Miebach, C.
Osaka J. Math.
47 (2010), 331–352

QUOTIENTS OF BOUNDED HOMOGENEOUS DOMAINS BY CYCLIC GROUPS

CHRISTIAN MIEBACH

(Received August 15, 2008)

Abstract

Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and let φ be an automorphism of D which generates a discrete subgroup Γ of $\mathrm{Aut}_{\mathcal{O}}(D)$. It is shown that the complex space D/Γ is Stein.

1. Introduction

Let $D \subset \mathbb{C}^n$ be a bounded domain of holomorphy and let φ be an automorphism of D such that the cyclic group $\Gamma := \langle\varphi\rangle := \{\varphi^k;\, k \in \mathbb{Z}\}$ is a discrete subgroup of the automorphism group $\mathrm{Aut}_{\mathcal{O}}(D)$. It follows that Γ acts properly on D and hence that the quotient $X := D/\Gamma$ is a complex space. In this situation one would like to know conditions on D or φ which guarantee that X is a Stein space.

Since the group Γ is cyclic, it is either finite or isomorphic to $\mathbb{Z}$. In the first case it is a classical result that Steinness of D implies Steinness of X. Therefore we assume that Γ is infinite cyclic. In the case that D is biholomorphically equivalent to the unit ball $\mathbb{B}_n$ it is proven in [4] and [5] that $X = D/\langle\varphi\rangle$ is Stein for hyperbolic and parabolic automorphisms φ. We will generalize this result to arbitrary bounded homogeneous domains.

Theorem. *Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain. Let φ be an automorphism of D such that the group $\Gamma = \langle\varphi\rangle$ is a discrete subgroup of $\mathrm{Aut}_{\mathcal{O}}(D)$. Then the quotient $X = D/\Gamma$ is a Stein space.*

The main steps of the proof are as follows. Since the group $\mathrm{Aut}_{\mathcal{O}}(D)$ has only finitely many connected components, we may assume that φ is contained in $G = \mathrm{Aut}_{\mathcal{O}}(D)^0$. By Kaneyuki's theorem the group G is isomorphic to the identity component of a real-algebraic group. Hence, every element $\varphi \in G$ may be written as $\varphi = \varphi_{\mathsf{e}}\varphi_{\mathsf{h}}\varphi_{\mathsf{u}}$ where φ_{e} is elliptic, φ_{h} is hyperbolic, φ_{u} is unipotent and where these elements commute. It can be shown that the group $\Gamma' := \langle\varphi_{\mathsf{h}}\varphi_{\mathsf{u}}\rangle$ is again discrete in G. Since the groups Γ and Γ' differ by the compact torus generated by φ_{e}, the quotient $X' = D/\Gamma'$ is Stein if and only if X

2000 Mathematics Subject Classification. 32M10, 32E10.
The author is supported by SFB/TR 12 of the Deutsche Forschungsgemeinschaft.

is Stein. Consequently we may work with the group Γ' which has the advantage of being contained in a maximal split solvable subgroup S of G which acts simply transitively on D. Exploiting the structure theory of S we obtain the existence of an S-equivariant holomorphic submersion $\pi : D \to D'$ onto a bounded homogeneous domain D' whose fibers are biholomorphically equivalent to the unit ball $\mathbb{B}_m$. If Γ' acts properly on D' we are in position to use an inductive argument to prove Steinness of X while if Γ' stabilizes every π-fiber we use the fact that the quotients $\mathbb{B}_m/\Gamma'$ are already known to be Stein.

This paper is organized as follows. In the first section we provide the necessary background on bounded homogeneous domains and their automorphism groups. In the second section we establish the existence of a Jordan–Chevalley decomposition in G and reduce the problem to discrete subgroups of S. In Section 3 we study in detail the unit ball $\mathbb{B}_n$ and obtain a new proof of the fact that $\mathbb{B}_n/\Gamma$ is Stein. Afterwards we prove the existence of the S-equivariant submersion $\pi : D \to D'$ which allows us to prove the main result in the last section.

2. Background on bounded homogeneous domains

We review several facts from the theory of bounded homogeneous domains. For further details we refer the reader to [20] and [15] and the references therein.

2.1. The automorphism group of a bounded homogeneous domain. Let $D \subset \mathbb{C}^n$ be a bounded domain. A theorem of H. Cartan ([3]) states that the group $\mathrm{Aut}_{\mathcal{O}}(D)$ of holomorphic automorphisms of D is a real Lie group with respect to the compact open topology such that its natural action on D is differentiable and proper. We write G for the connected component of $\mathrm{Aut}_{\mathcal{O}}(D)$ which contains the identity. We identify the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with the Lie algebra of complete holomorphic vector fields on D.

DEFINITION 2.1. The bounded domain D is called homogeneous if $\mathrm{Aut}_{\mathcal{O}}(D)$ acts transitively on it.

REMARK. (1) Let D be a bounded homogeneous domain and let $z_0 \in D$ be a base point. Since $D \cong \mathrm{Aut}_{\mathcal{O}}(D)/\mathrm{Aut}_{\mathcal{O}}(D)_{z_0}$ is connected, the (compact) isotropy group $\mathrm{Aut}_{\mathcal{O}}(D)_{z_0}$ meets every connected component of $\mathrm{Aut}_{\mathcal{O}}(D)$. This shows that $\mathrm{Aut}_{\mathcal{O}}(D)$ has at most finitely many connected components.
(2) If D is homogeneous, then $G = \mathrm{Aut}_{\mathcal{O}}(D)^0$ acts transitively on D, too.

From now on we assume that the bounded domain $D \subset \mathbb{C}^n$ is homogeneous. It follows from [2] that the group G is semi-simple (and then in particular real-algebraic) if and only if D is symmetric. For arbitrary homogeneous domains the group G is semi-algebraic by Kaneyuki's theorem.

Theorem 2.2 ([14]). *There exists a faithful representation ρ of G such that $\rho(G) \subset \mathrm{GL}(N, \mathbb{R})$ is the identity component of a real-algebraic group. In particular, $\mathfrak{g}$ is isomorphic to an algebraic Lie algebra.*

Recall that a real Lie algebra $\mathfrak{s}$ is called split solvable if it is solvable and if the eigenvalues of $\mathrm{ad}(\xi)$ are real for every $\xi \in \mathfrak{s}$. A Lie group is called split solvable if it is simply-connected and if its Lie algebra is split solvable. If G is semi-simple, the Iwasawa decomposition $K \times A \times N \to G$ exhibits G as diffeomorphic to the product of its maximal compact subgroup K and its maximal split solvable subgroup $S := AN \cong A \ltimes N$. The following theorem of Vinberg generalizes this decomposition to the group $G = \mathrm{Aut}_{\mathcal{O}}(D)^0$ for arbitrary bounded homogeneous domains D.

Theorem 2.3 ([25]). *Let H be the connected component of a real-algebraic group. Then there exist a maximal compact subgroup K and a maximal split solvable subgroup S of H such that the map $K \times S \to H$, $(k, s) \mapsto ks$, is a diffeomorphism. Each maximal split solvable subgroup of H is conjugate to S by an inner automorphism of H.*

REMARK. Let $K \times S \to G$ be the decomposition of G from Theorem 2.3. Then S acts simply transitively on D.

For later use we collect some properties of split solvable Lie groups.

Theorem 2.4. *Let S be a split solvable Lie group.*
(1) *The group S is isomorphic to a closed subgroup of the group of upper triangular matrices in $\mathrm{GL}(N, \mathbb{R})$.*
(2) *The exponential map $\exp\colon \mathfrak{s} \to S$ is a diffeomorphism.*
(3) *Every connected subgroup of S is closed and simply-connected.*
(4) *For each element $g \in S$ the group $\{g^k;\ k \in \mathbb{Z}\}$ is a discrete subgroup of S isomorphic to $\mathbb{Z}$.*
(5) *Let $S' \subset S$ be a connected subgroup and let $(S')^{\mathbb{C}} \subset S^{\mathbb{C}}$ be their universal complexifications in the sense of* [10]. *Then the homogeneous space $S^{\mathbb{C}}/(S')^{\mathbb{C}}$ is biholomorphic to $\mathbb{C}^{\dim S - \dim S'}$.*

Proof. The first three statements are classical (see for example [26]). The fourth assertion is a direct consequence of the second one. A proof of the last assertion can be found in [9]. □

2.2. Siegel domains and the grading of $\mathfrak{g}$. In this subsection we will describe the notion of Siegel domains of the first and of the second kind. Our motivation for the study of these domains comes from the fact that each bounded homogeneous domain can be realized as a Siegel domain ([24]). In addition we discuss the grading of $\mathfrak{g}$ which has been introduced in [16].

Let V be a finite-dimensional real vector space and let $\Omega \subset V$ be a regular cone, i.e. an open convex cone which does not contain any affine line.

DEFINITION 2.5. The tube domain $D := D(\Omega) := \{z \in V^{\mathbb{C}};\ \mathrm{Im}(z) \in \Omega\} = V + i\Omega$ is called the Siegel domain of the first kind associated with Ω.

