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NON-LOCAL ELLIPTIC PROBLEM IN HIGHER DIMENSION

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Abstract

Non-local elliptic problem, $-\Delta v = \lambda(e^v / (\int_{\Omega} e^v dx)^p)$ with Dirichlet boundary condition is considered on n -dimensional bounded domain Ω with $n \geq 3$ for $p > 0$. If Ω is the unit ball, $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, we have infinitely many bendings in λ of the solution set in $\lambda - v$ plane. Finally if Ω is an annulus domain and $p \geq 1$, we show that a solution exists for all $\lambda > 0$.

1. Introduction

In this paper we consider the following elliptic equation with non-local term:

$$(1) \quad \begin{cases} -\Delta v = \lambda \frac{e^v}{(\int_{\Omega} e^v dx)^p} & x \in \Omega, \\ v = 0 & x \in \partial\Omega, \end{cases}$$

where λ, p are positive constants and Ω is a bounded domain in \mathbf{R}^n with smooth boundary $\partial\Omega$. Actually, usual Gel'fand problem, in the theories of thermonic emission ([5]), isothermal gas sphere ([4]), and gas combustion ([1]), is formulated as the nonlinear eigenvalue problem

$$(2) \quad \begin{cases} -\Delta v = \sigma e^v & x \in \Omega, \\ v = 0 & x \in \partial\Omega, \end{cases}$$

with a constant $\sigma > 0$. Problems (1) and (2) are equivalent through the relation

$$\sigma = \frac{\lambda}{(\int_{\Omega} e^v dx)^p},$$

and hence some features of the solution set

$$\mathcal{C} = \{(\lambda, v) \mid v = v(x) \text{ is a classical solution of (1) for } \lambda > 0\}$$

resemble those of the solution set for (2), denoted by \mathcal{S} . (1) is the non-local stationary problem of

$$\begin{cases} v_t = \Delta v + \lambda \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p} & x \in \Omega, t > 0, \\ v = 0 & x \in \partial\Omega, t > 0, \\ v|_{t=0} = v_0(x) & x \in \Omega. \end{cases}$$

Such problems are studied in ([2]). They arise in the study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates ([3]) and Ohmic heating ([10], [11]). We note that if $p = 1$, the motivation to study (1) is the Keller-Segel system ([9]) which describes the chemotactic aggregation of cellular slime molds given by

$$(3) \quad \begin{cases} \varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v) & x \in \Omega, t \in (0, T), \\ \tau v_t = \Delta v + u & x \in \Omega, t \in (0, T), \\ \frac{\partial u}{\partial v} - u \frac{\partial v}{\partial v} = v = 0 & x \in \partial\Omega, t \in (0, T), \\ u|_{t=0} = u_0(x) \geq 0 & x \in \Omega, \\ v|_{t=0} = v_0(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, τ, ε are positive constants, and v is the outer unit normal vector, respectively. In the stationary state of (3), it is reduced to (1) ([20]). In fact, since $u = u(x, t) \geq 0$ and

$$\frac{d}{dt} \int_{\Omega} u dx = \frac{1}{\varepsilon} \int_{\Omega} \nabla \cdot (\nabla u - u \nabla v) dx = 0,$$

the total mass is conserved, that is, $\|u(\cdot, t)\|_1 = \|u_0\|_1$. Here and henceforth, $\|\cdot\|_p$ denotes the standard L^p norm. The system of equations (3) has a Lyapunov function

$$J(u, v) = \int_{\Omega} \left(u(\log u - 1) - uv + \frac{1}{2} |\nabla v|^2 \right) dx$$

and it holds that

$$\varepsilon \frac{d}{dt} J(u, v) + \tau \varepsilon \|v_t\|_2^2 + \int_{\Omega} u |\nabla(\log u - v)|^2 dx = 0.$$

It implies that in the stationary state

$$\log u - v = \log \sigma$$

holds for some constant $\sigma > 0$. In other words

$$u = \sigma e^v$$

holds. Putting $\lambda = \|u_0\|_1$ we have

$$\sigma = \frac{\lambda}{\int_{\Omega} e^v dx}$$

by mass conservation. Thus the second equation in (3) implies that

$$0 = \Delta v + \lambda \frac{e^v}{\int_{\Omega} e^v dx}$$

with Dirichlet boundary condition.

