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The Thickening of Combinatorial n-Manifolds in (n+1)-Space

By Hiroshi NOGUCHI

1. Introduction

The Schönflies conjecture for dimension \( n \) is the following statement:

Let a combinatorial \((n-1)\)-sphere \( S^{n-1} \) be piecewise linearly imbedded in Euclidean \( n \)-space \( R^n \). Then the closure of the bounded component of \( R^n - S^{n-1} \) is a combinatorial \( n \)-cell. For \( n \leq 3 \) this has been affirmatively proved, see Alexander [1], Graeub [2] and Moise [5].

The purpose of this paper is to prove the following (Theorem 3 in section 6): Let a combinatorial, closed (= compact and without boundary), orientable \( n \)-manifold \( M^n \) be imbedded as a subcomplex of a combinatorial, orientable \((n+1)\)-manifold \( W^{n+1} \) without boundary. Let \( U(M^n, W^{n+1}) \) be a regular neighborhood of \( M^n \) in \( W^{n+1} \). Suppose that the Schönflies conjecture is true for dimension \( \leq n \). Then there is a piecewise linear homeomorphism into \( \theta: M^n \times J \to W^{n+1} \) such that \( \theta(x,0) = x \) for all \( x \in M^n \) and such that \( \theta(M^n \times J) = U(M^n, W^{n+1}) \), where \( J \) is the interval \(-1 \leq s \leq 1\). (The regular neighborhood \( U(M^n, W^{n+1}) \) in this paper is necessarily a closed neighborhood of \( M^n \) in \( W^{n+1} \) in the sense of the set-theory, see Definition 1 in section 3. The simplicial subdivision of \( M^n \) gives, in the usual way [3], p. 35, a simplicial subdivision of \( M^n \times J \); and the mapping \( \theta \) is to be piecewise linear relative to such an induced simplicial subdivision of \( M^n \times J \).

In fact, the above theorem is a consequent of the following main theorem (Theorem 2 in section 5): Let a combinatorial, closed \( n \)-manifold \( M_i^n \) be imbedded as a subcomplex of a combinatorial, oriented (= orientable, oriented) \((n+1)\)-manifold \( W_i^{n+1} \) without boundary, \( i = 1, 2 \). Let \( U(M_i^n, W_i^{n+1}) \) be a regular neighborhood of \( M_i^n \) in \( W_i^{n+1} \), and \( \phi: M_i^n \to M_2^n \) be a piecewise linear homeomorphism onto. Suppose that the Schönflies conjecture is true for dimension \( \leq n \). Then there is a piecewise linear homeomorphism onto \( \psi: U(M_i^n, W_i^{n+1}) \to U(M_2^n, W_2^{n+1}) \) such that \( \psi|M_i^n = \phi \), and such that the oriented image of oriented \( U(M_i^n, W_i^{n+1}) \) is the oriented \( U(M_2^n, W_2^{n+1}) \), where the orientation of \( U(M_i^n, W_i^{n+1}) \) is induced by that of \( W_i^{n+1} \). Another application of Theorem 2 is Theorem 4 in section 6.

In the proofs of these theorems, we shall make extensive use of
combinatorial methods and results of J. H. C. Whitehead [7] and V. K. A. M. Gugenheim [3], [4]. In particular, the following (Theorem 1 in section 3) is a modification of results of Whitehead. Let a finite polyhedron $P$ be imbedded as a subcomplex of a combinatorial manifold $W$ without boundary, and let $U_i(P, W)$ be regular neighborhoods of $P$ in $W$, $i = 1, 2$. Then there is a piecewise linear homeomorphism onto $\psi: W \rightarrow W$ such that $\psi(U_i(P, W)) = U_2(P, W)$ and such that $\psi|P =$ identity where $\psi$ is an orientation preserving piecewise linear homeomorphism onto if $W$ is orientable.

The exposition is as follows: In section 2 Definitions and notation will be explained. In section 3 a modification of the regular neighborhood of Whitehead and Theorem 1 will be given. Section 4 will prepare the preliminary lemmas and notation needed in the latter. Section 5 will be devoted to prove Theorem 2. In section 6 applications of Theorem 2 will be stated.

In his delightful paper “Embeddings of spheres”, Bull. Amer. Math. Soc., vol. 65 (1959), pp. 59–65, Professor B. Mazur mentioned an unpublished lemma of mine. The present paper is the revised version of the manuscript in question.

It is a pleasure to express my gratitude to Professor V. K. A. M. Gugenheim for many useful suggestions, and to Professors E. E. Moise and J. R. Stallings for their advice during revising the paper.

2. Definition and Notation

By a simplex we shall always mean a closed Euclidean simplex and the word complex will mean a closed, rectilinear, locally finite, simplicial complex of some Euclidean space. If $K$ is a complex, then $|K|$ denotes the point-set which is the union of the simplices of $K$. Such a set $|K|$ will be called a polyhedron, and $K$ will be called a simplicial subdivision of the polyhedron $|K|$. $K'$ and $K''$ will stand for the first and second barycentric subdivisions of $K$. Let $K$ be a $q$-complex. We say that $K$ is homogeneous (see, [6], p. 48), if every $p$-simplex ($p < q$) is a face of at least one $q$-simplex. Then $\partial K$ denotes its boundary (modulo 2), that is, the totality of all $(q-1)$-simplices which are incident to an odd number of $q$-simplices; and if $P = |K|$, then $\partial P$ denotes the polyhedron $|\partial K|$. The point set $P - \partial P$ will be called the interior of $P$, and will be denoted by $Int P$. A polyhedron will be called finite if it has a simplicial subdivision which is a finite complex.

Let $K$ be a complex and $\Delta$ one of its simplex. The set of all simplices of $K$ having $\Delta$ as a face is called the star of $\Delta$ in $K$, whose polyhedron is denoted by $St(\Delta, K)$ and is called the star set of $\Delta$ in $K$. The set of simplices of $K$ which are faces opposite $\Delta$ in some simplex
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of the star of \( \Delta \) in \( K \) is called the link of \( \Delta \) in \( K \), whose polyhedron is denoted by \( Lk(\Delta, K) \) and is called the link set of \( \Delta \) in \( K \). Let \( x \) be a point of \( |K| \). We denote by \( St(x, K) \) the point set of points of all simplices of \( K \) containing \( x \) and by \( Lk(x, K) \) the point set of points of simplices of \( St(x, K) \) not containing \( x \). If \( x \) is a vertex of \( K \), these definitions coincide with those given just above, see [4], p. 134. Let \( L \) be a subcomplex of \( K \). Then \( N(L, K) \) will stand for the point set of points of all simplices of \( K \) meeting \( |L| \), and will be called the star neighborhood of \( L \) in \( K \), see [7], p. 251.

