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NOTES ON TRIVIAL SOURCE MODULES

Dedicated to Professor S. Endo on his 60th birthday

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1. Introduction

Let G be a finite group and k an algebraically closed field of characteristic p . An indecomposable kG -module with a vertex Q is said to be a *weight module* if its Green correspondent with respect to $(G, Q, N_G(Q))$ is simple. Let B be a block of kG . Alperin [1] conjectured that the number of the weight modules belonging to B equals that of the simple modules in B . If this is the case and a defect group of B a TI set, then it can be shown under some additional assumption that the socles of weight modules are simple, which in turn determine the isomorphism classes of the weight modules; this holds if G is a simple group with a cyclic Sylow p -subgroup. This rather surprising property has been known to hold for finite groups of Lie type of characteristic p . However little is known about general properties of weight modules. In the final section we shall study solvable groups that have only simple weight modules.

Throughout this paper G denotes a finite group and k an algebraically closed field of prime characteristic p . For a kG -module M , $\text{hd}(M)$, $\text{soc}(M)$ and $P(M)$ denote the head, socle and projective cover of M respectively. If N is a kG -module, $N|M$ indicates that N is isomorphic to a direct summand of M , and (N, M) denotes the multiplicity of N as a summand of M . We fix a block B of kG and let D be its defect group. $\text{IRR}(B)$ denotes a full set of non-isomorphic simple modules in B , $l(B)$ its cardinality and $\text{WM}(B)$ a full set of non-isomorphic weight modules belonging to B . Let f be the Green correspondence with respect to (G, D, H) , where $H = N_G(D)$. If $\text{WM}(B|D)$ denotes the subset of $\text{WM}(B)$ consisting of the weight modules with vertices D and b the Brauer correspondent of B in kH , then f induces a bijection between $\text{WM}(B|D)$ and $\text{IRR}(b)$.

The author thanks the referee for improving the proof of Proposition 4 below.

2. Weight modules over blocks with TI defect groups

To begin with, we quote the following as a preliminary lemma.

Lemma 1 (Robinson [8]). *Let T be a subgroup of G . Let M (resp. N) be*

a simple kG (resp. kT)-module. Then we have $(P(M), N^G) = (P(N), M_{|H})$.

Throughout this section D is assumed to be a non-trivial TI subgroup of G , i.e., $D \cap D^x = 1$ if $x \in G \setminus H$. Let $\text{IRR}(B) = \{M_1, \dots, M_r\}$, $\text{IRR}(b) = \{W_1, \dots, W_e\}$ and $n_i = \dim_k W_i$. We set $\text{WM}(B|D) = \{V_i = f^{-1}(W_i); 1 \leq i \leq e\}$. Note that $\text{WM}(B) = \text{WM}(B|D)$. In fact, let $V \in \text{WM}(B)$ and $Q = vx(V)$. We may assume that $D \supset Q$. If $D > Q$, then $N_D(Q) > Q$. On the other hand, since D is a TI set, it follows that $H \supset N_G(Q)$ and hence $N_D(Q)$ is normal in $N_G(Q)$. So, $N_G(Q)/Q$ fails to have a block of defect zero. This is a contradiction, since $f(V)$ is simple and projective as an $N_G(Q)/Q$ -module.

Lemma 2. $M_{i|H} = f(M_i) \oplus N_i$, where $N_{i|D}$ is projective and $f(M_i)_{|D}$ has no projective summand.

Proof. If L is an indecomposable component of N_i with vertex P , then P lies in $\mathfrak{Y}(D, H)$, where

$$\mathfrak{Y}(D, H) = \{Q; Q \subset D^x \cap H, x \in G \setminus H\}.$$

By the Mackey decomposition theorem we have

$$(L \otimes_P kH)_{|D} = \bigoplus_{y \in P \setminus H/D} (L \otimes_P y) \otimes_{P^y \cap D} kD.$$

There is $x \in G \setminus H$ such that $P \subset D^x \cap H$. Hence for any $y \in H$, we have

$$P^y \cap D \subset D^{xy} \cap D \cap H = 1, \text{ as } xy \in G \setminus H.$$

Therefore $(L \otimes_P kH)_{|D}$ is projective. Since $L|L \otimes_P kH$, $L_{|D}$ is also projective.