REMARK. The assumption that D is a tube domain over a regular cone is quite strong. Although the unit ball in $\mathbb{C}^n$ is biholomorphically equivalent to a tube domain over a convex domain in $\mathbb{R}^n$, it can not be realized as a Siegel domain of the first kind.

The automorphism group $G(\Omega)$ of Ω is defined by

$$G(\Omega) := \{g \in \mathrm{GL}(V);\ g(\Omega) = \Omega\}.$$

Since the condition $g(\Omega) = \Omega$ is equivalent to $g(\overline{\Omega}) = \overline{\Omega}$, the group $G(\Omega)$ is closed in $\mathrm{GL}(V)$ and hence a Lie group. We embed $G(\Omega)$ into the automorphism group of $D = D(\Omega)$ by $g \mapsto \varphi_g$ with $\varphi_g(z) = gz$.

Let W be a finite-dimensional complex vector space. A map $\Phi\colon W \times W \to V^{\mathbb{C}}$ is called Ω-Hermitian if the following holds:

(1) For all $w' \in W$ the map $w \mapsto \Phi(w, w')$ is complex-linear.
(2) We have $\Phi(w', w) = \overline{\Phi(w, w')}$ for all $w, w' \in W$.
(3) We have $\Phi(w, w) \in \overline{\Omega}$ for all $w \in W$, and $\Phi(w, w) = 0$ if and only if $w = 0$.

REMARK. If $V = \mathbb{R}$ and $\Omega = \mathbb{R}^{>0}$, then an Ω-Hermitian form is the same as a positive definite Hermitian form on W.

DEFINITION 2.6. Given Ω and Φ as above, the domain

$$D := D(\Omega, \Phi) := \{(z, w) \in V^{\mathbb{C}} \times W;\ \mathrm{Im}(z) - \Phi(w, w) \in \Omega\}$$

is called the Siegel domain of the second kind associated to Ω and Φ.

Proposition 2.7. *Every Siegel domain of the first or second kind is convex and biholomorphically equivalent to a bounded domain. Hence, each Siegel domain D is a domain of holomorphy and its automorphism group is a real Lie group acting properly on D.*

Proof. Convexity of Siegel domains is elementary to check. For a proof of the fact that D is biholomorphically equivalent to a bounded domain we refer the reader to [20]. □

Theorem 2.8 ([24]). *Every bounded homogeneous domain can be realized as a Siegel domain of either the first or the second kind.*

Let $D = D(\Omega, \Phi)$ be a Siegel domain. As usual we write G for the connected component of the identity in $\mathrm{Aut}_{\mathcal{O}}(D)$. Let us introduce linear coordinates z_k, $1 \le k \le \dim_{\mathbb{C}} V^{\mathbb{C}}$, in $V^{\mathbb{C}}$ and w_α, $1 \le \alpha \le \dim_{\mathbb{C}} W$, in W. It follows from the definition that $\mathfrak{g}$ contains the vector field

$$\delta := \sum_k z_k \frac{\partial}{\partial z_k} + \frac{1}{2} \sum_\alpha w_\alpha \frac{\partial}{\partial w_\alpha}.$$

Theorem 2.9 ([16]). *The Lie algebra $\mathfrak{g}$ admits a decomposition*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_\lambda$ is the eigenspace of $\mathrm{ad}(\delta)$ for the eigenvalue λ. Then the following holds.
(1) *We have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ for all $\lambda, \mu \in \{\pm 1, \pm 1/2, 0\}$.*
(2) *The translation vector fields $\partial/\partial z_k$, $1 \le k \le \dim_{\mathbb{C}} V^{\mathbb{C}}$, form a basis of $\mathfrak{g}_{-1}$. Consequently, we have $\dim \mathfrak{g}_{-1} = \dim_{\mathbb{C}} V^{\mathbb{C}}$.*
(3) *The elements of $\mathfrak{g}_{-1/2}$ are of the form*

$$2i \sum_k \Phi_k(w, c) \frac{\partial}{\partial z_k} + \sum_\alpha c_\alpha \frac{\partial}{\partial w_\alpha} \quad (c \in \mathbb{C}^{\dim_{\mathbb{C}} W}).$$

Consequently, $\dim \mathfrak{g}_{-1/2} = 2 \dim_{\mathbb{C}} W$, and $\mathfrak{g}_{-1/2} = \{0\}$ if and only if D is a Siegel domain of the first kind.
(4) *The Lie subalgebra $\mathfrak{g}_0$ consists of all elements of the form*

$$\sum_{k,l} a_{kl} z_k \frac{\partial}{\partial z_l} + \sum_{\alpha,\beta} b_{\alpha\beta} w_\alpha \frac{\partial}{\partial w_\beta},$$

where the matrix $A := (a_{kl})$ lies in the Lie algebra of $G(\Omega)$ and $B := (b_{\alpha\beta}) \in \mathfrak{gl}(W)$ fulfills

$$A\Phi(w, w') = \Phi(Bw, w') + \Phi(w, Bw')$$

for all $w, w' \in W$.
(5) *The subalgebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0$ is the Lie algebra of the group of affine automorphisms of D.*

Theorem 2.9 allows us to find a particularly nice maximal split solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$.

Proposition 2.10. *Let $\mathfrak{s}_0$ be a maximal split solvable subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{s} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{s}_0$ is a maximal split solvable subalgebra of $\mathfrak{g}$.*

Proof. This is the content of Proposition 2.8 in [15]. □

2.3. Normal j-algebras. We have seen that every bounded homogeneous domain D is diffeomorphic to a split solvable Lie algebra $\mathfrak{s}$. Transferring the complex structure and the Bergman metric of D to $\mathfrak{s}$ we obtain the notion of a normal j-algebra which was introduced by Pyateskii-Shapiro. We follow the exposition in [12]. Complete proofs and further details can be found in [20].

DEFINITION 2.11. A normal j-algebra is a pair $(\mathfrak{s}, j)$ of a split solvable Lie algebra $\mathfrak{s}$ and a complex structure j on $\mathfrak{s}$ such that

$$[\xi, \xi'] + j[j\xi, \xi'] + j[\xi, j\xi'] - [j\xi, j\xi'] = 0 \tag{2.1}$$

for all $\xi, \xi' \in \mathfrak{s}$. In addition, we demand the existence of a linear form $\omega \in \mathfrak{s}^*$ such that

$$\langle \xi, \xi' \rangle_\omega := \omega([j\xi, \xi'])$$

defines a j-invariant inner product on $\mathfrak{s}$.

REMARK. If we extend the complex structure j on $\mathfrak{s}$ to a left invariant complex structure J on the simply-connected group S, then condition (2.1) guarantees that S is a complex manifold with respect to J.

Let us describe the fine structure of a normal j-algebra $(\mathfrak{s}, j)$ via a root space decomposition. Since $\mathfrak{s}$ is solvable, its derived algebra $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ is nilpotent. Let $\mathfrak{a}$ denote the orthogonal complement of $\mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_\omega$. Hence, we obtain $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ and one can show that $\mathfrak{a}$ is a maximal Abelian subalgebra consisting of semi-simple elements of $\mathfrak{s}$. The dimension $r := \dim \mathfrak{a}$ is called the rank of $\mathfrak{s}$. Since $\mathfrak{s}$ is split solvable, we can form the root space decomposition

$$\mathfrak{s} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{s}_\alpha, \tag{2.2}$$

where we write $\mathfrak{s}_\alpha := \{\xi \in \mathfrak{s};\ [\eta, \xi] = \alpha(\eta)\xi\}$ for $\alpha \in \mathfrak{a}^*$ and $\Delta := \Delta(\mathfrak{s}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\};\ \mathfrak{s}_\alpha \neq \{0\}\}$.

Proposition 2.12. *Let $(\mathfrak{s}, j)$ be a normal j-algebra.*
(1) *The root space decomposition* (2.2) *is orthogonal with respect to $\langle \cdot, \cdot \rangle_\omega$.*
(2) *There exist r linearly independent roots $\alpha_1, \ldots, \alpha_r$ such that all other roots are of the form*

$$\frac{1}{2}\alpha_k \ (1 \leq k \leq r) \quad \textit{and} \quad \frac{1}{2}(\alpha_l \pm \alpha_k) \ (1 \leq k < l \leq r).$$

Note that not all possibilities have to occur.

(3) *Let* $(\eta_1, \dots, \eta_r)$ *be the basis of* $\mathfrak{a}$ *dual to* $(-\alpha_1, \dots, -\alpha_r)$ *and set* $\xi_k := -j\eta_k$. *Then we have* $\mathfrak{s}_{\alpha_k} = \mathbb{R}\xi_k$ *for all* $1 \le k \le r$.
(4) *For all* $1 \le k < l \le r$ *we have* $j\mathfrak{s}_{(\alpha_l-\alpha_k)/2} = \mathfrak{s}_{(\alpha_l+\alpha_k)/2}$.
(5) *For all* $1 \le k \le r$ *we have* $j\mathfrak{s}_{\alpha_k/2} = \mathfrak{s}_{\alpha_k/2}$.