As usual two complexes \( K_1, K_2 \) are combinatorially equivalent if \( K_1 \) and \( K_2 \) have isomorphic simplicial subdivisions \( L_1, L_2 \). In this case, we shall say that the polyhedra \( |K_1| \) and \( |K_2| \) are equivalent. By a \( q \)-cell we shall mean a polyhedron equivalent to \( q \)-simplex, by a \( q \)-sphere one equivalent to the boundary of \( (q + 1) \)-simplex. When polyhedra \(|K_1| \) and \(|K_2| \) are equivalent, there is a homeomorphism

\[
\phi : |K_1| \leftrightarrow |K_2|
\]

which maps each simplex of \( L_1 \) linearly onto the corresponding simplex of \( L_2 \). This \( \phi \) is simplicial relative to \( L_1 \) and \( L_2 \), and piecewise linear relative to the original complexes \( K_1 \) and \( K_2 \). All mappings used in this paper will be piecewise linear homeomorphisms. Thus, whenever we mention a homeomorphism, it should be understood that we mean a piecewise linear homeomorphism. If the mapping is onto, this will be indicated by a double-headed arrow, as in the displayed formula above. If \( P, Q \) are polyhedra and \( \phi : P \rightarrow Q \) is a homeomorphism of \( P \) into \( Q \), and \( \partial P \) is well defined, then \( \partial \phi \) denotes the homeomorphism \( \phi|\partial P \).

A complex \( K \) is called the combinatorial \( q \)-manifold if for each point \( x \) of \( |K| \), \( St(x, K) \) is the \( q \)-cell, alternatively \( Lk(x, K) \) is the \((q-1)\)-cell if \( x \in |\partial K| \) and \( Lk(x, K) \) is the \((q-1)\)-sphere if \( x \in \text{Int} \ |K| \). (See [3], p. 31) A polyhedron is called the combinatorial \( q \)-manifold if it has a simplicial subdivision which is a combinatorial \( q \)-manifold. Whenever we mention a manifold, it should be understood that we mean a combinatorial, connected manifold. We shall call a finite manifold closed if it has no boundary. For the sake of convenience, a polyhedron \( P \) in a polyhedron \( Q \) will stand for the polyhedron \( P \) being piecewise linearly imbedded as a subcomplex of a simplicial subdivision of the polyhedron \( Q \).

If a polyhedron \( M \) is an orientable \( q \)-manifold, we shall denote by \( \langle M \rangle \) the oriented manifold obtained by assigning one of the possible orientations; \( M \) with the opposite orientation will be denoted by \( -\langle M \rangle \). As a matter of convention, \( 1 \langle M \rangle, -1 \langle M \rangle \) will mean \( \langle M \rangle, -\langle M \rangle \) respectively. If \( N \subset M \) is an orientable \( q \)-manifold, we shall write
\langle N \rangle \subset \langle M \rangle$ if $\langle N \rangle$ has been oriented by giving to each of $q$–simplices the orientation of $\langle M \rangle$. If $\partial M$ is not empty and orientable, by $\partial \langle M \rangle$ we shall denote the oriented $\partial M$ obtained by giving to each of its $(q-1)$–simplices the orientation coherently induced by that of the oriented $q$–simplex $\langle \Delta \rangle \subset \langle M \rangle$ which is incident to the former. Let $\langle M \rangle$, $\langle N \rangle$ be oriented $q$–manifolds and $\phi: M \to N$ be a homeomorphism. If the orientation of $\langle N \rangle$ and the orientation induced by $\phi$ and that of $\langle M \rangle$ are identical, we shall write $\phi: \langle M \rangle \to \langle N \rangle$, and denote the oriented image of $M$ by $\phi \langle M \rangle$.

Let $P, Q \subset M$ be polyhedra, $M$ be an orientable manifold and $\phi: M \to M$ be an orientation preserving homeomorphism such that $\phi P = Q$. In this case, we shall say that $P, Q$ are congruent in $M$.

By $I$ and $J$ we shall denote the linear intervals $0 \leq t \leq 1$ and $-1 \leq s \leq 1$ respectively. We shall denote by $Cl_x X$ or $Cl X$ the closure of $X$ in $Y$. Let $X, Y$ be point sets of some Euclidean space. We shall denote by $XY = YX$ the join of $X$ and $Y$, that is, the set of points $tx + (1-t)y$ where $x \in Y, y \in Y$ and $t \in I$, using vector notation.

3. The Regular Neighborhood

Let $P$ be a finite polyhedron in an $m$–manifold $V$. The regular neighborhood of $P$ in $V$, defined by Whitehead [7], p. 297, is an $m$–manifold $U(P, V)$ contained in $V$ and containing $P$, which contracts geometrically into $P$. The following results of Whitehead are necessary in this paper, see [7], pp. 293–296.

(1) $N(K', L')$ is a regular neighborhood of $P$ in $V$ where $K, L$ are simplicial subdivisions of $P, V$ and where $K$ is a subcomplex of $L$.

(2) If $P$ is a cell, then $U(P, V)$ is an $m$–cell.

The regular neighborhood defined above is not necessarily a neighborhood in the point-set theoretic sense and Theorem 1 in this section does not hold for this regular neighborhood. Therefore we shall put some restrictions to it as follows.

Definition 1. Let $P$ be a finite polyhedron in an $m$–manifold $W$ without boundary. The regular neighborhood $U(P, W)$ of $P$ in $W$ means an $m$–manifold contained in $W$ and containing $P$ in the interior, which contracts geometrically into $P$.

In sections 3 and 4 however we shall use the regular neighborhood defined by Whitehead which will be called the regular neighborhood in the weak sense there.

Lemma 1. The properties (1) and (2) above mentioned still hold for the regular neighborhood.
Proof. Since the regular neighborhood is also the regular neighborhood in the weak sense, it is enough to prove (1) that \( N(K'', L'') \) contains \( P \) in the interior where \( K, L \) are simplicial subdivisions of \( P, W \) and where \( K \) is a subcomplex of \( L \). Let \( x \) be a point \( P \). Then \( St(x, L'') \) is an \( m \)-cell containing \( x \) in the interior, for \( W \) is an \( m \)-manifold without boundary. Since \( \text{Int} \ St(x, L'') \) is open in \( W \) and contained in \( \text{Int} \ N(K'', L'') \), \( P \) is contained in \( \text{Int} \ N(K'', L'') \). The property (2) follows immediately from the property (2) for the regular neighborhood in the weak sense.

Let \( N \) be a \( q \)-manifold and \( C \) a \( q \)-cell such that

\[
N \cap C = \partial N \cap \partial C = F,
\]
a \((q-1)\)-cell. We shall say that \( N \) and \( C \) have regular contact in \( F \). In this situation, a transformation

\[
N \Rightarrow N \cup C,
\]
or the resultant of a finite sequence of such transformations will be called the regular expansion of \( N \), see [7], p. 291. Then, suppose that \( N \) is in an \( m \)-manifold \( W \) without boundary. Let \( D \subseteq N \) be a \( q \)-cell such that

\[
\partial N \cap \partial D \supseteq F.
\]
Let \( G \subseteq W \) be an \( m \)-cell containing \( C \cup D \) in the interior, and

\[
\theta : G \rightarrow R \text{ be a homeomorphism such that } \theta(C \cup D) = \Delta,
\]
a \( q \)-simplex in Euclidean \( m \)-space \( R \). Then we call \( C \) a flat attachment to \( N \), see [3], p. 33.

