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Large-time behavior of solutions of scalar viscous conservation law with non-convex flux

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January 8

Preface

In this thesis, we study the large-time behavior of solutions to an initial boundary value problem on the half line for scalar viscous conservation law, where the data on the boundary and also at the far field are prescribed. In the case where the flux function is convex and the corresponding Riemann problem for the hyperbolic part admits the transonic rarefaction wave (which means that its characteristic speed changes the sign), it is known by the work of Liu-Matsumura-Nishihara [17]('98) that the solution tends toward an asymptotic state which consists of the linear superposition of the stationary solution and the rarefaction wave of the hyperbolic part. In this thesis, based on the results by Hashimoto-Matsumura [5]('07), we first show that even for a quite wide class of flux functions which are not necessarily convex, such superposition of the stationary solution and the rarefaction wave is asymptotically stable, provided the rarefaction wave is weak. We also show the decay rate estimates of the solutions toward the asymptotic state, based on the recent arguments in Hashimoto-Kawashima-Ueda [4]('08). These proofs are given by a technical L^2-weighted energy method motivated by the works Matsumura-Mei [23]('97), Kawashima-Matsumura [8]('94), Matsumura-Nishihara [27]('94) and etc. We furthermore show that this technical weighted energy method can be applied to the problem of asymptotic stability of stationary solutions to an initial boundary value problem on the half line for damped wave equation with convection term. This problem has been intensively investigated by Kawashima-Nakamura-Ueda [13]('08) and Ueda [35]('08), and they showed that the stationary solution is asymptotically stable, provided that the function of convection term (which corresponds to flux function of conservation law) is convex and satisfies, so to speak, "sub-characteristic condition" in the whole state space. In this thesis, by applying the above mentioned weighted energy method, we show that the asymptotic stability of the stationary solution holds even for a wide class of functions of convection term which are not necessarily convex and satisfy the sub-characteristic condition only at far field.

One-dimensional motion of various physical quantities is described by system of nonlinear partial differential equations in terms of the physical quantities and their flux functions which describes their transportation

in a one-dimensional direction. The system consists of first order hyperbolic system together with some physical viscosity terms of second order. If the diffusive effects are neglected, the hyperbolic system is called "conservation law", and if not, the whole system with viscosity terms is called "viscous conservation law". A typical example for conservation law is the Euler equation which describes the motion of perfect gas, and one for viscous conservation law is a system of equations for viscous and heat-conductive compressible gas (often called compressible Navier-Stokes equation). Especially, the Burgers equation is well known as the simplest example of scalar viscous conservation law which describes a motion of viscous gas. One of the basic mathematical problems for the viscous conservation law is the Cauchy problem with the initial data whose far field states are given. Since this setting is very important and basic from both mathematical and physical point of view, there have been many works on the global solutions in time, and in particular, their large-time behaviors (cf. [2], [21], [24], [25], [34], etc.). All these results show that the large time behaviors of solutions of the Cauchy problem are basically same as that of the corresponding Riemann problem to the hyperbolic part of the system. The Riemann problem is the Cauchy problem for the hyperbolic system with the initial data of a step function (called Riemann data) whose constant states are given by the far field states. This problem is proposed in the Riemann's paper of 1860, as an elementary problem to investigate the both microscopic and macroscopic behaviors of solutions of hyperbolic system. Since then, the theory of conservation law has been greatly developed under the condition that all the characteristic fields are either genuinely nonlinear or linearly degenerate (this condition is satisfied by many standard examples like the Euler equation, and for scalar conservation law, this condition corresponds to that the flux function is convex or just linear)(cf. [1], [15], [33], etc.). The theory implies that the elementary nonlinear waves for the Riemann problem are given by dilation invariant solutions (called Riemann solutions): shock waves, rarefaction waves, and contact discontinuities, and the linear combinations of these basic waves. Since the hyperbolic system is regarded as an idealization when the diffusive effects are neglected, it is of great importance to study the large-time behavior of solutions of the corresponding viscous conservation law toward the viscous versions

of these elementary nonlinear waves. On this line of research, a pioneering work was done by Il'in-Oleinik [7]('60) for scalar viscous conservation law under the condition that the flux function is convex. They clarified the relation between the Riemann solution for the hyperbolic part and the asymptotic behavior of the solution of the viscous conservation law. Namely, they showed that if the Riemann problem admits the rarefaction wave, the solution to the Cauchy problem for the viscous conservation law tends toward the rarefaction wave itself, and if the Riemann problem admits the shock wave with discontinuity, the solution tends toward the corresponding traveling wave solution, so called, "viscous shock wave" which is smoothed by the viscosity effect. Since their proof was given by the maximum principle, the extension of their results to systems had been open for a long time till Matsumura-Nishihara [24]('85) and Goodman [2]('86) independently developed a L^2-energy method which is applicable to systems. Since then, so much progress has been achieved on the asymptotic stability of each wave pattern for quite general perturbation for the compressible Navier-Stokes equation and general system of viscous conservation law including the non-genuinely nonlinear systems. The examples of system some of whose characteristic fields are non-genuinely nonlinear (non-convex in a sense) appear in the fields of visco-elasticity, multi-phase flow, traffic flow, etc. Through these researches, many new techniques like weighted energy estimates and point wise estimates by Green functions have been developed(cf. [6], [9], [16], [19], [26], [31], [34], [36], etc.).

On the other hand, the initial boundary value problem (IBVP) on the half line, where the data both on the boundary and at the far field are prescribed, is also a basic and important problem, because they are expected to describe the interactions of the elementary non-linear waves and the boundary. In these cases, the influence of viscosity is expected to emerge not only in smoothing effect on discontinuous shock wave, but also in forming a stationary solution, so to speak, "boundary layer". Roughly speaking, the boundary layer solution is a stationary solution which is formed by inconsistency of the boundary value of incoming Riemann solutions with the boundary condition. So, in the IBVP, we should take into account not only viscous versions of nonlinear hyperbolic waves but also stationary solutions. On this line of research, Liu-Yu

[20] ('97) first investigated the solution of the Burgers equation by using the maximum principle, and later Liu-Nishihara [18]('97) and Nishihara [30]('01) studied the general scalar viscous conservation law with convex flux functions by a weighted L^2-energy method, provided that the corresponding Riemann solution is either incoming or outgoing to the boundary (Nishihara-Liu [18] also investigated the case where the flux function is non-convex, and the corresponding Riemann solution consists of an incoming or outgoing shock wave). They showed that for the IBVP, when the Riemann solution is incoming, the stationary solution is asymptotically stable, and when the Riemann solution is outgoing, the corresponding rarefaction wave or viscous shock wave is asymptotically stable. So the cases where the Riemann solution consists of a transonic rarefaction wave, or the flux is non-convex were widely left open. On these backgrounds, in 1998, Liu-Matsumura-Nishihara [17] succeeded in treating the case where the corresponding Riemann problem admits the transonic rarefaction wave under the condition that the flux function is convex. Motivated by this work and the recent arguments on weighted energy estimates, new developments for the cases of non-convex flux function and their applications to damped wave equation, which are our main theme in the present thesis, have been done as we summarized above. Finally in this preface, since the studies on the IBVPs for the systems of compressible Navier-Stokes equation also recently have been developed much by Matsumura-Nishihara [28], Matsumura-Mei [22], Kawashima-Nishibata-Zhu [10], Huang-Matsumura-Shi [6], etc., where all characteristic fields are genuinely nonlinear or linearly degenerate, we do hope that the arguments for non-convex flux functions in the present thesis could be extended to other physical systems which have a non-genuinely nonlinear characteristic field.

The thesis is organized as follows. The initial boundary value problem for viscous conservation law is discussed in the Chapter 1. We give the precise statements of our main theorems in the Section 1.1. Recalling the arguments on the stationary solution in the Section 1.2 and rarefaction wave in the Section 1.3, we reformulate the problem in the Section 1.4. We introduce a new weight function in the Section 1.4, and establish the *a priori* estimates for the asymptotic stability. We give the decay rate of convergence for the convex flux in the Section 1.7, and for the non-

convex flux in the Section 1.8. Finally an application of these arguments to damped wave equation is discussed in the Chapter 2.

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Contents

1 Scalar viscous conservation law **8**

1.1 Introduction and main theorems 8
1.2 Stationary solution 14
1.3 Rarefaction wave and smooth approximation 15
1.4 Reformulation of the problem 18
1.5 Weight function 21
1.6 *A priori* estimate 25
1.7 Decay rate estimate I; convex flux 32
1.8 Decay rate estimate II; non-convex flux 38

2 Damped wave equation **44**

2.1 Introduction and main theorems 44
2.2 Reformulation of the problem; case $u_- < u_+ = 0$ 47
2.3 *A priori* estimate I 49
2.4 A case including rarefaction wave 54
2.5 Anti-derivative method; case $u_+ = 0 < u_-$ 56
2.6 *A priori* estimate II. 61
2.7 Reconsideration to the case of $u_- < u_+ = 0$ 63

Chapter 1

Scalar viscous conservation law

1.1 Introduction and main theorems

In this chapter, we study the following initial-boundary value problem on the half line for scalar viscous conservation law:

$$\begin{cases} u_t + f(u)_x = u_{xx}, & x > 0,\ t > 0, \\ u(0,t) = u_-, & t > 0, \\ \lim\limits_{x\to\infty} u(x,t) = u_+, & t > 0, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.1.1}$$

where the flux f is a given C^2 function of u satisfying $f(0) = f'(0) = 0$, $u_\pm$ are given constants, and the initial data u_0 is assumed to satisfy $u_0(0) = u_-$ and $\lim\limits_{x\to\infty} u_0(x) = u_+$ as the compatibility conditions.

We are interested in the large time behavior of the solution which is determined by the shape of the flux $f(u)$ and the given constants $u_\pm$. It is known that the asymptotic behavior is closely related to the solution of the corresponding Riemann problem for the hyperbolic part (cf. [17], [18]):

$$\begin{cases} u_t + f(u)_x = 0, & x \in R,\ t > 0, \\ u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0. \end{cases} \end{cases} \tag{1.1.2}$$

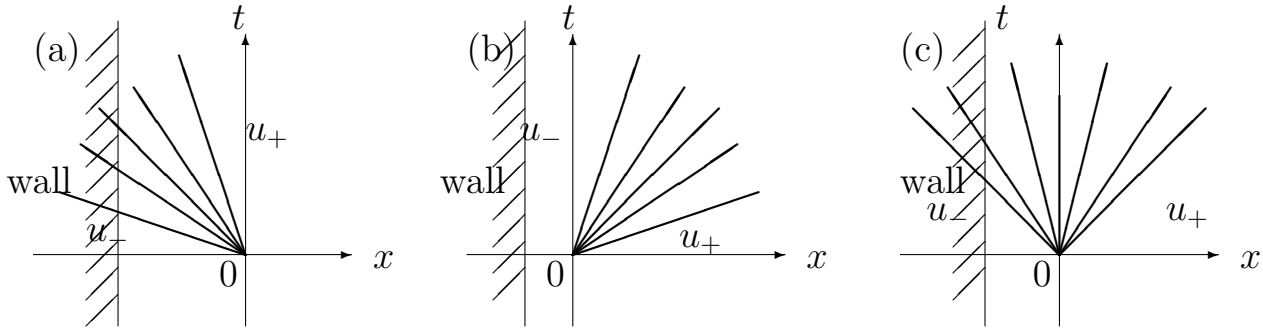
In the case where the flux f is convex

$$f''(u) > 0 \quad \text{for} \quad u \in R, \tag{1.1.3}$$

and the Riemann Problem (1.1.2) has the rarefaction wave solution, Liu-Matsumura-Nishihara [17]('98) showed that depending on the signs of

the characteristic speeds $f'(u_\pm)$, the large-time behavior of the solution is classified into the three cases:

(a) $f'(u_-) < f'(u_+) \le 0$, (equivalently $u_- < u_+ \le 0$),
(b) $0 \le f'(u_-) < f'(u_+)$, (equivalently $0 \le u_- < u_+$),
(c) $f'(u_-) < 0 < f'(u_+)$, (equivalently $u_- < 0 < u_+$),



More precisely, they showed the following. In the case (a) where all the characteristic speeds of the rarefaction wave are non-positive, the solution of (1.1.1) eventually tends toward the stationary solution ϕ which connects u_- to u_+, where $\phi = \phi(x)$ is defined by the solution of the corresponding stationary problem to (1.1.1) :

$$\begin{cases} f(\phi)_x = \phi_{xx}, \quad x > 0, \\ \phi(0) = u_-, \quad \phi(+\infty) = u_+. \end{cases} \tag{1.1.4}$$

In the case (b) where all the characteristic speeds of the rarefaction wave are non-negative, the solution of (1.1.1) eventually tends toward the rarefaction wave ψ^R itself which connects u_- to u_+, where $\psi^R = \psi^R(\frac{x}{t})$ is concretely given by

$$\psi^R(\frac{x}{t}) = \begin{cases} u_-, & x \le f'(u_-)t, \\ (f')^{-1}(\frac{x}{t}), & f'(u_-)t \le x \le f'(u_+)t, \\ u_+, & x \ge f'(u_+)t. \end{cases} \tag{1.1.5}$$

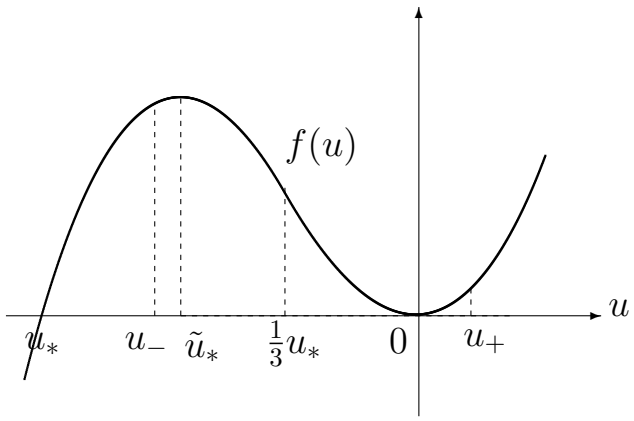
In the case (c) where the rarefaction wave is transonic, that is, the characteristic speeds change the sign, the solution of (1.1.1) eventually tends toward the linear superposition of the stationary solution ϕ connecting u_- to 0 and the rarefaction wave ψ^R connecting 0 to u_+.

Focusing on the most interesting case (c), $u_- < 0 < u_+$, we can naturally expect that the superposition $\phi + \psi^R$ is asymptotically stable for

more general flux $f(u)$ which is convex for $u > 0$ but not necessarily convex for $u < 0$ as long as positive for $u \neq 0$, because even for such general flux the stationary solution ϕ connecting u_- to 0 is easily seen to exist. There have been only a few results on such cases without the convexity condition. Nagase (2000, [29]) studied in her master thesis the case where the flux $f(u)$ satisfies

$$\begin{aligned} &f(0) = f'(0) = 0, \\ &\exists u_* < 0 \text{ s.t. } f(u_*) = 0, \text{ and } f(u) > 0, \quad u \in (u_*, \infty),\ u \neq 0, \\ &\exists \bar{u}_* \in (u_*, 0) \text{ s.t. } f''(\bar{u}_*) = 0, \\ &f'''(u) > 0, \quad u \in R. \end{aligned} \tag{1.1.6}$$

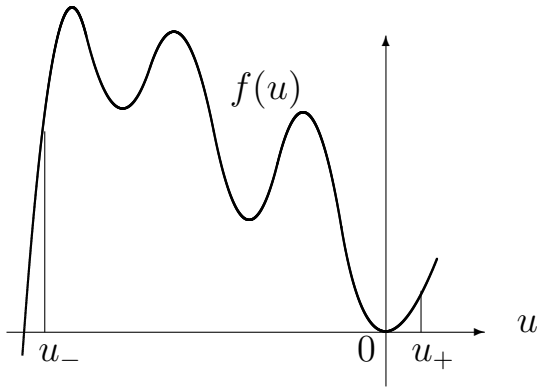
Here, it is noted that the condition (1.1.6) implies $f''(u) \gtrless 0$ for $u \gtrless \bar{u}_*$, and $\exists \tilde{u}_* \in (u_*, \bar{u}_*)$ s.t. $f'(\tilde{u}_*) = 0$. In this case, the superposition $\phi + \psi^R$ is expected to be asymptotically stable for $u_- \in (u_*, 0)$. A typical example which satisfies (1.1.6) is $f(u) = u^2(u - u_*)$, where $\bar{u}_*$ and $\tilde{u}_*$ are given by $u_*/3$ and $2u_*/3$ respectively.



Then she showed that for sufficiently small $\varepsilon > 0$, if $u_- \in (\tilde{u}_* - \varepsilon, 0)$ and $0 < u_+ < \varepsilon$, the superposition $\phi + \psi^R$ is asymptotically stable. The proof is given by using a L^2-weighted energy method as in the previous works ([8], [18], [27]) where in order to show the asymptotic stability of viscous shock profile for non-convex state equations, they manipulate a weight function constructed by viscous shock profile itself. She also used a suitable weight function constructed by the stationary solution ϕ. However the case $u_- \in (u_*, \tilde{u}_* - \varepsilon)$ had been left open. In this chapter, based on the arguments in Hashimoto-Matsumura [5]('07), it is shown that we can solve this open question and even can make the condition

(1.1.6) much weaker as

$$\begin{aligned} &f(0) = f'(0) = 0, \quad f''(0) > 0, \\ &f(u) > 0, \quad u \in [u_-, 0). \end{aligned} \tag{1.1.7}$$



We also assume that the initial data u_0 satisfies

$$u_0 - u_+ \in H^1, \quad u(0) = u_-. \tag{1.1.8}$$

Noting that the conditions $f \in C^2$ and $f''(0) > 0$ imply the existence of positive constants r and ν satisfying

$$f''(u) \geq \nu > 0, \quad |u| \leq r, \tag{1.1.9}$$

we further assume

$$u_- < 0 < u_+ \leq r. \tag{1.1.10}$$

Under these assumptions, we show that if u_+ is positive but sufficiently small, then the superposition $\phi + \psi^R$ is asymptotically stable. Here, we define the rarefaction wave for the initial boundary problem (1.1.1) by the restriction of ψ^R on the half line $\psi^R(\frac{x}{t})|_{x>0}$, and write it again as ψ^R without confusion. Now we are ready to state our result on the asymptotic stability of $\phi + \psi^R$.

Theorem 1.1.1 (asymptotic stability)**.** *Assume* (1.1.7), (1.1.8), *and* (1.1.10). *Then, there exists a positive constant ε such that, if $u_+ \leq \varepsilon$ and $\| u_0 - \phi - \psi^R(\cdot)\|_{H^1} \leq \varepsilon$, then the initial boundary value problem* (1.1.1) *has a unique global solution in time u satisfying*

$$\begin{cases} u - u_+ \in C([0,\infty); H^1), \\ u_x \in L^2(0,T; H^1) \quad ({}^\forall T > 0) \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{x>0}|u(x,t)-\phi(x)-\psi^R(\frac{x}{t})|=0. \tag{1.1.11}$$

For the proof, we employ a technique in Matsumura-Mei [23] to obtain the *a priori* estimate of the solution, where they manipulate not only a weight function but also a transformation of the unknown functions in order to prove the asymptotic stability of viscous shock profile for a system of visco-elasticity with a non-convex nonlinearity.

Next, we state the decay rate of convergence of the global solution in time toward the asymptotic state $\phi+\psi^R$. In the case of a one-dimensional whole space, the decay rate toward the rarefaction waves was first investigated by Harabetian [3]. For the half space problem, Kawashima-Nishibata-Nishikawa and Kawashima-Nakamura-Ueda showed the decay rate for the case (a) in [11] and [13] respectively. In [11], the decay rate toward the non-degenerate stationary solutions is considered, and the rate toward the degenerate stationary solutions is shown in [13]. On the other hand, the convergence rate for the case (b) was considered by Nakamura [32]. However the case (c) had been left open even for the convex flux function till the recent results in Hashimoto-Kawashima-Ueda [4]. Based on the arguments on weighted energy methods and interpolation inequalities in [4], we show the decay rate of convergence for the case (c). When the flux is convex, it is noted that the smallness conditions on u_+ and initial perturbations are not necessary by the results in [17]. Then our second theorem on the decay rate is stated as follows.