Finally, we set $\delta := \eta_1 + \cdots + \eta_r$ and write $\mathfrak{s}_\lambda$ for the eigenspace of $\mathrm{ad}(\delta)$ with eigenvalue $\lambda \in \mathbb{R}$. Then we obtain the grading

$$\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_{-1/2} \oplus \mathfrak{s}_0$$

of $\mathfrak{s}$ where

$$\begin{aligned} \mathfrak{s}_{-1} &= \bigoplus_{k=1}^{r} \mathfrak{s}_{\alpha_k} \oplus \bigoplus_{1\le k<l\le r} \mathfrak{s}_{(\alpha_l+\alpha_k)/2}, \\ \mathfrak{s}_{-1/2} &= \bigoplus_{k=1}^{r} \mathfrak{s}_{\alpha_k/2}, \\ \mathfrak{s}_0 &= \mathfrak{a} \oplus \bigoplus_{1\le k<l\le r} \mathfrak{s}_{(\alpha_l-\alpha_k)/2} \end{aligned} \tag{2.3}$$

hold.

Next we explain how the domain D can be recovered from $(\mathfrak{s}, j)$. Let S be the simply-connected Lie group with Lie algebra $\mathfrak{s}$ and let S_0 be the analytic subgroup whose Lie algebra is given by $\mathfrak{s}_0$. We define $\xi := \xi_1 + \cdots + \xi_r$ and $\Omega := \mathrm{Ad}(S_0)\xi$. One can show that Ω is a regular cone in $\mathfrak{s}_{-1}$.

Since $\mathfrak{s}_{-1/2}$ is invariant under j, we may consider $(\mathfrak{s}_{-1/2}, j)$ as a complex vector space. Then the map $\Phi\colon \mathfrak{s}_{-1/2} \times \mathfrak{s}_{-1/2} \to \mathfrak{s}_{-1}^{\mathbb{C}}$,

$$\Phi(\xi, \xi') := \frac{1}{4}([j\xi, \xi'] + i[\xi, \xi'])$$

is an Ω-Hermitian form on $\mathfrak{s}_{-1/2}$. Hence, we obtain the associated Siegel domain

$$D_{\mathfrak{s}} := \{(\xi, \xi') \in \mathfrak{s}_{-1}^{\mathbb{C}} \times \mathfrak{s}_{-1/2};\ \mathrm{Im}(\xi) - \Phi(\xi', \xi') \in \Omega\}.$$

Let S_- be the analytic subgroup of S corresponding to $\mathfrak{s}_{-1} \oplus \mathfrak{s}_{-1/2}$. Then the group $S = S_- \rtimes S_0$ acts by affine maps on $D_{\mathfrak{s}}$ via

$$(\exp(\xi + \xi'), s)\cdot(z, w) := (\mathrm{Ad}(s)z + \xi + 2i\,\Phi(\mathrm{Ad}(s)w, \xi') + i\,\Phi(\xi', \xi'), \mathrm{Ad}(s)w + \xi'), \tag{2.4}$$

where $\xi \in \mathfrak{s}_{-1}$, $\xi' \in \mathfrak{s}_{-1/2}$, $s \in S_0$ and $(z, w) \in \mathfrak{s}_{-1}^{\mathbb{C}} \times \mathfrak{s}_{-1/2}$ hold. One can show that this action is simply transitive on $D_{\mathfrak{s}}$ which implies that $D_{\mathfrak{s}}$ is biholomorphically equivalent to a bounded homogeneous domain.

Theorem 2.13. *The construction described above yields a one-to-one correspondence between equivalence classes of homogeneous bounded domains and isomorphism classes of normal j-algebras.*

Finally we note the following corollary of (2.4).

Lemma 2.14. *The group $S^{\mathbb{C}}$ acts transitively on $\mathfrak{s}_{-1}^{\mathbb{C}} \times \mathfrak{s}_{-1/2}$.*

3. Cyclic groups acting on bounded homogeneous domains

We carry out the first step towards a proof of Steinness of $X = D/\Gamma$ by showing that it is enough to assume that the cyclic group Γ lies in a maximal split solvable subgroup of G.

3.1. Reduction to automorphisms in G. Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain and let $\varphi \in \mathrm{Aut}_{\mathcal{O}}(D)$ be such that the subgroup $\Gamma := \langle\varphi\rangle := \{\varphi^m;\, m \in \mathbb{Z}\}$ is discrete in $\mathrm{Aut}_{\mathcal{O}}(D)$. Since every discrete subgroup is also closed, this implies that Γ acts properly on D, and hence that $X := D/\Gamma$ is a complex space.

Since the group Γ is cyclic, it is either finite or isomorphic to $\mathbb{Z}$. In the first case it is classical that Steinness of D implies Steinness of X (see for example [7]). Therefore we will assume in the following that Γ is isomorphic to $\mathbb{Z}$. Since every proper $\mathbb{Z}$-action is automatically free, the quotient X is a complex manifold in this case.

Recall that the group $\mathrm{Aut}_{\mathcal{O}}(D)$ has only finitely many connected components which implies that $\Gamma^0 := \Gamma \cap G$ is a normal subgroup of finite index in Γ. Since D/Γ is Stein if and only if D/Γ^0 is so, we may assume without loss of generality that φ is contained in $G = \mathrm{Aut}_{\mathcal{O}}(D)^0$.

3.2. Jordan–Chevalley decomposition. In this subsection we will explain how Kaneyuki's Theorem 2.2 implies the existence of the Jordan–Chevalley decomposition in G.

Let us quickly review the Jordan–Chevalley decomposition. If H is a real-algebraic group, then every element $h \in H$ can be uniquely written as $h = h_s h_u = h_u h_s$ where $h_s \in H$ is semi-simple and $h_u \in H$ is unipotent. Following [17] we decompose the semi-simple part h_s further as $h_s = h_e h_h$ where the eigenvalues of $h_e \in H$ lie in the unit circle in $\mathbb{C}$ and where $h_h \in H$ has only positive real eigenvalues. We call h_e the elliptic and h_h the hyperbolic part of h. Note that the elements h_e, h_h and h_u commute.

Lemma 3.1. *Let $H \subset \mathrm{GL}(N, \mathbb{R})$ be a real-algebraic group and let $h \in H^0$ be given. If $h = h_e h_h h_u$ is the multiplicative Jordan decomposition of h in H, then we have $h_e, h_h, h_u \in H^0$.*

Proof. Let $h = h_s + h_n$ be the additive Jordan decomposition in $\mathbb{R}^{N\times N}$. As is well known the matrices h_s and h_n can be expressed as polynomials in h. Furthermore,

the multiplicative Jordan decomposition of h is then given by $h = h_s h_u$ with $h_u = I_N + h_s^{-1} h_n$. Since H is real-algebraic, we have $h_s, h_u \in H$ and the matrices h_s and h_u depend continuously on h. Moreover, the matrices h_e and h_h lie in H and depend continuously on h, too.

If $h \in H^0$ holds, we find a continuous curve $t \mapsto h(t) \in H^0$, $t \in [0, 1]$, with $h(0) = I_N$ and $h(1) = h$. Forming the multiplicative Jordan decomposition $h(t) = h_e(t)h_h(t)h_u(t)$ we obtain continuous curves $t \mapsto h_e(t) \in H$, $t \mapsto h_h(t) \in H$ and $t \mapsto h_u(t) \in H$. Because of $h_e(0) = h_h(0) = h_u(0) = I_N$ the claim follows. □

Since Kaneyuki's theorem asserts that there exists a faithful representation $\rho\colon G \to \mathrm{GL}(N, \mathbb{R})$ such that $\rho(G) = H^0$ for a real-algebraic subgroup $H \subset \mathrm{GL}(N, \mathbb{R})$, we obtain the following notion of Jordan–Chevalley decomposition in G.

DEFINITION 3.2. We say that an element $g \in G$ is elliptic, hyperbolic or unipotent if the element $\rho(g)$ has this property.

Proposition 3.3. *Every element $g \in G$ may be uniquely written as $g = g_e g_h g_u$ where g_e is elliptic, g_h is hyperbolic and g_u is unipotent and where these three elements commute with each other.*

The following proposition generalizes Propositions 2.3 and 2.5 of [17].

Proposition 3.4. *Every elliptic element of G is conjugate to an element in the maximal compact subgroup K, while every element $g \in G$ with $g_e = e$ is conjugate to an element in the maximal split solvable group S.*

Proof. The claim follows from the facts that elliptic elements generate compact groups, that elements with trivial elliptic part generate split solvable groups and that maximal compact respectively split solvable groups are conjugate. □

3.3. Reduction to automorphisms with trivial elliptic part. In this subsection we will show that it is enough to consider automorphisms $\varphi \in G$ whose elliptic part vanishes.