**Lemma 2.** Let \( N \) be an \( m \)-manifold in an \( m \)-manifold \( W \) without boundary, and \( N \) and an \( m \)-cell \( C \subset W \) have regular contact in an \((m-1)\)-cell \( F \). Then \( C \) is a flat attachment to \( N \).

Proof. Let \( D \) be a regular neighborhood \( U(F, N) \) in the weak sense. By the property (2) of Whitehead, \( D \) is an \( m \)-cell in \( N \) and \( \partial N \cap \partial D \supseteq F \). Since \( C \) and \( D \) have regular contact in \( F \), \( C \) and \( C \cup D \) are equivalent, see [3], p. 35, and \( C \cup D \) is an \( m \)-cell. By (2) of Lemma 1, a regular neighborhood \( U(C \cup D, W) = G \), say, is an \( m \)-cell containing \( C \cup D \) in the interior. Let \( \theta' : G \rightarrow R \) be a homeomorphism. This is possible, for \( G \) is an \( m \)-cell. By Theorem 3 in [3], p. 32, there is a homeomorphism \( \phi : R \leftrightarrow R \) such that \( \phi \theta'(C \cup D) = \Delta \), an \( m \)-simplex in \( R \). Therefore \( \theta = \phi \theta' : G \rightarrow R \) is a homeomorphism such that \( \theta(C \cup D) = \Delta \). Hence \( C \) is a flat attachment to \( N \).
Lemma 3. Let \( P \) be a finite polyhedron in an \( m \)-manifold \( W \) without boundary. Let \( U_i(P, W) \) and \( U_2(P, W) \) be regular neighborhoods of \( P \) in \( W \) such that \( U_i(P, W) \) expands regularly into \( U_2(P, W) \). Then there is a homeomorphism \( \psi : W \to W \) such that

\[
\psi(U_i(P, W)) = U_2(P, W) \quad \text{and} \quad \psi|P = \text{identity}
\]

where \( \psi \) is an orientation preserving homeomorphism if \( W \) is orientable.

Proof. Let \( N_1, \ldots, N_k \) be a sequence of \( m \)-manifolds in \( W \) such that \( N_i = U_i(P, W) \), \( N_h = U_2(P, W) \) and \( N_{i-1} = N_i \cup C_i \) is a regular expansion where \( N_i \) and an \( m \)-cell \( C_i \) have regular contact in an \((m-1)\)-cell \( F_i \) (\( i = 2, \ldots, k \)). By Lemma 2, \( C_i \) is a flat attachment to \( N_{i-1} \). Namely there are \( m \)-cell \( G_i \subseteq W \) containing \( C_i \) and \( D_i = U(F_i, N_{i-1}) \) in the interior, and a homeomorphism \( \theta_i : G_i \to \mathbb{R}^m \) such that \( \theta_i(C_i \cap D_i) = \Delta \), an \( m \)-simplex. By Theorem 6 in [3], pp. 48-49, there is a homeomorphism

\[
\eta_i : \theta_i G_i \leftrightarrow \theta_i G_i
\]

such that

\[
\eta_i|\theta_i(\partial G_i \cup (\text{Cl}(N_{i-1} \cap D_i) \setminus G_i)) = \text{identity and} \quad \eta_i \theta_i D_i = \Delta.
\]

Then \( \psi_i : W \to W \) defined by taking

\[
\psi_i|\text{Cl}(W \setminus G_i) = \text{identity and} \quad \psi_i|G_i = \theta_i^{-1} \eta_i \theta_i
\]

is a homeomorphism such that

\[
\psi_i N_{i-1} = N_i \quad \text{and} \quad \psi_i|\text{Cl}(N_{i-1} \setminus D_i) = \text{identity},
\]

where \( \psi_i \) is an orientation preserving homeomorphism if \( W \) is orientable.

In this situation, \( D_i = U(F_i, N_{i-1}) \) will be taken so that \( D_i \) does not meet \( P \). This is possible, because \( P \subseteq \text{Int} N_i \subseteq \text{Int} N_{i-1} \) and by the property (2) of Whitehead if we give a sufficiently fine simplicial subdivision to \( N_{i-1} \), then \( D_i \) may be arbitrarily near \( F_i \) which is contained in \( \partial N_{i-1} \). Then \( P \subseteq \text{Cl}(N_{i-1} \setminus D_i) \) and \( \psi_i|P = \text{identity} \).

Hence \( \psi : W \to W \) defined by taking

\[
\psi = \psi_h \cdots \psi_2
\]

is the required homeomorphism.

Theorem 1. Let \( P \) be a finite polyhedron in an manifold \( W \) without boundary. Then for any two regular neighborhoods \( U_i(P, W) \) and \( U_2(P, W) \) of \( P \) in \( W \) there is a homeomorphism \( \psi : W \to W \) such that

\[
\psi(U_i(P, W)) = U_2(P, W), \psi|P = \text{identity},
\]
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where $\psi$ is an orientation preserving homeomorphism if $W$ is orientable.

Proof. Let $K, L$ be simplicial subdivisions of $P, W$ where $K$ is a subcomplex of $L$ and where each of $U_i(P, W)$, considering it as subcomplex of $L$, contracts formally into $K$. Then by Whitehead [7], p. 296, we have the following

$$U_i(P, W) \Rightarrow N(U_i(P, W), L'') \Leftarrow N(K'', L''),$$

where $i=1, 2$ and $\Rightarrow$ means the regular expansion.

By the property (1) of Whitehead and Definition 1, $N(U_i(P, W), L'')$ is a regular neighborhood of $U_i(P, W)$ in $W$ and a regular neighborhood of $P$ in $W$. By Lemma 3 we have homeomorphisms $\psi_i, \rho_i : W \leftrightarrow W$ such that

$$\psi_iU_i(P, W) = N(U_i(P, W), L''),
\rho_iN(U_i(P, W), L'') = N(K'', L''),$$

and

$$\psi_i|P = \rho_i|P = \text{identity},$$

where $\psi_i, \rho_i$ are orientation preserving homeomorphisms if $W$ is orientable. Therefore

$$\psi = \psi_2^{-1}\rho_2^{-1}\rho_1\psi_1 : W \leftrightarrow W$$

is the required homeomorphism.

4. Preliminaries for Thickening

Let $M$ be a closed $n$-manifold in an $m$-manifold $W$ without boundary, where $n \leq m$.

