which will be denoted by $G_0$. In the case when $\Pi \cap G_0 \ni z$, $\tau_i (z)$ are defined already, and in the case when $\Pi \cap \setminus G_0 \ni z \in \Pi$ we define $\tau_i$ as the limit of $\tau_i (z_n): z_n \in \Pi \cap \setminus G_0$. From the continuity of $f(z)$ we easily recognize that these $\tau_i (z): z \in \Pi \cap \setminus D'$ satisfies all the conditions of Lemma 1.

Proof of the lemma 1. For a positive measure set $\Pi$, we know that the set of linearly density point of \(^6\) $\Pi$ with respect to a fixed direction, has the same measure as that of $\Pi$.

Now let us denote by $X$ and $Y$ axes the two half lines of the angles associated with the fixed directions $\frac{n_1}{NP}$ and $\frac{n_2}{NP}$. These axes intersect perpendicularly each other. If we denote by $\Pi^*$ the set of points of linearly density of $\Pi$ with respect to $X$, and $Y$ directions simultaneously, then

\[ \text{mes} |\Pi - \Pi^*| = 0 \]

By Egoroff's theorem for any small number $\varepsilon$ and $\eta$ we can find a positive measure set $\Pi^{**}$ of $\Pi^*$ and a positive number $\delta$ such that if

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\(^6\) Saks: loc. cit. p. 54.
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l is the line containing a point \( \Pi^{**} \) at least parallel to X or Y axis and its length is smaller than \( \delta \) then

\[
\frac{\text{mes } l \cap \Pi^*}{l} > 1 - \frac{\varepsilon}{2}, \quad \text{mes } |\Pi^* - \Pi^{**}| < \eta : \delta < \frac{1}{P}
\]

If \( z_0 \) is a point of \( \Pi^{**} \), let us trace X, Y axes going through \( z_0 \) and denote by \( V_s(z_0) \) the circular neighbourhood of \( z_0 \) with centre \( z_0 \) and diameter \( s \), where

\[
s \leq \frac{\delta}{2(1-\varepsilon)^3 \csc \left( \frac{1}{2P - \frac{1}{2NP}} \right) \left[ 1 + \frac{1}{1-\varepsilon} \tan \left( \frac{1}{2P + \frac{1}{2NP}} \right) \cot \left( \frac{1}{2P - \frac{1}{2NP}} \right) \right]}
\]

Then \( \frac{|f(z) - f(z_0)|}{|z - z_0|} < MP : z \in \Pi \cap V_s(z_0) : M \) depends only on \( P \) and \( N \).

Take \( p_1 \) so that \( |z_0 p_1| = s \csc \left( \frac{1}{2P - \frac{1}{2NP}} \right) \delta \) then there exists a point \( p'_1 : p'_1 \in \Pi^* \), \( |p'_1 z_0| < \frac{1}{1-\varepsilon} |p_1 z_0| \) on the left hand of \( p_1 \), and take \( \tau_1(p'_1) \) which intersects with Y axis at \( p_2 \) and denote by \( \theta_1 \), the angle made X axis and \( \tau_1(p'_1) \).

As \( |z_2 p| < \delta \), there exists a point \( p'_2 \in \Pi \) such as \( |p'_2 z_0| < \frac{1}{1-\varepsilon} |p_2 z_0| \) and trace \( \tau_2(p'_2) \) intersecting with X axis at \( p_3 \), we shall name as follow

- \( p_5 \) = the intersecting point of \( \tau_1(p'_1) \) and \( \tau_2(p'_2) \)
- \( \theta_2 \) = angle \( p'_1 p_5 p_3 \)
- \( \theta_3 \) = angle \( p'_2 p_5 p_3 \)

For \( N \geq 2 \) we have

\[
0 < \frac{1}{4P} < \frac{1}{2P} - \frac{1}{2NP} \leq \theta_1 \leq \frac{1}{2P} + \frac{1}{2NP} \leq \frac{3}{4} \leq \frac{\pi}{2}
\]

\[
0 < \frac{\pi}{2} - \frac{1}{2P} - \frac{1}{2NP} \leq \theta_2 \leq \frac{\pi}{2} - \frac{1}{2P} + \frac{1}{2NP} \leq \frac{\pi}{2}
\]

\[
\frac{1}{P} - \frac{1}{NP} \leq \theta_3 \leq \frac{1}{P} + \frac{1}{NP}
\]

Let \( z \) be a point of \( \Pi \) lying on the periphery of \( V_s(z_0) \), then \( \tau_1(z) \) exists which has a point \( \zeta \) with \( \tau_2(p'_2) \) in common and as \( z_0 \in \Pi^{**} \), \( \Pi \) there exists \( \tau_1(z_0) \) which has a common point \( \zeta_0 \) with \( \tau_2(p'_2) \).
Let

\[ \theta_4 = \text{angle } z_0^\prime \xi_0, \quad \theta_5 = z_0 \xi_0 p_3, \]

then

\[
\begin{align*}
\frac{\pi}{2} - \frac{1}{2P} - \frac{1}{NP} \leq \theta_4 & \leq \frac{\pi}{2} - \frac{1}{2P} + \frac{1}{NP}, \\
\pi - \frac{1}{2P} - \frac{1}{NP} & \leq \theta_5 \leq \pi - \frac{1}{2P} + \frac{1}{NP}.
\end{align*}
\]

\[ |z_0 p_1| = s \csc \left( \frac{1}{2P} - \frac{1}{2NP} \right) \leq \delta; \quad |p_1' z_0| = \frac{1}{1 - e}; \quad p'_i \in \Pi^* \]

\[ |z_2 p'_2| \leq |z_0 p_2| \left( \tan \theta_1 \right) \frac{1}{1 - e}; \quad |z_0 p_3| = |z_0 p'_2| \tan \theta_1; \quad |p_2 p'_2| = \frac{\varepsilon}{1 + e} |z_0 p_2| = \frac{\varepsilon}{1 + e} |z_0 p_1| \tan \theta_1 \leq \frac{\varepsilon}{1 + e} s \csc \left( \frac{1}{2P} - \frac{1}{2NP} \right) \tan \left( \frac{1}{2P} + \frac{1}{2NP} \right) \]

\[ |p_2 p'_2| = \frac{|p_2 p'_2| \sin \theta_3}{\cos \theta_1} \leq \delta; \quad |z_0^\prime| = \frac{(s + z_0 p_3)}{\sin \left( \frac{\pi}{2} - \theta_3 \right)} \frac{(s + z_0 p_3)}{\sin \left( \frac{\pi}{2} - \theta_3 \right)} \cos \left( \frac{\pi}{2} - \frac{1}{1 - e} \right) \]

Thus

\[ |z_0^\prime|, \ |z_0^\prime p_1|, \ |p_2 p'_2|, \ |p_3 p'_3|, \ |p_4 p'_4|, \ |p_5 p'_5|, \ |p_6 p'_6|, \ |z_5|, \ |z_6| \leq \delta \]

and all \(<K_i z - z_0|\ i=1, 2, 3\ and \ all \ K_i < +\infty\ depend \ only \ on \ P \ and \ N.\]

In the same manner we proceed with \(\tau_2(p_1')\), etc., in the half plane under the \(X\) axis, and \(\tilde{p}_2, \tilde{p}_3, \ldots\ etc.\ are \ denoted \ as \ in \ the \ former \ and \ p_2 p_3\ and \ \tilde{p}_2 \tilde{p}_3\ intersect \ at \ p'_*\, then \ p_1, p_5, p'_5, \tilde{p}_5\ and \ p_1' \ forms \ a \ quasi \ parallelogram \ \square' .\]

Finally

\[
\begin{align*}
|f(z') - f(z_0)| & \leq |f(z_0) - f(z'_0)| + |f(z'_0) - f(p'_2)| + |f(p'_2) - f(p_5)| \\
+ |f(p_5) - f(p'_i)| + |f(p'_i) - f(p_5)| & + |f(p_5) - f(p'_2)| + |f(p'_2) - f(z'_0)| \\
+ |f(z'_0) - f(z)| & \leq M.P |z - z_0|,
\end{align*}
\]

where \(M\) depends only on \(P\) and \(N\) whenever \(z \in V_i(z_0) \setminus \Pi\).