Let $\varphi = \varphi_e \varphi_h \varphi_u$ be the Jordan–Chevalley decomposition of φ and set $\varphi' := \varphi_h \varphi_u$ as well as $\Gamma' := \langle \varphi' \rangle$. By Proposition 3.4 we may assume that the group Γ' is contained in the split solvable subgroup S of G. This implies in particular that Γ' is a closed subgroup of G. Thus we may consider the complex manifold $X' := D/\Gamma'$. We will show that X is Stein if and only if X' is Stein.

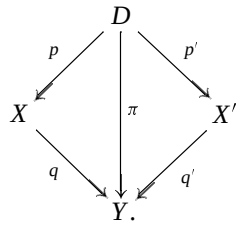
The closure T of the group generated by φ_e is a compact torus in G. Since φ' and φ_e commute, we conclude that Γ and Γ' lie in the centralizer $\mathcal{Z}_G(T)$. Consequently, the sets $T\Gamma$ and $T\Gamma'$ are subgroups of G.

Lemma 3.5. *We have $T\Gamma = T\Gamma'$, and the action of $T\Gamma'$ on D is proper. Hence, $Y := D/(T\Gamma) = D/(T\Gamma')$ is a Hausdorff topological space.*

Proof. The identity $T\Gamma = T\Gamma'$ is elementary to check.

Since T is compact, the T-action on X' is proper, hence the product group $T \times \Gamma'$ acts properly on D. Since the element φ' has by definition trivial elliptic part, the group $T\Gamma'$ is isomorphic to $T \times \Gamma'$. □

Since the groups Γ and Γ' are normal in $T\Gamma = T\Gamma'$, the torus T acts properly on X and X' and we obtain the following commutative diagram:



The following proposition is the main result of this subsection.

Proposition 3.6. *The manifold X is Stein if and only if X' is Stein. Hence, we can restrict our attention to automorphisms with trivial elliptic part.*

Proof. In a first step we investigate how T-invariant functions on X' induce T-invariant functions on X. For this let $f\colon X' \to \mathbb{R}$ be any smooth function which is invariant under T. It follows that the pull-back $(p')^* f\colon D \to \mathbb{R}$ is smooth and $T\Gamma'$-invariant. Since Γ is a normal subgroup of $T\Gamma'$, we obtain a T-invariant smooth function $\tilde{f}\colon X \to \mathbb{R}$.

Since the above diagram commutes, f and $\tilde{f}$ induce the same continuous function on Y. By compactness of T this implies that if f is an exhaustion, then $\tilde{f}$ is also an exhaustion. Moreover, if f is strictly plurisubharmonic, then $(p')^* f$ is strictly plurisubharmonic and hence $\tilde{f}$ is strictly plurisubharmonic.

If X' is Stein, then there exists a strictly plurisubharmonic exhaustion function on X'. Since T is compact, we can assume that this function is T-invariant. By the above arguments, we obtain a strictly plurisubharmonic exhaustion function on X. Hence, X is Stein.

The converse is proved similarly. □

4. Example: The unit ball in $\mathbb{C}^n$

In this section we discuss the automorphism group and the normal j-algebra of the unit ball $\mathbb{B}_n := \{z \in \mathbb{C}^n;\ \|z\| < 1\}$ in $\mathbb{C}^n$. It has been proven in [4] and [5] that

Table 1. Automorphisms of $\mathbb{H}_n$ generating $B_n \cong A \ltimes N_n$.

One-parameter group of automorphisms	Vector field
$(z, w) \mapsto (z+t, w)$	$\zeta = \partial/\partial z$
$(z, w) \mapsto (z+2itw_k+it^2, w_1, \ldots, w_k+t, \ldots, w_{n-1})$	$\xi_k = 2iw_k\,\partial/\partial z + \partial/\partial w_k$ $(1 \le k \le n-1)$
$(z, w) \mapsto (z+2tw_k+it^2, w_1, \ldots, w_k+it, \ldots, w_{n-1})$	$\eta_k = 2w_k\,\partial/\partial z + i\,\partial/\partial w_k$ $(1 \le k \le n-1)$
$(z, w) \mapsto (e^t z, e^{t/2} w)$	$\delta = z\,\partial/\partial z + (1/2)\sum_{k=1}^{n-1} w_k\,\partial/\partial w_k$

the quotient manifold $\mathbb{B}_n/\langle\varphi\rangle$ is Stein for hyperbolic and parabolic automorphisms $\varphi \in \mathrm{Aut}_{\mathcal{O}}(\mathbb{B}_n)$. We will give here a different proof of this fact.

4.1. The automorphism group of the unit ball. Let us first describe the full automorphism group of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. For this we embed $\mathbb{C}^n$ into the complex projective space $\mathbb{P}_n(\mathbb{C})$ by $(z_1, \ldots, z_n) \mapsto [z_1 : \cdots : z_n : 1]$. The image of $\mathbb{B}_n$ under this embedding is given by

$$D := \{[z_1 : \cdots : z_{n+1}] \in \mathbb{P}_n(\mathbb{C});\ |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 < 0\}.$$

Consequently, the group $\mathrm{SU}(n, 1)$, acting as a subgroup of $\mathrm{SL}(n+1, \mathbb{C})$ by projective transformations on $\mathbb{P}_n(\mathbb{C})$, leaves D invariant. Hence, we obtain a homomorphism $\Phi\colon \mathrm{SU}(n, 1) \to \mathrm{Aut}_{\mathcal{O}}(\mathbb{B}_n)$. One can show that Φ is a surjective homomorphism of Lie groups whose kernel coincides with the (finite) center of $\mathrm{SU}(n, 1)$ (see for example [1]).

In order to find explicit formulas for the automorphisms of $\mathbb{B}_n$ belonging to a maximal split solvable subgroup B_n of $G = \mathrm{Aut}_{\mathcal{O}}(\mathbb{B}_n)$ we make use of the realization of $\mathbb{B}_n$ as the Siegel domain

$$\mathbb{H}_n := \{(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1};\ \mathrm{Im}(z) - \|w\|^2 > 0\}.$$

From Theorem 2.9 we obtain $2n$ one parameter subgroups of automorphisms of $\mathbb{H}_n$ which generate the group B_n. These are listed together with their corresponding complete holomorphic vector fields in Table 1.

4.2. The normal j-algebra of the unit ball. Let $\mathfrak{b}_n$ be the Lie algebra of the group B_n. Its derived algebra $\mathfrak{n}_n := [\mathfrak{b}_n, \mathfrak{b}_n]$ is given by

$$\mathfrak{n}_n = \mathbb{R}\zeta \oplus \mathbb{R}\xi_1 \oplus \cdots \oplus \mathbb{R}\xi_{n-1} \oplus \mathbb{R}\eta_1 \oplus \cdots \oplus \mathbb{R}\eta_{n-1},$$

while $\mathfrak{a} := \mathbb{R}\delta$ is maximal Abelian consisting of semi-simple elements of $\mathfrak{b}_n$. One computes directly that the only non-vanishing commutators are

$$[\delta, \xi_k] = -\frac{1}{2}\xi_k, \quad [\delta, \eta_k] = -\frac{1}{2}\eta_k, \quad [\delta, \zeta] = -\zeta, \quad [\xi_k, \eta_k] = 4\zeta.$$

In particular, $\mathfrak{n}_n$ is a $(2n-1)$-dimensional Heisenberg algebra with center $\mathbb{R}\zeta$. Choosing the base point $z_0 = (i, 0) \in \mathbb{H}_n$ we obtain via the isomorphism $\mathfrak{b}_n \to \mathfrak{b}_n \cdot z_0 = T_{z_0}\mathbb{H}_n$ the following complex structure j on $\mathfrak{b}_n$:

$$j\zeta = \delta, \quad j\xi_k = \eta_k.$$

These data describe the normal j-algebra $(\mathfrak{b}_n, j)$ of the unit ball $\mathbb{B}_n$.

In the rest of this subsection we will prove several technical facts which lead to a proof of Steinness of $\mathbb{B}_n/\Gamma$.

Lemma 4.1. *Let $\xi \in \mathfrak{n}_n$ be arbitrary. Then there exists an n-dimensional Abelian subalgebra $\mathfrak{n}'_n$ of $\mathfrak{n}_n$ which contains ξ.*

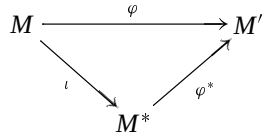
Proof. We proof the claim by induction over n. For $n = 1$ the subalgebra $\mathfrak{n}_1$ itself is one-dimensional and Abelian. Hence, let $n > 1$ and let us assume that the claim holds for $n-1$. We write $\xi = \xi' + \xi''$ according to the decomposition $\mathfrak{n}_n = \mathfrak{n}_{n-1} \oplus \mathbb{R}\xi_{n-1} \oplus \mathbb{R}\eta_{n-1}$. By our induction hypotheses there exists an $(n-1)$-dimensional Abelian subalgebra $\mathfrak{n}'_{n-1}$ of $\mathfrak{n}_{n-1}$ containing ξ'. Then $\mathfrak{n}'_n := \mathfrak{n}'_{n-1} \oplus \mathbb{R}\xi''$ has the required properties. □

As a consequence we obtain the following

Proposition 4.2 ([27], [11]). *Let N'_n be the analytic subgroup of B_n with Lie algebra $\mathfrak{n}'_n$. Then every N'_n-orbit in $\mathbb{B}_n$ is totally real and $\mathbb{B}_n$ is biholomorphically equivalent to a tube domain D in $\mathbb{C}^n$ such that N'_n acts by translations on D.*

For the proof we have to review parts of the theory of (universal) globalizations of local holomorphic actions. We use [8] as a general reference.