Theorem 1.1.2 (decay rate for convex flux)**.** *Assume* (1.1.3), (1.1.8), $u_-<0<u_+$, *and* $u_0-\phi-\psi^R(\cdot)\in L^1$. *Then the initial-boundary value problem* (1.1.1) *has a unique global solution in time u satisfying*

$$\begin{cases} u-u_+\in C([0,\infty);H^1), \\ u_x\in L^2(0,T;H^1) \quad (^\forall T>0) \end{cases}$$

and the decay rate estimates

$$\|(u-\phi-\psi^R)(t)\|_{L^p}\le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\log^2(2+t) \quad (1\le p<\infty),$$
$$\|(u-\phi-\psi^R)(t)\|_{L^\infty}\le C_\epsilon(1+t)^{-\frac{1}{2}+\epsilon} \quad (\forall\epsilon>0).$$

In the case where the flux functions are not necessarily convex as the type (1.1.7), we obtain the following.

Theorem 1.1.3 (decay rate for non-convex flux)**.** *Assume* (1.1.7), (1.1.8) *and* (1.1.10)*, and also* $u_0 - \phi - \psi^R(\cdot) \in L^1$*. Then, there exists a positive constant* ϵ *such that, if* $u_+ \le \epsilon$ *and* $\|u_0 - \phi - \psi^R(\cdot)\|_{H^1} \le \epsilon$*, then the unique global solution in time* u *obtained by the Theorem* 1.1.1 *satisfies the decay rate estimates*

$$\|(u - \phi - \psi^R)(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \log^2(2+t) \quad (1 \le p < \infty),$$
$$\|(u - \phi - \psi^R)(t)\|_{L^\infty} \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon} \quad (\forall \epsilon > 0).$$

The rest of this chapter is organized as follows. We state the properties of the stationary solution and rarefaction wave in the Section 1.2 and Section 1.3 respectively. We make a reformulation of the problem in the Section 1.4. In the Section 1.5, making use of the linearized equation with $u_+ = 0$, we explain the essence how to construct our weight function. Then we prove *a priori* estimate in the Section 1.6, which completes the proof of Theorem 1.1.1. Finally, we state the proof of the Theorem 1.1.2 and 1.1.3 in the Section 1.7 and Section 1.8 respectively.

Notation. We denote by C generic positive constants unless they need to be distinguished. For function spaces, $L^p = L^p((0,\infty))$ and $H^k = H^k((0,\infty))$ denote the usual L^p-Lebesgue space of square integrable functions and k-th order Sobolev space on the half line $(0,\infty)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$, respectively. We also denote by $H^1_0 = H^1_0((0,\infty))$ the space of functions $f \in H^1$ with $f(0) = 0$, as a subspace of H^1.

1.2 Stationary solution

In this section, we recall the properties of the stationary solution ϕ which is given by the solution to the boundary value problem for the ordinary differential equation:

$$\begin{cases} f(\phi)_x = \phi_{xx}, \quad x > 0, \\ \phi(0) = u_-, \quad \phi(+\infty) = u_+. \end{cases} \tag{1.2.1}$$

In what follows, we assume $u_+ = 0$ and

$$\begin{cases} f(0) = 0, \quad u_- < 0, \\ f(u) > 0, \quad (u \in [u_-, 0)), \\ f''(0) = 0. \end{cases} \tag{1.2.2}$$

or

$$\begin{cases} f(0) = 0, \quad u_- < 0, \\ f(u) > 0, \quad (u \in [u_-, 0)), \\ f'(0) < 0. \end{cases} \tag{1.2.3}$$

If we integrate the equation of (1.2.1) once, it is easy to see (1.2.1) is equivalent to the problem :

$$\begin{cases} \phi_x = f(\phi), \quad x > 0, \\ \phi(0) = u_-. \end{cases} \tag{1.2.4}$$

When the condition (1.2.2) holds, the stationary solution is called "degenerate", when (1.2.3), "non-degenerate". Then we have the following lemmas which are proved in the same way as in [2] and [3], so we omit the proof. The statement for the degenerate stationary solution is

Lemma 1.2.1. *Assume $u_- < u_+ = 0$ and* (1.2.2)*. Then, the boundary value problem* (1.2.1) *which is equivalent to* (1.2.4) *has a unique solution $\phi \in C^3([0, \infty))$ satisfying*

$$\begin{cases} u_- < \phi(x) < 0 \text{ and } \phi_x(x) > 0, \quad x > 0, \\ |\phi(x)| \leq C(1+x)^{-1}, \quad x \geq 0. \end{cases} \tag{1.2.5}$$

On the other hand, the statement for the non-degenerate stationary solution is

Lemma 1.2.2. *Assume $u_- < u_+ = 0$ and (1.2.3). Then, the boundary value problem (1.2.1) which is equivalent to (1.2.4) has a unique solution $\phi \in C^3([0,\infty))$ satisfying for some $\alpha > 0$ and $C > 0$,*

$$\begin{cases} u_- < \phi(x) < 0 \text{ and } \phi_x(x) > 0, \quad x > 0, \\ |\phi(x)| \le C\exp(-\alpha x), \quad x \ge 0. \end{cases} \tag{1.2.6}$$

1.3 Rarefaction wave and smooth approximation

In this section, we recall the properties of the rarefaction wave. We start with the Riemann problem for the invicid Burgers equation:

$$\begin{cases} w_t + (\frac{1}{2}w^2)_x = 0, \quad x \in R,\ t > 0, \\ w(x,0) = \begin{cases} w_+, & x > 0, \\ w_-, & x < 0. \end{cases} \end{cases} \tag{1.3.1}$$

If $w_- < w_+$, the Riemann problem (1.3.1) has the weak solution $w^R(x/t)$, so called "rarefaction wave" which is concretely given by

$$w^R(\frac{x}{t}; w_-, w_+) = \begin{cases} w_-, & x \le w_- t, \\ \frac{x}{t}, & w_- t \le x \le w_+ t, \\ w_+, & x \ge w_+ t. \end{cases} \tag{1.3.2}$$

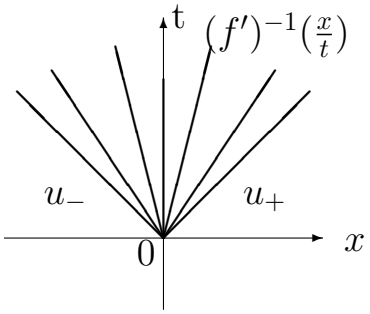
We next consider the Riemann problem for more general flux of convex function $f(u)$:

$$\begin{cases} u_t + f(u)_x = 0, \quad x \in R,\ t > 0, \\ u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0. \end{cases} \end{cases} \tag{1.3.3}$$

Then it is known that if $u_- < u_+$, the Riemann problem (1.3.3) has the rarefaction wave ψ^R connecting u_- to u_+ which is given by

$$\begin{aligned} \psi^R(\tfrac{x}{t}; u_-, u_+) &= (f')^{-1}(w^R(\tfrac{x}{t}; f'(u_-), f'(u_+)) \\ &= \begin{cases} u_-, & x \le f'(u_-), \\ (f')^{-1}(\frac{x}{t}), & f'(u_-) \le x \le f'(u_+)t, \\ u_+, & x \ge f'(u_+)t. \end{cases} \end{aligned} \tag{1.3.4}$$

We note that for our arguments on the initial boundary value problem (1.1.1), we use the rarefaction wave $\psi^R(\frac{x}{t}; -u_+, u_+)$ so that it satisfies the boundary condition $u(x,0) = 0$, and we write the restriction of this ψ^R on the half line $\psi^R(\frac{x}{t})|_{x>0}$ again as ψ^R without confusion.



In the proof of the Theorem 1.1.1, we make a smooth approximation of the rarefaction wave ψ^R as in the previous papers ([9, 25, 26]). Because the non-smoothness of ψ^R causes a trouble in the process of handling the second derivative of the solution in *a pripri* estimate. Following the arguments in [25], define a smooth approximation $w(x,t)$ of $w^R(\frac{x}{t}; -w_+, w_+)(w_+ = f'(u_+))$ by the solution of the Cauchy problem

$$\begin{cases} w_t + (\frac{1}{2}w^2)_x = 0, \quad x \in R,\ t > 0, \\ w(x,0) = f'(u_+)\tanh x, \quad x \in R, \end{cases} \tag{1.3.5}$$

and a smooth approximation $\psi(x,t)$ of $\psi^R(\frac{x}{t}; -u_+, u_+)|_{x>0}$ by

$$\psi(x,t) = (f')^{-1}(w(x,t)), \quad x \geq 0. \tag{1.3.6}$$

Then we have the next lemma which is proved in the same way as in [25].

Lemma 1.3.1. *Assume* (1.1.7) *and* $0 < u_+ \leq r$. *Then we have the following:*
1) $\psi(x,t)$ *is the smooth solution of the initial boundary value problem*

$$\begin{cases} \psi_t + f(\psi)_x = 0, \quad x > 0,\ t > 0, \\ \psi(0,t) = 0,\ \lim\limits_{x\to\infty} \psi(x,t) = u_+, \quad t > 0, \\ \psi(x,0) = (f')^{-1}(f'(u_+)\tanh x), \quad x > 0. \end{cases} \tag{1.3.7}$$

2) $0 < \psi(x,t) < u_+$ *and* $\psi_x(x,t) > 0, \quad x > 0,\ t > 0.$

3) *For $1 \le p \le \infty$, there exists a positive constant C_p such that*

$$\begin{aligned}
&\| \psi_x(t) \|_{L^p} \le C_p \min(u_+, u_+^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}),\\
&\| \psi_{xx}(t) \|_{L^p} \le C_p \min(u_+, (1+t)^{-1}),\\
&\| \psi_{xxx}(t) \|_{L^p} \le C_p \min(u_+, (1+t)^{-1}).
\end{aligned}$$

4) $\lim\limits_{t\to\infty} \sup\limits_{x>0} |\psi(x,t) - \psi^R(\frac{x}{t})| = 0.$

In the proof of Theorem 1.1.2 and Theorem 1.1.3, we employ another way to make a smooth approximation studied in [14], because the former way is not useful enough for deriving the estimate of difference of ψ^R and ψ. We define $\omega(x,t)$ as the solution of the following Cauchy problem:

$$\begin{cases}
\omega_t + \omega\omega_x = \omega_{xx}, \quad x \in R,\ t > -1,\\
\omega(x,-1) = \begin{cases} f'(u_+), & x > 0,\\ -f'(u_+), & x < 0. \end{cases}
\end{cases} \tag{1.3.8}$$

We can get the explicit formula of $\omega(x,t)$ by using the Hopf-Cole transformation for the Burgers equation. Then, we define a smooth approximation $\psi(x,t)$ of the rarefaction wave $\psi^R(x/t)$ as

$$\psi(x,t) = (f')^{-1}(\omega(x,t)), \quad x \ge 0.$$

The function $\psi(x,t)$ is well-defined since $f(u)$ is strictly convex on $u \in [0,u_+]$ for any $u_+ > 0$ under the condition (1.1.3), or for $0 < u_+ \le r$ under (1.1.7). Then we have the next lemma which is similar to Lemma 1.3.1. For the proof, refer to [14].

Lemma 1.3.2. *Assume either* (1.1.3) *with* $u_+ > 0$ *or* (1.1.7) *with* $0 < u_+ \le r$*, then we have the following:*
1) $\psi(x,t)$ *is the smooth solution of the initial boundary value problem*

$$\begin{cases}
\psi_t + f(\psi)_x = \psi_{xx} + \frac{f'''(\psi)}{f''(\psi)}\psi_x^2, \quad x > 0,\ t > 0,\\
\psi(x,0) = \psi_0(x) := (f')^{-1}(w(x,0)), \quad x > 0.
\end{cases}$$

2) $0 < \psi(x,t) < u_+$ *and* $\psi_x(x,t) > 0, \quad x > 0,\ t > 0.$
3) *For $1 \le p \le \infty$, there exists a positive constant C_p such that*

$$\begin{aligned}
&\| \psi_x(t) \|_{L^p} \le C_p \min(u_+, (1+t)^{-1+\frac{1}{p}}),\\
&\| \psi_{xx}(t) \|_{L^p} \le C_p \min(u_+, (1+t)^{-1}).
\end{aligned}$$

4) *For $1 \le p \le \infty$, there exists a positive constant C_p such that*

$$\|\psi(t) - \psi^R(\cdot/t)\|_{L^p} \le C_p(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}.$$

1.4 Reformulation of the problem

In this section, in order to prove Theorem 1.1.1, we reformulate the problem (1.1.1) with respect to the deviation of u from the asymptotic state $\phi + \psi^R$. Now if we put

$$\Phi(x,t) = \phi(x) + \psi(x,t) \tag{1.4.1}$$

as the expected asymptotic state, it follows from the definitions of ϕ and ψ that Φ approximately satisfies the equation of (1.1.1) as

$$\Phi_t + f(\Phi)_x - \Phi_{xx} = -F(\phi,\psi), \tag{1.4.2}$$

where

$$F(\phi,\psi) = -(f'(\phi+\psi) - f'(\phi))\phi_x - (f'(\phi+\psi) - f'(\psi))\psi_x + \psi_{xx}. \tag{1.4.3}$$

Define the deviation v of u from Φ by

$$v(x,t) = u(x,t) - \Phi(x,t). \tag{1.4.4}$$

Then the problem (1.1.1) is reformulated in terms of v in the form

$$\begin{cases} v_t + \{f(\Phi+v) - f(\Phi)\}_x - v_{xx} = F(\phi,\psi), \quad x > 0,\ t > 0, \\ v(0,t) = 0, \quad t > 0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x) - \psi(x,0), \quad x > 0, \end{cases} \tag{1.4.5}$$

where we can see $v_0 \in H_0^1$ by the assumption (1.1.8). The theorem for the reformulated problem (1.4.5) we shall prove is

Theorem 1.4.1. *Assume* (1.1.7),(1.1.8), *and* (1.1.10). *Then, there exists a positive constant ε such that, if $\|v_0\|_{H^1} \le \varepsilon$ and $0 < u_+ \le \varepsilon$, then the initial boundary value problem* (1.4.5) *has a unique global solution in time v satisfying*

$$\begin{cases} v \in C([0,\infty); H_0^1), \\ v_x \in L^2(0,\infty; H^1), \\ \lim\limits_{t\to\infty} \sup\limits_{x>0} |v(x,t)| = 0. \end{cases} \tag{1.4.6}$$

If we note

$$\|v_0\|_{H^1} = \| u_0 - \phi - \psi(\cdot, 0)\|_{H^1} \leq \| u_0 - \phi - \psi^R(\cdot)\|_{H^1} + C|u_+|$$

and particularly 4) of the lemma 1.1.5, Theorem 1.1.1 is a direct consequence of the Theorem 1.4.1. The Theorem 1.4.1 itself is proved by combining the local existence theorem together with the *a priori* estimate as in the previous papers. To state the local existence theorem precisely, we define the solution set for any interval $I \subset R$ and constant $M > 0$ by

$$X_M(I) = \{v \in C(I; H_0^1); v_x \in L^2(0, T; H^1), \sup_{t \in I} \|v(t)\|_{H^1} \leq M\},$$

and also generalize the initial boundary value problem for any constant $\tau \geq 0$ as

$$\begin{cases} v_t + \{f(\Phi + v) - f(\Phi)\}_x - v_{xx} = F(\phi, \psi), \quad x > 0, \ t > \tau, \\ v(0, t) = 0, \quad t > \tau, \\ v(x, \tau) = v_\tau(x), \quad x > 0, \quad v_\tau \in H_0^1. \end{cases} \tag{1.4.7}$$

Then we state the local existence theorem.

Proposition 1.4.2 (local existence)**.** *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ which is independent of τ such that if $\|v_\tau\|_{H^1} \leq M$, the initial boundary value problem* (1.4.5) *has a unique solution $v \in X_{2M}([\tau, \tau + t_0])$.*

It is noted that the case $\tau = 0$ is enough to prove, and then the problem (1.4.7) is reduced to the integral equation

$$v(x,t) = \int_0^\infty G(x, y, t - \tau) v_\tau(y) dy + \\ + \int_\tau^t \int_0^\infty G(x, y; t - s)(-(f(\phi + v) - f(\phi))_x + F(\phi, \psi))(s)\, dyds,$$

where $G(x, y; t)$ is the Green kernel of the Dirichlet zero boundary value problem for the linear heat equation on the half line, which is concretely given by

$$G(x, y; t) = \frac{1}{\sqrt{4\pi t}}(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}).$$

Since we can prove the Proposition 1.4.2 by a standard iterative method, we omit the proof. Next, let us state the *a priori* estimate which is essential for the proof of Theorem 1.4.1.

Proposition 1.4.3 (*a priori* estimate)**.** *Under the condition* (1.1.7)*, there exist positive constants* ε *and* C *such that if* $0 < u_+ < \varepsilon$ *and* $v \in X_\varepsilon([0,T])$ *is the solution of the problem* (1.4.7) *for some* $T > 0$*, then it holds*

$$\begin{aligned} \|v(t)\|_{H^1}^2 + \int_0^t (\|\sqrt{\Phi_x} v(s)\|_{L^2}^2 + \|v_x(s)\|_{H^1}^2)\, ds \\ \le C(\|v_0\|_{H^1}^2 + |u_+|^{\frac{1}{6}}), \quad t \in [0,T]. \end{aligned} \tag{1.4.8}$$

Here we should note $\Phi_x = \phi_x + \psi_x > 0$ by the Lemmas 1.2.1 and 1.3.1. The proof of the Proposition 1.4.3 is given in the Sections 1.5 and 1.6. Once the Propositions 1.4.2 and 1.4.3 are proved, the Theorem 1.4.1 is proved in a standard way as in the previous works. In fact, combining the local existence and *a priori* estimate, we can first prove the global existence of the solution in time by choosing $\|v_0\|_{H^1}$ and u_+ suitably small. Then we can see the estimate (1.4.8) holds even for $t \in [0,\infty)$, that is,

$$\sup_{t\ge 0} \|v(t)\|_{H^1}, \ \int_0^\infty \|v_x(t)\|_{H^1}^2\, dt < \infty. \tag{1.4.9}$$

By using the equation and the estimate (1.4.9), we can also have (cf. (1.6.24),(1.6.27))

$$\int_0^\infty |\frac{d}{dt}\|v_x(t)\|_{L^2}^2|\, dt < \infty$$

which implies

$$\lim_{t\to\infty} \|v_x(t)\|_{L^2} = 0. \tag{1.4.10}$$

Using Sobolev's embedding lemma, the estimates (1.4.9) and (1.4.10), we can easily have

$$\sup_{x>0} |v(x,t)| \le \sqrt{2}\|v(t)\|_{L^2}^{\frac{1}{2}} \|v_x(t)\|_{L^2}^{\frac{1}{2}} \le C\|v_x(t)\|_{L^2}^{\frac{1}{2}} \to 0, \quad t \to \infty,$$

which shows the asymptotic behavior of the solution. Thus, we can show the proof of the Theorem 1.4.1 by the Propositions 1.4.2 and 1.4.3.