In the case when \(z \in \Pi\), we make \(s'\) so small that quasi parallelogram \(\square'\) associated with \(s'\) and \(z_0\) may be contained in \(V_i(z_0)\) completely, then we have the same conclusion for any point of \(z'\) lying on the circumference of \(\square'\), that is

\[
\left| \frac{f(z') - f(z)}{z' - z_0} \right| \leq M'.P : z' \in \square', \text{ s' periphery } \cap V_i \setminus D' : M' = M'(M.P)
\]

If \(z' \in \square' \setminus (V_i \setminus \Pi) \setminus D'\) \(\frac{f(z') - f(z_0)}{z' - z_0}\) is regular.
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By the maximum principle of analytic functions

\[ \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \leq MP : M'' = \max(M, M') : \text{if } z \in V_s(z_0) \cap \mathbb{D}. \]

Since \( \varepsilon \) and \( \eta \) any positive numbers, by Stepanoff’s theorem \( f(z) \) is totally derivable almost everywhere.

Remark. When \( N \geq 1 \) the proof is valid too with no essential alteration.

6. Lemma 2. When at \( z = z_0 \), \( f(z) \) is totally derivable and satisfies \( K'''' \), then \( f(z) \) is monogenic at \( z = z_0 \).

\[ f(z_1) - f(z_0) = (A_1 + iA_2)(x_1 - x_0) + (B_1 + iB_2)(y_1 - y_0) + \varepsilon(z_1) |z_1 - z_0| = s(z_1) \]

\[ + \varepsilon(z) |z_1 - z_0| : \lim_{z_1 \to z_0} \varepsilon(z_1) = 0 : z_i = x_i + y_i : i = 1.0 \]

\[ = \sqrt{(A_1 + iA_2) \cos \theta + (B_1 + iB_2) \sin \theta + 2 \sin \theta \cos \theta (A_1 B_1 + A_2 B_2)} \]

\( A_1, A_2, B_1 \) and \( B_2 \) constants for \( \theta \); \( i = 1.2.3 \ (\text{mod } \pi) \)

We easily have the relation \( A_1 = \pm B_2, A_2 = \mp B_1 \), but from the latter condition of \( K'''' \) we have \( A_1 = B_2, A_2 = -B_1 \). Therefore \( f(z) \) is monogenic, in the case of \( K'''' \) will be proved in the same manner.

7. Lemma 3. A continuous function \( f(x) \) is defined in the closed interval \([a, b]\) and there is a closed set \( F \). \( [a, b] - F = \bigcup I_i : I_i = (a_i, b_i) \) are intervals contiguos to \( F \), with satisfying the following conditions

1° \( \left| \frac{f(z_i) - f(z_j)}{z_i - z_j} \right| \leq M : \text{if } z_i, z_j \in F \)

2° \( f'(x) \) exists almost everywhere and \( \sum I_i \int_{I_i} |f'(x)| dx < +\infty \)

3° For each interval : \( I_i = (a_i, b_i) \), \( f(x) \) is absolutely continuous

then \( \int_a^b f(x) dx = f(b) - f(a) \).

Let us denote the upper and lower bound of \( F \) by \( a' \) and \( b' \) and

\( \bar{f}(x) = f(x) = f(x) \) if \( x \in F \) or \( x < a' \) or \( x > b' \)

\( \bar{f}(x) = \mu f(a_i) + \lambda f(c_i) : x = \frac{\mu a_i + \lambda b_i}{\mu + \lambda} \), if \( x \in F' \) and \( a'<x<b' \)

where \( \lambda, \mu > 0 \)

After elementary calculation we have
Consequently $\tilde{f}(x)$ has the property $N$ of Lusin, and from $2^o \tilde{f}(x)$ is integrable. We denote the upper lower relative to $F$ derivatives by $f'_p(x)$ or $\tilde{f}_p(x)$ and when two are equal, by $f'_p(x)$.

Then $f'_p(x) = f'_p(x) = f'(x)$ almost everywhere in $F$, where

$$\lim f'(x) \leq \lim f'_p(x) \leq \lim \tilde{f}'_p(x) \leq \lim \tilde{f}'(x).$$

From $2^o \int |f'(x) - \tilde{f}'(x)| dx + \sum \int |f'(x) - \tilde{f}'(x)| dx = 0$, it follows

$$\tilde{f}(b) - \tilde{f}(a) = f(b) - f(a) = \int_a^b f'(x) dx$$

8. Proof of the theorem 1'.

We have only to show that $f(x)$ is holomorphic in $D'$, for it follows that $\Pi$ is empty set.

Let us take $\xi$, and $\eta$ axies which are perpendicular to $\frac{n_1}{NP}$ and $\frac{n_2}{NP}$ directions respectively and denote by $\alpha$ and $\beta$ the angles made between $\xi$ and $\eta$ and $X$ axis, then we have

$$x-x_0 = \xi \cos \alpha + \eta \cos \beta, \quad y-y_0 = \xi \sin \alpha + \eta \sin \beta,$$

$$\pi > \pi - \frac{1}{P} - \frac{1}{2NP} > [\tau_1(z_1) \wedge \tau_2(z_2)] > \frac{1}{P} - \frac{1}{NP} > 0 \quad (2)$$

Take a so small parallelogram $\square$, in $D'$ whose four sides are parallel $\xi$ or $\eta$ axis, of which the diameter is smaller than

$$\frac{1}{P} \sin \left( \frac{1}{P} - \frac{1}{NP} \right) \quad (3)$$

We shall prove that $f(x)$ is holomorphic in this parallelogram. If $z_1, z_2$ have the same $\xi$ coordinates and both in $\Pi \cap D'$ then $\tau_2(z_1)$ and $\tau_1(z_2)$ exist which have a point $z_3$ in common. From (2) and (3)

$$|z_1-z_3| < \frac{1}{P}, \quad |z_2-z_3| < \frac{1}{P}, \quad |z_1-z_3| + |z_1-z_2| < M |z_1-z_2| :$$

$$M = M (P, N)$$

We see directly that $\frac{|f(z_1) - f(z_2)|}{z_1-z_2} \leq M \cdot P$ in the same manner of Lemma 2, if $z \in \square \cap D' - \Pi$ then $f(z)$ is regular, therefore $U$, and $V$ absolutely continous with respect to $\xi$. From condition $S$ and change of variables,
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\[ \iint_{D'} |f'(z)|^2 d\xi d\eta = \iint_{D'} |f'(z)|^2 dxdy \leq \iint_{\Omega} s(w) dU \cdot dV < +\infty \]

By the theorem of Fubini

\[ \int_{\eta'} |f'(z)| d\xi < +\infty \text{ for almost } \eta \text{ and } |f''(z)| \frac{\partial U}{\partial \xi}, \frac{\partial V}{\partial \xi} \]

by Lemma 3

\[ \int_{\xi_1}^{\xi_2} \frac{\partial U}{\partial \xi} d\xi = U(\xi_2) - U(\xi_1), \quad \int_{\xi_1}^{\xi_2} \frac{\partial V}{\partial \xi} d\xi = V(\xi_2) - V(\xi_1) \text{ for almost } \eta \]

Similarly we have for \( \eta \) axis.

\[ \int_{\eta_1}^{\eta_2} \frac{\partial U}{\partial \eta} d\eta = U(\eta_2) - U(\eta_1), \quad \int_{\eta_1}^{\eta_2} \frac{\partial V}{\partial \eta} d\eta = V(\eta_2) - V(\eta_1) \text{ for almost } \xi \]

Denoting by \( C \) the circumference of \( \square \)

\[ \int f(z)dz = \iint (-U_x \cos \alpha + V_y \sin \alpha + U_z \cos \beta - V_z \sin \beta) d\xi d\eta \]

\[ + i \iint (-V_x \cos \alpha - U_y \sin \alpha + V_z \cos \beta + U_y \beta) d\xi d\eta \]

\[ = \iint (U_x - V_y) dxdy + i \iint (U_y + V_z) dxdy = 0, \]

because \( f(z) \) is monogene almost everywhere in \( D \).

Finally we conclude that \( f(z) \) is holomorphic in \( D' \), from which follows that \( f(z) \) is holomorphic in \( D \).

\( \S \ 2 \)

9. In this paragraph we intend to enlarge the results in the preceedings, in the wide sense.

We denote by \( f(z) = w \), a continuous function defined in a domain of the \( z \)-plane.

Proposition 1. If \( f(z) \) satisfies the following conditions.