Let M be a complex manifold endowed with a local holomorphic action of a complex Lie group L. A globalization of this local action consists in an open holomorphic embedding ι of M into a (possibly non-Hausdorff) complex manifold M^* on which L acts holomorphically such that ι is locally equivariant and $M^* = L \cdot \iota(M)$. A globalization M^* is called universal if for every locally L-equivariant map $\varphi\colon M \to M'$ into an L-manifold M' there exists a unique L-equivariant map $\varphi^*\colon M^* \to M'$ such that the diagram



commutes. By remark in §3 in [8] the universal globalization of a local L-action on M exists if and only if any globalization exists.

Proof of Proposition 4.2. Since the group N_n' is Abelian and $\mathbb{H}_n$ is hyperbolic, every N_n'-orbit in $\mathbb{H}_n$ must be totally real.

Let $(N_n')^{\mathbb{C}}$ be the universal complexification of N_n'. Since $(N_n')^{\mathbb{C}}$ acts by affine-linear transformations on $\mathbb{C}^n$, the universal globalization $\mathbb{H}_n^*$ of the local $(N_n')^{\mathbb{C}}$-action on $\mathbb{H}_n$ exists. Since every N_n'-orbit is totally real and of maximal dimension n, every $(N_n')^{\mathbb{C}}$-orbit in $\mathbb{H}_n^*$ is open. Thus $\mathbb{H}_n^* \cong (N_n')^{\mathbb{C}}/(N_n')_z^{\mathbb{C}}$ is homogeneous and in particular Hausdorff. Moreover, $\mathbb{H}_n$ is biholomorphically equivalent to a N_n'-invariant domain in this homogeneous space. Because of $\dim \mathbb{H}_n^* = n = \dim(N_n')^{\mathbb{C}}$ the isotropy $(N_n')_z^{\mathbb{C}}$ is discrete. Since $(N_n')^{\mathbb{C}} \cong \mathbb{C}^n$ is simply-connected, we may apply Lemma 2.1 of [13] in order to conclude that $\mathbb{H}_n^*$ is simply-connected which implies $(N_n')_z^{\mathbb{C}} = \{e\}$. Hence the claim follows. □

Lemma 4.3. *Let $\xi = \xi_{\mathfrak{a}} + \xi_{\mathfrak{n}_n} \in \mathfrak{a} \oplus \mathfrak{n}_n = \mathfrak{b}_n$ be an element with $\xi_{\mathfrak{a}} \neq 0$. Then there exists an element $g \in B_n$ with $\mathrm{Ad}(g)\xi \in \mathfrak{a}$.*

Proof. We prove the lemma by induction over n. If $n = 1$, we identify the Lie algebra $\mathfrak{b}_1$ with $\left\{\begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}; t, s \in \mathbb{R}\right\}$. If $\xi = \begin{pmatrix} t & s \\ 0 & -t \end{pmatrix}$ with $t \neq 0$ is given, one verifies that $g = \begin{pmatrix} 1 & s/2t \\ 0 & 1 \end{pmatrix} \in B_1$ fulfills $\mathrm{Ad}(g)\xi \in \mathfrak{a}$.

Let $n > 1$ and let us assume that the claim is proven for $n-1$. We write $\xi = \xi' + \xi''$ according to $\mathfrak{b}_n = \mathfrak{b}_{n-1} \oplus (\mathbb{R}\xi_{n-1} \oplus \mathbb{R}\eta_{n-1})$. Since $\xi_{\mathfrak{a}}' = \xi_{\mathfrak{a}} \neq 0$ (and in particular $\xi' \neq 0$), our induction hypothesis implies the existence of an element $g \in B_{n-1}$ such that $\mathrm{Ad}(g)\xi' \in \mathfrak{a}$ holds. Since we have

$$[\xi', \xi''] = [\xi_{\mathfrak{a}}', \xi''] + [\xi_{\mathfrak{n}_{n-1}}', \xi''] = -\lambda\xi''$$

for some $\lambda \neq 0$, the subspace $\mathbb{R}\,\mathrm{Ad}(g)\xi' \oplus \mathbb{R}\,\mathrm{Ad}(g)\xi'' \subset \mathfrak{a} \oplus \mathfrak{n}_n$ is a subalgebra of $\mathfrak{b}_n$ isomorphic to $\mathfrak{b}_1$. Since $\mathrm{Ad}(g)\xi' \neq 0$, there is an element g' in the corresponding subgroup with $\mathrm{Ad}(g')(\mathrm{Ad}(g)\xi' + \mathrm{Ad}(g)\xi'') = \mathrm{Ad}(g'g)\xi \in \mathfrak{a}$. Hence, the lemma is proven. □

Lemma 4.4. *The subspace $\mathfrak{b}_n' := \mathfrak{a} \oplus \mathbb{R}\xi_1 \oplus \cdots \oplus \mathbb{R}\xi_{n-1}$ is an n-dimensional subalgebra of $\mathfrak{b}_n$ such that every orbit of the corresponding subgroup B_n' of B_n is totally real in $\mathbb{H}_n$.*

Proof. Using the commutator relations one checks directly that $\mathfrak{b}_n'$ is a subalgebra of $\mathfrak{b}_n$.

To prove the second claim note that for $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$ we have $T_{(z,w)}(B_n' \cdot (z, w)) = \mathfrak{b}_n' \cdot (z, w) = \mathbb{R}\delta(z, w) \oplus \mathbb{R}\xi_1(z, w) \oplus \cdots \oplus \mathbb{R}\xi_{n-1}(z, w)$. Elementary considerations show that this real subspace of $\mathbb{C}^n$ is totally real if and only if the the matrix whose columns

are given by the above vector fields has non-zero determinant. Since one computes

$$\det\begin{pmatrix} 2z & 2iw_1 & 2iw_2 & \cdots & 2iw_{n-1} \\ w_1 & 1 & 0 & \cdots & 0 \\ w_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ w_{n-1} & 0 & \cdots & 0 & 1 \end{pmatrix} = (-1)^{n-1}(2z - 2iw_1^2 - \cdots - 2iw_{n-1}^2),$$

the orbit of $B_n' \cdot (z, w)$ fails to be totally real if and only if $z = i\sum_{k=1}^{n-1} w_k^2$ holds. Because of

$$\operatorname{Im}\left(i\sum_{k=1}^{n-1} w_k^2\right) - \|w\|^2 = -2\sum_{k=1}^{n-1} \operatorname{Im}(w_k)^2 \leq 0$$

such a point does not lie in $\mathbb{H}_n$ which proves the claim. □

We have established the following fact.

Corollary 4.5. *Let $\xi \in \mathfrak{b}_n$ be an arbitrary element. Then there exists an n-dimensional subalgebra $\mathfrak{b}_n'$ of $\mathfrak{b}_n$ containing ξ such that the corresponding group B_n' has only totally real orbits in $\mathbb{H}_n$.*

The same argument as in the proof of Proposition 4.2 applies to show the following

Proposition 4.6. *Let $\xi \in \mathfrak{b}_n$ be arbitrary. Then there exists a subgroup $B_n' \subset B_n$ containing* $\exp(\xi)$ *such that $\mathbb{H}_n$ is biholomorphically equivalent to a B_n'-invariant domain in $(B_n')^{\mathbb{C}}$ where $(B_n')^{\mathbb{C}}$ acts by left multiplication on itself.*

4.3. Quotients of the unit ball. Let φ be an automorphism of the unit ball $\mathbb{B}_n$ which generates a discrete subgroup $\Gamma \subset G$. The following proposition gives a sufficient condition for $X = \mathbb{B}_n/\Gamma$ to be Stein.