We next show that $f(M_i)_{|D}$ is projective-free. Actually, this is a general fact. Note that $f(M_i)$ belongs to b and b has the normal defect group D . So, it suffices to show that if L is a non-projective indecomposable b -module, then $L_{|D}$ is projective-free. But since L is D -projective, this is a routine work, using Mackey decomposition.

Lemma 3. $\text{Hom}_{kG}(M_i, V_j) \simeq \text{Hom}_{kH}(f(M_i), W_j)$ for all i, j .

Proof. There is an isomorphism

$$\text{Hom}_{kG}(M_i, V_j) / \text{Tr}_{\mathfrak{X}}^G(M_i, V_j) \simeq \text{Hom}_{kH}(f(M_i), W_j) / \text{Tr}_{\mathfrak{X}}^H(f(M_i), W_j),$$

where $\mathfrak{X} = \mathfrak{X}(D, H) = \{Q; Q \subset D^x \cap D, x \in G \setminus H\}$. However, since D is a TI set, we have $\mathfrak{X} = \{1\}$. And if M and V are non-projective indecomposable and if one of them is simple, then $\text{Tr}_1^G(M, V) = 0$, whence the result follows.

Proposition 4. *Let ε be the block idempotent of B . Then we have*

$$(k_D)^G \varepsilon \simeq \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{i=1}^r a_i P(M_i), \text{ with } a_i = (kD, M_{iD}).$$

Proof. Let

$$k[H/D] = \sum_{i=1}^m n_i W_i$$

be an indecomposable decomposition. Note that no W_j belongs to b if $j \geq e+1$. Since D is a TI set, we have

$$W_i^G = f^{-1}(W_i) \oplus (\text{projectives}).$$

Moreover we know by Green's theorem that $V_j = f^{-1}(W_j)$ does not belong to B if $j \geq e+1$. Thus

$$(k_D)^G = (k_B^H)^G = k[H/D]^G = \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{j=e+1}^m n_j V_j \oplus (\text{projectives}),$$

whence we have

$$(k_D)^G \varepsilon = \bigoplus_{i=1}^e n_i V_i \oplus \bigoplus_{i=1}^r a_i P(M_i), \text{ with } a_i \geq 0$$

and by Lemma 1, $a_i = (kD, M_{iD})$ for $i = 1, 2, \dots, r$.

Theorem 5. *Assume that D is a TI set and that $\text{hd}(f(M_i))$ is simple for all i . Then we have the following:*

- (1) $l(B) \geq l(b)$;
- (2) *the equality sign in the above holds if and only if $\text{soc}(V_i)$ is simple for all i ($1 \leq i \leq e$), in which case we have that*

$$\text{soc}(V_i) \simeq \text{soc}(V_j) \text{ if and only if } V_i \simeq V_j.$$

Proof. From the assumption we may set $\text{hd}(f(M_i)) = W_{\tau(i)}$ ($1 \leq i \leq r$, $1 \leq \tau(i) \leq e$). By lemma 3 we find easily that

- (i) $M_i | \text{soc}(V_{\tau(i)})$ with multiplicity one.
- (ii) If $M_i | \text{soc}(V_j)$, then $j = \tau(i)$.

Now, the second assertion yields that the map

$$\tau: \{1, 2, \dots, r\} \rightarrow \{1, 2, \dots, e\}$$

is a surjection. In fact, for an arbitrary V_j , take M_i such that $M_i | \text{soc}(V_j)$. Then $j = \tau(i)$. Thus τ is surjective. In particular, we have that $r \geq e$.

To show the second part of the theorem, suppose that $\text{soc}(V_j)$ is simple for all j . Then $M_i = \text{soc}(V_{\tau(i)})$ and hence τ is a bijection. Therefore we have $r = e$. If, conversely, $r = e$, then τ is a bijection. This implies by (ii) above that $\text{soc}(V_i)$ must be simple and $V_i \simeq V_j$ if and only if $\text{soc}(V_i) \simeq \text{soc}(V_j)$.