1° \( f(z) \) is continuous and for almost \( y, \text{ app } U \), \( \text{ app } V \) and for almost \( x, \text{ app } U_x, \text{ app } V_y \) exist except at most enumerable set, relative \( x, \text{ and } y \) axis respectively.

2° \( \iint |\text{ app } U_x| dxdy, \iint |\text{ app } U_y| dxdy, \iint |\text{ app } V_x| dxdy, \iint |\text{ app } V_y| dxdy < \infty \)

3° $\text{app } U_x = \text{app } V_y$, $\text{app } U_y = -\text{app } V_x$ almost everywhere in $D$, then $f(z)$ is holomorphic in $D$.

From Fubini's theorem for almost $y \int_{y=y_0}^{y} \text{app } U_x \, dx$ and 1°) follows that $[U(x,y)]$ is function A. C. G. 8) We define $U(x,y) = \int_{y=y_0}^{x} \text{app } U_x(x,y) \, dx$, then $U - \tilde{U}$ is a function A. C. G, therefore $\tilde{U}_x = \text{app } \tilde{U}_y = \text{app } U_x$ almost everywhere with respect to $x$, so we have $U - \tilde{U} = \text{const.}$, it follows that $U(b) - U(a) = \tilde{U}(b) - \tilde{U}(a)$; $a > a_0$, after all we have $U(b) - U(a) = \int_{a}^{b} \text{app } U_x(x,y) \, dx$.

In the same way as in the proof of the theorem 1, for any square in $D$. $\int_{c}^{d} f(z) \, dz = \iint_{D} (\text{app } U_x - \text{app } V_y) \, dx dy + i \iint_{D} (\text{app } V_x + \text{app } U_y) \, dx dy = 0$.

**Proposition 2.** If $f(z)$ satisfies the following conditions

1° $\text{app } U_x$, $\text{app } U_y$, $\text{app } V_x$ and $\text{app } V_y$ exist except at most at enumerable point in $D$, and further 2° conditions $S$ is satisfied, then $f(z)$ is holomorphic.

Denote by $E(n_1, n_2)$ for any given $\varepsilon_0$ the set : $n_t$ are integers.

$$E \left[ \text{mes \cdot line } \frac{f(z + h) - f(z)}{h} \leq n_1; \ 0 < h < \frac{1}{n_1} \right] \geq (1 - \varepsilon_0) \frac{1}{n_1} : \ h = \text{real}$$

$$E \left[ \text{mes \cdot line } \frac{f(z + ih) - f(z)}{h} \leq n_2; \ 0 < h < \frac{1}{n_2} \geq (1 - \varepsilon_0) \frac{1}{n_2} \right].$$

If $f(z)$ is not holomorphic in $D$, we can find a portion $\Pi$ defined by $D'$ in which $E(n_1^0, n_2^0)$ is dense, and by taking limit, $\Pi$ is contained in the closure of a certain $E(n_1^0, n_2^0)$ completely. We term this operation $B$.

If $\Pi$ is defined by $D'$ from condition 1°) $\text{app } U_x$, $\text{app } U_y$, $\text{app } V_x$, and $\text{app } V_y$ exist, therefore, they are $\leq \text{Max}(n_1^0, n_2^0)$ in absolute value. $f(z)$ is regular, if $z \in D' - \Pi$.

From proposition 1 we conclude that $f(z)$ is holomorphic in $D$.

**10. Proposition 3.** If $f(x)$ is a continuous function defined in a closed interval $[a, b]$, and if there is a closed set $F \subseteq [a, b], I_i = (a_i, b_i)$ denoting the intervals contiguous satisfying the following conditions.

1°) $\int_{I_i} f'(z) \, dx = f(b_i) - f(a_i)$ for each interval and $\sum_{I_i} \int_{I_i} |f'(x)| \, dx < \infty$.

---

8) see τ).
2°) \( f'_r(x) \) exists except at most at enumerable set and \( \int \left| f'_r(x) \right| \, dx < \infty \), then \( f(b) - f(a) = \sum \int_{I_i} f'(x) \, dx + \int f'_r(x) \, dx \).

Proof. If \( x \in F \) and \( x \) is isolated from \( F \), \( f_F(x) \) loses its meaning, but the set where \( x \) is isolated, is at most enumerable, therefore \( f'_r(x) \) has finite value everywhere in \( F \) except at most enumerable set in \( F \), we define a function such as

\[
\bar{f}(x) = \frac{\lambda f(a_i) + \mu f(b_i)}{\lambda + \mu} \quad \text{if} \quad x \in I_i = (a_i, b_i), \quad x = \frac{\lambda a_i + \mu b_i}{\lambda + \mu}, \quad \lambda, \mu > 0.
\]

When \( |f'_r(x)| < K \); \( |K| < \infty \), there exists a sequence \( x_i \) converging to \( x \), \( x_i \in F \) and there is number \( \delta \) exists so that

\[
\text{if} \quad x_i \in (x \pm \delta) \cap F
\]

a) In the case when \( x_i, x \in F \) \( |x - x_i| < \delta \) follows \( K - \varepsilon < \frac{f(x_i) - f(x)}{x_i - x} \)

\[
< K + \varepsilon.
\]

b) In the case when \( x \in F \), and \( x_i \in F \)

b,1) \( F \ni x_i > x_i \) = lower bound of \( (x - \delta) \cap F \)

b,2) \( F \ni x_i < x_i \) = upper bound of \( (x + \delta) \cap F \), there exists a \( I_i = (a_i, b_i) \in x_i \)

from this it is clear \( \frac{\bar{f}(x_i) - \bar{f}(x)}{x_i - x} < K + \varepsilon. \)

2) If \( x \in F \) \( |f(x)| < M \) (because \( f(x) \) is continuous in closed interval, there exists an interval \( I_i = (a_i, b_i) \ni x_i, x_j \) therefore for \( x_i, x_j \)

\[
\left| \frac{f(x_i) - f(x_j)}{x_i - x_j} - \frac{f(b_i) - f(a_i)}{b_i - a_i} \right| \leq \frac{2M}{b_i - a_i} < \infty, \quad M = \max |f(x)|; \quad x \in [a, b].
\]

Finally all \( \bar{f}(x) \) has finite Dini's derivatives everywhere except at most enumerable set, from 2°) \( \bar{f}(x) \) is an absolutely continuous function, on the other hand \( \bar{f}'(x) = \bar{f}'(x) = f'_r(x) \) almost everywhere in \( F \), then

\[
f(b) - f(a) = \sum \int_{I_i} \bar{f}'(x) \, dx + \int f'(x) \, dx = \sum \int f'(x) \, dx + \int f'_r(x) \, dx.
\]

11. Theorem 2. \( f(z) \) is a continuous \(^9\) function in \( D \), and \( D \) is

expressed in the form \( D = \sum E_i + H \), where \( H \) is an enumerable set, and satisfies the following conditions.

1°) For each \( E_i \ni z \) two lines (fixed direction) denoted by \( \tau_i \), issuing from \( z \), correspond, and for \( z' \in E_i \setminus \tau_i \) and \( |z' - z| < \delta(z) \)

\[
\tau_i B_{E_i} = \lim_{\tau \to z} \frac{f(\xi) - f(z)}{\xi - z}
\]

exist except at most enumerable set in \( E_j \), and when two \( \tau_i B \) exist, \( \tau_i B_{E_j} \) almost everywhere in \( E_j \), and \( S \) is satisfied, then \( f(z) \) is holomorphic in \( D \). (Of course on \( \tau_i(z) \setminus E_j \), when \( z \) is isolated from \( \tau_i(x) \setminus E_j \), relative derivative loses its meaning)

Generality will not be lost by assuming that the two fixed directions are that of \( x \) and \( y \) axis. \( H_j \) denotes the set of \( E_j \) where (1°) is not satisfied.

Then \( D = \sum E_j + H_j + H \).

Denote by \( E_{\ast} \) the set \( E \) satisfying the following conditions

1°) \( E_{\ast} \left[ \left| \frac{f(z+h) - f(z)}{h} \right| < P \right] \) if \( z, z+h \in E_{\ast} \) : \( 0 < h < \frac{1}{P} \):

\( h = \) real or imaginary

2°) \( \text{dist}(z, \text{boundary of } D) \geq \frac{1}{P} \)

\( E_{\ast} = \sum_{\ast} E_{\ast} \), \( D = \sum_{\ast} E_{\ast} + H_{\ast} + H \).