Proposition 4.7. *Let Ω be a domain in a Stein manifold M. Then Ω is Stein if and only if some covering of Ω is Stein.*

Proof. Let us assume that there is a covering of Ω which is Stein. It follows from [23] that the universal covering $p\colon \tilde{\Omega} \to \Omega$ is then Stein, too. If Ω is not Stein, then there exists a Hartogs figure (H, P) in M such that $H \subset \Omega$ and $P \not\subset \Omega$ hold (see [6]). Since H is simply connected, each component of $p^{-1}(H)$ is mapped biholomorphically onto H by p. Let $\tilde{H}$ be a component of $p^{-1}(H)$ and write $s := (p|_{\tilde{H}})^{-1}\colon H \to \tilde{H}$. Since by assumption $\tilde{\Omega}$ is a Stein manifold, we can embed it as a closed submanifold

into some $\mathbb{C}^N$. Hence, the map s extends to a map $s\colon P \to \tilde{\Omega} \subset \mathbb{C}^N$. Thus the composition $p \circ s\colon P \to \Omega \subset M$ is defined. Since $(p \circ s)|_H = \mathrm{id}_H$ holds, the continuation principle shows $p \circ s = \mathrm{id}_P$, which contradicts our assumption $P \not\subset \Omega$. □

Theorem 4.8. *Let $\varphi \in G$ be any automorphism generating a discrete subgroup Γ of G. Then the quotient $X = \mathbb{B}_n/\Gamma$ is a Stein space.*

Proof. By virtue of Proposition 3.6 we can assume that φ lies in the maximal split solvable subgroup $B_n \subset G$. Then we find an n-dimensional closed subgroup $B_n' \subset B_n$ containing Γ such that each B_n'-orbit is totally real in $\mathbb{B}_n$. By Proposition 4.6 we may embed $\mathbb{B}_n$ as a B_n'-invariant domain into $(B_n')^{\mathbb{C}}$ where $(B_n')^{\mathbb{C}}$ acts by left multiplication on itself. Let $\mathbb{C}_\Gamma$ be the complex one parameter subgroup of $(B_n')^{\mathbb{C}}$ which contains Γ. Since $(B_n')^{\mathbb{C}}/\Gamma$ is a $\mathbb{C}^*$-principal bundle over $(B_n')^{\mathbb{C}}/\mathbb{C}_\Gamma \cong \mathbb{C}^{n-1}$, we conclude that $(B_n')^{\mathbb{C}}/\Gamma$ is a Stein manifold. Therefore the claim follows from Proposition 4.7. □

5. Existence of equivariant holomorphic submersions

In [20] the j-invariant ideals of a normal j-algebra are investigated. For the sake of completeness we indicate how the root space decomposition of a normal j-algebra may be used to find a j-invariant ideal which is isomorphic to the normal j-algebra of the unit ball.

5.1. Existence of j-invariant ideals isomorphic to the unit ball. Let $(\mathfrak{s}, j)$ be a normal j-algebra with gradation $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_{-1/2} \oplus \mathfrak{s}_0$. We define $\mathfrak{s}' := \mathfrak{s}'_{-1} \oplus \mathfrak{s}'_{-1/2} \oplus \mathfrak{s}'_0$ by

$$\mathfrak{s}'_{-1} := \bigoplus_{k=1}^{r-1} \mathfrak{s}_{\alpha_k} \oplus \bigoplus_{1 \le k < l \le r-1} \mathfrak{s}_{(\alpha_l + \alpha_k)/2},$$

$$\mathfrak{s}'_{-1/2} := \bigoplus_{k=1}^{r-1} \mathfrak{s}_{\alpha_k/2},$$

$$\mathfrak{s}'_0 := \mathbb{R}\eta_1 \oplus \cdots \oplus \mathbb{R}\eta_{r-1} \oplus \bigoplus_{1 \le k < l \le r-1} \mathfrak{s}_{(\alpha_l - \alpha_k)/2},$$

i.e. $\mathfrak{s}'$ is the direct sum of all root spaces in which the roots α_r, $(1/2)\alpha_r$ or $(1/2)(\alpha_r \pm \alpha_k)$ $(1 \le k \le r-1)$ do not appear.

Lemma 5.1. *The subspace $\mathfrak{s}'$ is a j-invariant subalgebra of $\mathfrak{s}$. Moreover, there exists an $\omega' \in (\mathfrak{s}')^*$ such that $(\mathfrak{s}', j')$ is a normal j-algebra where $j' := j|_{\mathfrak{s}'}$.*

Proof. The fact that $\mathfrak{s}'$ is closed under the Lie bracket follows from the properties of the root space decomposition and j-invariance is a direct consequence of Proposition 2.12 (3)–(5). Setting $\omega' := \omega|_{\mathfrak{s}'}$ the claim follows. □

Let $\pi\colon \mathfrak{s} \to \mathfrak{s}$ be the orthogonal projection onto $\mathfrak{s}'$ with respect to $\langle \cdot\,,\, \cdot \rangle_\omega$.

Lemma 5.2. *The map π is a homomorphism of normal j-algebras algebras whose kernel is given by*

$$\mathfrak{b} := \mathfrak{s}_{\alpha_r} \oplus \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r+\alpha_k)/2} \oplus \mathfrak{s}_{\alpha_r/2} \oplus \mathbb{R}\eta_r \oplus \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r-\alpha_k)/2},$$

and hence induces an isomorphism $\mathfrak{s}/\mathfrak{b} \cong \mathfrak{s}'$. In particular, $\mathfrak{b}$ is a j-invariant ideal in $\mathfrak{s}$ and thus inherits the structure of a normal j-algebra.

Proof. Using properties of the root space decomposition one checks directly that the map π preserves the Lie brackets. The kernel of π is given by the orthogonal complement of $\mathfrak{s}'$ in $\mathfrak{s}$ with respect to $\langle \cdot\,,\, \cdot \rangle_\omega$ which in turn coincides with $\mathfrak{b}$ by Proposition 2.12 (1). Since $\mathfrak{s}'$ and $\mathfrak{b}$ are j-invariant, it follows that $\pi \circ j = j \circ \pi$ holds. This finishes the proof. □

Lemma 5.3. *The normal j-algebra $(\mathfrak{b}, j)$ is isomorphic to the normal j-algebra of the unit ball.*

Proof. One computes directly

$$[\mathfrak{b}, \mathfrak{b}] = \mathfrak{s}_{\alpha_r} \oplus \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r+\alpha_k)/2} \oplus \mathfrak{s}_{\alpha_r/2} \oplus \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r-\alpha_k)/2}.$$

Hence, $\mathbb{R}\eta_r$ is maximal Abelian in $\mathfrak{b}$ and in particular $\mathfrak{b}$ has rank one. The claim will follow if we show that $[\mathfrak{b}, \mathfrak{b}]$ is a Heisenberg algebra. For this one checks that $\mathfrak{s}_{\alpha_r}$ is the center of $[\mathfrak{b}, \mathfrak{b}]$ and that the Lie bracket

$$[\cdot\,,\, \cdot]\colon \mathfrak{b}_{-1/2} \times \mathfrak{b}_{-1/2} \to \mathfrak{s}_{\alpha_r}$$

defines a symplectic form on

$$\mathfrak{b}_{-1/2} := \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r+\alpha_k)/2} \oplus \mathfrak{s}_{\alpha_r/2} \oplus \bigoplus_{k=1}^{r-1} \mathfrak{s}_{(\alpha_r-\alpha_k)/2}.$$ □

Lemma 5.4. *Let $\hat{\pi}\colon S \to S' \cong S/B_m$ be the homomorphism on the group level. The short exact sequence $1 \to B_m \to S \to S' \to 1$ splits, i.e. S is isomorphic to $S' \ltimes B_m$.*

Proof. The claim follows from the fact that $\mathfrak{s}' \hookrightarrow \mathfrak{s}$ is a homomorphism of Lie algebras and a section to π. □

5.2. Geometric realization of the fibration. In this subsection we view π as a map $\mathfrak{s}_{-1}^{\mathbb{C}} \times \mathfrak{s}_{-1/2} \to (\mathfrak{s}'_{-1})^{\mathbb{C}} \times \mathfrak{s}'_{-1/2}$ by restriction and $\mathbb{C}$-linear extension.

Lemma 5.5. *The map π maps $D_{\mathfrak{s}}$ into $D_{\mathfrak{s}'}$.*

Proof. First we note that π maps the base point $\xi_0 = \xi_1 + \cdots + \xi_r$ onto the base point $\xi'_0 = \xi_1 + \cdots + \xi_{r-1}$. Since $\pi\colon \mathfrak{s} \to \mathfrak{s}'$ is a homomorphism of Lie algebras, it gives rise to a unique morphism $\hat{\pi}\colon S \to S'$ between the corresponding Lie groups such that

$$\pi(\mathrm{Ad}(s)\xi) = \mathrm{Ad}(\hat{\pi}(s))\pi(\xi)$$

holds. Since π also respects the grading of $\mathfrak{s}$ and $\mathfrak{s}'$, we conclude that π maps the cone $\Omega = \mathrm{Ad}(S_0)\xi_0$ onto the cone $\Omega' = \mathrm{Ad}(S'_0)\xi'_0$. Since the Ω-Hermitian form Φ is determined by the complex structure J and the Lie bracket of $\mathfrak{s}$ which both are respected by π, we obtain $\Phi'(\pi(\xi), \pi(\xi')) = \pi\Phi(\xi, \xi')$. This proves the claim. □

Choosing the base point $z_0 := (i\xi_0, 0) \in D_{\mathfrak{s}}$ we obtain the diffeomorphism $S \to D_{\mathfrak{s}}$, $s \mapsto s \cdot z_0$. Equipping S with the left invariant extension J of j this diffeomorphism becomes biholomorphic (see Lemma 1.2 in [12]). Let B be the normal subgroup of S with Lie algebra $\mathfrak{b}$ and let S' be the analytic subgroup with Lie algebra $\mathfrak{s}'$. Note that S' is isomorphic to S/B via $\hat{\pi}\colon S \to S'$. The base point $z'_0 := \pi(z_0)$ yields the isomorphism $S' \to D_{\mathfrak{s}'}$. Now we are in position to prove the main result of this section.