REMARK 1. If the Alperin conjecture is true, we always have $l(B) = l(b)$ when D is a TI set.

3. Weight modules for the symmetric group S_p

In this section we assume that $G = S_p$ is the symmetric group on p letters. If D is a Sylow p -subgroup of G , then D has order p and $C_G(D) = D, H/D \simeq (Z/(p))^*$, the group of units of $Z/(p)$. In particular, it follows that $b = kH$ is the block of kH . Let us write

$$\text{IRR}(b) = \{W_0, \dots, W_{p-2}\}, \text{ where } \dim_k W_i = 1 \ (0 \leq i \leq p-2).$$

If B denotes the principal block of G , then B is a unique block of kG of positive defect and $l(B) = p-1$. The decomposition matrix of B is known. It can be displayed as follows, see James [5].

	φ_0	φ_1	φ_2	\dots	φ_{p-2}
$\chi_0 = (p) = 1_G$	1				
$\chi_1 = (p-1, 1)$	1	1			0
$\chi_2 = (p-2, 1^2)$		1	1		
\vdots			\ddots	\ddots	
$\chi_{p-2} = (2, 1^{p-2})$	0			1	1
$\chi_{p-1} = (1^p)$					1

Since $\chi_i = \varphi_{i-1} + \varphi_i$ and $\deg \chi_i = {}_{p-1}C_i$, we find via induction that $\deg \varphi_i = {}_{p-2}C_i$ ($0 \leq i \leq p-2$). So we can label the simple modules in B such that

$$\text{IRR}(B) = \{M_0, \dots, M_{p-2}\}, \text{ with } m_i = \dim_k M_i = {}_{p-2}C_i.$$

Here we note the following facts on binomial coefficients ${}_n C_i$.

Lemma 6.

$$(1) \quad m_i = {}_{p-2}C_i = \begin{cases} i + 1 \pmod p, & \text{if } i \text{ is even;} \\ p - i - 1 \pmod p, & \text{if } i \text{ is odd.} \end{cases}$$

(2) Suppose that $n \geq 4$. If $2 \leq i \leq n-2$, then ${}_n C_i \geq n+2$.

Now, since H/D is abelian, every principal indecomposable module over kH has dimension p , and thus every non-projective indecomposable module has dimension smaller than p . In particular it follows that $\dim_k f(M_i) < p$. By Lemma 2, we can write

$$M_{i|D} = f(M_i)_{|D} \oplus a_i kD.$$

For $i=0, 1, p-3$ or $p-2$, we have that $m_i < p$ and so $a_i = 0$. This is true for all i , provided $p \leq 5$. Suppose $p > 5$. If $2 \leq i \leq p-4$, then $m_i \geq p$ by Lemma 6(2) and hence $a_i > 0$. This, together with Lemma 6(1) yields that $\dim_k f(M_i) = i+1$ or $p-i-1$ according as whether i is even or odd ($2 \leq i \leq p-4$). Thus we have:

$$a_i = \begin{cases} (m_i - i - 1)/p, & \text{if } i \text{ is even;} \\ (m_i - (p - i - 1))/p, & \text{if } i \text{ is odd.} \end{cases}$$

Now we have the following result by Lemma 1 and Proposition 4.

Proposition 7. Let $WM(B) = \{V_0, \dots, V_{p-2}\}$, where $V_i = f^{-1}(W_i)$, and let $\{U_j; 1 \leq j \leq q\}$ be the set of simple kG -modules belonging to the blocks of defect zero. Then we have

$$(k_D)^G \simeq \bigoplus_{i=0}^{p-2} V_i \oplus \bigoplus_{i=2}^{p-4} a_i P(M_i) \oplus \bigoplus_{i=1}^q (\dim_k U_i / p) U_i.$$

4. Socles of weight modules

In view of Theorem 5, it seems to be natural to consider the following situation:

(#) Every weight module belonging to B has a simple socle, and for $U, V \in WM(B)$, we have

$$\text{soc}(U) \simeq \text{soc}(V) \quad \text{if and only if } U \simeq V.$$

We first remark that

Proposition 8. If G is a simple group with a cyclic Sylow p -subgroup, the condition (#) holds for every block B .