If \( f(z) \) is not holomorphic in \( D \), by operation \( B \) we can find a portion \( \Pi \) defined by \( D' \) in which a certain \( E_{\ast} \) is dense, we conclude by taking limit of \( \tau_i(z_n) : z_n \in E_{\ast} \), \( \lim z_n = z \). For any \( z \in D' \setminus \Pi \) 1°) and 2°) is satisfied,

\( f(z) \) is regular : if \( z \in D' - \Pi \),

\[
\left| \frac{\partial f}{\partial x} \right| _{\Pi}, \left| \frac{\partial f}{\partial y} \right| _{\Pi} \leq P : \text{if } z \in \Pi .
\]

By using Fubini's theorem about \( S \) condition \( \int_{D' - \Pi} |f'(z)| \, dx \, dy < \infty \) and proposition 3, we conclude that for almost all \( y \)

\[
U(x, y) \equiv U(x_1, y) = \left[ \int_{x_1}^{x_2} U'_{D' - \Pi}(x, y) \, dx \right] + \int_{\Pi} U'(x, y) \, dx , \text{etc.}
\]
and further \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \), etc. almost everywhere in \( \Pi \). Finally we have
\[
\oint_{\partial} f(z) \, dz = 0
\]

12. Proposition 4. \( w = f(z) \) is approximately monogene except at most enumerable set and condition \( S \) is satisfied in \( D \), then \( f(z) \) is holomorphic in \( D \).

If \( f(z) \) is not holomorphic in \( D \), we can find by \( B \) operation a portion \( \Pi \) defined by \( D' \), there exist a certain \( \epsilon_0 \) and \( r_0 \) and \( M_0 \) not depending on \( z \in \Pi \).

\[
f(z) \text{ is regular, if } z \in D' - \Pi
\]

1°) \( \text{mes } | \frac{f(z + \epsilon e^{i\theta}) - f(z)}{\epsilon} | < \epsilon_0 \), \( 0 < \theta < 2\pi :\)

where \( |A| = M_0, \ h < h_0 < r_0 : \) if \( z \in \Pi \).

We have only to show that \( f(z) \) is holomorphic for any small square in \( D' \) for this purpose, we take a square with its diametre smaller than \( \frac{r_0}{2} \), then for \( z_1, z_2 \in \Pi \) we find a cercle \( C(z_1) \) and, \( C(z_2) \) their diametre \( |z_1 - z_2| \), in which

(1°) is satisfied and \( \text{mes } |C(z_1) \setminus C(z_2)| > \frac{\pi}{3} |z_1 - z_2|^2 < (1 - \epsilon_0) |z_1 - z_2|^2 \pi \) therefore there exists at least a point \( z_3 \in C(z_1) \setminus C(z_2) \)

\[
\left| \frac{f(z_1) - f(z_3)}{z_1 - z_3} \right| \leq P, \ \left| \frac{f(z_2) - f(z_3)}{z_2 - z_3} \right| \leq P
\]

and so \( \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \leq 2MP : \) if \( z_1, z_2 \in \Pi ; \ M = M(A, \epsilon_0) \)

On the other hand \( f(z) \) is approximately monogene

\[
f(z_1) - f(z_2) = (A_1 + iA_2)(x_2 - x_1) + (B_1 + iB_2)(y_2 - y_1) + \epsilon(z_2)|z_2 - z_1| : \]

\[
\lim_{z_2 \to z_1} \epsilon(z_2) = 0
\]

(approximately totally derivable)

but directions are fixed

\[
f(z_2) - f(z_1) = (A_1 + iA_2)(x_2 - x_1) + (B_1 + iB_2)(y_2 - y_1) + \epsilon(z_2)|z_2 - z_1|
\]

then we have \( (A_1 + iA_2) = \text{app } f_x, \ (B_1 + iB_2) = \text{app } f_y \) almost everywhere in \( \Pi \). Finally from the theorem 2, \( f(z) \) is holomorphic in \( D \).

§ 3

We give the simplest proof under a little change of the conditions of the theorem 1.
We denote by \( l(z) \) straight line passing through \( z \) and denote it by \( B(z) \) or \( \lim l(z) \).

**Theorem 3 (Menchoff)**

1. If \( f(z) \) is a continuous function with the following conditions.
   1°) To every point except at most at enumerable points, correspond two lines passing through \( z \), \( \langle l_1^0 \cap l_2^0 \rangle \equiv 0 \) (mod \( \pi \)).
   2°) \( B_1 = B_2 \).

Then \( f(z) \) is holomorphic in \( D \).

Or more generally \( \lim l_1 A \), \( \lim l_2 A < \infty \) and two sequences on them

\[
\lim_{n \to \infty} B(q_1^n) = \lim_{n \to \infty} B(q_2^n).
\]

We prove this theorem as an application of following Pompeiu's theorem. A complex function \( f(z) \), continuous in an open set \( D \), is regular in \( D \), if it is monogene at almost all the point \( D \) and if further

\[
\lim_{n \to \infty} \frac{|f(z+h) - f(z)|}{h} < \infty \quad \text{at each point except at most enumerable set.}
\]

**Proof.** It is not regular in \( D \) we can find as in the case of theorem 1, the portion \( \Pi \) defined by \( D' \) and followingly conditioned.

1°) \( f(z) \) is regular, if \( z \in D' - \Pi \)

2°) \( l_1^0, l_2^0 \) are fixed direction \( \langle l_1^0 \cap l_2^0 \rangle \leq \frac{1}{2NP} : N \geq 2 \)

3°) \( \frac{1}{P} - \frac{1}{NP} < \langle l_1^0 \cap l_2^0 \rangle \leq \pi - \frac{1}{P} + \frac{1}{NP} \)

\[ \text{dist (z, boundary of } D) \geq \frac{1}{P} \]

4°) \( \left| f(\xi) - f(z) \right| \leq P \) if \( \xi \in l(z) \), \( 0 \leq \left| \xi - z \right| \leq \frac{1}{P} : i = 1, 2 \)

If we associate a sector \( S(z) \) (fixed direction and fixed opening angle) to each point \( z \) of the plane set \( \Pi \), of which \( z \) is the vertex of the sector \( S(z) \). It is clear that the set of \( z \) which is isolated from \( S(z) \cap \Pi \) is at most enumerable.

Let \( R \) be a subset of \( \Pi \), which is isolated from \( \Pi \) in any one of four sectors, then \( R \) is at most enumerable.

**Lemma 1.** Let us denote by \( V_s(z) \) the circular neighbourhood of \( z \) with the centre at \( z \) and the radius \( s \).

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Sufficient Condition for a Function to be Holomorphic

\[ s < \frac{1}{2P} \times \sin \left( \frac{\theta}{P - NP} \right) , \]

then

\[ \lim_{z'' \to z} \frac{f(z'') - f(z)}{z'' - z} \text{ is bounded if } z'' \in \Pi \cap V_s(z) : z \in \Pi \cap \Pi' . \]

**Proof.** We take a point \( z'' \in \Pi \cap V_s(z) \), then exist two \( l_1(z'') \) and \( l_2(z'') \), which intersect with \( l_s(z) \) and \( l_1(z) \) at points \( p_1 \) and \( p_2 \), and denote the angle \( \theta_0 = \text{angle } z'' p_2 z \), \( \theta_1 = \text{angle } p_2 z z'' \) then

\[ 0 \leq \theta_1 < 1 \frac{1}{P} \frac{1}{PN} \]

denote the angle \( \theta_0 = \text{angle } z z'' \) then accordingly

\[ |z'' - p_2| + |p_2 - z| < |z'' - z| \left( \sin \frac{\theta_1 + \theta_2}{\sin \theta_3} \right) \left( 1 \frac{1}{P} \frac{1}{NP} \right) \]

\[ K_i = K(P, N) \]

We directly see that \( \frac{f(z'') - f(z)}{z'' - z} \leq P.M \) if \( z'' \in V_s(z) \cap \Pi \) in the same way as in Theorem 1, where \( M \) depends only on \( P \) and \( N \).

From that \( z \) is not contained in \( R \), there exists \( z' \) such as

\[ |z' - z| \leq s, \text{ } z' \in S_t(z) \cap \Pi \]

and two lines \( l(z') \) exist which intersect \( l(z) \) at \( p_1 \), and \( p_2 \) where \( S_t(z) \)

is a sector of which vertex is \( z \) and its half line is the half line \( l'_{z} \)

and \( l'_{p} \) and its opening angle sufficiently small given number \( \varepsilon_0 \).