1.5 Weight function

In this section, we explain how to make the weight function which plays an essential role in our technical L^2-energy method. For simplicity of explanation, noting that v and u_+ is sufficiently small in the *a priori* estimate, we take the linearized equation of the problem (1.4.7) with $u_+ = 0$ (accordingly, $\Phi(x,t) = \phi(x)$)

$$\begin{cases} v_t + \{f'(\phi)v\}_x - v_{xx} = 0, \quad x > 0,\ t > 0, \\ v(0,t) = 0, \quad t > 0, \\ v(x,0) = v_0(x), \quad x > 0, \quad v_0 \in H^1_0. \end{cases} \tag{1.5.1}$$

Let $v \in C([0,T]; H^1_0) \cap L^2(0,T; H^2)$ be a solution of (1.5.1). The most typical way to have the L^2 estimate which does not depend on T is one to multiply the equation of (1.5.1) by v and integrate it with respect to x over $(0,\infty)$. Then, the integration by parts gives

$$\frac{d}{dt}\int_0^\infty \frac{1}{2}v^2 dx + \frac{1}{2}\int_0^\infty f''(\phi)\phi_x v^2 dx + \int_0^\infty v_x^2\, dx = 0. \tag{1.5.2}$$

If we note $\phi_x > 0$, the estimate of (1.5.2) works well in the case $f'' > 0$, but not in the case f is not convex because f'' changes its sign. In order to overcome this difficulty, we try to apply a weighted energy method as in ([8], [18], [27]) where to show the asymptotic stability of viscous shock profile for non-convex state equations, a weight function w is manipulated as a function of the viscous shock wave itself. Take a weight funtion $w(\phi)$ as a function of ϕ, and multiply the equation of (1.5.1) by vw and integrate it over $(0,\infty)$. Then, noting the relation $\phi_x = f(\phi)$, we have

$$\begin{aligned} \frac{d}{dt}\int_0^\infty \frac{1}{2}w(\phi)v^2 dx + \int_0^\infty (\frac{1}{2}f''w - f'w' - \frac{1}{2}fw'')(\phi)\phi_x v^2\, dx \\ + \int_0^\infty w(\phi)v_x^2\, dx = 0. \end{aligned} \tag{1.5.3}$$

Under the condition (1.5), Nagase [29] succeeded in making a positive and smooth weight function $w(u)$ for $u \in [u_-, 0],\ u_- \in (\tilde{u}_* - \varepsilon, 0)$ so that it holds in (1.5.3)

$$(\frac{1}{2}f''w - \frac{1}{2}w''f - f'w')(u) > 0, \quad u \in [u_-, 0]. \tag{1.5.4}$$

Roughly speaking, she constructed $w(u)$ as almost identically constant in the region $f'' > 0$, and $-f' + constant.$ in the remaining region, and then patch them up on the whole $[u_-, 0]$. In fact, even in the region $f'' < 0$, the term $-w''f/2 - f'w'(= f'''w/2 + f'f'')$ in (1.5.4) is positive as long as $f' < 0$ and $f''' > 0$, and so plays a nice role to control the negative term $f''w$ and show the positivity of (1.5.4), which is a basic technical idea in [29]. However this choice of weight function can not be easily extended to a region $f'' < 0$ and $f' > 0$ because the term $-f'w'(= f'f'')$ becomes negative and so causes a problem, which is a main reason why the case $u_- \in (u_*, \tilde{u}_* - \varepsilon)$ has been left open. In order to overcome this difficulty, we employ a technique in Matsumura-Mei [23] where they manipulated not only a weight function but also a transformation of the unknown functions in order to prove the asymptotic stability of viscous shock profile for a system of visco-elasticity with a non-convex nonlinearity. Following their technique, we introduce a new unknown function $\tilde{v}$ by

$$v(x,t) = \chi(\phi(x))\tilde{v}(x,t), \tag{1.5.5}$$

where $\chi(u)$ is a positive and smooth function on $[u_-, 0]$. Substitute (1.5.5) into (1.5.1), then we have the equation of $\tilde{v}$ as in the form

$$\tilde{v}_t + \frac{1}{\chi}(f'\chi\tilde{v})_x - \frac{1}{\chi}(\chi\tilde{v})_{xx} = 0. \tag{1.5.6}$$

Multiply (1.5.6) by $w\tilde{v}$ and integrate it over $(0, \infty)$, then we have

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty \frac{1}{2}w\tilde{v}^2 dx + \int_0^\infty \{(\frac{1}{2}f''w - f'w' - \frac{1}{2}fw'')+ \\ &+\frac{\chi'}{\chi^2}(\chi f'w + \chi fw' - \chi' fw)\}\phi_x\tilde{v}^2 dx + \int_0^\infty w\tilde{v}_x^2\, dx = 0. \end{aligned} \tag{1.5.7}$$

So we may choose positive functions $\chi(u)$ and $w(u)$ on $[u_-, 0]$ so that

$$\{\frac{1}{2}f''w - \frac{1}{2}w''f - f'w' + \frac{\chi'}{\chi^2}(\chi f'w + \chi fw' - \chi' fw)\}(u) \tag{1.5.8}$$

is positive on $[u_-, 0]$. Now let us choose $\chi = w$, then (1.5.8) becomes

$$\frac{1}{2}(f''w - fw''). \tag{1.5.9}$$

Hence it is enough to seek a positive weight function $w(u)$ which makes (1.5.9) positive on $[u_-, 0]$. Under the condition (1.1.7), we choose the

function $w(u)$ by

$$w(u) = f(u) + \delta g(u), \quad u \in [u_-, r] \tag{1.5.10}$$

where δ is a positive constant and

$$g(u) = -u^{2m} + r^{2m}, \quad u \in [u_-, r],\ m \geq 1. \tag{1.5.11}$$

Here it is noted that the interval $[u_-, 0]$ is extended to $[u_-, r]$ to treat the case $u_+ > 0$, and that the constant δ and the integer m are properly chosen later.

Lemma 1.5.1 (weight function)**.** *Under the condition* (1.1.7)*, if we take δ sufficiently small and m sufficiently large, the functions $\frac{1}{2}(f''(u)w(u) - f(u)w''(u))$ and $w(u)$ are positive for $u \in [u_-, r]$.*

Proof. First it is easy by the condition (1.1.7) to see that there exists a positive constant ν such that $f''(u) \geq \nu$ for $|u| \leq r$, and $w(u) \geq \nu$ for $u \in [u_-, -r]$. Substituting (1.5.10) into (1.5.9), we have

$$\begin{aligned} f''w - fw'' &= f''(f + \delta g) - f(f + \delta g)'' \\ &= \delta(f''g - fg''). \end{aligned} \tag{1.5.12}$$

We divide the interval $[u_-, r]$ into $[u_-, -r]$ and $[-r, r]$. For $u \in [u_-, -r]$, substituting (1.5.11) into (1.5.12), we obtain

$$\begin{aligned} \delta(f''g - fg'') &= \delta(f''(-u^{2m} + r^{2m}) + 2m(2m-1)fu^{2(m-1)}) \\ &= 2m(2m-1)\delta u^{2(m-1)}\{\frac{-1 + |\frac{r}{u}|}{2m(2m-1)}u^2 f'' + f\}. \end{aligned} \tag{1.5.13}$$

Because f'' is bounded, $|r/u| \leq 1$ and $f(u) \geq \nu$ for $u \in [u_-, -r]$, we can choose m sufficiently large so that

$$\frac{-1 + |\frac{r}{u}|}{2m(2m-1)}u^2 f''(u) + f(u) \geq \frac{1}{2}\nu, \quad u \in [u_-, -r]. \tag{1.5.14}$$

Therefore, (1.5.13) and (1.5.14) imply

$$\delta(f''g - fg'') \geq m(2m-1)\delta r^{2(m-1)}\nu > 0, \quad u \in [u_-, -r]. \tag{1.5.15}$$

For $|u| \leq r$, we further divide the interval to $|u| \leq r/2$ and $r/2 \leq |u| \leq r$. For $|u| \leq r/2$, since $f'' > 0, g > 0$ and $f \geq 0$, $g'' \leq 0$, it clearly holds

$$\delta(f''g - fg'') > 0, \quad |u| \leq \frac{r}{2}. \tag{1.5.16}$$

On the other hand, for $r/2 \le |u| \le r$, since $f'' > 0, g \ge 0$ and $f > 0$, $g'' < 0$, it easily holds

$$\delta(f''g - fg'') > 0, \quad \frac{r}{2} \le |u| \le r. \tag{1.5.17}$$

Thus, it follows from (1.5.15), (1.5.16) and (1.5.17) that

$$(f''w - fw'')(u) = \delta(f''g - fg'')(u) > 0, \quad u \in [u_-, r]. \tag{1.5.18}$$

Next, we prove $w(u) = f(u) + \delta g(u)$ to be positive. To do that, we again divide the interval $[u_-, r]$ to $[u_-, -r]$, $r/2 \le |u| \le r$ and $|u| \le r/2$. Noting $f \ge \nu$ and g is bounded for $u \in [u_-, -r]$, we can take δ sufficiently small so that

$$w(u) = f(u) + \delta g(u) \ge \frac{\nu}{2} > 0, \quad u \in [u_-, -r]. \tag{1.5.19}$$

Because $f > 0$, $g \ge 0$ for $r/2 \le |u| \le r$ and $f \ge 0$, $g > 0$ for $|u| \le r/2$, we can easily have

$$w(u) = f(u) + \delta g(u) > 0, \quad |u| \le r. \tag{1.5.20}$$

Thus (1.5.18),(1.5.19) and (1.5.20) complete the proof of the Lemma 1.5.1. $\square$

1.6 *A priori* estimate

In this section, we give the proof of the Proposition 1.4.3. First, put

$$N(T) = \sup_{0\le t\le T} \|v(t)\|_{H^1},$$

and then we suppose $N(T) \le 1$ throughout this section. Now, motivated by the argument in the Section 1.5, we introduce a new unknown function $\tilde{v}$ by

$$v(x,t) = w(\Phi(t,x))\tilde{v}(x,t), \tag{1.6.1}$$

where $\Phi(t,x) = \phi(x) + \psi(t,x)$ and $w = f + \delta g$ is the weight function in the Lemma 1.5.1. Since the Lemmas 1.2.1 and 1.3.1 imply $\Phi(x,t) \in [u_-, r],\ x \ge 0,\ t \ge 0$, we note that $w(\Phi(x,t))$ is well defined as weight function by the Lemma 1.5.1, that is, smooth and satisfies

$$\nu \le w(\Phi(x,t)) \le C, \quad x \ge 0,\ t \ge 0 \tag{1.6.2}$$

for some positive constants ν and C. Substituting (1.6.1) into the equation of (1.4.5), we get

$$(w(\Phi)\tilde{v})_t + (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))_x - (w(\Phi)\tilde{v})_{xx} = F(\phi,\psi). \tag{1.6.3}$$

Multiplying $\tilde{v}$ by (1.6.3) and integrating it over $(0,\infty)$, we have

$$\begin{aligned}(\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}^2 dx)_t + \int_0^\infty \frac{1}{2}w'(\Phi)\psi_t\tilde{v}^2 dx \\ + \int_0^\infty -(f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))\tilde{v}_x dx \\ + \int_0^\infty (w(\Phi)\tilde{v})_x\tilde{v}_x dx = \int_0^\infty \tilde{v}F dx.\end{aligned} \tag{1.6.4}$$

We rewrite the third term on the left hand side of (1.6.4) as

$$\begin{aligned}&\int_0^\infty -(f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))\tilde{v}_x\, dx \\ &= -\int_0^\infty \{\int_0^{\tilde{v}} f(\Phi + w(\Phi)\eta) - f(\Phi) d\eta\}_x \\ &\quad - \int_0^{\tilde{v}} (f'(\Phi + w(\Phi)\eta) - f'(\Phi))\Phi_x \\ &\qquad\qquad + f'(\Phi + w(\Phi)\eta)w'(\Phi)\Phi_x\eta\, d\eta dx\end{aligned} \tag{1.6.5}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^{\tilde{v}} (f'(\Phi + w(\Phi)\eta) - f'(\Phi))\, d\eta\, \Phi_x\, dx \\
&\qquad + \int_0^\infty \int_0^{\tilde{v}} f'(\Phi + w(\Phi)\eta) w'(\Phi)\eta\, d\eta\, \Phi_x\, dx \\
&=: I_1 + I_2.
\end{aligned} \tag{1.6.6}
$$

We further rewrite I_1 and I_2 by the Taylor's formula as

$$
\begin{aligned}
I_1 &= \int_0^\infty \int_0^{\tilde{v}} f''(\Phi) w(\Phi)\eta + O(\eta^2)\, d\eta \Phi_x dx \\
&= \int_0^\infty \frac{1}{2} f''(\Phi) w(\Phi) \tilde{v}^2 \Phi_x + O(\tilde{v}^3)\Phi_x dx.
\end{aligned} \tag{1.6.7}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^\infty \int_0^{\tilde{v}} f'(\Phi) w'(\Phi)\eta + O(\eta^2)\, d\eta \Phi_x dx \\
&= \int_0^\infty \frac{1}{2} f'(\Phi) w'(\Phi) \tilde{v}^2 \Phi_x + O(\tilde{v}^3)\Phi_x dx.
\end{aligned} \tag{1.6.8}
$$

Hence, substituting (1.6.7) and (1.6.8) into (1.6.5), we have

$$
\begin{aligned}
&\int_0^\infty - (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))\tilde{v}_x\, dx \\
&\qquad = \int_0^\infty \frac{1}{2}(f'' w + f' w')(\Phi)\tilde{v}^2 \Phi_x\, dx + \int_0^\infty O(\tilde{v}^3)\Phi_x\, dx.
\end{aligned} \tag{1.6.9}
$$

We also rewrite the fourth term on the left hand side of (1.6.4) as

$$
\begin{aligned}
&\int_0^\infty (w(\Phi)\tilde{v})_x \tilde{v}_x\, dx \\
&= \int_0^\infty w\tilde{v}_x^2 + w'\Phi_x \tilde{v}\tilde{v}_x\, dx \\
&= \int_0^\infty w\tilde{v}_x^2 - \frac{1}{2} w'\Phi_{xx}\tilde{v}^2 - \frac{1}{2} w''\Phi_x^2 \tilde{v}^2\, dx \\
&=: \int_0^\infty w\tilde{v}_x^2\, dx + I_3 + I_4.
\end{aligned} \tag{1.6.10}
$$

Now, recalling the relation $\Phi_t + f(\Phi)_x - \Phi_{xx} = -F$, we further rewrite

I_3 as

$$\begin{aligned} I_3 &= \int_0^\infty -\frac{1}{2}w'\Phi_{xx}\tilde{v}^2\,dx \\ &= \int_0^\infty \frac{1}{2}w'(-\Phi_t - f'(\Phi)\Phi_x - F)\tilde{v}^2\,dx \\ &= \int_0^\infty (-\frac{1}{2}w'f'(\Phi)\Phi_x\tilde{v}^2 - \frac{1}{2}w'\psi_t\tilde{v}^2 - \frac{1}{2}w'F\tilde{v}^2)\,dx \end{aligned} \tag{1.6.11}$$

and I_4 as

$$\begin{aligned} I_4 &= \int_0^\infty -\frac{1}{2}w''\Phi_x^2\tilde{v}^2\,dx \\ &= \int_0^\infty -\frac{1}{2}w''(\phi_x+\psi_x)\Phi_x\tilde{v}^2\,dx \\ &= \int_0^\infty -\frac{1}{2}w''(f(\Phi)+\phi_x+\psi_x - f(\phi+\psi))\Phi_x\tilde{v}^2\,dx \\ &= \int_0^\infty -\frac{1}{2}w''f(\Phi)\Phi_x\tilde{v}^2 \\ &\qquad\qquad - \frac{1}{2}w''\Phi_x\tilde{v}^2\{f(\phi)-f(\phi+\psi)+\psi_x\}\,dx \\ &= \int_0^\infty -\frac{1}{2}w''f(\Phi)\Phi_x\tilde{v}^2\,dx + \int_0^\infty O(|\psi|+|\psi_x|)\Phi_x\tilde{v}^2\,dx. \end{aligned} \tag{1.6.12}$$

Substituting (1.6.11) and (1.6.12) into (1.6.10), we have

$$\begin{aligned} &\int_0^\infty (w(\Phi)\tilde{v})_x\tilde{v}_x\,dx \\ &\quad = \int_0^\infty -\frac{1}{2}(w''f+w'f')(\Phi)\Phi_x\tilde{v}^2\,dx \\ &\quad + \int_0^\infty O(|\psi|+|\psi_x|)\Phi_x\tilde{v}^2\,dx \\ &\qquad\qquad + \int_0^\infty (-\frac{1}{2}w'\psi_t\tilde{v}^2 + w\tilde{v}_x^2 - \frac{1}{2}w'F\tilde{v}^2)\,dx. \end{aligned} \tag{1.6.13}$$

Thus, by (1.6.9) and (1.6.13), (1.6.4) reads

$$\begin{aligned} &(\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}^2dx)_t + \int_0^\infty \frac{1}{2}(wf''-w''f)(\Phi)\Phi_x\tilde{v}^2\,dx + \int_0^\infty w\tilde{v}_x^2\,dx \\ &= \int_0^\infty (\tilde{v}F + \frac{1}{2}w'F\tilde{v}^2)\,dx + \int_0^\infty O(\tilde{v}+|\psi|+|\psi_x|)\Phi_x\tilde{v}^2\,dx. \end{aligned} \tag{1.6.14}$$

Noting that the Sobolev's embedding lemma and Lemma 2 easily imply

$$\int_0^\infty O(\tilde{v}+|\psi|+|\psi_x|)\Phi_x\tilde{v}^2\,dx \le C(N(T)+|u_+|)\int_0^\infty \Phi_x\tilde{v}^2\,dx \quad (1.6.15)$$

and also the Sobolev's embedding lemma and Young's inequality imply

$$\begin{aligned}|\int_0^\infty (\tilde{v}F+\frac{1}{2}w'F\tilde{v}^2)\,dx| &\le C\int_0^\infty |\tilde{v}\|F|\,dx\\ &\le C\|\tilde{v}\|_{L^2}^{\frac{1}{2}}\|\tilde{v}_x\|_{L^2}^{\frac{1}{2}}\|F\|_{L^1}\\ &\le \frac{1}{2}\|\tilde{v}_x\|_{L^2}^2+C\|F\|_{L^1}^{\frac{4}{3}},\end{aligned} \quad (1.6.16)$$

we can estimate (1.6.14) due to the Lemma 1.5.1 as

$$\begin{aligned}&(\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}^2\,dx)_t+\nu\int_0^\infty \Phi_x\tilde{v}^2dx+\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}_x^2\,dx\\ &\le C(N(T)+|u_+|)\int_0^\infty \Phi_x\tilde{v}^2\,dx+C\|F\|_{L^1}^{\frac{4}{3}}\end{aligned} \quad (1.6.17)$$

for a positive constant ν. Therefore, taking $N(T)+|u_+|$ suitably small, we have

$$\begin{aligned}&(\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}^2\,dx)_t+\frac{\nu}{2}\int_0^\infty \Phi_x\tilde{v}^2dx+\frac{1}{2}\int_0^\infty w(\Phi)\tilde{v}_x^2\,dx\\ &\le C\|F\|_{L^1}^{\frac{4}{3}}.\end{aligned} \quad (1.6.18)$$

Using the positivity of w and the fact

$$\begin{aligned}\|v_x\|_{L^2}^2=\|(w\tilde{v})_x\|_{L^2}^2&=\|w_x\tilde{v}+w\tilde{v}_x\|_{L^2}^2\\ &\le C(\|\sqrt{\Phi_x}\tilde{v}\|_{L^2}^2+\|\tilde{v}_x\|_{L^2}^2),\end{aligned} \quad (1.6.19)$$

and integrating (1.6.18) with respect to t over $(0,t)$, we have

$$\begin{aligned}\|v(t)\|_{L^2}^2+\int_0^t(\|\sqrt{\Phi_x}v(\tau)\|_{L^2}^2+\|v_x(\tau)\|_{L^2}^2)\,d\tau&\\ \le C(\|v_0\|_{L^2}^2+\int_0^t\|F(\tau)\|_{L^1}^{\frac{4}{3}}\,d\tau).&\end{aligned} \quad (1.6.20)$$

Next, we proceed to the estimate of v_x. Multiplying $-v_{xx}$ by the equation of (1.6.3) and integrating it with respect to x over $(0,\infty)$, we

have

$$
\begin{aligned}
(\frac{1}{2}\int_0^\infty v_x^2\,dx)_t + \int_0^\infty v_{xx}^2\,dx &= -\int_0^\infty Fv_{xx}\,dx \\
&+ \int_0^\infty (f'(\Phi+v)(\Phi_x+v_x) - f'(\Phi)\Phi_x)v_{xx}\,dx.
\end{aligned}
\tag{1.6.21}
$$

We estimate the right hand side of (1.6.21) as

$$
|\int_0^\infty Fv_{xx}\,dx| \le \frac{1}{4}\|v_{xx}\|_{L^2}^2 + C\|F\|_{L^2}^2, \tag{1.6.22}
$$

and

$$
\begin{aligned}
&|\int_0^\infty (f'(\Phi+v)(\Phi_x+v_x) - f'(\Phi)\Phi_x)v_{xx}\,dx| \\
&\qquad \le \int_0^\infty C(|v|\Phi_x + |v_x|)|v_{xx}|\,dx \\
&\qquad \le \frac{1}{4}\|v_{xx}\|_{L^2}^2 + C(\|\sqrt{\Phi_x}v\|_{L^2}^2 + \|v_x\|_{L^2}^2).
\end{aligned}
\tag{1.6.23}
$$