In fact we know that a Sylow p -subgroup is a TI set (Blau[2]) and that $l(B) = l(b)$, hence the result follows from Theorem 5.

On the other hand, we have the following, as is shown on pp.370–371 in

Alperin [1].

Proposition 9 (Alperin). *Let G be a finite group of Lie type of characteristic p . Then the condition (#) holds for every block B .*

Before proceeding let us recall that a simple module is a weight module if and only if it has trivial source (Okuyama [7]).

Now, for the rest of this paper we assume that G is solvable. In this case the Alperin conjecture has been proved by Okuyama.

DEFINITION. A solvable group G is said to be p' -supersolvable if all of its chief composition factors of order prime to p are cyclic.

Proposition 10. *If G is p' -supersolvable, every simple module has trivial source. Hence $WM(B) = IRR(B)$ for every block B .*

Proof. Let G be a counter-example of minimum order and let V be a simple kG -module with source not isomorphic to k . Let K be a maximal abelian normal p' -subgroup of G and W a simple summand of $V_{|K}$. By Fong's reduction and the minimality of G , W must be G -invariant. So W is faithful as K -module and hence K must be central. If $O_p(G/K) = 1$, G/K has a cyclic normal p' -subgroup, say M/K . Then M is abelian, contradicting the choice of K . Thus $O_p(G/K) > 1$, which implies that $O_p(G) > 1$, since K is central. This is a contradiction. The second statement is clear since the number of weight modules belonging to B equals $l(B)$.

Now we give a definition:

DEFINITION. A finite group is said to be a CR1-group if all of its characteristic abelian subgroups are cyclic.

We say that the group G involves a group T provided there are subgroups $L \triangleright M$ of G such that $L/M \simeq T$. For a prime number q , let us denote a Sylow q -subgroup of G by G_q .

Theorem 11. *Let G be a solvable group and suppose that G_q involves no non-abelian CR1-group for each prime q different from p . Then the conclusion of Proposition 10 holds.*

Proof. We shall show that every simple kG -module has trivial source by the induction on the order of G . We may assume $O_p(G) = 1$. Let K be the Fitting subgroup of G , so we have that $C_G(K) \subset K$. If K is cyclic, $\text{Aut}(K)$ is abelian and

so is G/K . Thus G is supersolvable and the result follows from Proposition 10. If K is non-cyclic, our assumption implies that G has a non-cyclic abelian normal q -subgroup, say L , for some prime q . Let V be a simple kG -module and W a simple summand of $V|_L$. If the inertial group of W is proper, the result follows by induction. If W is G -invariant, $N = \text{Ker}(W)$ is a non-trivial normal subgroup of G . Then we get the result by applying the inductive hypothesis to G/N .

REMARK 2. The CR1- q -groups are classified (Gorenstein [3], Chap. 5). In particular, a non-abelian CR1- q -group contains D_3 or Q_3 if $q=2$, while it contains $M(q)$ if q is odd, where

$$M(q) = \langle x, y, z; x^q = y^q = z^q = 1, [x, z] = [y, z] = 1, [x, y] = z \rangle,$$

which has order q^3 and exponent q .

REMARK 3. One may show that the following q -group Q involves no non-abelian CR1- q -group:

$$Q = \langle x, y; x^{q^a} = y^{q^b} = 1, x^y = x^{1+q^{a-1}} \rangle,$$

where $a \geq 2$, $b \geq 1$, and $a \geq 3$ if $q=2$.

In fact every proper subgroup of Q is abelian (cf. Huppert [4] III, Aufgaben 22). So it suffices to show that Q has no factor group isomorphic to D_3 , Q_3 or $M(q)$, which will be easily done.

REMARK 4. Let $G = \langle \sigma \rangle$ be the semidirect product, where σ is an automorphism of the quaternion group Q_3 of order 3. Then kG has a simple module whose source is not trivial, where k is of characteristic 3. On the other hand, if G is the symmetric group S_4 , every simple kG -module has trivial source, k being the same as above. In both groups the Sylow 2-subgroups are CR1-groups.

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