Then, diameter of \( (zp_2z'p_1) \leq \frac{1}{P} \left( \frac{f(\xi) - f(z)}{\xi - z} \right) \leq PM : \) if \( \xi \)

lies on the circumference of \( (zp_2z'p_1) \), which can be proved as usual.

Finally \( \frac{f(z') - f(z)}{z' - z} \leq PM : \) if \( z' \in V_s(z) \cap (zp_2z'p_1) \).

In the long run we conclude that

\[ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \leq PM : \text{ if } z \in \Pi' \cap D'. \]

When \( z \) is contained in \( D' - \Pi', f(z) \) is regular, so

\[ \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \leq \infty \]
at every point except at most enumerable set, from condition 2°, \( f(z) \) must be monogene almost everywhere in \( D' \). By the theorem of Pompeiu \( f(z) \) is holomorphic in \( D' \).

Remark. It is clear that this method is applicable when \( K'' \) is under the condition that three lines issuing from \( z \) never lie on the same side of any line passing through \( z \).

When four lines issuing from \( z \), we can this apply without any satisfied further condition. It is important not that \( \lim B(z) \) exist, but that \( \lim A < \infty \).

15. We know what effect the number of \( \tau_i(z) \) of which \( \lim \tau_i A < \infty \) has on the condition of regularity.

1) two lines, condition \( S. \quad B_1 = B_2 \)
2) three lines condition \( S. \quad A_1 = A_2 = A_3 \)
3) two lines passing through or four lines issuing from \( z. \quad B_1 = B_2 \)
4) two lines (fixed direction) relative or approximate derivative conditions \( S. \)

§ 4

16. Invariance of angles. The properties studied in the preceding paragraphs are quantitative relations between the behaviours of \( z \) and \( w \) in the sense of segment's ratio or its extended meaning. Nevertheless on the contrary this property is not direct relation between them but it only tells us the indirectly, in the other word, it means the connection of quantatives (angles) defined by pairs \( (z, y) \) and \( (U, V) \).

Property \( K' \)

With \( z=z_0 \) three half lines \( \tau_i(z) : i=1.2.3 \) issuing from \( z_0 \) are associated and any Jordan curve \( J \) terminating in \( z_0 \) with one of \( \tau_i(z_0) \) as its tangent, has its image \( f(J) \) with a half line \( T_<(w_0) : w_0 = f(z_0) \) issuing from \( w_0 \) as its tangent in the \( w \)-plane.

\[ [\tau_1(z) \wedge \tau_2(z)] = [T_<(w) \wedge T_<(w)] \equiv 0 \mod \pi \quad i.f. = 1.2.3 \]

Menchoff proved the following theorem \(^{11}\).

**Theorem 4.** *If \( w = f(z) \) is univalent and continuous function defined in a domain of the \( z \)-plane and if it has \( K' \) at every point except at most enumerable point, then \( f(z) \) is holomorphic in \( D \).*

For the purpose to make this theorem remain valid, in the case when \( f(z) \) is not univalent, we take a little changed property \( K' \) as it follows.

Sufficient Conditions for a Function to be Holomorphic

Property \(K^r\kappa\)

With \(z = z_0\) three lines \(l_i(z)\) : \(i = 1, 2, 3\) passing through \(z\) are associated having its image \(f(l_i(z))\) in the \(w\)-plane which has a tangent \(T_i\) in the neighbourhood of \(w\) and at \(w = f(z)\)

\[
[l_i \wedge l_j] = [T_i \wedge T_j] \equiv 0 \pmod{\pi} \quad i, j = 1, 2, 3
\]

17. Theorem 4'. If \(w = f(z)\) is a continuous function which has \(K^r\kappa\) at every points except at most enumerable points, and further if condition \(S\) is satisfied in \(D\), then \(f(z)\) is holomorphic in \(D\).

Let us denote by \(T_i(w)\) the tangent of \(f(l_i(z))\) at \(w\) : \(i = 1, 2, 3\) and by \(G(P, N, n_1, n_2, n_3)\) the set conditioned followingly.

\[
\begin{align*}
1° & \quad - \frac{1}{2NP} \left( \frac{l_i(z)}{NP} \right) \left( \frac{n_i}{2NP} \right) - \frac{1}{2NP} \left( \frac{T_i \wedge n_i}{2NP} \right) \\
2° & \quad \frac{1}{P} \left( rac{l_i(z)}{NP} \right) \left( \frac{n_i}{2NP} \right) - \frac{1}{P} \left( \frac{T_i \wedge T_j}{2NP} \right) \\
3° & \quad [l_i \wedge l_j] = [T_i \wedge T_j] \\
4° & \quad - \frac{1}{2NP} \left( \frac{T_i(w)}{NP} \right) \left( \frac{T_j(w_0)}{NP} \right) - \frac{1}{2NP} \\
5° & \quad \text{dist (z, boundary of } D) \geq \frac{1}{P} \quad N \geq 4
\end{align*}
\]

Then \(D = \sum G(P, N, n_1, n_2, n_3) + H\)

where \(P, N, n_1, n_2, n_3\) are all integers, and \(H\) is enumerable set.

If \(f(z)\) where not holomorphic in \(D\), we can find the portion \(\Pi\) defined by a certain open set \(D'\), and in \(\Pi\) a certain \(G(P^0, N^0, n_1^0, n_2^0, n_3^0)\) is dense. In the case when \(z \in G_0 \cap \Pi\), \(l_i(z)\) are defined already, in the case when \(z \in G_0 \cap \Pi\), we can define \(l_i(z)\) by the limit of \(l_i(z_n)\): \(\lim z_n = z : z_n \in G_0 \cap \Pi\), then by the continuity conditions \(1°, \ldots, 5°\) are satisfied

\[
\begin{align*}
1° & \quad - \frac{1}{2NP} \left( \frac{\Delta_i \wedge l_i(z)}{2NP} \right) - \frac{1}{2NP} \left( \frac{\Delta_i \wedge T_i(w)}{2NP} \right) \\
2° & \quad \frac{1}{P} - \frac{1}{NP} \left( \frac{\Delta_i \wedge \Delta_j}{NP} \right) - \left( \frac{1}{P} - \frac{1}{NP} \right) \\
4° & \quad - \frac{1}{2NP} \left( \frac{T_i(w)}{NP} \right) \left( \frac{T_j(w_0)}{NP} \right) - \frac{1}{2NP} \\
\end{align*}
\]

where \(\Delta_i\) and \(\bar{\Delta}_i\) are all fixed directions in the \(z\) or \(w\)-plane respectively.
18. **Lemma 1.** \( f(z) \) is totally derivable almost everywhere in \( \Pi \).

Let us denote by \( \Delta_{1,2} \) the half line of the angle made by \( \Delta_1 \) and \( \Delta_2 \) (\( \Delta_i \) are fixed directions) which is named \( X \)-axis the other axis perpendicular to this axis will be named \( Y \)-axis, and denote by \( l(y) \) the line passing through \( y \) and parallel to \( X \)-axis. In the same way the half line of \( \Delta_1 \) and \( \Delta_2 \) and the other will be named \( U \) and \( V \)-axis, (this is possible by rotation of the coordinates).

**Remark 1.**

If \( |z_i - z_k| < \min \left( \frac{1}{P} \cos(X^\Delta_i), \frac{1}{P} \cos(X^\Delta_2) \right) \) and \( z_i, z_k \in l(y) \cap \Pi \)

then \( \tan(\Delta_2 - \frac{1}{NP}) \leq \frac{V(z_i) - V(z_k)}{U(z_i) - U(z_k)} \leq \tan(\Delta_1 + \frac{1}{NP}) \).

**Proof.** If it were not so, there is at least one point where the branches of \( l_{i}(z_k) \) and \( l_{2}(z_k) \) intersects. But their images \( f \) (branch of \( l_{i}(z_k) \)) and \( f \) (branch of \( l_{2}(z_k) \)) do not intersect, this is impossible.

This follows clearly that \( U(x, y) \) is monoton increasing function of \( x : x \in l(y) \cap \Pi \), accordingly if \( x \in l(y) \cap \Pi \), then \( U(x, y) \) and \( V(x, y) \) are functions of bounded variation on \( l(y) \cap \Pi \). But on the other hand from condition \( S \int_{D' - \Pi} |f'(z)|^2 \, dx \, dy < \infty \). We see directly that \( U(x, y) \) and \( V(x, y) \) are functions of bounded variation on \( l(y) \cap D' \) for almost \( y \), consequently \( (w=U+iV) \) is a rectifiable curve for almost \( y \) as a function of \( x \).