Substituting (1.6.22) and (1.6.23) into (1.6.21), we have

$$
\begin{aligned}
(\frac{1}{2}\int_0^\infty v_x^2\,dx)_t + \frac{1}{2}\int_0^\infty v_{xx}^2\,dx \\
\le C(\|F\|_{L^2}^2 + \|\sqrt{\Phi_x}v\|_{L^2}^2 + \|v_x\|_{L^2}^2).
\end{aligned}
\tag{1.6.24}
$$

Integrating (1.6.24) with respect to t over $(0,t)$ and combining it with the estimate (1.6.20), we obtain

$$
\begin{aligned}
\|v_x\|_{L^2}^2 + \int_0^t \|v_{xx}\|_{L^2}^2\,d\tau \\
\le C(\|v_0\|_{H^1}^2 + \int_0^t (\|F\|_{L^1}^{\frac{4}{3}} + \|F\|_{L^2}^2)\,d\tau).
\end{aligned}
\tag{1.6.25}
$$

Thus by (1.6.20) and (1.6.25), we have

$$
\begin{aligned}
\|v(t)\|_{H^1}^2 + \int_0^t (\|\sqrt{\Phi_x}v(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{H^1}^2)\,d\tau \\
\le C(\|v_0\|_{H^1}^2 + \int_0^t \|F(\tau)\|_{L^1}^{\frac{4}{3}} + \|F(\tau)\|_{L^2}^2\,d\tau).
\end{aligned}
\tag{1.6.26}
$$

Following the arguments in [17], we finally estimate the right hand side

of (1.6.26) by using the Lemmas 1.2.1 and 1.3.1, and eventually can show

$$\begin{aligned} \|F(t)\|_{L^1}^{\frac{4}{3}} &\le C|u_+|^{\frac{1}{6}}(1+t)^{-\frac{7}{6}}\log^{\frac{4}{3}}(2+t), \\ \|F(t)\|_{L^2}^2 &\le C|u_+|^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{1.6.27}$$

Noting

$$|F(\Phi)| \le C(|\psi\phi_x| + |\psi_x\phi| + |\psi_{xx}|), \tag{1.6.28}$$

we only show the estimates of $\|\psi\phi_x\|_{L^1}$ and $\|\psi\phi_x\|^2$ because the other terms can be obtained in the same way. Using the fact that the Lemma 1.2.1 implies

$$|\phi_x(x)| \le \frac{C}{(1+x)^2}, \quad x > 0 \tag{1.6.29}$$

and the decay estimates of ψ in the Lemma 1.3.1, we have

$$\begin{aligned} \|\psi\phi_x\|_{L^1} &\le C\int_0^\infty \frac{\psi}{(1+x)^2}\,dx \\ &\le C\int_0^t \frac{\psi}{(1+x)^2}\,dx + C\int_t^\infty \frac{\psi}{(1+x)^2}\,dx \\ &\le C([-\frac{\psi}{(1+x)}]_0^t + \int_0^t \frac{\psi_x}{(1+x)}\,dx) + C\|\psi\|_{L^\infty}\int_t^\infty \frac{1}{(1+x)^2}\,dx \\ &\le C\|\psi_x\|_{L^\infty}\log(1+t) + C|u_+|(1+t)^{-1} \\ &\le C\|\psi_x\|_{L^\infty}^{\frac{1}{8}}\|\psi_x\|_{L^\infty}^{\frac{7}{8}}\log(2+t) + C|u_+|^{\frac{1}{8}}|u_+|^{\frac{7}{8}}(1+t)^{-1} \\ &\le C|u_+|^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\log(2+t), \end{aligned} \tag{1.6.30}$$

$$
\begin{aligned}
\|\psi\phi_x\|_{L^2}^2 &\le C\int_0^\infty \frac{\psi^2}{(1+x)^4}\,dx \\
&\le C\int_0^t \frac{\psi^2}{(1+x)^4}\,dx + C\int_t^\infty \frac{\psi^2}{(1+x)^4}\,dx \\
&\le C([-\frac{\psi^2}{3(1+x)^3}]_0^t + \int_0^t \frac{\psi\psi_x}{3(1+x)^3}\,dx) + C|u_+|^2(1+t)^{-3} \\
&\le C\|\psi_x\|_{L^\infty}\int_0^t \frac{\psi}{(1+x)^3}\,dx + C|u_+|^2(1+t)^{-3} \\
&\le C\|\psi_x\|_{L^\infty}^2\int_0^t \frac{1}{(1+x)^2}\,dx + C|u_+|^2(1+t)^{-3} \\
&\le C|u_+|^{\frac{1}{2}}(1+t)^{-\frac{3}{2}},
\end{aligned}
\tag{1.6.31}
$$

and similarly

$$
\begin{aligned}
&\|\psi_x\phi\|_{L^1} + \|\psi_{xx}\|_{L^1} \le C|u_+|^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\log(2+t), \\
&\|\psi_x\phi\|_{L^2}^2 + \|\psi_{xx}\|_{L^2}^2 \le C|u_+|^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}.
\end{aligned}
\tag{1.6.32}
$$

Hence (1.6.30),(1.6.31) and (1.6.32) prove the estimate (1.6.27). Then substituting (1.6.27) into (1.6.26), we finally have the desired *a priori* estimate for suitably small $N(T)+|u_+|$

$$
\begin{aligned}
\|v(t)\|_{H^1}^2 + \int_0^t (\|\sqrt{\Phi_x}v(\tau)\|_{L^2}^2 + \|v_x^2(\tau)\|_{H^1})\,d\tau \\
\le C(\|v_0\|_{H^1}^2 + |u_+|^{\frac{1}{6}}), \quad t\in[0,T].
\end{aligned}
\tag{1.6.33}
$$

Thus the proof of the Proposition 1.4.3 is completed. □

1.7 Decay rate estimate I; convex flux

In this section, we give the proof of the Theorem 1.1.2. Let us recall the Theorem 1.1.2.

Theorem 1.7.1 (decay rate for convex flux)**.** *Assume* (1.1.3), (1.1.8), $u_- < 0 < u_+$, *and also* $u_0 - \phi - \psi^R(\cdot) \in L^1$. *Then the initial-boundary value problem* (1.1.1) *has a unique global solution in time* u *satisfying*

$$\begin{cases} u - u_+ \in C([0,\infty); H^1), \\ u_x \in L^2(0,T; H^1), \quad ({}^\forall T > 0), \end{cases}$$

and the decay rate estimates

$$\|(u - \phi - \psi^R)(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \log^2(2+t), \quad (1 \le p < \infty),$$
$$\|(u - \phi - \psi^R)(t)\|_{L^\infty} \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}, \quad ({}^\forall \epsilon > 0).$$

As in the Section 1.4, we put

$$\Phi(x,t) = \phi(x) + \psi(x,t)$$

as the expected asymptotic state, where we employed the smooth approximation for the rarefaction wave in the Lemma 1.3.2. It follows from the definitions of ϕ and ψ that Φ satisfies the following equation:

$$\Phi_t + f(\Phi)_x - \Phi_{xx} = F(\phi, \psi),$$

where

$$F(\phi,\psi) = (f'(\phi+\psi) - f'(\phi))\phi_x + (f'(\phi+\psi) - f'(\psi))\psi_x + \frac{f'''(\psi)}{f''(\psi)}\psi_x^2.$$

By using Lemma 1.2.1 and Lemma 1.3.1, the direct computations give the estimates of $F(\phi,\psi)$ as follows.

Lemma 1.7.2. *$F(\phi,\psi)$ satisfies the following L^1- and L^2-estimates:*

$$\|F\|_{L^1} \le C(1+t)^{-1}\log(2+t),$$
$$\|F\|_{L^2} \le C(1+t)^{-2}.$$

For the proof, refer to the Section 1.6. Define the deviation v of u from Φ by

$$v(x,t) = u(x,t) - \Phi(x,t),$$

then the problem (1.1.1) is reformulated in terms of v in the form

$$\begin{cases} v_t + \{f(\Phi+v) - f(\Phi)\}_x - v_{xx} = -F(\phi,\psi), \quad x>0,\ t>0, \\ v(0,t) = 0, \quad t>0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x) - \psi(x,0), \quad x>0, \end{cases} \tag{1.7.1}$$

where we emphasize that $v_0 \in H^1_0$ under the assumptions in the Theorem 1.7.1. The theorem for the reformulated problem (1.7.1) we shall prove is the following.

Theorem 1.7.3. *Assume* (1.1.3), $u_- < 0 < u_+$ *and also* $v_0 \in H^1_0 \cap L^1$. *Then the problem* (1.7.1) *has a unique global solution* v *satisfying* $v \in C([0,\infty); H^1_0)$ *and* $v_x \in L^2(0,\infty; H^1)$, *and the decay rate estimates*

$$\begin{aligned} &\|v(t)\|_{L^p} \le C_p(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\log^2(2+t) \quad (1\le p<\infty), \\ &\|v(t)\|_{L^\infty} \le C_\epsilon(1+t)^{-\frac{1}{2}+\epsilon}, \quad (\forall \epsilon>0). \end{aligned}$$

If we note the Lemma 1.3.2, the Theorem 1.7.1 is a direct consequence of Theorem 1.7.3.

Next, we proceed to the proof of Theorem 1.7.3. Theorem 1.7.3 is given by using the following a priori decay rate estimate for the initial-boundary value problem (1.7.1).

Proposition 1.7.4 (decay rate estimate I)**.** *Assume* (1.1.3), $u_- < 0 < u_+$ *and also* $v_0 \in H^1_0 \cap L^1$. *Then, the solution of* (1.7.1) *satisfies*

$$\begin{aligned} (1+t)^\alpha \|v\|^p_{L^p} &+ \int_0^t (1+\tau)^\alpha \|\Phi_x^{1/p} v\|^p_{L^p} d\tau \\ &+ \int_0^t (1+\tau)^\alpha \|(|v|^{\frac{p}{2}-1}v)_x\|^2_{L^2} d\tau \\ &\le C\|v_0\|^p_{L^p} + CM^p(1+t)^{\alpha-\frac{p-1}{2}}\log^{2p}(2+t) \end{aligned} \tag{1.7.2}$$

and

$$\begin{aligned} (1+t)^\alpha \|v_x\|^2_{L^2} &+ \int_0^t (1+\tau)^\alpha \|v_{xx}\|^2_{L^2}\, d\tau \\ &\le C\|v_0\|^2_{L^2} + C(1+t)^{\alpha-\frac{1}{2}}\log^4(2+t), \end{aligned} \tag{1.7.3}$$

for $2 \le p < \infty$ *and* $\alpha > \frac{p-1}{2}$.

In order to derive the decay estimate in Proposition 1.7.4, we start with the L^1 estimate. To this end, we introduce $a_\delta(v)$ and $A_\delta(v)$ as follows:

$$a_\delta(v) := (\rho_\delta * sgn)(v) = \int_{-\infty}^{\infty} \mathrm{sgn}(y)\rho_\delta(v-y)dy, \quad A_\delta(v) := \int_0^v a_\delta(\eta)d\eta,$$

where sgn is a usual signature function defined by

$$\mathrm{sgn}(v) := \begin{cases} -1 & \text{for} \quad v < 0, \\ 0 & \text{for} \quad v = 0, \\ 1 & \text{for} \quad v > 0, \end{cases}$$

and ρ_δ denotes the Friedrichs mollifier defined by

$$\rho_\delta := \frac{1}{\delta}\rho\Big(\frac{v}{\delta}\Big),$$

where ρ is a smooth non-negative function which has a compact support and satisfies $\int_{-\infty}^{\infty}\rho(x) = 1$. Then, the solution of (1.4.7) satisfies the following L^1-estimate.

Proposition 1.7.5 (L^1-estimate)**.** *Assume* (1.1.3), $u_- < 0 < u_+$ *and also* $v_0 \in H_0^1 \cap L^1$. *Then the solution of* (2.2.2) *satisfies*

$$\|v(t)\|_{L^1} \le C(\|v_0\|_{L^1} + 1)\log^2(2+t). \tag{1.7.4}$$

Proof. Multiplying $a_\delta(v)$ by (2.2.2), we have

$$\begin{aligned} A_\delta(v)_t + a_\delta(v)\{f(\Phi+v) - f(\Phi)\}_x - a_\delta(v)v_{xx} \\ = -a_\delta(v)F(\phi,\psi). \end{aligned} \tag{1.7.5}$$

We rewrite the second and third terms on the left hand side of (1.7.5) as

$$\begin{aligned} &a_\delta(v)\{f(\Phi+v) - f(\Phi)\}_x - a_\delta(v)v_{xx} \\ &= \Big\{a_\delta(v)\big(f(\Phi+v) - f(\Phi)\big) \\ &\qquad - \int_0^v a_\delta'(\eta)\big(f(\Phi+\eta) - f(\Phi)\big)d\eta - a_\delta(v)v_x\Big\}_x \\ &\qquad + \int_0^v a_\delta'(\eta)\big(f'(\Phi+\eta) - f'(\Phi)\big)\Phi_x d\eta + a_\delta'(v)v_x^2. \end{aligned} \tag{1.7.6}$$

We note that the second and third terms of the right hand side of (1.7.6) are positive since the relation $f'' > 0$, $\Phi_x > 0$ and $a_\delta'(v) > 0$. Using Lemma 1.7.2, this yields the following inequality

$$\Big|\int_0^t\int_0^\infty a_\delta(v)F(\phi,\psi)dxd\tau\Big| \le \int_0^t \|F\|_{L^1}d\tau \le C\log^2(2+t).$$

Finally, integrating (1.7.5) over $(0,\infty)$ and $(0,t)$ by the above estimates and making $\delta \to 0$ afterward, we have the desired estimate (1.7.4). □

Proof of Proposition 1.7.4. The proof is obtained by using the L^p energy method in [13] which makes use of interpolation inequalities. We first show the estimate (1.7.2). Multiplying (1.7.1) by $|v|^{p-2}v$, we have

$$\begin{aligned}\Big(\frac{1}{p}|v|^p\Big)_t + \big\{\mathcal{F} - |v|^{p-2}vv_x\big\}_x + \frac{4(p-1)}{p^2}\big|\big(|v|^{\frac{p}{2}-1}v\big)_x\big|^2 \\ + (p-1)\int_0^v \big(f'(\Phi+\eta) - f'(\Phi)\big)|\eta|^{p-2}d\eta\ \Phi_x \\ = -F(\phi,\psi)|v|^{p-2}v,\end{aligned} \tag{1.7.7}$$

where

$$\mathcal{F} := \big(f(\Phi+v) - f(\Phi)\big)|v|^{p-2}v - (p-1)\int_0^v \big(f(\Phi+\eta) - f(\Phi)\big)|\eta|^{p-2}d\eta.$$

By using the strict convexity of the flux f, the last integral on the left hand side of (1.7.7) is estimated as

$$\int_0^v (f'(\Phi+\eta) - f'(\Phi))|\eta|^{p-2}d\eta \geq \frac{c_0}{p}|v|^p,$$

where c_0 is a certain positive constant. Now we integrate (1.7.7) over $(0,\infty)$ together with the above estimate, we obtain

$$\begin{aligned}\Big(\frac{1}{p}\|v\|_{L^p}^p\Big)_t + \frac{4(p-1)}{p^2}\|(|v|^{\frac{p}{2}-1}v)_x\|_{L^2}^2 + \frac{c_0(p-1)}{p}\|\Phi_x^{1/p}v\|_{L^p}^p \\ \leq \|v\|_{L^\infty}^{p-1}\|F\|_{L^1}.\end{aligned} \tag{1.7.8}$$

Multiplying $(1+t)^\alpha$ by (1.7.8) and integrating over $(0,t)$, we have

$$\begin{aligned}&\frac{1}{p}(1+t)^\alpha\|v\|_{L^p}^p + \frac{4(p-1)}{p^2}\int_0^t (1+\tau)^\alpha\|V_x\|_{L^2}^2 d\tau \\ &\qquad + \frac{c_0(p-1)}{p}\int_0^t (1+\tau)^\alpha\|\Phi_x^{1/p}v\|_{L^p}^p d\tau \\ &\qquad \leq \frac{1}{p}\|v_0\|_{L^p}^p + \frac{\alpha}{p}\int_0^t (1+\tau)^{\alpha-1}\|v\|_{L^p}^p d\tau \\ &\qquad\qquad + \int_0^t (1+\tau)^\alpha\|F\|_{L^1}\|v\|_{L^\infty}^{p-1}\ d\tau.\end{aligned} \tag{1.7.9}$$

Here we put $V := |v|^{\frac{p}{2}-1}v$. By using Lemma 1.7.2, Lemma 1.8.14 and the interpolation inequalities

$$\|v\|_{L^\infty} \le C\|V_x\|_{L^2}^{\frac{2}{p+1}}\|v\|_{L^1}^{\frac{1}{p+1}}, \quad \|v\|_{L^p}^p \le C\|V_x\|_{L^2}^{\frac{2(p-1)}{p+1}}\|v\|_{L^1}^{\frac{2p}{p+1}} \quad \text{for} \quad 0 < p < \infty,$$

we rewrite the second and third terms of the right hand side of (1.7.9) as

$$\begin{aligned} &\frac{\alpha}{p}\int_0^t (1+\tau)^{\alpha-1}\|v\|_{L^p}^p d\tau + \int_0^t (1+\tau)^{\alpha}\|F\|_{L^1}\|v\|_{L^\infty}^{p-1}\, d\tau \\ &\le \varepsilon\int_0^t (1+\tau)^{\alpha}\|V_x\|_{L^2}^2 d\tau + C_{\varepsilon,p}M^p(1+t)^{\alpha-\frac{p-1}{2}}\log^{2p}(2+t), \end{aligned} \tag{1.7.10}$$

for any $\varepsilon > 0$ and $\alpha > (p-1)/2$. Here we defined $M := \|v_0\|_{L^1} + 1$. Substituting (1.7.10) into (1.7.9) and choosing ε suitably small, we have the desired estimate (1.7.2). In particular, we have

$$\|v\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\log^2(2+t). \tag{1.7.11}$$

Next, we proceed to the estimate of v_x. Multiplying $-v_{xx}$ by the equation of (1.7.1) and integrating it with respect to x over $(0,\infty)$, we have

$$\begin{aligned} &\Big(\frac{1}{2}\|v_x\|_{L^2}^2\Big)_t + \|v_{xx}\|_{L^2}^2 \\ &= \int_0^\infty \big(f'(\Phi+v)(\Phi_x+v_x) - f'(\Phi)\Phi_x\big)v_{xx}\,dx - \int_0^\infty F v_{xx}\,dx. \end{aligned} \tag{1.7.12}$$

We estimate the right hand side of (1.7.12) as

$$\begin{aligned} &\Big|\int_0^\infty \big(f'(\Phi+v)(\Phi_x+v_x) - f'(\Phi)\Phi_x\big)v_{xx}\,dx\Big| \\ &\quad\le C\int_0^\infty (\Phi_x|v| + |v_x|)|v_{xx}|\,dx \\ &\quad\le \frac{1}{4}\|v_{xx}\|_{L^2}^2 + 4C\big(\|\sqrt{\Phi_x}v\|_{L^2}^2 + \|v_x\|_{L^2}^2\big), \\ &\Big|\int_0^\infty F v_{xx}\,dx\Big| \le \frac{1}{4}\|v_{xx}\|_{L^2}^2 + 4\|F\|_{L^2}^2. \end{aligned} \tag{1.7.13}$$

Substituting (1.7.13) into (1.7.12), we have

$$\frac{d}{dt}\|v_x\|_{L^2}^2 + \|v_{xx}\|_{L^2}^2 \le C\big(\|F\|_{L^2}^2 + \|\sqrt{\Phi_x}v\|_{L^2}^2 + \|v_x\|_{L^2}^2\big). \tag{1.7.14}$$

Multiply (1.7.14) by $(1+t)^\alpha$ and integrate the resultant inequality with respect to t over $(0,t)$, we have

$$\begin{aligned}
&(1+t)^\alpha \|v_x\|_{L^2}^2 + \int_0^t (1+\tau)^\alpha \|v_{xx}\|_{L^2}^2 \, d\tau \\
&\quad \le \|v_{0,x}\|_{L^2}^2 + \alpha \int_0^t (1+\tau)^{\alpha-1} \|v_x\|_{L^2}^2 \, d\tau \\
&\qquad + C \int_0^t (1+\tau)^\alpha \big(\|F\|_{L^2}^2 + \|\sqrt{\Phi_x} v\|_{L^2}^2 + \|v_x\|_{L^2}^2 \big) d\tau.
\end{aligned} \tag{1.7.15}$$

By using (1.7.2) and Lemma 1.7.2, the right hand side of (1.7.15) is estimated from above by

$$C\big(\|v_0\|_{L^2}^2 + \|v_{0,x}\|_{L^2}^2 \big) + C(1+t)^{\alpha-\frac{1}{2}} \log^4(2+t)$$

for $\alpha > 1/2$. Then we have the desired estimate (1.7.3). In particular, we have

$$\|v_x\|_{L^2} \le C(1+t)^{-\frac{1}{4}} \log^2(2+t). \tag{1.7.16}$$

Finally, we proof the second inequality of Theorem 1.7.3. From the Gagliardo-Nirenberg inequality, it follows that

$$\|v\|_{L^\infty} \le C \|v_x\|_{L^2}^\theta \|v\|_{L^q}^{1-\theta}, \qquad 1 \le q < \infty, \quad \theta = \frac{2}{q+2}. \tag{1.7.17}$$

Applying the inequality (1.7.16), (1.7.17) is rewritten as

$$\begin{aligned}
&\|v\|_{L^\infty} \\
&\le C(1+t)^{-\frac{\theta}{4}} \log^{2\theta}(2+t) \cdot (1+t)^{-\frac{1}{2}(1-\frac{1}{q})(1-\theta)} \log^{2(1-\theta)}(2+t) \\
&= C(1+t)^{-\frac{1}{2}(1-\theta)} \log^2(2+t).
\end{aligned} \tag{1.7.18}$$

Choose q as $\theta/2 = 1/(q+2) < \epsilon$ for any positive constant ϵ, then the right hand side of (1.7.18) is estimated as

$$\|v\|_{L^\infty} \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}.$$

This completes the proof. □

1.8 Decay rate estimate II; non-convex flux

In this section, we give the proof of the Theorem 1.1.3. Let us recall the Theorem 1.1.3 where the flux is not necessarily convex.