Proposition 5.6. *The following diagram commutes*:

$$\begin{array}{ccc} S & \xrightarrow{\cong} & D_{\mathfrak{s}} \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ S' & \xrightarrow[\cong]{} & D_{\mathfrak{s}'}. \end{array}$$

It follows that $\pi\colon D_{\mathfrak{s}} \to D_{\mathfrak{s}'}$ is an S-equivariant holomorphic submersion whose fibers are isomorphic to the unit ball.

Proof. We have to show that

$$\pi(s \cdot z_0) = \tilde{\pi}(s) \cdot z'_0$$

holds for all $s \in S$. In the proof of Lemma 5.5 we have already seen that this holds true for $s \in S_0$. Using the explicit formula (2.4) for the S-action on $D_{\mathfrak{s}}$ one verifies the claim for the whole group S. □

REMARK. Let $\pi\colon D \to D'$ be the S-equivariant holomorphic submersion whose fibers are biholomorphically equivalent to $\mathbb{B}_m$. It follows from Proposition 5.6 that π is

a smooth principal bundle with group B_m. If π was a holomorphic fiber bundle, then by a result of Royden ([21]) it would be holomorphically trivial, i.e. $D \cong D' \times \mathbb{B}_m$. This shows that π admits in general no local holomorphic trivializations.

6. Proof of the main theorem

6.1. Equivariant fiber bundles. In this subsection we present an auxiliary result concerning the quotient of an equivariant fiber bundle. Since it seems to be hard to find an explicit reference for it, we give a proof here.

Proposition 6.1. *Let $p\colon B \to X$ be a fiber bundle with typical fiber F and structure group S. Let G be a group acting on B by bundle automorphisms. We assume that the induced G-action on X is free and proper. Then G acts freely and properly on B, and hence we obtain the commutative diagram*

$$\begin{array}{ccc} B & \longrightarrow & B/G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ X & \longrightarrow & X/G. \end{array}$$

The induced map $\bar{p}\colon B/G \to X/G$ is again a fiber bundle with the same typical fiber and structure group.

Proof. We prove first that the map $\bar{p}\colon B/\Gamma \to X/\Gamma$ admits local trivializations. To see this let $U \subset X$ be an open set such that there exists a trivialization $\varphi\colon U \times F \to p^{-1}(U)$. Shrinking U if necessary, we may assume that there exists a slice for the G-action on $\hat{U} := G \cdot U$, i.e. that $\hat{U}$ is G-equivariantly isomorphic to $G \times S$ where G acts on $G \times S$ by $g \cdot (g', x) := (gg', x)$. It follows that $p^{-1}(S)$ is a slice for the G-action on $p^{-1}(\hat{U})$ (see [19]), hence we obtain a G-equivariant isomorphism $p^{-1}(\hat{U}) \cong_G G \times p^{-1}(S)$. Therefore the map

$$\hat{\varphi}\colon \hat{U} \times F \to p^{-1}(\hat{U}), \quad \hat{\varphi}(g \cdot x, y) := g \cdot \varphi(x, y),$$

with $g \in G$ and $x \in S$ is well-defined and hence a G-equivariant trivialization. This implies that the map $\bar{p}\colon B/G \to X/G$ admits local trivializations.

Since G acts by bundle automorphisms on B, the transition functions between different G-equivariant local trivializations $\hat{\varphi}$ and $\hat{\psi}$ induce isomorphisms of F given by the action of the structure group S. Thus the structure group of the fiber bundle $\bar{p}\colon B/G \to X/G$ is again given by S. □

EXAMPLE. Let G be a complex Lie group and let $H_1 \subset H_2 \subset G$ be closed complex subgroups. According to Theorem 7.4 in [22] the natural map $G/H_1 \to G/H_2$

is a holomorphic fiber bundle with fiber H_2/H_1. The structure group of this bundle is given by $H_2/(H_1)_0$ where $(H_1)_0$ denotes the largest subgroup of H_1 which is normal in H_2. In particular, if H_2 is connected, then the structure group is connected. Moreover, the maps $g'H_1 \mapsto gg'H_1$, $g \in G$, are bundle automorphisms of $G/H_1 \to G/H_2$. Hence, we may apply Proposition 6.1 to any subgroup G' of G which acts properly and freely on G/H_2 to obtain the quotient bundle

$$G'\backslash G/H_1 \to G'\backslash G/H_2.$$

6.2. Globalizing the submersion. Let D be a homogeneous Siegel domain and let $\pi\colon D \to D'$ be the S-equivariant holomorphic submersion whose fibers are biholomorphically equivalent to the unit ball $\mathbb{B}_m$. By Lemma 2.14 $S^{\mathbb{C}}$ acts transitively on $S^{\mathbb{C}} \cdot D = \mathbb{C}^n$, hence we obtain $S^{\mathbb{C}} \cdot D = \mathbb{C}^n \cong S^{\mathbb{C}}/S^{\mathbb{C}}_{z_0}$. Since $\mathbb{C}^n$ is simply-connected, it follows that $S^{\mathbb{C}}_{z_0}$ is connected. This implies that $D^* := S^{\mathbb{C}} \cdot D$ is the universal globalization of the local $S^{\mathbb{C}}$-action on D. Similarly, $(D')^* := (S')^{\mathbb{C}} \cdot D' = \mathbb{C}^{n-m}$ is the universal globalization of the local $(S')^{\mathbb{C}}$-action on D'.

Proposition 6.2. *There exists a unique $S^{\mathbb{C}}$-equivariant map $\pi^*\colon D^* \to (D')^*$ which exhibits D^* as a holomorphic fiber bundle over $(D')^*$ with typical fiber $\mathbb{H}^*_m = B^{\mathbb{C}}_m \cdot \mathbb{H}_m = \mathbb{C}^m$. The structure group is a connected complex Lie group.*

Proof. Since the submersion $\pi\colon D \to D'$ is S-equivariant, we have $(D')^* = (S')^{\mathbb{C}} \cdot D' = S^{\mathbb{C}} \cdot D' = \mathbb{C}^{n-m}$. Since moreover $D^* \cong S^{\mathbb{C}}/S^{\mathbb{C}}_{z_0}$ and $(D')^* \cong S^{\mathbb{C}}/S^{\mathbb{C}}_{z'_0}$, the existence of π^* follows from the fact that $S^{\mathbb{C}}_{z_0}$ is contained in $S^{\mathbb{C}}_{z'_0}$. It is then immediate that $\pi^*\colon D^* \to (D')^*$ is unique. Since D^* and $(D')^*$ are simply-connected, the groups $S^{\mathbb{C}}_{z_0}$ and $S^{\mathbb{C}}_{z'_0}$ are connected. Hence, it follows from Theorem 7.4 in [22] that D^* is a holomorphic fiber bundle over $(D')^*$ with fiber $S^{\mathbb{C}}_{z'_0}/S^{\mathbb{C}}_{z_0}$ such that the structure group is a connected complex Lie group. Since B_m is a normal subgroup of S, it lies in the S-isotropy of each point in D'. Hence, $B^{\mathbb{C}}_m$ is a normal subgroup of $S^{\mathbb{C}}_{z'_0}$. Because of $S^{\mathbb{C}}_{z_0} \cap B^{\mathbb{C}}_m = (B^{\mathbb{C}}_m)_{z_0}$ the inclusion $B^{\mathbb{C}}_m \hookrightarrow S^{\mathbb{C}}_{z'_0}$ induces an isomorphism $B^{\mathbb{C}}_m/(B^{\mathbb{C}}_m)_{z_0} \to S^{\mathbb{C}}_{z'_0}/S^{\mathbb{C}}_{z_0}$ which proves that the fibers of π^* are isomorphic to $\mathbb{H}^*_m = B^{\mathbb{C}}_m \cdot \mathbb{H}_m \cong B^{\mathbb{C}}_m/(B^{\mathbb{C}}_m)_{z_0}$. □

Corollary 6.3. *We have $(\pi^*)^{-1}(D') = B^{\mathbb{C}}_m \cdot D$. Hence, the restricted map $\pi^*\colon B^{\mathbb{C}}_m \cdot D \to D'$ is a holomorphic fiber bundle with typical fiber $\mathbb{H}^*_m$.*