19. **Remark 2.** \( U \) and \( V \) are bounded variation, therefore they are derivable with respect to \( x \) almost everywhere in \( l(y) \cap D' \), and from remark 1

\[
\left| \frac{\partial V}{\partial x} \right| \leq M \left( \frac{\partial U}{\partial x} \right) \quad i=1,2
\]

almost everywhere in \( l(y) \cap \Pi \).

Let us denote by \( E_k \) the set satisfying the following condition on \( l(y) \cap \Pi \) and denote by \( E(y) \) the set \( l(y) \cap \Pi \)

\[
\tan \frac{K-1}{NP} \leq \frac{\partial x}{\partial U} \leq \tan \frac{K}{NP} \quad \text{if } z \in E_k : K < NP : N \geq 2
\]

\[
\text{mes } E(y) = |l(y) \cap \Pi| = \sum \text{mes } E_k.
\]

To prove the total derivability, we have only to show that
almost everywhere in $\Pi$, we assume that $\lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} \right| < \infty$ in a positive measure set $\Pi^0 : \Pi^0 \subseteq \Pi$.

$$\text{mes} |\Pi \cap \sum_y E(y)| = \sum_y \sum E_y = \text{mes} \Pi > \text{mes} \Pi^0 \geq d > 0,$$

therefore there is a certain $m$ such as least

$$\text{mes} |\Pi^0 \cap \sum_y E_m| > \frac{d}{NP}.$$

If $z \in \Pi^0 \cap \sum_y E_m$, then $\tan \frac{m-1}{NP} < \frac{\partial V}{\partial x} \leq \tan \frac{m}{NP}$.

By Egoroff's theorem for any positive number $\delta$. There exist $\delta$ and a closed subset $\Pi'$ such as

$$\text{mes} |\Pi^0 - \Pi'| < \epsilon$$

If $z \in \Pi'$ and $h$ is real number and $|h| < \delta$ then

$$-\frac{1}{NP} \leq \frac{m}{NP} (\frac{V(z+h) - V(z)}{U(z+h) - U(z)}) \leq \frac{1}{NP}.$$

From $\Pi'$, we take a set $\Pi^2$ which is linearly density with respect to any line $l(y)$ and by Egoroff's theorem we can find a subset $\Pi^2$ of $\Pi'$ such as

If length of $l(y) < \delta$, then $\frac{\text{mes} |l(y) \cap \Pi'|}{\text{mes} l(y)} > 1 - \frac{\epsilon}{2}$.

We denote by $\theta_1$ and $\theta_2$ the angles which is made $\Delta_1$ and $\Delta_2$ with $X$-axis we can assume that $-\frac{2}{NP} > \left( \frac{\theta_1 - \frac{m}{NP}}{2NP} \right)$, $\frac{\pi}{2} - \frac{2}{NP} > \left( \frac{m - \theta_2}{NP} \right)$

by choosing adequate $\Delta_1$, $\Delta_2$ among $\Delta_1$, $\Delta_2$, $\Delta_3$, and now let $\delta$ be smaller than

$$\min \left( \frac{1}{1-\epsilon} \frac{1}{P} \sin \left( \theta_1 - \frac{1}{NP} \right), \frac{1}{1-\epsilon} \frac{1}{P} \sin \left( \theta_2 + \frac{1}{NP} \right) \right)$$

20. Remark 3. Maximal and minimal quasi parallelogram in the $z$-plane with centre $z$ and radius $h$.

If $z \in \Pi^2$, then $\lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} \right| = \infty$. Therefore there exists $z_n = z+h$ such as $\left| \frac{f(z+h) - f(z)}{h} \right| \geq M$ for any large number $M$. 

We write the circle with centre at $z$ and radius

$$h \begin{cases} < \frac{1}{2P} \sin \left( \theta_1 + \frac{1}{2NP} \right) \\ < \frac{1}{2P} \sin \left( \theta_2 - \frac{1}{2NP} \right) \end{cases}.$$ 

We can find $\alpha_1$ and $\alpha'_1$ on $l(y) \cap \Pi$ satisfying the following conditions

$$\max \begin{cases} h \sec \left( \theta_2 + \frac{1}{2NP} \right) \\ h \sec \left( \theta_1 - \frac{1}{2NP} \right) \end{cases} \leq \begin{cases} |z - \alpha_1| \\ |z - \alpha'_1| \end{cases} \leq \max \begin{cases} h \sec \left( \theta_2 + \frac{1}{2NP} \right) \frac{1}{1 - \varepsilon} \\ h \sec \left( \theta_1 - \frac{1}{2NP} \right) \frac{1}{1 - \varepsilon} \end{cases}.$$ 

From $\alpha_1$ and $\alpha'_1$ we trace $l_1(\alpha_1)$ and $l_2(\alpha_1)$ and $l_1(\alpha'_1)$ and $l_2(\alpha'_1)$. These lines forms a quasi parallelogram. This will be called maximal quasi parallelogram $\square$ with centre at $z$ and radius $h$.

Next we can find $\beta_1$ and $\beta'_1$ on $l(y) \cap \Pi$ satisfying following conditions

$$\min \begin{cases} h \sec \left( \theta_2 - \frac{1}{2NP} \right) \\ h \sec \left( \theta_1 + \frac{1}{2NP} \right) \end{cases} \leq \begin{cases} |z - \beta_1| \\ |z - \beta'_1| \end{cases} \leq \min \begin{cases} \frac{1}{1 - \varepsilon} h \cos \left( \theta_2 - \frac{1}{2NP} \right) \\ \frac{1}{1 - \varepsilon} h \cos \left( \theta_1 + \frac{1}{2NP} \right) \end{cases}$$ 

and we trace $l_i(\beta)$ in the same manner as in the preceding, we call this quasi minimal parallelogram $\square$ with centre $z$ radius $h$.

Evidently

$$z_{\max} \leq \begin{cases} \square \leq \square \end{cases} \leq \begin{cases} K_1, \quad \text{area} \leq K_2 \end{cases} \quad (4)$$

$$K_1, K_2 = K(P, N)$$

21. Remark 4. Outer minimal, and outest parallelogram in the $w$-plane and their property.

In general, let us denote the image of $p$ by $\bar{p}$ in the $w$-plane.
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From \( \bar{\alpha}_1 \) and \( \bar{\alpha}'_1 \) we trace lines \( L_i(\bar{\alpha}_1) \) and \( L_i(\bar{\alpha}'_1) \) etc,

- direction \( L_1(\bar{\alpha}_1) = \bar{\Delta}_1 + \frac{1}{2NP} \)
- direction \( L_2(\bar{\alpha}_1) = \bar{\Delta}_2 - \frac{1}{2NP} \)

- direction \( L_1(\bar{\alpha}'_1) = \pi - \bar{\Delta}_1 - \frac{1}{2NP} \)
- direction \( L_2(\bar{\alpha}'_1) = \pi - \bar{\Delta}_2 + \frac{1}{2NP} \)

These \( L_i \) form a parallelogram named outer \( \square \). From \( \bar{\alpha}_1 \) and \( \bar{\alpha}'_1 \) we trace lines \( L_i \) so that

- direction \( L_1(\bar{\alpha}_1) = \bar{\Delta}_1 - \frac{1}{2NP} \)
- direction \( L_2(\bar{\alpha}_1) = \bar{\Delta}_2 + \frac{1}{2NP} \)

- direction \( L_1(\bar{\alpha}'_1) = \pi - \bar{\Delta}_1 + \frac{1}{2NP} \)
- direction \( L_2(\bar{\alpha}'_1) = \pi - \bar{\Delta}_2 - \frac{1}{2NP} \)

This is named outer minimal parallelogram \( \square \).

As \( \alpha_1 \cdot \alpha'_1 \in \Pi \). So \( \tan \frac{m-1}{NP} \leq \frac{\partial V}{\partial U} \leq \tan \frac{m}{NP} \), \( \bar{\beta}_1, \bar{\beta}'_1 \in \square \).

From \( \bar{\alpha}_1, \bar{\alpha}'_1 \) we make images \( f(l_i(\alpha_i)) \), etc this forms a quasi parallelogram with four curves \( \square (\alpha_1, \alpha'_1) \).