Notation 1. By $K$ and $L$ we shall denote simplicial subdivisions of $M$ and $W$ respectively where $K$ is a subcomplex of $L$. By $\Delta$ we shall denote a simplex of $L'$ and then $x$ will denote the barycenter of $\Delta$. If $\Delta$ is an $(n-q)$-simplex of $K'$, we shall denote by $\nabla$ the $q$-cell dual to $\Delta$ in $K'$ with the simplicial subdivision $Y$ which is a subcomplex of $K''$, and by $\Box$ we shall denote the $q+(m-n)$-cell dual to $\Delta$ in $L'$ with the simplicial subdivision $Z$ which is a subcomplex of $L''$. Let us denote the $q$-skeleton of $K'$ by $(K')^q$ where $(K')^{-1}$ means the empty set. By $\mathcal{R}^q$ we shall denote the polyhedron of the $q$-cellcomplex which consists of all the dual cells $\nabla$ and by $\mathcal{R}^{q+(m-n)}$ the polyhedron of the $q+(m-n)$-cellcomplex which consists of all the dual cells $\Box$, where $\Delta$ ranges over $K'-(K')^{n-q-1}$. 
Lemma 4. Let $\Delta$ be an $(n-q)$-simplex of $K'$. Then

$$\bigcup_j \square_j = N(\partial Y, \partial Z)$$

and $\bigcup_j \square_j$ is a regular neighborhood of the $(q-1)$-sphere $\partial V$ in the $(q+(m-n)-1)$-sphere $\partial \square$, where $\Delta_j$ ranges over the $(n-q+1)$-simplices of $K'$ incident to $\Delta$.

Proof. As a matter of convenience $N$ will stand for $N(\partial Y, \partial Z)$. If $\Delta_\alpha, \ldots, \Delta_\beta$ are simplices of $K'$ which have $\Delta$ as a proper face and $\Delta_\alpha \subset \ldots \subset \Delta_\beta$, then by the proof of Theorem II of [6], p. 230, the join $x_\alpha \cdots x_\beta$ is a simplex of $\partial Y$ and conversely every simplex of $\partial Y$ is such a join. Similarly a join $x_\alpha \cdots x_\beta$ is a simplex of $\partial Z$ if and only if the simplices $\Delta_\beta, \ldots, \Delta_\alpha$ are in $L'$, which have $\Delta$ as a proper face and $\Delta_\beta \subset \ldots \subset \Delta_\alpha$.

By the definition of $N$, a simplex $B=x_\alpha \cdots x_\beta$ of $\partial Z$ is in $N$ if and only if there is a simplex $A=x_\alpha \cdots x_\alpha$ of $\partial Y$ such that $AB$ is a simplex of $\partial Z$. Let $\Delta_\alpha \subset \ldots \subset \Delta_\beta$. Then $\Delta_\alpha$ is a simplex of $K'$ having $\Delta$ as a proper face, and there is an $(n-q+1)$-simplex $\Delta_\beta$ of $K'$, incident to $\Delta_\alpha$, which is a face of $\Delta_\beta$. Then the simplex $x_\alpha \cdots x_\alpha$ is in the complex $Z_j$, and $\bigcup_j \square_j \supset N$. Conversely every $(q+(m-n)-1)$-simplex $C$ of $\square_j$ is written by $x_\alpha \cdots x_\alpha$ where $\Delta_\alpha, \ldots, \Delta_\alpha$ are simplices of $L'$, $1 \leq i \leq q+(m-n)-1$, such that $\Delta_\beta \subset \Delta_\beta \subset \ldots \subset \Delta_\beta$. Since $x_\alpha$ is a vertex of $\partial Y$, $C$ is in $N$. Since $Z_j \subset L''$ is a $(q+(m-n)-1)$-homogeneous complex, $\bigcup_j \square_j \supset N$. Therefore $\bigcup_j \square_j = N$.

Let $p$ be a point in $\partial V$. Then $St(p, \partial Z)$ is a $(q+(m-n)-1)$-cell containing $p$ in the interior, for $\partial \square$ is a $(q+(m-n)-1)$-sphere. Since $Int St(p, \partial Z) \subset Int N$, we have $\partial V \subset Int N$.

It remains to prove that $N$ is a regular neighborhood of $\partial V$ in $\partial \square$ in the weak sense. To show this we first prove the following three assertions (see, [7], p. 293).

(a) None of the simplices and its interior of $\partial Z-\partial Y$ has all its vertices in $\partial V$.

(b) If a simplex $A$ of $\partial Z$ does not meet $\partial V$, then $\partial V \setminus Lk(A, \partial Z)$ is a cell (possibly the empty set).

(c) If $B$ is a simplex of $\partial Z$, then the complexes $\partial Y \setminus Lk(B, \partial Z)$ and $Lk(B, \partial Z)$ also satisfy the conditions (a) and (b).

Proof of (a). If a simplex $x_\alpha \cdots x_\alpha$ of $\partial Z$ or its interior has all its vertices in $\partial V$, then the simplices $\Delta_\alpha, \ldots, \Delta_\alpha$ are in $K'$. Hence the simplex $x_\alpha \cdots x_\alpha$ and its interior are in $\partial V$, proving (a).

Proof of (b). Let $A=x_\alpha \cdots x_\alpha$ where $\Delta_\alpha, \ldots, \Delta_\alpha$ are simplices of $L'$
having $\Delta$ as a proper face and $\Delta_a \subset \cdots \subset \Delta_s$. Since $A$ does not meet $\partial V$, there does not exist $i$ among $a, \cdots, \alpha$ such that $\Delta_i$ is in $K'$. In particular $\Delta_s$ is not in $K'$.

Suppose that $\partial V \cap Lk(A, \partial Z)$ is not empty. Since $Lk(A, \partial Z) = \bigcup_{B} Lk(A, B)$ where $B$ ranges over all simplices of $\partial Z$ having $A$ as a face, there is a $B$ for which $\partial V \cap Lk(A, B)$ is not empty. Let $Lk(A, B)=C$, a simplex of $\partial Z$. If $\Delta_s \subset \Delta$, and $\Delta_i$ is in $K'$, then $\Delta_s$ is also in $K'$. Then $C = x_c \cdots x_{cs} \cdots x_e$, where $\Delta_c, \cdots, \Delta_e$ are simplices of $K'$ having $\Delta$ as a proper face and $\Delta_c, \cdots, \Delta_v$ are not simplices of $K'$ but simplices of $L'$ such that $\Delta_c \subset \cdots \subset \Delta_s \subset \Delta_f \subset \cdots \subset \Delta_v$. Then $\partial V \cap Lk(A, B) = \partial V \cap C = x_c \cdots x_e$ which is not empty. Since $B$ is a simplex of $L'$ having $A, C$ as faces, $\Delta_e$ is a face of $\Delta_a$, which is in $K'$ and has $\Delta$ as a proper face.