Proof. Let $\tilde{z} \in (\pi^*)^{-1}(D')$ be given. By definition of D^* there exist a $g \in S^{\mathbb{C}}$ and a $z \in D$ such that $\tilde{z} = g \cdot z$ hold. Since S' acts transitively on D', we find a $g' \in S'$ such that $g \cdot \pi(z) = \pi^*(\tilde{z}) = g' \cdot \pi(z)$ holds. Since S' acts freely on D', we conclude $g(g')^{-1} \in B^{\mathbb{C}}_m$. This shows that $\tilde{z} = g(g')^{-1} \cdot (g' \cdot z) \in B^{\mathbb{C}}_m \cdot D$ holds. The converse inclusion follows from the fact that π^* is $B^{\mathbb{C}}_m$-invariant. □

Corollary 6.4. *Let B'_m be a subgroup of B_m having only totally real orbits in $\mathbb{H}_m$. Then the universal globalization of the local $(B'_m)^{\mathbb{C}}$-action on D is a $(B'_m)^{\mathbb{C}}$-principal bundle over D'.*

Proof. Since D' is a contractible Stein domain, we may apply Grauert's Oka principle to the bundle $B_m^{\mathbb{C}} \cdot D \to D'$ in order to obtain a $B_m^{\mathbb{C}}$-equivariant biholomorphism $B_m^{\mathbb{C}} \cdot D \to D' \times \mathbb{H}_m^*$. Then Proposition 4.6 implies that the universal globalization of the local $(B'_m)^{\mathbb{C}}$-action on D is isomorphic to $D' \times (B'_m)^{\mathbb{C}}$ which proves the claim. □

6.3. Proof of the main theorem. Finally we are in position to prove that our main result.

Theorem 6.5. *Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain. Let φ be an automorphism of D such that the group $\Gamma = \langle \varphi \rangle$ is a discrete subgroup of* $\mathrm{Aut}_{\mathcal{O}}(D)$. *Then the quotient $X = D/\Gamma$ is a Stein space.*

Proof. Due to our reduction steps in Section 3 we may assume that φ is contained in a maximal split solvable subgroup $S \subset G$. Moreover, we assume that D is realized as a Siegel domain in $\mathbb{C}^n$ such that S acts by affine-linear transformations on D.

We prove this theorem by induction on $n = \dim_{\mathbb{C}} D$. If $n = 1$, then D is biholomorphically equivalent to the unit disc in $\mathbb{C}$ and the claim follows.

Let $n > 1$ and let us assume that the claim is proven for every $n' < n$. Let $\pi\colon D \to D'$ be the S-equivariant holomorphic submersion with fibers isomorphic to $\mathbb{H}_m$ onto the homogeneous Siegel domain $D' \subset \mathbb{C}^{n-m}$. In the first step we consider the case that Γ is contained in the normal subgroup B_m, hence that π is Γ-invariant. By Corollary 4.5 there exists an m-dimensional subgroup B'_m of B_m which contains Γ such that every B'_m-orbit in $\mathbb{H}_m$ is totally real. Applying Corollary 6.4 we see that the universal globalization D^* of the local $(B'_m)^{\mathbb{C}}$-action on D is a $(B'_m)^{\mathbb{C}}$-principal bundle over D' which must be holomorphically trivial. Therefore D^*/Γ is a Stein manifold and Proposition 4.7 applies to show that X is Stein.

If Γ is not contained in B_m, then we obtain a proper Γ-action on D'. Since Γ normalizes the group $B_m^{\mathbb{C}}$, it follows that Γ acts by bundle automorphisms on the holomorphic fiber bundle $B_m^{\mathbb{C}} \cdot D \to D'$. We conclude from Proposition 6.1 that the quotient bundle

$$(B_m^{\mathbb{C}} \cdot D)/\Gamma \to D'/\Gamma$$

is a holomorphic fiber bundle with fiber $\mathbb{H}_m^*$ and connected structure group $S_{z_0}^{\mathbb{C}}/(S_{z_0}^{\mathbb{C}})_0$. Since the base is Stein by our induction hypothesis, a result of Matsushima and Morimoto (Theorem 6 in [18]) implies that $(B_m^{\mathbb{C}} \cdot D)/\Gamma$ is Stein, hence that X is Stein. This finishes the proof. □

ACKNOWLEDGEMENTS. I would like to thank Prof. Dr. K. Oeljeklaus for many helpful and encouraging discussions on the topics presented here as well as for several invitations to the Université de Provence (Aix-Marseille I) where this paper has been written.

References

[1] D.N. Akhiezer: Lie Group Actions in Complex Analysis, Aspects of Mathematics, E27, Vieweg, Braunschweig, 1995.
[2] A. Borel: *Kählerian coset spaces of semisimple Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 1147–1151.
[3] H. Cartan: Sur les Groupes de Transformations Analytiques, (Exposés mathématiques IX.) Actual. scient. et industr. **198**, Hermann, Paris, 1935.
[4] C. De Fabritiis: *A family of complex manifolds covered by* Δ_n, Complex Variables Theory Appl. **36** (1998), 233–252.
[5] C. De Fabritiis and A. Iannuzzi: *Quotients of the unit ball of* $\mathbb{C}^n$ *for a free action of* $\mathbb{Z}$, J. Anal. Math. **85** (2001), 213–224.
[6] F. Docquier and H. Grauert: *Leisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.
[7] H. Grauert and R. Remmert: Theory of Stein Spaces, Classics in Mathematics, Translated from the German by Alan Huckleberry, Reprint of the 1979 translation, Springer, Berlin, 2004.
[8] P. Heinzner and A. Iannuzzi: *Integration of local actions on holomorphic fiber spaces*, Nagoya Math. J. **146** (1997), 31–53.
[9] A.T. Huckleberry and E. Oeljeklaus: *Homogeneous spaces from a complex analytic viewpoint*; in Manifolds and Lie Groups (Notre Dame, Ind., 1980), Progr. Math. **14**, Birkhäuser, Boston, Mass, 1981, 159–186.
[10] G. Hochschild: The Structure of Lie Groups, Holden-Day, San Francisco, 1965.
[11] A. Iannuzzi: *Characterizations of G-tube domains*, Manuscripta Math. **98** (1999), 425–445.
[12] H. Ishi: *On symplectic representations of normal j-algebras and their application to Xu's realizations of Siegel domains*, Differential Geom. Appl. **24** (2006), 588–612.
[13] A. Iannuzzi, A. Spiro and S. Trapani: *Complexifications of holomorphic actions and the Bergman metric*, Internat. J. Math. **15** (2004), 735–747.
[14] S. Kaneyuki: *On the automorphism groups of homogeneuous bounded domains*, J. Fac. Sci. Univ. Tokyo Sect. I **14** (1967), 89–130.
[15] S. Kaneyuki: Homogeneous Bounded Domains and Siegel Domains, Lecture Notes in Math. **241**, Springer, Berlin, 1971.
[16] W. Kaup, Y. Matsushima and T. Ochiai: *On the automorphisms and equivalences of generalized Siegel domains*, Amer. J. Math. **92** (1970), 475–498.
[17] B. Kostant: *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 413–455.
[18] Y. Matsushima and A. Morimoto: *Sur certains espaces fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France **88** (1960), 137–155.
[19] R.S. Palais: *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323.
[20] I.I. Pyateskii-Shapiro: Automorphic Functions and the Geometry of Classical Domains, Translated from the Russian. Mathematics and Its Applications **8**, Gordon and Breach, New York, 1969.
[21] H.L. Royden: *Holomorphic fiber bundles with hyperbolic fiber*, Proc. Amer. Math. Soc. **43** (1974), 311–312.

[22] N. Steenrod: The Topology of Fibre Bundles, Princeton Mathematical Series **14** Princeton Univ. Press, Princeton, N.J., 1951.

[23] K. Stein: *Überlagerungen holomorph-vollständiger komplexer Räume*, Arch. Math. (Basel) **7** (1956), 354–361.

[24] È.B. Vinberg, S.G. Gindikin and I.I. Pjateckiĭ-Šapiro: *Classification and canonical realization of complex homogeneous bounded domains*, Trudy Moskov. Mat. Obšč. **12** (1963), 359–388.

[25] È.B. Vinberg: *The Morozov-Borel theorem for real Lie groups*, Dokl. Akad. Nauk SSSR **141** (1961), 270–273.

[26] È.B. Vinberg (ed.): Lie Groups and Lie Algebras, III, Encyclopaedia Math. Sci. **41**, Springer, Berlin, 1994.

[27] P.C. Yang: *Geometry of tube domains*; in Complex Analysis of Several Variables (Madison, Wis., 1982), Proc. Sympos. Pure Math. **41**, Amer. Math. Soc. Providence, RI, 1984, 277–283.

Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
D - 44780 Bochum
Germany
Telefon: +49 - 234 - 32 - 23329
Fax: +49 - 234 - 32 - 14498
e-mail: christian.miebach@ruhr-uni-bochum.de

Current address:
LATP-UMR (CNRS) 6632
CMI-Université d'Aix-Marseille I, 39
rue Joliot-Curie
F-13453 Marseille Cedex 13
France
e-mail: miebach@cmi.univ-mrs.fr