Theorem 1.8.1 (non-convex case)**.** *Assume* (1.1.7), (1.1.8) *and* (1.1.10), *and also* $u_0 - \phi - \psi^R(\cdot) \in L^1$. *Then, there exists a positive constant* ϵ *such that, if* $u_+ \le \epsilon$ *and* $\|u_0 - \phi - \psi^R(\cdot)\|_{H^1} \le \epsilon$, *then the unique global solution in time* u *obtained by the* Theorem 1.1.1 *satisfies the decay rate estimates*

$$\|(u - \phi - \psi^R)(t)\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \log^2(2+t), \quad (1 \le p < \infty),$$
$$\|(u - \phi - \psi^R)(t)\|_{L^\infty} \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}, \quad ({}^\forall \epsilon > 0).$$

The smooth approximation of the rarefaction wave and reformulation of the problem are same as in Section 1.7. The theorem for the reformulated problem (1.7.1) we shall prove is following.

Theorem 1.8.2. *Assume* (1.1.7), (1.1.8) *and* (1.1.10), *and also* $v_0 \in L^1$. *Then, there exists a positive constant* ϵ *such that, if* $u_+ \le \epsilon$ *and* $\|v_0\|_{H^1} \le \epsilon$, *then the unique global solution in time* v *obtained by the* Theorem 1.4.1 *satisfies the decay rate estimates*

$$\|v(t)\|_{L^p} \le C_p(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \log^2(2+t) \quad (1 \le p < \infty),$$
$$\|v(t)\|_{L^\infty} \le C_\epsilon (1+t)^{-\frac{1}{2}+\epsilon}, \quad ({}^\forall \epsilon > 0).$$

As in the last section, the Theorem 1.8.2 is given by the following *a priori* decay estimate.

Proposition 1.8.3 (decay rate estimate II)**.** *Assume* (1.1.7), (1.1.8) *and* (1.1.10), *and also* $v_0 \in L^1$. *Then, the solution of* (1.7.1) *satisfies*

$$\begin{aligned}(1+t)^\alpha \|v\|_{L^p}^p + \int_0^t (1+\tau)^\alpha \|\Phi_x^{1/p} v\|_{L^p}^p d\tau \\ + \int_0^t (1+\tau)^\alpha \|(|v|^{\frac{p}{2}-1}v)_x\|_{L^2}^2 d\tau \\ \le C\|v_0\|_{L^p}^p + CM^p (1+t)^{\alpha - \frac{p-1}{2}} \log^{2p}(2+t),\end{aligned} \tag{1.8.1}$$

and for $2 \le p < \infty$, $\alpha > (p-1)/2$

$$\begin{aligned}(1+t)^{\alpha}\|v_x\|_{L^2}^2 + \int_0^t (1+\tau)^{\alpha}\|v_{xx}\|_{L^2}^2\,d\tau \\ \le C\|v_0\|_{L^2}^2 + C(1+t)^{\alpha-\frac{1}{2}}\log^4(2+t).\end{aligned} \tag{1.8.2}$$

To treat the non-convex condition (1.1.7), we apply the weight function $w(\Phi)$ which is defined in Lemma 1.5.1. In order to apply the weight function w, we introduce a new unknown function $\tilde{v}$ by

$$v(x,t) = w(\Phi(t,x))\tilde{v}(x,t), \tag{1.8.3}$$

where $\Phi(t,x) = \phi(x) + \psi(t,x)$. Substituting (1.8.3) into the equation of (1.7.1), we get

$$\big(w(\Phi)\tilde{v}\big)_t + \big\{f(\Phi + w(\Phi)\tilde{v}) - f(\Phi)\big\}_x - \big(w(\Phi)\tilde{v}\big)_{xx} = -F(\phi,\psi). \tag{1.8.4}$$

We introduce the general energy inequality for (1.8.4) in the following Proposition.

Proposition 1.8.4 (general energy inequality)**.** *Assume that* $s(\eta)$ *is a smooth function of* η *satisfying* $s(0) = 0$ *and monotone increasing, and define*

$$S(\tilde{v}) := \int_0^{\tilde{v}} s(\eta)\ d\eta, \qquad \mathcal{S}(\tilde{v}) := \int_0^{\tilde{v}} s'(\eta)\eta\ d\eta.$$

Then there exists positive constants ε *and* c *such that if* $N(T) + |u_+| < \varepsilon$, *then the solution of* (1.8.4) *satisfies the following energy inequality.*

$$\begin{aligned}&\frac{d}{dt}\int_0^{\infty} w(\Phi)S(\tilde{v})dx + c\int_0^{\infty}\Phi_x\mathcal{S}(\tilde{v})dx + \int_0^{\infty} w(\Phi)s'(\tilde{v})\tilde{v}_x^2 dx \\ &\quad\le \int_0^{\infty} s(\tilde{v})F(\phi,\psi)dx + \int_0^{\infty}\big|w'(\Phi)F(\phi,\psi)\mathcal{S}(\tilde{v})\big|dx.\end{aligned} \tag{1.8.5}$$

Here we note $S(\tilde{v}) \ge 0$ and $\mathcal{S}(\tilde{v}) \ge 0$ by the definition.

Proof. Multiplying $s(\tilde{v})$ by (1.8.4), we have

$$s(\tilde{v})\big(w(\Phi)\tilde{v}\big)_t + \mathcal{G}_x + \Phi_x\mathcal{H} + s'(\tilde{v})(w(\Phi)\tilde{v})_x\tilde{v}_x = s(\tilde{v})F(\phi,\psi). \tag{1.8.6}$$

where

$$\begin{aligned}\mathcal{G} &:= s(\tilde{v})\big(f(\Phi + w(\Phi)\tilde{v}) - f(\Phi)\big) \\ &\qquad - \int_0^{\tilde{v}} s'(\eta)\big(f(\Phi + w(\Phi)\eta) - f(\Phi)\big)d\eta - s(\tilde{v})(w(\Phi)\tilde{v})_x, \\ \mathcal{H} &:= \int_0^{\tilde{v}} s'(\eta)\Big(f'(\Phi + w(\Phi)\eta) - f'(\Phi) + f'(\Phi + w(\Phi)\eta)w'(\Phi)\eta\Big)d\eta.\end{aligned}$$

For the first term on the left hand side, we rewrite as

$$\begin{aligned}s(\tilde{v})\big(w(\Phi)\tilde{v}\big)_t &= w(\Phi)_t \int_0^{\tilde{v}} (s(\eta)\eta)_\eta d\eta + s(\tilde{v})w(\Phi)\tilde{v}_t \\ &= w(\Phi)_t \int_0^{\tilde{v}} \big(s'(\eta)\eta + s(\tilde{\eta})\big)d\eta + w(\Phi)s(\tilde{v})\tilde{v}_t \\ &= w(\Phi)_t \mathcal{S}(\tilde{v}) + \big\{w(\Phi)S(\tilde{v})\big\}_t.\end{aligned} \tag{1.8.7}$$

We also rewrite the forth term on the left hand side of (1.8.6) as

$$\begin{aligned}s'(\tilde{v})\big(w(\Phi)\tilde{v}\big)_x\tilde{v}_x &= w(\Phi)_x s'(\tilde{v})\tilde{v}\tilde{v}_x + w(\Phi)s'(\tilde{v})\tilde{v}_x^2 \\ &= w(\Phi)_x \mathcal{S}_\delta(\tilde{v})_x + w(\Phi)s'(\tilde{v})\tilde{v}_x^2 \\ &= \big\{w(\Phi)_x \mathcal{S}_\delta(\tilde{v})\big\}_x - w(\Phi)_{xx}\mathcal{S}_\delta(\tilde{v}) \\ &\quad + w(\Phi)s'(\tilde{v})\tilde{v}_x^2.\end{aligned} \tag{1.8.8}$$

By noting that

$$\begin{aligned}&f'(\Phi + w(\Phi)\eta) - f'(\Phi) + f'(\Phi + w(\Phi)\eta)w'(\Phi)\eta \\ &\quad = f''(\Phi)w(\Phi)\eta + f'(\Phi)w'(\Phi)\eta + O(\eta^2),\end{aligned} \tag{1.8.9}$$

we rewrite $\mathcal{H}$ as follows

$$\mathcal{H} = \big(f''(\Phi)w(\Phi) + f'(\Phi)w'(\Phi)\big)\mathcal{S}_\delta(\tilde{v}) + \int_0^{\tilde{v}} s'(\eta)O(\eta^2)d\eta. \tag{1.8.10}$$

Substituting (1.8.7), (1.8.8) and (1.8.10) into (1.8.6), we have

$$\begin{aligned}&\big\{w(\Phi)S(\tilde{v})\big\}_t \\ &+ \big\{w(\Phi)_t - w(\Phi)_{xx} + \big(f''(\Phi)w(\Phi) + f'(\Phi)w'(\Phi)\big)\Phi_x\big\}\mathcal{S}_\delta(\tilde{v}) \\ &+ w(\Phi)s'(\tilde{v})\tilde{v}_x^2 + \big\{\mathcal{G} + w(\Phi)_x\mathcal{S}_\delta(\tilde{v})\big\}_x \\ &+ \Phi_x \int_0^{\tilde{v}} s'(\eta)O(\eta^2)d\eta = s(\tilde{v})F(\phi,\psi).\end{aligned} \tag{1.8.11}$$

Recalling the relation $\Phi_{xx} = \Phi_t + f(\Phi)_x - F$ and $w(\Phi)_{xx} = w''(\Phi)\Phi_x^2 + w'(\Phi)\Phi_{xx}$, we further rewrite the second term on the left hand side of (1.8.11) as

$$\begin{aligned} &w(\Phi)_t - w(\Phi)_{xx} + \big(f''(\Phi)w(\Phi) + f'(\Phi)w'(\Phi)\big)\Phi_x \\ &= \big(f''(\Phi)w(\Phi) - w''(\Phi)\Phi_x\big)\Phi_x + w'(\Phi)F(\phi,\psi) \\ &= \big(f''(\Phi)w(\Phi) - w''(\Phi)f(\Phi)\big)\Phi_x + \big(f(\Phi) - \Phi_x\big)w''(\Phi)\Phi_x \\ &\qquad\qquad + w'(\Phi)F(\phi,\psi). \end{aligned} \tag{1.8.12}$$

Therefore, substituting (1.8.12) into (1.8.11) and integrating it over $(0,\infty)$, we have

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty w(\Phi)S(\tilde{v})\,dx \\ &\qquad + \int_0^\infty (f''w - fw'')\Phi_x \mathcal{S}(\tilde{v})dx + \int_0^\infty w(\Phi)s'(\tilde{v})\tilde{v}_x^2 dx \\ &\qquad \le \int_0^\infty s(\tilde{v})F(\phi,\psi)\,dx + \int_0^\infty \big|w'(\Phi)F(\phi,\psi)\mathcal{S}(\tilde{v})\big|dx \\ &\qquad + \int_0^\infty O(|u_+| + |\tilde{v}|)\Phi_x \mathcal{S}(\tilde{v})dx. \end{aligned} \tag{1.8.13}$$

Make $N(T) + |u_+|$ sufficiently small and note Lemma 1.5.1, then we obtain the desired inequality (1.8.5). □

Next, we show the following L^1-estimate of v.

Proposition 1.8.5 (L^1-estimate)**.** *Assume* (1.1.7), (1.1.8) *and* (1.1.10), *and also* $v_0 \in L^1$. *Then the solution of* (1.7.1) *satisfies*

$$\|v(t)\|_{L^1} \le C(\|v_0\|_{L^1} + 1)\log^2(2+t). \tag{1.8.14}$$

Proof. Substituting $s(\tilde{v}) = a_\delta(\tilde{v})$ into (1.8.5), we have the following estimate.

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty w(\Phi)A_\delta(\tilde{v})dx + c\int_0^\infty \Phi_x \mathcal{A}_\delta(\tilde{v})dx + \int_0^\infty w(\Phi)a_\delta'(\tilde{v})\tilde{v}_x^2 dx \\ &\qquad \le \int_0^\infty a_\delta(\tilde{v})F(\phi,\psi)dx + \int_0^\infty \big|w'(\Phi)F(\phi,\psi)\mathcal{A}_\delta(\tilde{v})\big|dx, \end{aligned} \tag{1.8.15}$$

where

$$A_\delta(\tilde{v}) := \int_0^{\tilde{v}} a_\delta(\eta)\ d\eta, \qquad \mathcal{A}_\delta(\tilde{v}) := \int_0^{\tilde{v}} a_\delta'(\eta)\eta\ d\eta,$$

and a_δ is defined in the Section 1.7. For the remainder term, we can estimate as

$$\Big| \int_0^\infty a_\delta(\tilde{v}) F(\phi, \psi) dx \Big| \le \|F\|_{L^1},$$

$$\int_0^\infty \big| w'(\Phi) F(\phi, \psi) \mathcal{A}_\delta(\tilde{v}) \big| dx \le C \int_0^\infty |\tilde{v}| |F(\phi, \psi)| dx \le C \|F\|_{L^1}.$$

Integrate (1.8.15) over $(0, t)$ by the above estimates and make $\delta \to 0$ afterward, then we have

$$\|\tilde{v}\|_{L^1} \le \|\tilde{v}_0\|_{L^1} + C \int_0^t \|F\|_{L^1} d\tau \le C(\|\tilde{v}_0\|_{L^1} + 1) \log^2(2 + t).$$

By using the property $\nu \le w(\Phi) \le C$ for some positive constants ν and C again, we have the desired estimate (1.8.14). $\square$

Finally, we give the proof of Proposition 1.8.3.

Proof of Proposition 1.8.3. By substituting $s(\tilde{v}) = |\tilde{v}|^{p-2}\tilde{v}$ into (1.8.5), we have the following estimate.

$$\begin{aligned} &\frac{1}{p} \frac{d}{dt} \int_0^\infty w(\Phi) |\tilde{v}|^p dx + \frac{c(p-1)}{p} \int_0^\infty \Phi_x |\tilde{v}|^p dx \\ &\qquad + \frac{4(p-1)}{p^2} \int_0^\infty w(\Phi) |\tilde{V}_x|^2 dx \le C \|\tilde{v}\|_{L^\infty}^{p-1} \|F\|_{L^1}, \end{aligned} \tag{1.8.16}$$

where $\tilde{V} := |\tilde{v}|^{\frac{p}{2}-1}\tilde{v}$. Multiplying $(1+t)^\alpha$ by (1.8.16) and integrating it over $(0, t)$, we have

$$\begin{aligned} &\frac{1}{p}(1+t)^\alpha \|\tilde{v}\|_{L^p}^p + \frac{4(p-1)}{p^2} \int_0^t (1+\tau)^\alpha \|\Phi_x^{1/p} \tilde{v}\|_{L^p}^p d\tau \\ &\qquad + \frac{C(p-1)}{p} \int_0^t (1+\tau)^\alpha \|\tilde{V}_x\|_{L^2}^2 d\tau \\ &\le \frac{1}{p} \|\tilde{v}_0\|_{L^p}^p + \frac{\alpha}{p} \int_0^t (1+\tau)^{\alpha-1} \|\tilde{v}\|_{L^p}^p d\tau \\ &\qquad + C \int_0^t (1+\tau)^\alpha \|\tilde{v}\|_{L^\infty}^{p-1} \|F\|_{L^1} d\tau. \end{aligned} \tag{1.8.17}$$

By applying the same method as (1.7.9)-(1.7.10), we obtain the estimate

$$\begin{aligned} &(1+t)^\alpha \|\tilde{v}\|_{L^p}^p + \int_0^t (1+\tau)^\alpha \|\Phi_x^{1/p} \tilde{v}\|_{L^p}^p d\tau + \int_0^t (1+\tau)^\alpha \|(|\tilde{v}|^{\frac{p}{2}-1}\tilde{v})_x\|_{L^2}^2 d\tau \\ &\qquad \le C \|\tilde{v}_0\|_{L^p}^p + C M^p (1+t)^{\alpha - \frac{p-1}{2}} \log^{2p}(2+t). \end{aligned}$$

Using the positivity of w and the fact

$$\begin{aligned}\|(|v|^{\frac{p}{2}-1}v)_x\|_{L^2}^2 &= \|\frac{p}{2}|w\tilde{v}|^{\frac{p}{2}-1}(w\tilde{v})_x\|_{L^2}^2 \\ &\le \frac{p}{2}\||\tilde{v}|^{\frac{p}{2}-1}(w_x\tilde{v}+w\tilde{v}_x)\|_{L^2}^2 \\ &\le C_p\big(\|\Phi_x^{1/p}\tilde{v}\|_{L^p}^p + \|\tilde{V}_x\|_{L^2}^2\big),\end{aligned}$$

we have (1.8.3). In particular, we have

$$\|v\|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\log^2(2+t). \tag{1.8.18}$$

Finally, the estimate (1.8.2) can be obtained by the same way as (1.7.3). Thus the proof is completed. □

Chapter 2

Damped wave equation

2.1 Introduction and main theorems

In this Chapter 2, we show that the arguments on the weighted energy method developed in the Chapter 1 can be applied to the following initial boundary value problem to a damped wave equation with convection term on the half line:

$$\begin{cases} u_{tt} - u_{xx} + u_t + f(u)_x = 0, & x > 0,\ t > 0, \\ u(0,t) = u_-, & t > 0, \\ \lim\limits_{x\to\infty} u(x,t) = u_+, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x > 0, \end{cases} \tag{2.1.1}$$

where the function f which describes the convection (we also call " flux" as in the Chapter 1) is assumed to be C^2-function of u satisfying $f(0) = 0$, $u_\pm$ are given constants and the initial data u_0 is assumed to satisfy $u_0(0) = u_-$ and $\lim\limits_{x\to\infty} u_0(x) = u_+$ as the compatibility conditions. As for the initial condition, we assume that

$$u_0 - u_+ \in H^1, \quad u_1 \in L^2. \tag{2.1.2}$$

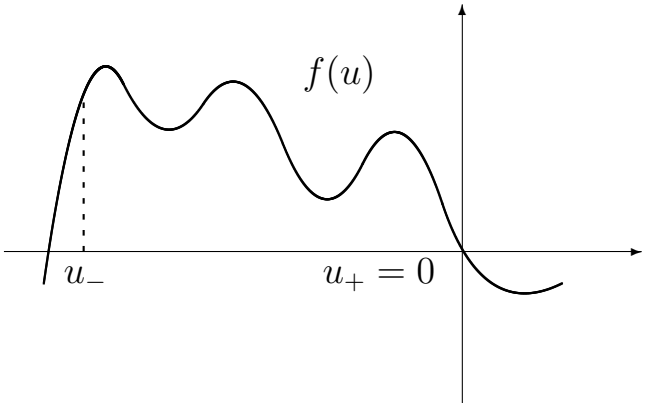
As in the Chapter 1, we are interested in the large-time behavior of the solution which is determined by the shape of the flux $f(u)$ and the given constants $u_\pm$. We first consider the case $u_- < u_+ = 0$. This problem has been intensively investigated by Kawashima-Nakamura-Ueda [13]('08) and Ueda [35]('08). In particular, Ueda [35] showed that if the flux $f(u)$ of (2.1.1) and $u_\pm$ satisfies

$$\begin{cases} u_- < u_+ = 0, & f(0) = 0, \\ f''(u) > 0, & u \in [u_-, 0], \\ |f'(u)| < 1, & u \in [u_-, 0], \end{cases} \tag{2.1.3}$$

then the solution of (2.1.1) tends toward the corresponding stationary solution $\phi(x)$ which is studied in the Section 1.2, provided the initial perturbation is suitably small. Here we note that the last condition of (2.1.3) is well-known as "sub-characteristic condition" in the theory of relaxation models of conservation law. In this chapter, we show that we can relax the condition (2.1.3) much as

$$\begin{cases} u_- < u_+ = 0, & f(0) = 0, \\ f''(0) > 0, & |f'(0)| < 1, \\ f(u) > 0 & (u \in [u_-, 0)), \end{cases} \tag{2.1.4}$$

where we emphasize that in the condition (2.1.4) we assume the convexity and the sub-characteristic condition for $f(u)$ only at the far field $u = u_+ = 0$ as long as $f(u)$ is positive for $u \in [u_-, 0)$.