Let $p \geq n - q + 1$ be the dimension of a face of $\Delta_a$ as follows. There is a $p$-face $\Delta^p$ having $\Delta$ as a proper face, which is in $K'$, and there is no $s$-face $\Delta^s$ ($s > p$) having $\Delta$ as a proper face, which is in $K'$. This is possible, because $\Delta_s$ is in $K'$ and $\Delta_a$ is not in $K'$, and both of which have $\Delta$ as a proper face. Suppose that there is an $r$-face $\Delta^r$ of $\Delta_a$, in $K'$, having $\Delta$ as a proper face, and that none of $\Delta^p$ and $\Delta^r$ is a face of the other. Then all vertices of the simplex $\Delta^p \Delta^r$, a face of $\Delta_a$, is in $K'$ and the simplex $\Delta^p \Delta^r$ is in $L'$. The dimension of $\Delta^p \Delta^r$ is at least $p + 1$. By the maximum property of $p$, $\Delta^p \Delta^r$ is not in $K'$. This contradicts the well known result [7], p. 294, that no simplex of $L'$ has all its vertices in $K'$. Therefore every face of $\Delta_a$ which is in $K'$ and has $\Delta$ as a proper face is a face of $\Delta^p$. Therefore every simplex of $\partial V \cap Lk(A, \partial Z)$ is the join $x_c \cdots x_h$ where $\Delta \subset \cdots \subset \Delta_s \subset \Delta_f \subset \cdots \subset \Delta_v$, the dimension of $\Delta_s \geq n - q + 1$ and $p \leq n$, and conversely.

By $\Delta_a$ we denote the $(p-n+q-1)$-simplex such that $\Delta^p = \Delta \Delta_a$. Then every simplex $x_c \cdots x_h$ is in $Lk(x, \Delta_a)$, where $x \Delta_a$ will be thought of as a subcomplex of $K''$. Conversely every simplex of $Lk(x, \Delta_a)$ is such a join. Hence $\partial V \cap Lk(A, \partial Z) = Lk(x, \Delta_a)$ which is a $(p-n+q-1)$-cell, because $x \Delta_a$ is the $(p-n+q)$-simplex containing $x$ on the boundary, proving (b).

Proof of (c). For (a) is obviously satisfied. If $A$ is a simplex of $Lk(B, \partial Z)$ not meeting $\partial V \cap Lk(B, \partial Z)$, then $AB$ is a simplex in $\partial Z$ not meeting $\partial V$ and $\partial V \cap Lk(AB, \partial Z)$ is a cell, by (b). Since $Lk(AB, \partial Z) = Lk(A, L(B, \partial Z))$ and $Lk(AB, \partial Z) \subset Lk(B, \partial Z)$,

$$\partial V \cap Lk(B, \partial Z) \cap Lk(A, L(B, \partial Z)) = \partial V \cap Lk(AB, \partial Z),$$

a cell, satisfying (b) and also proving (c).

Finally we shall prove that $N$ is a regular neighborhood of $\partial V$ in
in the weak sense (see, [7], p. 293–294). This will be proved by induction on the dimension \(q + (m-n) - 1\) of \(\partial \square\). This is trivial if \(q + (m-n) - 1 = 0\). By (a) and the definition of \(N\), \(N\) is a normal neighborhood of \(\partial Y\) (see, [7], p. 250). Since \(\partial V \subset N\) and \(Lk(A, N) = Lk(A, \partial Z) \cap N\), we have that \(\partial V \cap Lk(A, N) = \partial V \cap Lk(A, \partial Z)\), which is a cell, by (b), where \(A\) is a simplex of \(N\) not meeting \(\partial V\). Then \(N\) is a contractible neighborhood of \(\partial Y\), [7], p. 250. By Theorem 2 of [7], p. 250, \(N\) contracts into \(\partial Y\). It remains to prove that \(N\) is a manifold. Let \(b\) be a vertex in \(N\). If \(b\) is in \(\partial Y\), then \(Lk(b, N) = Lk(b, \partial Z)\) which is a \((q+(m-n)-2)\)-sphere, for \(\partial V\) is a \((q+(m-n)-1)\)-sphere. Suppose that \(b\) is not in \(\partial Y\). A simplex \(Ab\) of \(\partial Z\) meets \(\partial V\) if and only if \(A\) meets \(\partial V\). Therefore \(Lk(b, N) = N(\partial Y \cap Lk(b, \partial Z), Lk(b, \partial Z))\). By the hypothesis of induction and (c), \(Lk(b, N)\) is a regular neighborhood of the cell \(\partial V \cap Lk(b, \partial Z)\) in \(Lk(b, \partial Z)\) in the weak sense, and \(Lk(b, N)\) is a \((q+(m-n)-2)\)-cell, by the property (2) of Whitehead in section 3. Therefore \(N\) is a manifold, and a regular neighborhood of \(\partial V\) in \(\partial \square\) in the weak sense, completing the proof of Lemma 4.

**Definition 2.** Let us take a finite sequence \(\alpha = \Delta_1, \ldots, \Delta_a\) of successively incident simplices of \(K\) such that \(\Delta_1 = \Delta^*, \) a fixed \(n\)-simplex. We call \(\alpha\) the *way* in \(K\) to \(\Delta_a\). By <\(\Delta^*\)> we shall denote the oriented \(n\)-simplex. Since \(\Delta_1 = \Delta^*\), we have the well defined oriented simplex, written \(<\Delta_a>\), inductively such that \(<\Delta_i>\) is either \(\partial<\Delta_i>-<\Delta_{i-1}>\) or \(<\Delta_i> \subset \partial<\Delta_{i-1}>\) according the case.

Let \(M_i\) be a closed \(n\)-manifold in an oriented \(m\)-manifold \(<W_i>\) without boundary where \(n<m\) and \(i=1,2\). Let \(\phi: M_i \leftrightarrow M_2\) be a homeomorphism.

**Notation 2.** Using Notation 1, suppose that \(\phi\) is simplicial relative to the complexes \(K_1\) and \(K_2\) which are isomorphic under the isomorphism induced by \(\phi\). Then \(\phi\) is also simplicial relative to \(K_1'\) and \(K_2'\), and relative to \(K_1''\) and \(K_2''\). From now on by \(\Delta_i, \Delta_{ij}\) we denote simplices of \(K_i\) satisfying \(\phi \Delta_i = \Delta_2, \phi \Delta_{ij} = \Delta_{2j}\). Then \(\phi \nabla_i = \nabla_2\) and thus, \(\phi\) will induce a homeomorphism onto between the polyhedra \(S_i^n\) and \(S_2^n\). The correspondence between cells of \(S_i^n\) and cells of \(S_2^n\) induced by the correspondence between \(\square_i\) and \(\square_2\) is one-to-one. By \(<\Delta_i^*>\) we shall denote \(n\)-simplices with orientations such that \(\phi<\Delta_i^*> = <\Delta_2^*>\), which will keep fixed in the rest of the paper. Let \(\alpha\) be a way in \(K_1'\) to \(\Delta_i\), then the simplices of \(K_2'\) corresponding the simplices of the way in \(K_1'\) will be naturally thought of as a way in \(K_2'\) to \(\Delta_2\), which will be again denoted by \(\alpha\), and these are called the *ways* to \(\Delta_i\). It is well known [6], p. 249, that for the oriented simplex \(<\Delta_i>\) in the oriented manifold \(<W_i>\) there
is the oriented dual cell, written \(<\Pi/>_rt\), whose orientation is uniquely determined such that the intersection number of \(<\Delta f>\Lambda\) and \(<\Pi*>\Lambda\) is equal to 1 in \(<W_i>\).