It is evident \( \square \) \( \square (\alpha_1, \alpha'_1) \subseteq \square \). and from \( \frac{\pi}{2} - \frac{2}{NP} > \left( \theta_1 - \frac{m}{NP} \right) > \frac{2}{NP} \), \( \frac{\pi}{2} - \frac{2}{NP} > \left( \frac{m}{NP} - \theta_2 \right) > \frac{2}{NP} \) then

\[
\frac{1}{NP} [L_i \cdot \bar{\alpha}_1 \bar{\alpha}'_1], \text{ it follows that area } \square \text{, and area } \square = M |\bar{\alpha}_1 \bar{\alpha}'_1|^2;
\]

where

\[
0 < m^*(NP) \leq m^*(NP,m) \leq M^*(NP,m) \leq M^*(NP) \geq +\infty
\]

If \( z_n \in \Pi \) then \( w_n \in \square \) this is proved easily as in remark 1.

22. Remark 5.
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\[ \text{area } \square (\alpha, \bar{\alpha}) > K_3 \]

Case 1 \( z_n \in \Pi \cap \square - \square \), then \( f(z_n) \subset \square (\alpha_1, \bar{\alpha}_1') \)
because to \( z_n \) \( f(l(z_n)) \) correspond which must intersect the peripherie of \( \square (\alpha_1, \bar{\alpha}_1') \) to outer side

Case 2 \( z_n \in \Pi \cap \square - \square \)

\( \frac{f(z_n) - f(z)}{z_n - z} \) is regular, therefore the maximum of this absolute value is attained at the point \( p \) of \( \Pi \) or the peripherie of \( \square - \square \) therefore from case 1 or 2. There exists a point \( \xi_0 \) on the peripherie of \( \square \) such as \( |f(\xi_0) - f(z)| \geq M|\xi_0 - z| \), but in the \( z \)-plane \( |\xi_0 - z| > K_3 h \), or at a point of \( \Pi \) (this is case 1), accordingly in \( \square \), there are two point \( z, \xi_0 \) such as \( f(\xi_0) \) and \( f(z) \) \( \in \square, \) \( |f(\xi_0) - f(z)| > Mh \), this follows that

\[ \text{area of } \square > K_4 h^2 M : K_3, \quad K_4 : K_t = K(P, N) \]

23. Remark 6. If two maximal quasi parallelogram has no point in common in the \( z \)-plane, then corresponding two minimal outer parallelogram has no point in common in the \( w \)-plane,

Case 1 \( \square \), lies on one side of \( l_i(a_i) \). Let such \( l_i \) be \( l_i(a_i) \) then \( A_iB_i, C_iD_i \) lies on one side of \( A_iB_i \), if it were not so \( d_j \). \( A_iB_i \), opposite side, then \( a_i d_j \) intersect with \( l_i(a_i) \) or \( l_i(b_i) \), but \( A_iD_i \) or \( C_iD_i \) cannot intersects with \( A_iB_i \) or \( C_iD_i \) on its extension. This is impossible. Where \( A_i = f(a_i) \) etc.

Case 2 (not case 1) in this case, we can prove in the same way in 1 using the continuity of angle. Let \( \square \), be not contained in the angle \( a_i d_i c_i \), then \( D_i \) lies \( C_iD_iB_iD_i \) same side, therefore \( D_i \) is not contained in the angle \( A_iD_iB_i \), therefore minimal \( \square \), never overlappe.

Fig. 5

\[ \text{area } \square \]
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But

$$\frac{\text{area}}{\text{area}} < K_5$$

and from 4 area \( \geq K_6 h^2 \)

finally

$$\frac{\text{area}}{\text{area}} \geq K_7 M.$$ 

By Vitali’s covering theorem, we can find a sequence of \( \max z \) not overlapping each other and

$$\sum_{\max z} \text{mes} \geq d_0$$

\( \text{mes} f(D) > \text{mes} f(D') \sum_{\max z} \geq K_7 M \frac{d_0}{2} \), but \( M \to \infty \), this is a contradiction.

24. Lemma 2. If \( f(z) \) is totally derivable at \( z=z_0 \) and satisfies \( K' \) then \( f(z) \) is monogene at \( z=z_0 \)

\[ f(z)-f(z_0) = (A_1+iA_2)(x-x_0)+(B_1+iB_2)(y-y_0)+\varepsilon(z) |z-z_0| \]

\[ \lim_{z \to z_0} \varepsilon(z) = 0 \]

\[ \tan \Theta = \frac{A_2(x-x_0)+B_2(y-y_0)}{A_1(x-x_0)+B_1(y-y_0)} = \frac{A_2+B_2 \tan \theta_1}{A_1+B_1 \tan \theta_1} \]

\[ \tan \Theta - \tan \theta = \text{const for } \theta_i = 1, 2, 3. \]

Then we easily have \( A_1 = B_2 \), \( A_2 = -B_1 \).

25. Lemma 3. \( f(z) \) has property \( N \) on \( l(y) \) for almost all \( y \).

If it were not so there exists a positive measure set \( G \) on \( \Delta_z \) such that, for any \( y \in G \), \( |f(z)| \) are rectifiable and on which \( f(z) \) has not \( N \), this fact follows that there exists a set \( q(y) \) for line mes \( |q(y)| = 0 \) but \( f(q(y)) \) has line measure \( >0 \). By Lusin’s theorem there exists a such a perfect set as mes line \( q(y) \) of which any portion \( q \) of it line mes \( f(q(y)) > 0 \), of course \( q(y) \subseteq \Pi \) for \( D' - \Pi \ni z \), \( f(z) \) is regular accordingly absolutely continuous.

If \( z_k, z'_k \in l \cap \Pi \) and \( |z_k-z'_k| < \frac{1}{2P} \), then \( [w_k, w'_k \Delta_1] \) is contained in

\[ \left[ \begin{array}{c} \theta_1 + \frac{1}{2NP} \theta_2 \end{array} \right] \]

Let us denote the set of \( y \) such as

\[ G_m = \left\{ \lim \text{mes} f \left( \frac{E}{NP} \right) m \leq \left[ X-\text{axis} \wedge \tan^{-1} \frac{\partial V}{\partial x} \right] m + \frac{1}{NP} \right\} \lambda : \]

\( \text{mes} f(q(y)) = \lambda \).
Then there exists at least a set such as outer mes $G^w > \mu > 0$, which is denoted by $G_y$.

For any $y \in G_y$, let us divide $l(y)$ in equal length segments $\delta_i$: \( \delta_i \cap \delta_j = 0 \) and denote by $z_k$, $z_k'$ the ends of $\delta_i \cap \Pi$ and construct the parallelogram $\square$ formed by $l_i(z_k)$, $l_i(z_k')$ for $i = 1, 2, \ldots$. From mes $q(y) = 0$ follows $\sum_1^p \delta_i < \frac{\lambda}{A}$ for any large number $A$, and if $w_k = f(z_k)$, $w_k' = f(z_k')$, \( |w_k - w_k'| = \lambda_k \) are denoted then $\sum_1^p |w_k - w_k'| > \frac{\lambda}{2NP}$ for sufficiently large $p$, and

$$\frac{1}{NP} < [w_k - w_k' \cap T_m] < \frac{1}{NP} : \text{direction } T_m = \frac{m}{NP}.$$  

From the construction of $f(\square)$, we see that $f(\square)$ is a quasi parallelogram in the $w$-plane which has outer minimal parallelogram in its interior. These minimal parallelograms $\square_{\text{mini}}$ never overlapp, when corresponding maxima $\square_{\text{mini}}^w$ have no common point in the $z$-plane (see Lemma 1).

$$\text{area of } \sum_{\text{mini}} \square_{\text{mini}}^w > C\lambda_k^2 \quad (C \text{ depends only on } P \text{ and } N)$$

$$\sum_1^p \lambda_k > \frac{\lambda}{2NP} \quad \text{for sufficiently large } p.$$  

$\sum'$ and $\sum''$ means the summation over $k$ satisfying (1) or (2)

$$\lambda_k \geq \frac{A}{4NP} \delta_k(y) \quad (1)$$

$$\lambda_k \leq \frac{A}{4NP} \delta_k(y) \quad (2)$$

$$\frac{\lambda}{2NP} < \sum' = \sum' + \sum'' \quad \text{then } \sum' > \frac{\lambda}{4NP}$$

area of minimal parallelogram $\square_{\text{mini}}^w > \lambda_k^2 C \left(\frac{A}{4NP} \delta_k\right)^2 C$

$$\sum_1^p \text{area of } \square_{\text{mini}}^w > \sum' > \sum' \left(\frac{A}{4NP} \delta_k\right)^2 C > C \left(\frac{A}{4NP} \delta_k\right)^2 \frac{CA}{16NP} \delta_k \lambda$$

We denote by $s(K)$ the projection of parallelogram of which the diagonal is $z_k, z_k' = \delta_k$ on $\Delta_2$.