Now we are ready to state our first main theorem.

Theorem 2.1.1. *Assume* $u_- < u_+ = 0$, (2.1.2) *and* (2.1.4). *Then, there exists a positive constant* ϵ *such that, if* $\|u_0 - \phi\|_{H^1} + \|u_1\|_{L^2} \le \epsilon$, *then the initial-boundary value problem* (2.1.1) *has a unique global solution in time* u *satisfying*

$$\begin{cases} u - \phi \in C([0, \infty); H^1_0), \\ (u - \phi)_x, \ u_t \in L^2(0, \infty; L^2), \end{cases}$$

and the asymptotic behavior

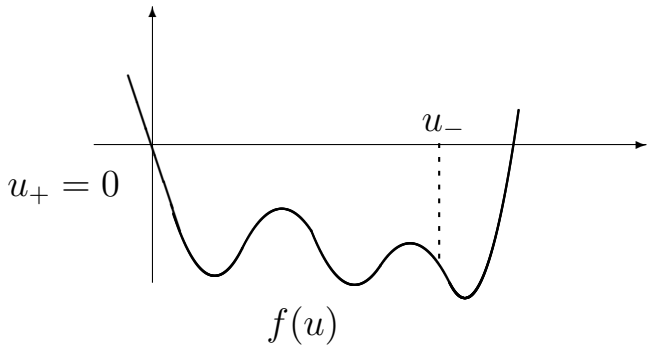
$$\lim_{t \to \infty} \sup_{x > 0} |u(x, t) - \phi(x)| = 0. \tag{2.1.5}$$

The proof is given in the Section 2.3 by a similar weighted energy method as in the Chapter 1. It is noted that the case $u_- < 0 < u_+$ is also treated in the same way as in the Chapter 1, that is, we can

prove that the superposition of stationary solution and rarefaction wave is asymptotically stable for suitably small u_+. This result is stated in the Section 2.4.

We next consider the case $u_+ = 0 < u_-$, which has been an open problem. By making use of anti-derivative method (cf. [18]), we show the asymptotic stability of the corresponding stationary solution under the assumptions on the flux as

$$\begin{cases} u_+ = 0 < u_-, & f(0) = 0, \\ 0 < |f'(0)| < 1, & \\ f(u) < 0 & (u \in (0, u_-]), \end{cases} \tag{2.1.6}$$



and on the initial data which is more restrictive than in the Theorem 2.1.1, that is,

$$\begin{aligned} &u_0 - \phi \in H^1 \cap L^1, \qquad u_1 \in L^2 \cap L^1, \\ &z_0 := -\int_x^\infty (u_0(y) - \phi(y))\, dy \in L^2, \quad z_1 := \int_x^\infty u_1(y)\, dy \in L^2. \end{aligned} \tag{2.1.7}$$

Our second theorem is

Theorem 2.1.2. *Assume $u_+ = 0 < u_-$, (2.1.6) and (2.1.7). Then, there exists a positive constant ϵ such that, if $\|z_0\|_{H^2} + \|z_1\|_{H^1} \le \epsilon$, then the initial-boundary value problem (2.1.1) has a unique global solution in time u satisfying*

$$u - \phi \in C^0([0,\infty); H_0^1) \cap C^1([0,\infty); L^2) \cap L^2(0,\infty; H^1) \tag{2.1.8}$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0. \tag{2.1.9}$$

The proof is given in the Section 2.6. It should be noted that although the initial data is more restrictive, the condition $f''(0) > 0$ is not needed. This suggests us that again for the case $u_- < u_+ = 0$, if we assume $f'(0) < 0$, we can show the asymptotic stability of the stationary solution ϕ for a restrictive class of initial data as (2.1.7) under the flux condition (2.1.2) without $f''(0) > 0$. The result is stated in the Section 2.7.

2.2 Reformulation of the problem; case $u_- < u_+ = 0$

The aim of this section is to reformulate the problem (2.1.1) for the case $u_- < u_+ = 0$. As in the arguments in the Chapter 1, we define the deviation v of u from ϕ by

$$v(x,t) = u(x,t) - \phi(x), \tag{2.2.1}$$

where ϕ is stationary solution connecting u_- and $u_+ = 0$ defined in the Section 1.2. Then the problem (2.1.1) is reformulated in terms of v in the form

$$\begin{cases} v_{tt} - v_{xx} + \{f(\Phi + v) - f(\Phi)\}_x + v_t = 0, \quad x > 0,\ t > 0, \\ v(0,t) = 0, \quad t > 0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x), \quad x > 0, \\ v_t(x,0) = v_1(x) := u_1(x), \quad x > 0. \end{cases} \tag{2.2.2}$$

where we can see $v_0 \in H_0^1$ and $v_1 \in L^2$ by the assumptions in the Theorem 2.1.1. In the case that $f(u)$ satisfies (2.1.4), the theorem for the reformulated problem (2.2.2) we shall prove is

Theorem 2.2.1. *Assume $u_- < u_+ = 0$, (2.1.2) and (2.1.4). Then there exists a positive constant ϵ such that, if $\|u_0 - \phi\|_{H^1} + \|u_1\|_{L^2} \le \epsilon$, then the problem (2.2.2) has a unique global solution in time v satisfying*

$$\begin{cases} v \in C^0([0,\infty); H_0^1) \cap C^1([0,\infty); L^2), \\ v_x, v_t \in L^2(0,\infty; L^2), \\ \lim\limits_{t\to\infty} \sup\limits_{x>0} |v(x,t)| = 0. \end{cases} \tag{2.2.3}$$

The proof of Theorems 2.2.1 is given in the Section 2.3. We note that main Theorem 2.1.1 is a direct consequence of Theorems 2.2.1. To state

the existence result of the solution v for (2.2.2), we define the solution space for any interval $I \subseteq R$ and $M > 0$ by

$$X_M(I) = \{v \in C^0(I; H_0^1(R_+));\ v_t \in C^0(I; L^2(R_+)),\\ \sup_{t\in I}(\|v(t)\|_{H^1} + \|v_t(t)\|_{L^2}) \le M\},$$

and also generalize the initial-boundary value problem for any constant $\tau \ge 0$ as

$$\begin{cases} v_{tt} - v_{xx} + v_t + \{f(\phi + v) - f(\phi)\}_x = 0, \quad x > 0,\ t > \tau, \\ v(0,t) = 0, \quad t > \tau, \\ v(x,\tau) = v_\tau(x), \quad x > 0, \quad (v_\tau \in H_0^1), \\ v_t(x,\tau) = v_{\tau,1}(x), \quad x > 0, \quad (v_{\tau,1} \in L^2). \end{cases} \tag{2.2.4}$$

Then we state the local existence theorem.

Proposition 2.2.2 (local existence)**.** *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ which is independent of τ such that if $\|v_\tau\|_{H^1} + \|v_{\tau,1}\|_{L^2} \le M$, the initial boundary value problem* (2.2.4) *has a unique solution $v \in X_{2M}([\tau, \tau + t_0])$.*

For the proof of the Proposition 2.2.2, it is noted that the case $\tau = 0$ is enough to prove, and then the problem (2.2.4) is easily reduced to the integral equation

$$\tilde{v}(t) = (\cos\wedge\, t)\tilde{v}_0 + \frac{\sin\wedge\, t}{\wedge}\ \tilde{v}_1 + \int_0^t \frac{\sin\wedge(t-s)}{\wedge}\ \tilde{h}(v(s))\ ds, \tag{2.2.5}$$

where $\tilde{v}$, $\tilde{v}_0$, $\tilde{v}_1$ and $\tilde{h}(v)$ are the odd extensions of v, v_0, v_1 and $h = -v_t - (f(\phi + v) - f(\phi))_x$ to the whole space $x \in R$, respectively, and

$$(\cos\wedge\, t)g := \mathcal{F}^{-1}\Big[\cos(|\xi|t)\hat{g}\Big], \quad (\frac{\sin\wedge\, t}{\wedge})g := \mathcal{F}^{-1}\Big[\frac{\sin(|\xi|t)}{|\xi|})\hat{g}\Big]. \tag{2.2.6}$$

Here $\hat{g}(\xi) := \mathcal{F}[g(x)](\xi)$ is the Fourier transformation with respect to x. Since we can prove the Proposition 2.2.2 by a standard iterative method, we omit the proof.

Next, let us state the *a priori* estimate which implies the Theorem 2.2.1 by combining the Proposition 2.2.2 (local existence).

Proposition 2.2.3 (*a priori* estimate I)**.** *Under the assumptions* (2.1.2) *and* (2.1.4)*, there exist positive constants ε and C such that if $v \in$*

$X_\varepsilon([0,T])$ is the solution of the problem (2.2.4) *for some $T>0$, then it holds*

$$\begin{aligned}\|v(t)\|_{H^1}^2+\|v_t(t)\|_{L^2}^2+\int_0^t(\|v_t\|_{L^2}^2+\|v_x(s)\|_{L^2}^2\\ +\|\sqrt{\phi_x}v(s)\|_{L^2}^2)\,ds\le C(\|v_0\|_{H^1}^2+\|v_1\|_{L^2}^2),\ t\in[0,T].\end{aligned}\tag{2.2.7}$$

2.3 *A priori* estimate I

In this section, we give the proof of the Proposition 2.2.3. First, put

$$N(T)=\sup_{0\le t\le T}(\|v(t)\|_{H^1}+\|v_t(t)\|_{L^2}),$$

and then we suppose $N(T)\le 1$, and also suppose $u_-<u_+=0$ and (2.1.4) throughout this section. Now, motivated by the argument in the Section 1.5, we introduce a new unknown function $\tilde v$ by

$$v(x,t)=w(\phi(x))\tilde v(x,t),\tag{2.3.1}$$

where $w=f+\delta g$ is the weight function in the Lemma 1.5.1. Substituting (2.3.1) into the equation of (2.2.2), we get

$$(w(\phi)\tilde v)_{tt}-(w(\phi)\tilde v)_{xx}+(f(\phi+w(\phi)\tilde v)-f(\phi))_x+(w(\phi)\tilde v)_t=0.\tag{2.3.2}$$

Lemma 2.3.1. *For sufficiently small $N(T)$ and positive constant c, it holds*

$$\begin{aligned}&\int_0^\infty(w(\phi)\tilde v_t\tilde v+\frac{1}{2}w(\phi)\tilde v^2+w(\phi)\tilde v_t^2)+w(\phi)\tilde v_x^2+\ c\phi_x\tilde v^2dx\\ &+\int_0^t\int_0^\infty(w(\phi)\tilde v_x^2+2(f'w-fw')\tilde v_t\tilde v_x+w(\phi)\tilde v_t^2)\\ &\qquad+c\phi_x\tilde v^2+O(|\tilde v|)\tilde v_t\tilde v_x\ dxd\tau\le C(\|\tilde v_0\|_{H^1}^1+\|\tilde v_1\|_{L^2}^2).\end{aligned}\tag{2.3.3}$$

Proof. Multiplying $\tilde v$ by (2.3.2). Then making use of the equality

$$(w(\phi)\tilde v)_{tt}=(w(\phi)\tilde v_t\tilde v)_t-w(\phi)\tilde v_t^2,\tag{2.3.4}$$

we obtain

$$\begin{aligned}&\frac{d}{dt}(w(\phi)\tilde v_t\tilde v+\frac{1}{2}w(\phi)\tilde v^2)+\mathcal{G}_x+\frac{1}{2}(f''w-fw'')(\phi)\phi_x\tilde v^2\\ &\qquad+O(\tilde v)\phi_x\tilde v^2+w(\phi)\tilde v_x^2-w(\phi)\tilde v_t^2=0,\end{aligned}\tag{2.3.5}$$

where

$$\begin{aligned}\mathcal{G} = -(w(\phi)\tilde{v}_x)_x\tilde{v} + \frac{1}{2}w'(\phi)\phi_x\tilde{v}^2 + \big(f(\phi + w(\phi)\tilde{v}) - f(\phi)\big)\tilde{v} \\ - \int_0^{\tilde{v}} f(\phi + w(\phi)\eta) - f(\phi)d\eta.\end{aligned} \tag{2.3.6}$$

Next, multiplying $\tilde{v}_t$ by (2.3.2), we have

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{2}w(\phi)\tilde{v}_t^2) + (-(w(\phi)\tilde{v})_x\tilde{v}_t)_x + w(\phi)_x\tilde{v}\tilde{v}_{tx} \\ &+ (f'(\phi + w(\phi)\tilde{v})(w(\phi)\tilde{v})_x \\ &+ \big(f'(\phi + w(\phi)\tilde{v}) - f'(\phi)\big)\phi_x)\tilde{v}_t + w(\phi)\tilde{v}_t^2 = 0\end{aligned} \tag{2.3.7}$$

For the third term on the left hand side, we rewrite as

$$\begin{aligned}w(\phi)_x\tilde{v}\tilde{v}_{tx} &= w(\phi)_x\tilde{v}\tilde{v}_{tx} + \frac{1}{2}(w(\phi)\tilde{v}_x^2)_t \\ &= (w_x\tilde{v}\tilde{v}_x + \frac{1}{2}w(\phi)\tilde{v}_x^2)_t - w_x\tilde{v}_t\tilde{v}_x \\ &= (w_x\tilde{v}\tilde{v}_x + \frac{1}{2}w(\phi)\tilde{v}_x^2)_t - w'f\tilde{v}_t\tilde{v}_x.\end{aligned} \tag{2.3.8}$$

We also rewrite the forth term on the left hand side as

$$\begin{aligned}&(f'(\phi + w(\phi)\tilde{v})(w(\phi)\tilde{v})_x + \big(f'(\phi + w(\phi)\tilde{v}) - f'(\phi)\big)\phi_x)\tilde{v}_t \\ &= \phi_x\big(\int_0^{\tilde{v}} f'(\phi + w\eta)w'\eta d\eta + \int_0^{\tilde{v}} f'(\phi + w\eta) - f'(\phi)d\eta\big)_t \\ &\qquad + f'(\phi + w\tilde{v})\tilde{v}_t w\tilde{v}_x \\ &= \phi_x\big(\int_0^{\tilde{v}} f'(\phi + w\eta)w'\eta d\eta + \int_0^{\tilde{v}} f'(\phi + w\eta) - f'(\phi)d\eta\big)_t \\ &\qquad f'(\phi)w\tilde{v}_t\tilde{v}_x + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x\end{aligned} \tag{2.3.9}$$

Substituting (2.3.8) and (2.3.9) into (2.3.7), we have

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}w(\phi)\tilde{v}_t^2 + \frac{1}{2}w(\phi)\tilde{v}_x^2 + w_x\tilde{v}\tilde{v}_x \\ &+ \phi_x\big(\int_0^{\tilde{v}} f'(\phi + w\eta)w'\eta d\eta + \int_0^{\tilde{v}} f'(\phi + w\eta) - f'(\phi)d\eta\big)\Big) \\ &\qquad + (-(w(\phi)\tilde{v})_x\tilde{v}_t)_x + (f'w - fw')\tilde{v}_t\tilde{v}_x \\ &\qquad\qquad + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x + w(\phi)\tilde{v}_t^2 = 0.\end{aligned} \tag{2.3.10}$$

Noting that $w_{xx} = w''f\phi_x + w'f'\phi_x$, we rewrite the third, fourth and fifth terms of the (2.3.10) as

$$\begin{aligned}
&w_x\tilde{v}\tilde{v}_x + \phi_x\Big(\int_0^{\tilde{v}} f'(\phi+w\eta)w'\eta d\eta \\
&\qquad\qquad + \int_0^{\tilde{v}} f'(\phi+w\eta) - f'(\phi)d\eta\Big)\Big) \\
=&w_x\tilde{v}\tilde{v}_x + \phi_x\int_0^{\tilde{v}} f'(\phi)w'\eta + f''(\phi+\theta w\eta)ww'\eta^2 d\eta \\
&+ \phi_x\int_0^{\tilde{v}} f''(\phi)w\eta + f'''(\phi+\theta w\eta)(w\eta)^2 d\eta \\
=&(\frac{1}{2}w_x\tilde{v}^2)_x + \frac{1}{2}(-w_{xx}\tilde{v}^2 + f'w'\phi_x\tilde{v}^2 + f''w\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2) \\
=&(\frac{1}{2}w_x\tilde{v}^2)_x + \frac{1}{2}(f''w - fw'')\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2.
\end{aligned} \tag{2.3.11}$$

Therefore, substituting (2.3.11) into (2.3.10), we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}w(\phi)\tilde{v}_t^2 + \frac{1}{2}w(\phi)\tilde{v}_x^2 \\
&+ (\frac{1}{2}w_x\tilde{v}^2)_x + \frac{1}{2}(f''w - fw'')\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2\Big) \\
&+ (-(w(\phi)\tilde{v})_x\tilde{v}_t)_x + (f'w - fw')\tilde{v}_t\tilde{v}_x \\
&\quad + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x + w(\phi)\tilde{v}_t^2 = 0.
\end{aligned} \tag{2.3.12}$$

We make a combination (2.3.12) $\times 2$ + (2.3.5), which yields the following equality

$$\begin{aligned}
&\frac{d}{dt}\Big(w(\phi)\tilde{v}_t\tilde{v} + \frac{1}{2}w(\phi)\tilde{v}^2 + w(\phi)\tilde{v}_t^2 + w(\phi)\tilde{v}_x^2 + (w_x\tilde{v}^2)_x \\
&\qquad\qquad + (f''w - fw'')\phi_x\tilde{v}^2 + O(|\tilde{v}|)\phi_x\tilde{v}^2\Big) \\
&+ (\mathcal{G} - 2(w(\phi)\tilde{v})_x\tilde{v}_t)_x \\
&+ \frac{1}{2}(f''w - fw'')(\phi)\phi_x\tilde{v}^2 + O(\tilde{v})\phi_x\tilde{v}^2 + w(\phi)\tilde{v}_x^2 \\
&+ 2(f'w - fw')\tilde{v}_t\tilde{v}_x + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x + w(\phi)\tilde{v}_t^2 = 0.
\end{aligned} \tag{2.3.13}$$

We integrate (2.3.13) over (0,t) $\times R$ and take suitably small $N(T)$ to get the desired inequality (2.3.3). $\square$

To show the positivity of the quadratic form $w(\phi)\tilde{v}_x^2 + 2(f'w - fw')(\phi)\tilde{v}_t\tilde{v}_x + w(\phi)\tilde{v}_t^2$ in (2.3.3), we introduce the following lemma.