**Lemma 5.** Let \(\alpha, \beta\) be the ways to \(\Delta_i\). If \(<\square_1>\alpha=\in<\square_1>\beta\). Then \(<\square_2>\alpha=\in<\square_2>\beta\), and if \(\Delta_i\) is a vertex and \(<\square_i>\alpha\subset\in<W_i>\), then \(<\square_2>\alpha\subset\in<W_2>\), where \(\in=1\) or \(-1\).

Proof. If \(<\square_i>\alpha=\in<\square_i>\beta\), then \(<\Delta_i>\alpha=\in<\Delta_i>\beta\). Since \(K_1\) and \(K_2\) are isomorphic under the correspondence \(\phi\Delta_1=\Delta_2, <\Delta_2>\alpha=\in<\Delta_2>\beta\) and then \(<\square_2>\alpha=\in<\square_2>\beta\). If \(<\square_i>\alpha\subset<\in<W_i>\) then \(<\Delta_i>\alpha=\in<\Delta_i>\). By the same reason mentioned above and \(\phi(\Delta_\ast)=<\Delta_\ast>\) we have that \(<\Delta_2>\alpha=\in<\Delta_2>\), and that \(<\square_2>\alpha\subset<\in<W_2>\).

5. The Proof of Theorem 2

**Lemma 6.** Let \(T\) be a \((q-1)\)-sphere in a \(q\)-sphere \(S\) and \(U(T, S)\) a regular neighborhood of \(T\) in \(S\). Suppose that the Schönflies conjecture is true for dimension \(q\). Then there is a homeomorphism \(\theta: T_\odot J\rightarrow S\) such that

\[\theta(T_\odot J) = U(T, S) \quad \text{and} \quad \theta(T_\odot 0) = T,\]

where \(T_\odot\) is a \((q-1)\)-sphere.

Proof. Let \(\Delta_a, \Delta_o\) and \(\Delta_b\) be \(q\)-simplices in \(S\) similarly situated with respect to a center of similitude in \(Int \Delta_a\) such that \(\Delta_a\subset Int \Delta_a\) and \(\Delta_o\subset Int \Delta_b\). By Corollary to Theorem 8 of [7], p. 260, \(Cl(\Delta_b-\Delta_a)\) is a regular neighborhood of \(\partial \Delta_o\) in \(S\). There is a homeomorphism \(\phi: \partial \Delta_o \times J \leftrightarrow Cl(\Delta_b-\Delta_a)\) such that \(\phi(\partial \Delta_o \times 0) = \partial \Delta_o\). By the assumption and Theorems 3 and 4 of [3], p. 32, there is an orientation preserving homeomorphism \(\psi_1: S \leftrightarrow S\) such that \(\psi_1(\partial \Delta_o) = T\). It is immediate that \(\psi_1(Cl(\Delta_b-\Delta_a))\) is a regular neighborhood of \(T\) in \(S\). By Theorem 1 in section 3 there is an orientation preserving homeomorphism \(\psi_2: S \leftrightarrow S\) such that \(\psi_2 \psi_1(Cl(\Delta_b-\Delta_a)) = U(T, S)\) and \(\psi_2| T = \text{identity}\). Putting \(T_\odot = \partial \Delta_o\) and \(\theta = \psi_2 \psi_1, \phi\), it completes the proof.

**Lemma 7.** Let \(<S_i>\) be an oriented \(q\)-sphere and \(T_i \subset S_i\) a \((q-1)\)-sphere where \(i=1, 2\). Suppose that the Schönflies conjecture is true for
dimension \( q \), and that there is a homeomorphism

\[ \phi: \langle U(T_1, S_1) \rangle \leftrightarrow \langle U(T_2, S_2) \rangle \]

such that \( \phi T_1 = T_2 \) where \( \langle U(T_1, S_1) \rangle \subseteq \langle S_1 \rangle \). Then there is a homeomorphism

\[ \psi: \langle S_1 \rangle \leftrightarrow \langle S_2 \rangle \]

such that \( \psi|U(T_1, S_1) = \phi \).

Proof. By Lemma 6 there are a \((q-1)\)-sphere \( T_0 \) and homeomorphisms \( \theta_i: T_0 \times I \leftrightarrow U(T_i, S_i) \) such that \( \theta_i(T_0 \times 0) = T_i \). By the assumption the \((q-1)\)-spheres \( \theta_i(T_0 \times 1) \) and \( \theta_i(T_0 \times -1) \) are congruent to the boundary of \( q \)-simplex in \( S_i \). Then \( Cl(S_i - U(T_i, S_i)) \) consists of two \( q \)-cells \( C_i \) and \( D_i \) such that \( \partial C_i = \theta_i(T_0 \times 1) \) and \( \partial D_i = \theta_i(T_0 \times -1) \). If we put \( \rho_c = \phi|\partial C_i \), then \( \rho_c(\partial C_i) \) is either \( \partial C_2 \) or \( \partial D_2 \), say \( \partial C_2 \). If we put \( \rho_d = \phi|\partial D_i \), then \( \rho_d(\partial D_i) = \partial D_2 \). By \( \phi(U(T_1, S_i)) = U(T_2, S_2) \), we have that

\[ \rho_c: \partial \langle C_1 \rangle \leftrightarrow \partial \langle C_2 \rangle \quad \text{and} \quad \rho_d: \partial \langle D_1 \rangle \leftrightarrow \partial \langle D_2 \rangle, \]

where \( \langle C_1 \rangle, \langle D_1 \rangle \subseteq \langle S_1 \rangle \).

By Lemma 3.12 of [3], p. 37, there are homeomorphisms

\[ \eta_c: \langle C_1 \rangle \leftrightarrow \langle C_2 \rangle, \quad \text{and} \quad \eta_d: \langle D_1 \rangle \leftrightarrow \langle D_2 \rangle \]

such that \( \partial \eta_c = \rho_c \) and \( \partial \eta_d = \rho_d \). Then \( \psi: \langle S_1 \rangle \leftrightarrow \langle S_2 \rangle \) defined by taking

\[ \psi|U(T_1, S_1) = \phi, \quad \psi|C_1 = \eta_c \quad \text{and} \quad \psi|D_1 = \eta_d \]

is the required homeomorphism.