We can find a sequence of intervals $I_i$ has no common point each other on $\Delta_2$.
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\[ \sum I_i > \frac{\mu}{8} \] follows \[ \sum s_k > \frac{\mu}{8} \] for large number \( m \).

This operation will be used for each \( I_i \) then we have

\[ \sum \sum \text{area of } \Box_{\min w} > C \sum \frac{A}{64NP} \delta \lambda > \frac{CA \lambda \mu}{128NP}. \]

This is a contradiction for \( A \to \infty \) and \( \text{mes } |f(D)| < +\infty \), the same fact occurs for another \( I(x) \), accordingly we can conclude

\[ U(x_1) - U(x_2) = \int_{x_1}^{x_2} \frac{\partial U}{\partial x} \, dx \]

for almost all \( y \), etc.,

then we can prove that \( \int_c f(z) \, dz \to 0 \) in the same manner as used in Theorem 1.

\[ \text{§ 5} \]

25. When the topological property of a regular function is characterized, this is called an inner transformation satisfying the following two fundamental conditions.

1° Light transformation: for any \( w \in f(D) \), \( f^{-1}(w) \) is totally disconnected, then \( f(z) \) is called a light transformation.

2° Open transformation: any open set is transformed into an open set.

Property \( K'' \). If at \( z = z_0 \), \( f(z) \) satisfies \( K' \) and further in the neighbourhood of \( z \), any Jordan curve issuing from \( z \) contained in the sector \( S_{\tau_i}(z) \) formed \( \tau_i \) and \( \tau_j \), has its image in the \( w \)-plane in the corresponding sector \( S_{\tau_j} \) which is not whole direction, then we call that \( f(z) \) has \( K'' \) at \( z = z_0 \).

In regarding that \( f(\tau_i(z)) \) has a tangent at \( w : w = f(z) \), there exists such \( \tau_0 \); if \( |\zeta - z| < \tau_0 : \zeta \in \tau_i(z) \) then \( f(\zeta) = f(z) \). We define \( \overline{S_{\tau_j}} = 2\pi - \overline{S_{\tau_j}} \) and \( T_{\tau_i} \) is the half line of \( T_i \) and \( T_j \).

We denote by \( G(N, P, n_1, n_2, n_3) \) the set satisfying the following conditions

1° \[ \frac{1}{2NP} \leq \left[ \tau_i \wedge \frac{n_i}{2NP} \right] < \pi - \frac{1}{2NP} \]

2° \[ \frac{1}{P} \leq \left[ \tau_i \wedge \tau_j \right] < \pi - \frac{1}{P} \]

3° \[ [T_i \wedge T_j] = [\tau_i \wedge \tau_j] \]
26. Menchoff proved the following theorem\textsuperscript{12)}

**Theorem 5.** If \( f(z) \) is topological and direct in \( D \), and \( \lim \arg \frac{f(\zeta) - f(z)}{\zeta - z} \) exists at every point except at most enumerable points, then \( f(z) \) is holomorphic in \( D \).

**Theorem 5'.** If \( f(z) \) is continuous (not necessarily univalent) and \( K'' \) is satisfied at every point except at most enumerable points, then \( f(z) \) is holomorphic in \( D \).

**Lemma 1.** \( f(z) \) is a light transformation in \( D \).

If \( f(z) \) were not so, there exists at least such a point of \( w \) as \( f^{-1}(w) \) is a continuum being clearly closed. A continuum is a perfect set, then there exists a portion \( \Pi \) of the continuum in which a certain \( G_0 \) is dense, therefore there is sequence of points converging to \( p \), and then there is also the subsequence of points converging to \( p \) in certain sector \( S(p) \) with the opening angle smaller than \( \frac{1}{2N}\pi \) and the vertex is \( p \). If we denote by \( q_{i,i+1} \) the intersection point of \( \tau_1(p_i) \) and \( \tau_2(p_{i+1}) \), then there exists at least a pair of \( p_i, p_{i+1} \) in \( S(p) \) satisfying conditions:

1° \( p_i, p_{i+1} \in S(p) \)

2° \( \text{dist } |q_{i,i+1}p_i| < \frac{1}{N}; \text{dist } |q_{i,i+1}p_{i+1}| < \frac{1}{N} \)

3° \( f(p_i) = f(q_{i,i+1}), f(p_{i+1}) = f(q_{i,i+1}) \)

4°. If length of \( \tau_1(p_i), \tau_2(p_{i+1}) < \frac{1}{N} \), then \( f(\tau_1(p_i)) \subset \overline{S}_i(f(p_i)) \)

\( f(\tau_2(p_{i+1})) \subset \overline{S}_i(f(p)), \) where the opening angle of \( \overline{S}_i \) is \( \frac{1}{N}\pi \)

and the half line of \( \overline{S}_i \) is \( T(\omega) = f(p) \) respectively.

But from \( f(p_i) = f(p_{i+1}) = f(p) \):

\( f(q_{i,i+1}) \subset \overline{S}_1 \cap \overline{S}_2 = f(p). \)

\( f(q_{i,i+1}) = f(p_{i+1}) = f(p_i) = f(p) \)

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This is a contradiction.

If \( f(z) \) is not holomorphic in \( D \), we can find a portion \( \Pi \) defined by \( D' \) which is completely contained in the closure of certain \( G_0 \).

**Lemma 2.** \( f(z) \) is an open transformation in \( D' \).

If \( z \in D' - \Pi \), \( f(z) \) is regular, therefore if \( f(z) \) were not an open transformation, then there exists such a point \( p \in \Pi \cap D' \) and an open set \( G \) as \( p \in \) interior of \( G \), and \( f(p) \in \) boundary of \( f(G) \).

We take a neighbourhood \( V(p) \) of \( p \): \( \text{dia}\ V(p) \leq \frac{1}{\delta} \); \( V(p) \subseteq G \subseteq D' \). Since \( f(z) \) is a light transformation \( f^{-1}f(p) \) is closed and disconnected. We take 3 points a. b. c. on \( \tau_i(p) \cap V(p) \) complement of \( f^{-1}f(p) \); \( i=1, 2, 3 \) and connect by the \( a, b, c \) in \( V(p) \cap \bar{S}_{i1}(p) \) complement of \( f^{-1}f(p) \), and so on about \( b, c \) and \( a, b, c \) in \( \bar{S}_{i2}(p), \bar{S}_{i3}(p) \) respectively to make a closed curve \( C \), then it is clear that

\[
\text{dist} (f(C), f(p)) \geq \delta \geq 0
\]

the order of \( f(C) \) with respect to \( f(p) \) is 1.

Hence \( f(p) \in \) boundary of \( f(G) \), then there exists another point \( q \) and another neighbourhood \( V'(f(p)) \); \( \text{dia}\ V'(f(p)) \leq \frac{\delta}{2} \), \( V'(f(p)) \ni q : f(G) \ni q \); \( \text{dist} (f(p), q) = \epsilon \leq \frac{\delta}{4} \), then

the order of \( f(C) \) with respect to \( q \) is 1.

In \( V(p) \) we deform continuously \( C \) into \( C' \); so that \( \text{dia}\ f(C') \leq \frac{\epsilon}{4} \) and enclosing \( p \), then

the order of \( f(C') \) with respect to \( q \) is 0.

This shows that \( q \) is covered by the schar of images of curves from \( C \) to \( C' \) in this deforming process, which contradicts that \( q \in f(G) \).

As \( f(z) \) is an inner transformation in \( D' \), therefore it is locally univalent and topological, consequently theorem 4 is applicable locally except enumerable points (branch point), finally \( f(z) \) is holomorphic in \( D \).

Remark. Theorem 5 is clearly contained in Theorem 5' therefore the condition of univalency of Menchoff's theorem is surplus.

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