Lemma 2.3.2. *Assume $f(u)$ satisfies a condition* (2.1.4)*, and $g(u)$ is a function which is defined in the* Lemma 1.5.1*, i.e.*

$$g(u) := -u^{2m} + r^{2m}. \tag{2.3.14}$$

Then, if $|f'(0)| < 1$, there exists a positive constant δ such that

$$\delta^2(f'(u)g(u) - f(u)g'(u))^2 < (f(u) + \delta g(u))^2, \quad u \in [u, 0]. \tag{2.3.15}$$

Proof. First, by the Lemma 1.5.1, we can choose and fix $\delta_0 > 0$ which implies $f(u) + \delta_0 g > 0$ for $u \in [u_-, 0]$. Then we assume $\delta < \delta_0$. We divide the interval $[u_-, 0]$ into $[-r', 0]$ and $[u_-, -r']$ for a positive constant $r'(\leq r)$. For $[-r', 0]$, it holds

$$\begin{aligned} &|\delta(f'(u)g(u) - f(u)g'(u))| \\ &\quad \leq \delta|f'(u)g(u)| + \delta|f(u)g'(u)| \\ &\quad \leq \delta(|f'(0)| + C|u|)g(u) + C\delta_0|f'(0)u||2mu^{2m-1}| \\ &\quad \leq \delta(|f'(0)| + Cr')g(u) + C\delta_0 m|r'|^{2m-1}|f'(0)u|. \end{aligned} \tag{2.3.16}$$

Because δ_0 and m are positive fixed constant, so we can choose r' sufficiently so small that $C\delta_0 m|r'|^{2m-1} < 1$. Then the last term of (2.3.16) is estimated as

$$\leq \delta(|f'(0)| + C|r'|)g(u) + |f'(0)u|. \tag{2.3.17}$$

Here, we note that by the assumption of $|f'(0)| < 1$, we further can choose r' sufficiently small such that $\|f'(0)| + C|r'| < 1$. On the other hand, because of the convexity of $f(u)$ in $u \in [-r', 0]$, we can see that

$$|f(u) + \delta g(u)| \geq |f'(0)u| + \delta g(u). \tag{2.3.18}$$

Therefore, (2.3.16)-(2.3.18) imply the desired inequality (2.3.15). For $u \in [u_-, r']$, it holds

$$\begin{aligned} &\delta^2(f'(u)g(u) - f(u)g'(u))^2 \\ &\qquad \leq \delta^2 \max_{u\in[u_-,0]}(|f'(u)g(u) - f(u)g'(u)|^2). \end{aligned} \tag{2.3.19}$$

On the other hand, we choose δ so small that

$$\begin{aligned} f + \delta g &\geq \min_{u_-\leq u\leq -r'} f(u) - \delta \max_{u_-\leq u\leq -r'} |g(u)| \\ &\geq \frac{1}{2} \min_{u_-\leq u\leq -r'} f(u), \end{aligned} \tag{2.3.20}$$

which implies

$$(f+\delta g)^2 \geq (\frac{1}{2}\min_{u_-\leq u\leq -r'} f(u))^2. \tag{2.3.21}$$

We further choose δ sufficiently small in (2.3.19) and (2.3.21), then we get the desired inequality (2.3.15). □

Finally, we prove the Proposition 2.2.3.

Proof. We define the quadratic forms A_1 and A_2 by the terms in (2.3.3) as

$$\begin{aligned} A_1 &:= w(\phi)\tilde{v}_t\tilde{v} + \frac{1}{2}w(\phi)\tilde{v}^2 + w(\phi)\tilde{v}_t^2, \\ A_2 &:= w(\phi)\tilde{v}_x^2 + 2(f'w - fw')\tilde{v}_t\tilde{v}_x + w(\phi)\tilde{v}_t^2. \end{aligned} \tag{2.3.22}$$

Calculate the discriminants of A_1, then we have

$$\mathcal{D}_{A_1} = w^2 - 4(\frac{1}{2}w)w = -w^2 < 0. \tag{2.3.23}$$

By this inequality, we can see that A_1 is a positive quadratic form with respect to $\tilde{v}$ and $\tilde{v}_t$. On the other hand, substituting the definition of w and using the Lemma 2.3.2, the discriminant of A_2 satisfies

$$\begin{aligned} \mathcal{D}_{A_2} &= (f'w - fw')^2 - w^2 \\ &= \big(f'(f+\delta g) - f(f'+\delta g')\big)^2 - (f+\delta g)^2 \\ &= \delta^2\big(f'g - fg'\big)^2 - (f+\delta g)^2 < 0. \end{aligned} \tag{2.3.24}$$

Then, A_2 is also a positive quadratic form with respect to $\tilde{v}_x$ and $\tilde{v}_t$. Therefore, there is a positive constant c such that

$$A_1 \geq c(\tilde{v}^2 + \tilde{v}_t^2), \qquad A_2 \geq c(\tilde{v}_t^2 + \tilde{v}_x^2). \tag{2.3.25}$$

Using (2.3.25) and the fact that

$$O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x \leq O(|\tilde{v}|)(\tilde{v}_t^2 + \tilde{v}_x^2), \tag{2.3.26}$$

and choosing $N(T)$ sufficiently small in (2.3.3), we finally obtain

$$\begin{aligned} &\int_0^\infty (\tilde{v}^2 + \tilde{v}_t^2 + \tilde{v}_x^2 + \phi_x\tilde{v}^2)dx + \int_0^t\int_0^\infty (\phi_x\tilde{v}^2 + \tilde{v}_x^2 + \tilde{v}_t^2)\, dxd\tau \\ &\qquad \leq C(\|\tilde{v}_0\|_{H^1}^2 + \|\tilde{v}_1\|_{L^2}^2). \end{aligned} \tag{2.3.27}$$

Thus we have the desired estimate (2.2.7). □

2.4 A case including rarefaction wave

In this section, we consider the case that $u_- < 0 < u_+$. In this case, we can expect that the solution tends to the superposition of stationary solution and rarefaction wave as in the Chapter 1. In fact, we obtain the similar result as Theorem 2.1.1.

Theorem 2.4.1. *Assume $u_- < 0 < u_+$, (2.1.2) and (2.1.4). Then, there exists a positive constant ϵ such that, if $u_+ \leq \epsilon$ and $\|u_0 - \phi - \psi^R(\cdot)\|_{H^1} + \|u_1\|_{L^2} \leq \epsilon$, then the initial-boundary value problem (2.1.1) has a unique global solution in time u satisfying*

$$\begin{cases} u - u_+ \in C([0,\infty); H^1), \\ u_x,\ u_t \in L^2(0,T;L^2) \quad ({}^\forall T > 0), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x) - \psi^R(\frac{x}{t})| = 0. \tag{2.4.1}$$

The proof is almost same as former section, so we state only the essential points. Put

$$\Phi(x,t) = \phi(x) + \psi(x,t),$$

and

$$v(x,t) = u(x,t) - \Phi(x,t).$$

Then the problem (2.1.1) is reformulated in terms of v in the form

$$\begin{cases} v_{tt} - v_{xx} + \{f(\Phi + v) - f(\Phi)\}_x + v_t = F, \quad x > 0,\ t > 0, \\ v(0,t) = 0, \quad t > 0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x) - \psi^R(x,0), \quad x > 0, \\ v_t(x,0) = v_1(x) := u_1(x). \quad x > 0. \end{cases} \tag{2.4.2}$$

where, F is defined by

$$F = -(f'(\phi+\psi) - f'(\psi))\psi_x - (f'(\psi+\phi) - f'(\phi))\phi_x + \psi_{xx} - \psi_{tt}.$$

Then, the theorem for the reformulated problem (2.4.2) we shall prove is

Theorem 2.4.2. *Assume $u_- < 0 < u_+$, (2.1.2) and (2.1.4). Then, there exists a positive constant ε such that, if $\|v_0\|_{H^1} + \|v_1\|_{L^2} \leq \varepsilon$ and*

$0 < u_+ \le \varepsilon$, *then the initial boundary value problem* (2.2.2) *has a unique global solution in time* v *satisfying*

$$\begin{cases} v \in C([0,\infty); H_0^1), \\ v_x \in L^2(0,\infty; L^2), \\ \lim\limits_{t\to\infty} \sup\limits_{x>0} |v(x,t)| = 0. \end{cases} \tag{2.4.3}$$

The *a priori* estimate for the Theorem 2.4.2 is as follows.

Proposition 2.4.3 (*a priori* estimate 2)**.** *Assume* $u_- < 0 < u_+$, (2.1.2) *and* (2.1.4)*. Then, there exist positive constants* ε *and* C *such that if* $0 < u_+ < \varepsilon$ *and* $v \in X_\varepsilon([0,T])$ *is the solution of the problem* (2.2.4) *for some* $T > 0$*, then it holds*

$$\begin{aligned} &\|v(t)\|_{H^1}^2 + \|v_t(t)\|_{L^2} + \|\sqrt{\Phi_x} v(s)\|_{L^2}^2 \\ &+ \int_0^t (\|v_t(s)\|_{L^2}^2 + \|v_x(s)\|_{L^2}^2 + \|\sqrt{\Phi_x} v(s)\|_{L^2}^2)\, ds \\ &\qquad \le C(\|v_0\|_{H^1}^2 + \|v_1\|_{L^2}^2 + |u_+|^{\frac{1}{6}}), \quad t \in [0,T]. \end{aligned} \tag{2.4.4}$$

Outline of Proof. As in the last section, substituting

$$v(x,t) = w(\Phi(t,x))\tilde{v}(x,t), \tag{2.4.5}$$

into the equation of (2.4.2), we get

$$(w(\Phi)\tilde{v})_{tt} - (w(\Phi)\tilde{v})_{xx} + (f(\Phi + w(\Phi)\tilde{v}) - f(\Phi))_x + (w(\Phi)\tilde{v})_t = F, \tag{2.4.6}$$

where $w = f + \delta g$ is the weight function defined in the Lemma 1.5.1. Multiply $\tilde{v} + 2\tilde{v}_t$ by (2.4.6) and integrate it over $(0,\infty) \times (0,t)$ with respect to x and t, we have

$$\int_0^\infty A_1\, dx\Big|_0^t + \int_0^t \int_0^\infty B_1\, dxds = \int_0^t \int_0^\infty -F(\tilde{v} + 2\tilde{v}_t)\, dxds, \tag{2.4.7}$$

where

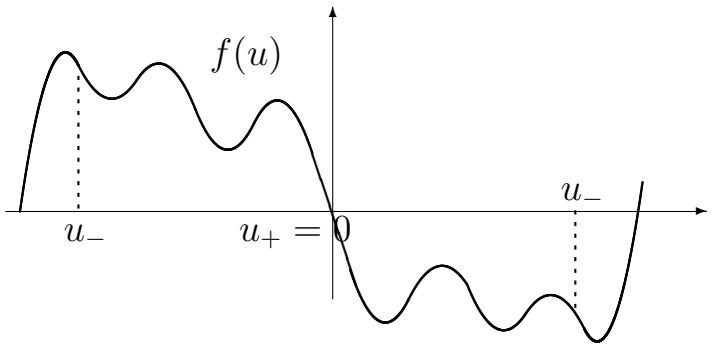
$$\begin{aligned} A_1 &= \big(\frac{1}{2} w\tilde{v}^2 + w\tilde{v}_t\tilde{v} + w\tilde{v}_t^2\big) + (f''w - fw'')\Phi_x\tilde{v}^2 + w\tilde{v}_x^2 \\ &\qquad + O(|u_+| + |\tilde{v}|)\Phi_x\tilde{v}^2 + O(|\bar{F}| + |u_+|)\tilde{v}^2, \\ B_1 &= w\tilde{v}_x^2 + 2(f'w - fw')\tilde{v}_t\tilde{v}_x + w\tilde{v}_t^2 + \frac{1}{2}(f''w - fw'')\Phi_x\tilde{v}^2 \\ &\quad + O(|\tilde{v}| + |u_+|)\tilde{v}_t\tilde{v}_x + O(|\psi_x|)\tilde{v}\tilde{v}_t + O(|u_+|)\Phi_x\tilde{v}^2 \\ &\quad + O(|\psi_{xx}| + |\psi_{xxx}| + |\bar{F}|)\tilde{v}^2 + O(|u_+|)\tilde{v}_t^2 + O(|u_+|)\tilde{v}_x^2. \end{aligned}$$

Starting with the energy equality (2.4.7), and making use of the argument on the positivity of A_1 and B_1 as in the last section and also on how to handle the right hand side of (2.4.7) which includes the rarefaction wave by the Lemma 1.2.1 and Lemma 1.3.1 as in the Chapter 1, we can prove the Proposition 2.4.3 for suitably small $N(T)+|u_+|$. We omit the details.

2.5 Anti-derivative method; case $u_+ = 0 < u_-$

In the former section, we assumed $u_- < u_+ = 0$ and the convexity for the flux $f(u)$ at the origin. In this section, we show that if we additionally assume $f'(0) < 0$, we can treat the case $u_+ = 0 < u_-$ and remove the convexity condition $f''(0) > 0$. To do that, we employ another approach, so called "anti-derivative method" (cf. [18]). For the flux function $f(u)$, we assume (2.1.6), that is,

$$f(0) = 0, \quad 0 < |f'(0)| < 1 \quad \textit{and} \quad f(u) < 0,\ u \in (0, u_-]. \tag{2.1.6}$$



Motivated by the argument in Liu-Nishihara ('97 [18]), put

$$z(x,t) = -\int_x^\infty v(y,t)dy, \tag{2.5.1}$$

where $v(x,t)$ is defined by (2.2.1). By integrating the initial boundary value problem (2.2.2) over $[x,\infty)$, we have the following problem in terms of z:

$$\begin{cases} z_{tt} - z_{xx} + \big(f(\phi + z_x) - f(\phi)\big) + z_t = 0, \quad x > 0,\ t > 0, \\ z_x(0,t) = 0, \quad t > 0, \\ z(x,0) = z_0(x) := -\int_x^\infty (u_0(y) - \phi(y))dy, \quad x > 0, \\ z_t(x,t) = z_1(x) := -\int_x^\infty u_1(y)dy, \quad x > 0, \end{cases} \tag{2.5.2}$$

where we assume (2.1.7) for the initial data, which implies $z_0 \in H^2$, $z_{0,x} \in H_0^1$ and $z_1 \in L^2$. The theorem for the reformulated problem (2.5.2) we shall prove is

Theorem 2.5.1. *Assume $u_+ = 0 < u_-$, (2.1.6) and (2.1.7). Then, there exists a positive constant ϵ such that, if $||z_0||_{H^2} + ||z_1||_{H^1} \leq \epsilon$, then the initial boundary value problem (2.5.2) has a unique global solution in time z satisfying*

$$\begin{cases} z \in C^0([0,\infty); H^2) \cap C^1([0,\infty); H^1), \\ z_x \in C^0([0,\infty); H_0^1), \\ z_x, z_t \in L^2(0,\infty; H^1), \\ \lim\limits_{t\to\infty} \sup\limits_{x>0} |z_x(x,t)| = 0. \end{cases} \tag{2.5.3}$$

To state the existence result of the solution precisely, we define the solution space for any interval $I \subseteq R$ and $M > 0$ by

$$X_M(I) = \{z \in C(I; H^2);\ z_t \in C(I; H^1),$$
$$z_x \in C(I; H_0^1),\ \sup_{t\in I}(||v(t)||_{H^2} + ||v_t||_{H^1}) \leq M\},$$

and also generalize the initial boundary value problem for any constant $\tau \geq 0$ as

$$\begin{cases} z_{tt} - z_{xx} + \big(f(\phi + z_x) - f(\phi)\big) + z_t = 0, \quad x > 0,\ t > \tau, \\ z_x(0,t) = 0, \quad t > \tau, \\ z(x,\tau) = z_\tau(x), \quad x > 0, \quad (z_\tau \in H^2,\ z_{\tau,x} \in H_0^1), \\ z_t(x,\tau) = z_{\tau,1}(x), \quad x > 0, \quad (z_{\tau,1} \in H^1). \end{cases} \tag{2.5.4}$$

Then we state the local existence theorem.

Proposition 2.5.2 (local existence)**.** *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ which is independent of τ such that if $||z_\tau||_{H^2} + ||z_{\tau,1}||_{H^1} \leq M$, the initial boundary value problem (2.5.4) has a unique solution $v \in X_{2M}([\tau, \tau + t_0])$.*

For the proof of the Proposition 2.5.2, as in the proof of the Proposition 2.2.2, it is noted that the case $\tau = 0$ is enough to prove, and the problem (2.5.4) with $\tau = 0$ can be reduced to the integral equation

$$\tilde{z}(t) = (\cos\wedge\, t)\tilde{z}_0 + \frac{\sin\wedge\, t}{\wedge}\ \tilde{z}_1 + \int_0^t \frac{\sin\wedge(t-s)}{\wedge}\ \tilde{h}(z(s))\ ds, \tag{2.5.5}$$

where $\tilde{z}$, $\tilde{z}_0$, $\tilde{z}_1$ and $\tilde{h}(z)$ are the even extensions of z, z_0, z_1 and $h = -(f(\phi + z_x) - f(\phi)) - z_t$ to the whole space $x \in R$. Since we can prove the Proposition 2.5.2 by a standard iterative method, we omit the proof.

Next, let us state the *a priori* estimate which is essential in this chapter.

Proposition 2.5.3 (*a priori* estimate)**.** *Assume* $u_+ = 0 < u_-$, (2.1.6) *and* (2.1.7). *Then, there exist positive constants* ε *and* C *such that if* $v \in X_\varepsilon([0,T])$ *is the solution of the problem* (2.5.2) *for some* $T > 0$, *then it holds*

$$\begin{aligned} &||z(t)||^2_{H^2} + ||z_t(t)||^2_{H^1} \\ &\quad + \int_0^t (||z_x(\tau)||^2_{H^1} + ||z_t(\tau)||^2_{H^1} + ||\sqrt{\phi_x} z(\tau)||^2_{L^2})\, d\tau \\ &\qquad \le C(||z_0||^2_{H^2} + ||z_1||^2_{H^1}), \quad t \in [0,T]. \end{aligned} \tag{2.5.6}$$

The first equation of (2.5.2) is rewritten as

$$z_{tt} - z_{xx} + f'(\phi) z_x + z_t = F, \tag{2.5.7}$$

where

$$F = -(f(\phi + z_x) - f(\phi) - f'(\phi) z_x) = O(|z_x|^2). \tag{2.5.8}$$

We introduce the weight function $w(u)$ for the estimate of z as

$$w(u) = \begin{cases} (e^{-Au} - 1)/f(u), & u \in (0, u_-], \\ -A/f'(0), & u = 0, \end{cases} \tag{2.5.9}$$

where A is a positive constant which is properly chosen later. Then we state a lemma which is used in the *a priori* estimate.

Lemma 2.5.4. *Assume* (2.1.6) *and* (2.5.9). *Then if we take* A *sufficiently large, it holds for* $u \in [0, u_-]$

$$\begin{aligned} &\text{(i) } w > 0, && \text{(ii) } \Big((fw)'\Big)^2 < w^2, \\ &\text{(iii) } (fw)''(\phi) > 0, && \text{(iv) } (fw)'(u_-) < 0. \end{aligned} \tag{2.5.10}$$

Since (iii) and (iv) are clear, we only prove (i) and (ii).