**Lemma 8.** Let \( M_i^n \) be a closed \( n \)-manifold in an oriented \((n+1)\)-manifold \( \langle W_i^{n+1} \rangle \) without boundary, \( i = 1, 2 \). Using Notation 2, let

\[ \phi: M_1^n \leftrightarrow M_2^n \]

be a homeomorphism which is simplicial relative to \( K_1 \) and \( K_2 \). Suppose that the Schönflies conjecture is true for dimension \( \leq n \). Then there is a homeomorphism

\[ \psi: \langle R_i^{n+1} \rangle \leftrightarrow \langle R_2^{n+1} \rangle \]

such that \( \psi|M_i^n = \phi \) where \( \langle R_i^{n+1} \rangle \subseteq \langle W_i^{n+1} \rangle \).

To prove the lemma we first prove the following;

(0). Let a homeomorphism \( \phi: M_1^n \leftrightarrow M_2^n \) be simplicial relative to \( K_1 \) and \( K_2 \). Then there is a homeomorphism

\[ \psi^\phi: R_i^n \leftrightarrow R_2^n \]

such that \( \psi^\phi|\partial_1^n = \phi \) and \( \psi^\phi|\partial_2^n = \langle \partial_2 \rangle \).

for each \( n \)-simplex \( \Delta_i \) of \( K_i \) and each way \( \alpha \) to \( \Delta_i \).
Proof of (0). Since \( \Delta_i \) is an \( n \)-simplex, \( \partial \square_i \) is a 0-sphere and we have a homeomorphism

\[
\psi^*_{\alpha} : \partial \langle \square_1 \rangle_\alpha \leftrightarrow \partial \langle \square_2 \rangle_\alpha \text{ for a way } \alpha.
\]

Since \( \nabla_i \) is the point such that \( \square_i = \nabla_i(\partial \square_i) \), we have a homeomorphism

\[
\psi^* : \langle \square_1 \rangle_\alpha \leftrightarrow \langle \square_2 \rangle
\]

such that

\[
\partial \psi^*_{\alpha} = \psi^*_{\alpha} \text{ and } \psi^*_{\alpha} \nabla_i = \nabla_2, \text{ by 3.11 of [3], p. 36.}
\]

Let \( \beta \) be another way to \( \Delta_i \), then \( \langle \square_1 \rangle_\beta = \epsilon \langle \square_1 \rangle_\alpha \) implies \( \langle \square_2 \rangle_\beta = \epsilon \langle \square_2 \rangle_\alpha \), by Lemma 5. Therefore we have that

\[
\psi^*_{\alpha} \langle \square_1 \rangle_\beta = \langle \square_2 \rangle_\beta.
\]

Thus we can put \( \psi' = \psi^*_{\alpha} \). Then \( \psi' : \mathcal{N}_1 \mapsto \mathcal{N}_2 \) defined by taking \( \psi'|_{\square_1} = \psi' \) is a homeomorphism such that

\[
\psi'|_{\partial \langle \square_1 \rangle_\alpha} = \phi \text{ and } \psi\langle \square_1 \rangle_\alpha = \langle \square_2 \rangle_\alpha
\]

for \( \Delta_i \) and \( \alpha \) to \( \Delta_i \), proving (0).

Next we shall prove the following:

\((q - 1) \rightarrow (q)\). Suppose that there is a homeomorphism

\[
\psi^{q-1} : \mathcal{N}_1^q \mapsto \mathcal{N}_2^q
\]

such that

\[
\psi^{q-1}|_{\mathcal{N}_1^{q-1}} = \phi \text{ and } \psi^{q-1}\langle \square_1 \rangle_\gamma = \langle \square_2 \rangle_\gamma
\]

for each \((n-q+1)\)-simplex \( \Delta_i \) of \( K'_i \) and for each way \( \gamma \) to \( \Delta_i \), and suppose that the Schönflies conjecture is true for dimension \( q \). Then there is a homeomorphism

\[
\psi^q : \mathcal{N}_1^{q+1} \mapsto \mathcal{N}_2^{q+1}
\]

such that

\[
\psi^q|_{\mathcal{N}_1^q} = \phi \text{ and } \psi^q\langle \square_1 \rangle_\alpha = \langle \square_2 \rangle_\alpha
\]

for each \((n-q)\)-simplex \( \Delta_i \) of \( K'_i \) and for each way \( \alpha \) to \( \Delta_i \).

Proof of \((q - 1) \rightarrow (q)\). By \( \Delta_{ij} \) we denote an \((n-q+1)\)-simplex of \( K'_i \) incident to an \((n-q)\)-simplex \( \Delta_i \). By \( \gamma \) we denote the way to \( \Delta_{ij} \) which is obtained from a way \( \alpha \) to \( \Delta_i \) adding \( \Delta_{ij} \) as the final term. Then \( \langle \square_{ij} \rangle_\gamma \subset \partial \langle \square_{ij} \rangle_\alpha \). Since \( \psi^{q-1}\langle \square_{ij} \rangle_\gamma = \langle \square_{ij} \rangle_\gamma \), we have that

\[
\psi^{q-1}\langle \bigcup_j \square_{ij} \rangle_\alpha = \langle \bigcup_j \square_{ij} \rangle_\alpha, \text{ where } \langle \bigcup_j \square_{ij} \rangle_\alpha \subset \partial \langle \square_{ij} \rangle_\alpha.
\]
By Lemma 4, $\bigcup_i \square_i$ is a regular neighborhood of the $(q-1)$-sphere $\partial \nabla$ in the $q$-sphere $\partial \Pi$. Then by the assumption and Lemma 7 there is a homeomorphism

$$\psi^* : \partial \langle \square_i \rangle_\alpha \leftrightarrow \partial \langle \square_2 \rangle_\alpha$$

such that

$$\psi^*|_{\bigcup_i \square_i} = \psi^{q-1}.$$

Let $x_i$ be the barycenter of $\Delta_i$, then $\square_i = x_i(\partial \square_i)$ and $\nabla_i = x_i(\partial \nabla_i)$, by Theorem II of [6], p. 230. By 3.11 of [3], p. 36, we have a homeomorphism

$$\psi : \langle \square_i \rangle_\alpha \leftrightarrow \langle \square_2 \rangle_\alpha$$

such that $\partial \psi^* = \psi^*$, and $\psi_0 \nabla_i$ is simplicial relative to $Y_1$ and $Y_2$, see Notation 1. Let $\beta$ be another way to $\Delta_i$, then we have that

$$\psi^* \langle \square_i \rangle_\alpha = \langle \square_2 \rangle_\alpha,$$

by Lemma 5.

Thus we can put $\psi^* = \psi$. Then $\psi^* : R_1^{q+1} \leftrightarrow R_2^{q+1}$ defined by taking

$$\psi^* \langle \square_i \rangle_\alpha = \psi$$

is a homeomorphism such that

$$\psi^*|_{\Pi^l} = \phi$$

and $\psi_0 \langle \square_i \rangle_\alpha = \langle \square_2 \rangle_\alpha$

for each $(n-q)$-simplex $\Delta_i$ and $\alpha$ to $\Delta_i$, proving $(q-1) \to (q)$.

**Proof of Lemma 8.** By assertions (0) and $(q-1) \to (q)$ there is a homeomorphism

$$\psi^* : R_1^{n+1} \leftrightarrow R_2^{n+1}$$

such that

$$\psi^*|_{\Pi^l} = \phi$$

and $\psi_0 \langle \square_i \rangle_\alpha = \langle \square_2 \rangle_\alpha$

for each 0-simplex $\Delta_i$ and each way $\alpha$ to $\Delta_i$. By Lemma 5 we have that if $\langle \square_i \rangle_\alpha \subset \langle W_1^{n+1} \rangle$ then $\langle \square_2 \rangle_\alpha \subset \langle W_2^{n+1} \rangle$. Therefore $\psi^* \langle \square_i \rangle = \langle \square_2 \rangle$, and

$$\psi^* \langle \Pi^{n+1} \rangle = \langle \Pi_2^{n+1} \rangle,$$

where $\langle \square_i \rangle \subset \langle W_1^{n+1} \rangle$.

If we put $\psi^* = \psi$, then this completes the proof of the lemma.

**Theorem 2.** Let $M_i^n$ be a closed $n$-manifold in an oriented $(n+1)$-manifold $\langle W_i^{n+1} \rangle$ without boundary where $i=1, 2$ and let

$$\phi : M_1^n \leftrightarrow M_2^n$$

be a homeomorphism. Suppose that the Schönflies conjecture is true for dimension $\leq n$. Then for any regular neighborhoods $U(M_i^n, W_i^{n+1})$ there is a homeomorphism
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$\psi : \langle U(M^n_1, W^{n+1}_1) \rangle \leftrightarrow \langle U(M^n_2, W^{n+1}_2) \rangle$ such that $\psi | M^n_i = \phi$ where $\langle U(M^n_1, W^{n+1}_1) \rangle \subset \langle W^{n+1}_i \rangle$.

Proof. Let $K_i, L_i$ be simplicial subdivisions of $M^n_i, W^{n+1}_i$ where $K_i$ is a subcomplex of $L_i$ and $\phi$ is simplicial relative to $K_i$ and $K_{i+1}$. By Lemma 8 there is a homeomorphism

$\psi' : \langle \mathcal{N}^{n+1}_i \rangle \leftrightarrow \langle \mathcal{N}^{n+1}_{i+1} \rangle$

such that $\psi' | M^n_i = \phi$ where $\langle \mathcal{N}^{n+1}_i \rangle \subset \langle W^{n+1}_i \rangle$.

Let $\Delta_{ij}$ be a 0-simplex of $K'_i$, then $\square_{ij} = N(\Delta_{ij}, L_i)$. On the other hand it is well known [7], p. 294, that $N(K', L_i) = \bigcup_j \square_{ij}$. Therefore we have that $N(K'_{i+1}, L_{i+1}) = \mathcal{N}^{n+1}_{i+1}$. Hence $\psi'$ is a homeomorphism

$\psi' : \langle N(K'_i, L'_i) \rangle \leftrightarrow \langle N(K'_{i+1}, L'_{i+1}) \rangle$

such that $\psi' | M^n_i = \phi$ where $\langle N(K'_i, L'_i) \rangle \subset \langle W^{n+1}_i \rangle$.

By Theorem 1 there are orientation preserving homeomorphisms

$\psi_i : W^{n+1}_{i+1} \leftrightarrow W^{n+1}_i$

such that

$\psi_i(U(M^n_i, W^{n+1}_i)) = N(K^n_{i+1}, L^n_i)$ and $\psi_i | M^n_i = \text{identity}$.

Then $\psi = \psi^{-1} \psi' \psi_i$ is the required homeomorphism.

6. Applications

**Theorem 3.** Let $M^n$ be an orientable, closed $n$-manifold in an orientable $(n+1)$-manifold $W^{n+1}$ without boundary. Let $U(M^n, W^{n+1})$ be a regular neighborhood of $M^n$ in $W^{n+1}$. Suppose that the Schönflies conjecture is true for dimension $\leq n$. Then there is a homeomorphism

$\theta : M^n \times J \rightarrow W^{n+1}$

where $J$ is the linear interval $-1 \leq s \leq 1$, such that

$\theta(x, 0) = x$ for each point $x$ of $M^n$

and such that

$\theta(M^n \times J) = U(M^n, W^{n+1})$.

Proof. Let us consider the Cartesian product $M^n \times R$ where $R$ is a Euclidean 1-space containing $J$. Then $M^n \times R$ is an orientable $(n+1)$-manifold without boundary. By Theorem 8 of [7], p. 260, and Definition
1, $M^n \times J$ is a regular neighborhood of $M^n \times 0$ in $M^n \times R$. A map $\phi: M^n \times 0 \to M^n$ defined by $\phi(x, 0) = x$ for each $x$ is a homeomorphism onto. If we give orientations to $M^n \times R$ and $W^{n+1}$, then by Theorem 2 we have a homeomorphism

$$\theta: M^n \times J \to W^{n+1}$$

which satisfies the theorem.

**Theorem 4.** Let $M^n$ be an orientable, closed $n$-manifold in an orientable $(n+1)$-manifold $W^{n+1}$ without boundary. Let

$$\phi: M^n \leftrightarrow M^n$$

be a homeomorphism which is onto isotopic to the identity (see, [3], p. 30). Suppose that the Schönflies conjecture is true for dimension $\leq n$. Then there is an orientation preserving homeomorphism

$$\psi: W^{n+1} \leftrightarrow W^{n+1} \text{ such that } \psi|M^n = \phi.$$

Proof. By Theorem 3 each point of a regular neighborhood $U(M^n, W^{n+1})$ will be denoted by a pair $(x, s)$ where $x$ is a point of $M^n$ and $s \in J$ and $(x, 0) = x$. Let $\phi_t: M^n \leftrightarrow M^n$, $t \in I$, be an onto isotopy between $\phi_0 = \phi$ and $\phi_1 =$-identity. Then $\psi: W^{n+1} \leftrightarrow W^{n+1}$ defined by taking

$$\psi(z) = z, \text{ if a point } z \in Cl(W^{n+1} - U(M^n, W^{n+1}))$$

and

$$\psi(z) = (\phi_{|s}(x), s), \text{ if } z \in U(M^n, W^{n+1}) \text{ and } z = (x, s)$$

is the required homeomorphism.

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