Proof of (i). We divide the interval $[0, u_-]$ into $[0, r]$ and $[r, u_-]$. For

$[0, r]$, it follows from the definition that

$$w(u) = \frac{1 - e^{-Au}}{|f(u)|} = \frac{|1 - (1 - Ae^{-A\theta u}u)|}{|f'(0)u + \frac{1}{2}f''(\theta u)u^2|} = \frac{|Ae^{-A\theta u}|}{|f'(0) + \frac{1}{2}f''(\theta u)u|}. \tag{2.5.11}$$

We can take $r > 0$ sufficiently small such that

$$\frac{1}{2}|f'(0)| \le |f'(0) + \frac{1}{2}f''(\theta\phi)\phi| \le \frac{3}{2}|f'(0)| \quad for \quad u \in [-r, 0]. \tag{2.5.12}$$

Then, for this r, by using (2.5.11), we derive

$$\frac{2Ae^{-Ar}}{3|f'(0)|} \le w(u) \le \frac{2A}{|f'(0)|}. \tag{2.5.13}$$

For $u \in [r, u_-]$, by using the fact that $w = (1 - e^{-Au})/|f(u)|$, we have

$$\frac{1 - e^{-Au_-}}{\max_{r\le u\le u_-}|f(u)|} \le w(u) \le \frac{1 - e^{-Ar}}{\min_{r\le u\le u_-}|f(u)|}. \tag{2.5.14}$$

Proof of (ii). We also divide the interval $[0, u_-]$ into $[0, r]$ and $[r, u_-]$. We prove $(|f'(w)|^2/w^2) < 1$ which is equivalent to (ii). On the interval $[0, r]$, it follows from the definition that

$$\begin{aligned} |\frac{(fw)'}{w}| &= |\frac{-f(u)Ae^{-Au}}{e^{-Au} - 1}| \\ &= \frac{|(f'(0)u + \frac{1}{2}f''(\theta u)u^2)Ae^{-Au}|}{|Ae^{-A\theta u}u|} \\ &= |f'(0) + \frac{1}{2}f''(\theta u)u|e^{-Au(1-\theta)} \\ &= |f'(0)| + \frac{1}{2}|f''(\theta u)|r. \end{aligned} \tag{2.5.15}$$

Because $|f'(0)| < 1$, we can choose r sufficiently small such that

$$|f'(0)| + \frac{1}{2}|f''(\theta u)|r < 1. \tag{2.5.16}$$

For $u \in [r, u_-]$, we have

$$|\frac{(fw)'}{w}| = |\frac{-Ae^{-Au}}{\frac{e^{-Au}-1}{f}}| \le \frac{(\max_{r\le u\le u_-}|f(u)|)Ae^{-Au}}{1 - e^{-Au}} = \frac{MA}{e^{Au} - 1}, \tag{2.5.17}$$

where $M := \max_{r \le u \le u_-} |f(u)|$. Hence, taking A sufficiently small, we can make

$$\frac{MA}{e^{Au} - 1} < 1. \tag{2.5.18}$$

Thus the proof of the Lemma 2.5.4 is completed. □

To estimate the derivatives of z, we also introduce another weight function. We define a weight function $w(u)$ by

$$w_2(u) := -f(u) + \delta. \tag{2.5.19}$$

Then, we have the following lemma for (2.5.19).

Lemma 2.5.5. *Assume* (2.1.6) *and* (2.5.19). *Then, if we take* $\delta > 0$ *sufficiently small, it follows* $(f'w_2 - fw_2')^2(u) \le w_2^2(u)$ *and there is a positive constant* ν *such that* $\nu \le w_2(u)$ *for* $[0, u_-]$.

Proof. We first see by (2.5.19) that the inequality $(f'w_2 - fw_2')^2(u) \le w_2^2(u)$ is equivalent to the inequality

$$|\delta f'(u)| < | - f(u) + \delta|. \tag{2.5.20}$$

To prove (2.5.20). we divide the interval $[0, u_-]$ into $[0, r]$ and $[r, u_-]$. For $[0, r]$, from the definition,

$$\begin{aligned} |\delta f'(u)| &= \delta |f'(0) + f''(\theta u)u| \\ &\le \delta |f'(0)| + \delta M |r|, \end{aligned} \tag{2.5.21}$$

where $M = \max_{u \in [0, u_-]} |f''(\theta u)|$. On the other hand, it follows

$$| - f(u) + \delta| \ge \delta. \tag{2.5.22}$$

Therefore, by taking r sufficiently small, the inequality (2.5.20) follows from (2.5.21) and (2.5.22). For $u \in [r, u_-]$, by taking sufficiently small δ, we have

$$\begin{aligned} |\delta f'(u)| &\le \delta \max_{u \in [0, u_-]} |f'(u)| \\ &\le \min_{r \le u \le u_-} |f(u)| + \delta \le | - f + \delta|. \end{aligned} \tag{2.5.23}$$

Thus the proof of the Lemma 2.5.5 is completed. □

2.6 *A priori* estimate II.

In this section, we give a rough sketch of the proof of the Proposition 2.5.3. Throughout of this section, we assume (2.1.6), (2.1.7) and

$$N(T) = \sup_{0\le t\le T} (\|z(t)\|_{H^2} + \|z_t(t)\|_{H^1})$$

is suitably small. We multiply (2.5.7) by $w(\phi)z$. Then making use of the equalities

$$f'(\phi)z_x wz = (\frac{1}{2}f'wz^2)_x - \frac{1}{2}(f''w + f'w')\phi_x z^2 \tag{2.6.1}$$

and

$$\begin{aligned} -z_{xx}wz &= -(z_x wz)_x + w'\phi_x(\frac{1}{2}z^2)_x + wz_x^2 \\ &= -(z_x wz - \frac{1}{2}w'\phi_x z^2)_x - \frac{1}{2}(w''f + w'f')\phi_x z^2 + wz_x^2, \end{aligned} \tag{2.6.2}$$

we have

$$\begin{aligned} &(\frac{1}{2}wz^2 + wz_t z)_t - wz_t^2 + (\frac{1}{2}f'wz^2 - wzz_x + \frac{1}{2}w'\phi_x z^2)_x \\ &\quad - \frac{1}{2}(f''w + 2f'w' + fw'')\phi_x z^2 + wz_x^2 = O(|z_x^2|)wz. \end{aligned} \tag{2.6.3}$$

Next, multiplying (2.5.7) by wz_t, we have

$$\begin{aligned} &\frac{1}{2}(wz_t^2 + wz_x^2)_t - (z_x wz_t)_x + wz_t^2 + f'(\phi)wz_x z_t \\ &\qquad + w'\phi_x z_x z_t = O(|z_x^2|)wz_t. \end{aligned} \tag{2.6.4}$$

We make a combination (2.6.3) $+2\times$(2.6.4) and integrate it over $(0,\infty)$ with respect to x, which yields the differential equality:

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty A_1\, dx + \int_0^\infty (A_2 + \bar{A}_2)\, dx \\ &\qquad + A_3 = \int_0^\infty O(|z_x^2|)(wz + 2wz_t)\, dx, \end{aligned} \tag{2.6.5}$$

where

$$
\begin{aligned}
A_1 &= wz_t^2 + wz_x^2 + wz_t z + \frac{1}{2}wz^2, \\
A_2 &= wz_t^2 + 2(fw)'z_t z_x + wz_x^2, \\
\bar{A_2} &= -\frac{1}{2}(fw)''\phi_x z^2, \\
A_3 &= -\frac{1}{2}\big(f(u_-)w(u_-)\big)'z(t,0)^2.
\end{aligned}
\tag{2.6.6}
$$

We can easily see that A_1 is a positive quadratic form. The condition which makes A_2, $\bar{A_2}$ and A_3 positive definite is

$$
\text{(i) } ((fw)'^2 < w^2, \quad \text{(ii) } (fw)''(\phi) > 0, \quad \text{(iii) } (fw)'(u_-) < 0, \tag{2.6.7}
$$

where we note that (i) is the condition for the discriminant of the quadratic form A_1, and $\phi_x < 0$ in (ii). By the definition of w and the Lemma 2.5.4, the above inequalities (i), (ii) and (iii) follow. Then, integrating (2.6.5) with respect to t over $(0,t)$, we have

$$
\begin{aligned}
\int_0^\infty (z^2 + z_t^2 + z_x^2)(t)\,dx + \int_0^t \int_0^\infty (z_x^2 + z_t^2 + \phi_x z^2)(s)\,dxds \\
+ \int_0^t z(0,s)^2 ds \le C(\|z_0\|_{H^1}^2 + \|z_1\|_{L^2}^2).
\end{aligned}
\tag{2.6.8}
$$

Next, we proceed to the estimates for the higher derivatives of z. Setting $z_x = v$ and $v = w_2(\phi)\tilde{v}$, we follow the same calculation (2.3.5)-(2.3.13) in the Section 2.3 for the estimates of z_x. Then we have the same inequality as (2.3.3) for some positive constant c :

$$
\begin{aligned}
&\int_0^\infty (A_1 + cw_2(\phi)\tilde{v}_x^2)\,dx + \int_0^t \int_0^\infty (A_2 + O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x)\,dxd\tau \\
&\le C\bigg(\int_0^\infty \phi_x \tilde{v}^2\,dx + \int_0^t \int_0^\infty \phi_x \tilde{v}^2\,dxd\tau + (\|\tilde{v}_0\|_{H^1}^2 + \|\tilde{v}_1\|_{L^2}^2)\bigg),
\end{aligned}
\tag{2.6.9}
$$

where

$$
\begin{aligned}
A_1 &= w_2(\phi)\tilde{v}_t\tilde{v} + \frac{1}{2}w_2(\phi)\tilde{v}^2 + w_2(\phi)\tilde{v}_t^2, \\
A_2 &= w_2(\phi)\tilde{v}_x^2 + 2(f'w_2 - fw_2')\tilde{v}_t\tilde{v}_x + w_2(\phi)\tilde{v}_t^2.
\end{aligned}
\tag{2.6.10}
$$

The first term and second term of the right hand side of (2.6.9) is estimated by (2.6.8). For the positivity of the left hand side of (2.6.9), we must make A_1 and A_2 positive quadratic form. If $w_2(u)$ is a positive function, we can see A_1 is a positive quadratic form. On the other hand, the discriminant of A_2 is

$$\mathcal{D}_{A_2} = (f'w_2 - fw_2')^2(u) - w_2^2(u), \tag{2.6.11}$$

and we can make A_2 a positive quadratic form by the Lemma 2.5.5. Then, there is a positive constant c such that

$$A_2 \geq c(\tilde{v}_x^2 + \tilde{v}_t^2). \tag{2.6.12}$$

Therefore, (2.6.9) implies

$$\begin{aligned} &\|\tilde{v}(t)\|_{L^2}^2 + \|\tilde{v}_t(t)\|_{L^2}^2 + \|\tilde{v}_x(t)\|_{L^2}^2 + \int_0^t (\|\tilde{v}_x\|_{L^2}^2 + \|\tilde{v}_t\|_{L^2}^2)(\tau)\ d\tau \\ &\leq C(\|\tilde{v}_0\|_{H^1}^2 + \|\tilde{v}_1\|_{L^2}^2 + \|z_0\|_{H^1}^2 + \|z_1\|_{L^2}^2), \end{aligned} \tag{2.6.13}$$

that is,

$$\begin{aligned} &\|z_x(t)\|_{L^2}^2 + \|z_{xt}(t)\|_{L^2}^2 + \|z_{xx}(t)\|_{L^2}^2 \\ &\quad + \int_0^t (\|z_{xx}\|_{L^2}^2 + \|z_{x,t}\|_{L^2}^2)(\tau)\, d\tau \leq C(\|z_0\|_{H^2}^2 + \|z_1\|_{H^1}^2). \end{aligned} \tag{2.6.14}$$

Combining (2.6.14) with (2.6.8), we finally have (2.5.6). Thus the proof of the Proposition 2.5.3 is completed. □

2.7 Reconsideration to the case of $u_- < u_+ = 0$

In this section, we reconsider the case $u_- < u_+ = 0$ in the Section 2.2. In this case, as we pointed out in the introduction, if we additionally assume $f'(0) < 0$, we can show the asymptotic stability of the stationary solution ϕ for a restrictive class of initial data as (2.1.7) under the flux condition (2.1.2) without $f''(0) > 0$. That is, we assume that $f(u)$ is C^2-function of u satisfying

$$f(0) = 0, \quad 0 < |f'(0)| < 1, \quad f(u) > 0 \quad (u \in [u_-, 0)). \tag{2.7.1}$$

Then we have the following

Theorem 2.7.1. *Assume $u_- < u_+ = 0$, (2.7.1) and (2.1.7). Then, there exists a positive constant ϵ such that, if $\|z_0\|_{H^2} + \|z_1\|_{H^1} \leq \epsilon$, then the initial boundary value problem (2.5.2) has a unique global solution in time z satisfying*

$$\begin{cases} z \in C^0([0,\infty); H^2) \cap C^1([0,\infty); H^1), \\ z_x \in C^0([0,\infty); H^1_0), \\ z_x, z_t \in L^2(0,\infty; H^1), \\ \lim\limits_{t\to\infty} \sup\limits_{x>0} |z_x(x,t)| = 0. \end{cases} \tag{2.7.2}$$

The proof is almost same as former section (2.5.4)-(2.6.14). Therefore, we state only the essential points. Multiply (2.5.7) by $w(z + 2z_t)$ where the weight function $w(\phi)$ is determined later, and integrate it over $(0,\infty)$, then we have the equality which corresponds to (2.6.5):

$$\frac{d}{dt}\int_0^\infty A_1\,dx + \int_0^\infty (A_2 + \bar{A}_2)\,dx + A_3 = \int_0^\infty O(|z_x^2|)(wz + 2wz_t)\,dx,$$

where

$$\begin{aligned} A_1 &= wz_t^2 + wz_x^2 + wz_t z + \frac{1}{2}wz^2, \\ A_2 &= wz_t^2 + 2(fw)' z_t z_x + wz_x^2, \\ \bar{A}_2 &= -\frac{1}{2}(fw)''\phi_x z^2, \\ A_3 &= -\frac{1}{2}\big(f(u_-)w(u_-)\big)' z(t,0)^2. \end{aligned}$$

It is noted that A_1 is same as (2.6.6), and the conditions which make A_2, $\bar{A}_2$ and A_3 positive definite are

$$\text{(i) } \big((fw)'\big)^2 < w^2, \quad \text{(ii) } (fw)''(\phi) < 0, \quad \text{(iii) } (fw)'(u_-) < 0, \tag{2.7.3}$$

where note $\phi_x > 0$ in $\bar{A}_2$ this time. In this case, instead of (2.5.9), we take the weight function w to be

$$w(u) = \begin{cases} (-e^{Au} + 1)/f(u), & u \in [u_-, 0), \\ -A/f'(0), & u = 0, \end{cases} \tag{2.7.4}$$

where A is a positive constant which is properly chosen later. As for this weight function, we have the following lemma whose proof is is given in the same way as the Lemma 2.5.4.

Lemma 2.7.2. *Assume* (2.7.1) *and* (2.7.4). *Then there exists a sufficiently large* A *such that* $w(u) > 0$ *and the conditions* (i), (ii) *and* (iii) *in* (2.7.3) *hold for* $u \in [u_-, 0]$.

Making use of this weight function $w(u)$, we can show the L^2-energy estimates for z and its first derivatives in the same way as for (2.6.8). Then, taking another weight function $w_2(u) = f(u) + \delta$ this time, we can also obtain the L^2-energy estimates for the higher derivatives as (2.6.14). Thus we can establish the desired *a priori* estimate to show the Theorem 2.7.1. We omit the details.

Bibliography

[1] R. Courant and K. Friedrichs, *Supersonic flow and shock waves*, Wiley-Interscience, 1948.

[2] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws*, Arch. Rational. Mech. Anal., **95** (1986), 325-344.

[3] E. Harabetian, *Rarafactions and large time behavior for parabolic equations and monotone schemes*, Comm. Math. Phys., **114** (1988), 527-536.

[4] I. Hashimoto, S. Kawashima and Y. Ueda, *Asymptotic decay toward the superposition of stationary solution and rarefaction waves for solutions of viscous conservation law in a half line*, preprint.

[5] I. Hashimoto and A. Matsumura, *Large time behavior of solutions to an initial boundary value problem on the half line for scalar viscous conservation law*, Methods Appl. Anal. **14** (2007), 45-60.

[6] F. Huang, A. Matsumura and X.Shi, *Viscous shock wave and boundary layer solution to an inflow problem for compressible viscous gas*, Comm. Math. Phys. **239** (2003), 261-285.

[7] A. M. Il'in and O. A. Oleinik, *Asymptotic behavior of the solutions of Cauchy problem for certain quasilinear equations for large time* (Russian), Mat. Sb. 51 (1960), 191-216.

[8] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Commun. Pure Appl. Math. **47**(1994), 1547-1569.

[9] S. Kawashima, A. Matsumura and K. Nishihara, *Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas*, Proc. Japan. Acad. **62**, Ser. A (1986), 249-252.

[10] S. Kawashima, S. Nishibata and P. Zhu, *Asymptiotic stbility of the stationary solution to the compressible Navier-Stokes equations in the half space*, Comm. Math. Phys. 240 (2003), 483-500.

[11] S. Kawashima, S. Nishibata and M. Nishikawa, *L^p energy method for multi-dimensional viscous conservation laws and application to the stability of planar waves*, J.Hyperbolic Differ. Equ. **1** (2004), 581-603.

[12] S. Kawashima, T. Nakamura, Y. Ueda, *Stability of degenerate stationary waves for viscous gases*, in preparation

[13] S. Kawashima, T. Nakamura and Y. Ueda, *Stability of planar stationary waves for damped wave equations with nonlinear convection in multi-dimensional half space*, Kinet. Relat. Models **1** (2008), 49–64.

[14] S. Kawashima and Y. Tanaka, *Stability of rarefaction waves for a model system of a radiating gas*, Kyushu J.Math. **58** (2004), 211-250.

[15] P. D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math. 10 (1957), 537-566.

[16] T.-P. Liu,*Nonlinear stability of shock waves for viscous conservation-laws*, Men. Amer. Math. Soc. **56** (1985).

[17] T.-P. Liu, A. Matsumura and K. Nishihara, *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal. **29** (1998), 293-308.

[18] T.-P. Liu and K. Nishihara, *Asymptotic behavior for scalar viscous conservation laws with boundary effect*, J. Differential Equations, **133** (1997), 296-320.

[19] T.-P. Liu and Z.-P.Xin, *Pointwize decay to contact discontinuities for systems of viscous conservation laws*, Asian J. Math. **1** (1997), 34-84.

[20] T.-P. Liu and S.-H. Yu, *Propagation of stationary viscous Burgers shock under the effect of boundary*, Arch. Rational Mech. Anal. **139** (1997), 57-82.

[21] A. Matsumura, *Asymptotic stability of traveling wave solutions for the one-dimensional viscous conservation laws*, Sugaku Expositions **11** (1998), 215-234.

[22] A. Matsumura and M. Mei, *Convergence to traveling fronts of solutions of the p-system with viscousity in the presence of a boundary*, Arch. Rational Mech. Anal. **146** (1999), 1-22.

[23] ————, *Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity*, Osaka J. Math. **34** (1997), 589-603.

[24] A. Matsumura and K. Nishihara, *On the stability of traveling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J.Appl.Math. **3**(1985), 17-25.

[25] ————, *Asymptotic toward the rarefaction wave of solutions of a one-dimensional model system for compressible viscous gas*, Japan J.Appl.Math. **3**(1986), 1-13.

[26] ————, *Global stability of the rarefaction wave of a one-dimentional model system for compressible viscous gas*, Comm. Math. Phys. **144** (1992), 325-335.

[27] ————, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Commun. Math. Phys. **165**(1994), 83-96.

[28] ————, *Global asymptotics toward the rarefaction wave for solutions of viscous p-system with boundary effect*, Quart. Appl. Math. **58** (2000), 69-83.

[29] J. Nagase, *Large time behavior of solutions to an initial-boundary value problem on the half space for the generalized Burgers equation*, Master thesis of Department of Mathematics, Guraduate School of Science, Osaka University, 2000.

[30] K. Nishihara, *Boundary effect on stationary viscous shock wave for scalar viscous conservation laws*, J. Math. Anal. Appl. **255** (2001), 535-550.

[31] K. Nishihara, T. Yang and H. Zhao, *Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations*, SIAM J. Math. Anal. **35** (2004), 1561-1597.

[32] T. Nakamura, *Asymptotic decay toward the rarefaction waves of solutions for viscous conservation laws in one-dimensional half space*, SIAM J. Math. Anal. **34** (2003), 1308-1317.

[33] J. Smoller, *Shock waves and reaction-diffusion Equations*, Springer-Verlag, New York-Berlin, 1983.

[34] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves*, Arch. Rational Math. Anal. **122** (1993), 53-103.

[35] Y. Ueda, *Asymptotic stability of stationary waves for damped wave equations with a nonlinear convection term*, to appear in Advances in Mathematical Sciences and Applications, 2008.

[36] K. Zumbrun, *Multidimentional stability of planar viscous shock waves*, Advances in the theory of shock waves, 307-516, Progr. Nonlinear Differential Equations Appl. **47**, Birkhauser Boston, MA, 2001.