We have known the result on \mathcal{C} when $n = 1, 2$ and Ω is the unit ball $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$. For $n = 1$ with $\Omega = B$, if $p \geq 1$, then (1) has a unique solution for all $\lambda > 0$. On the contrary if $0 < p < 1$, there exists $\bar{\lambda} > 0$ such that (1) has two solutions for $\lambda < \bar{\lambda}$, (1) has one solution for $\lambda = \bar{\lambda}$ and (1) has no solution for $\lambda > \bar{\lambda}$. For $n = 2$ with $\Omega = B$, if $p < 1$, then (1) has a unique solution for all $\lambda > 0$. If $p = 1$, then (1) has a unique solution for all $0 < \lambda < 8\pi$ but no solution for $\lambda \geq 8\pi$. On the contrary if $0 < p < 1$, there exists $\bar{\lambda} > 0$ such that (1) has two solutions for $\lambda < \bar{\lambda}$, (1) has one solution for $\lambda = \bar{\lambda}$ and (1) has no solution for $\lambda > \bar{\lambda}$. These facts are proven in [2].

We consider the $p = 1$ case in [13] and expand their results to the general $p > 0$ in this paper.

We also mention the structure of \mathcal{S} . For $n \geq 3$ with $\Omega = B$, \mathcal{S} is a one-dimensional open manifold with the end points in $(\sigma, v) = (0, 0)$ and $(\sigma, v) = (2(n - 2), 2 \log(1/|x|))$, respectively, the latter being a weak solution of (2). Moreover for $3 \leq n \leq 9$, \mathcal{S} bends infinitely many times with respect to σ around $\sigma = 2(n - 2)$. Morse indices increases by one whenever it bends. On the other hand if $n \geq 10$, no bending occurs. They are shown in [14], [15].

We define the section of \mathcal{C} cut by $\lambda > 0$ as follows:

$$\mathcal{C}^\lambda = \{v \in C^2(\Omega) \cap C(\bar{\Omega}) \mid v = v(x) \text{ solves (1)}\}.$$

The first theorem is concerned with the star-shaped domain, so that $x \cdot \nu > 0$ holds for each $x \in \partial\Omega$.

Theorem 1. *If Ω is star-shaped with respect to the origin with $n \geq 3$ and $p \leq 1$, then there is $\bar{\lambda} \in (0, +\infty)$ such that (1) has no solution for $\lambda > \bar{\lambda}$. Moreover, \mathcal{C}_0 is unbounded in $\lambda - v$ plane, where \mathcal{C}_0 stands for the connected component of \mathcal{C} satisfying $(0, 0) \in \bar{\mathcal{C}}_0$.*

The second theorem is concerned with the ball case.

Theorem 2. *If Ω is the unit ball $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$ with $n \geq 3$, then \mathcal{C} is a one-dimensional open manifold and can be parametrized as*

$$\mathcal{C} = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < +\infty\}$$

with the end points in $(\lambda, v) = (0, 0)$ and the weak solution

$$(\lambda, v) = \left(2\omega_n^p (n-2)^{1-p}, 2 \log\left(\frac{1}{|x|}\right) \right),$$

respectively, where ω_n denotes the area of the unit sphere in \mathbf{R}^n . Moreover for $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, \mathcal{C} bends infinitely many times with respect to λ around $\lambda = 2\omega_n^p (n-2)^{1-p}$. On the other hand if $n \geq 10$ and $p \leq 1$, no bending occurs.

The third theorem is on the spectral property of the linearized operator. To state the result, we define Morse index as follows. For given $(\lambda, v) \in \mathcal{C}$, the linearized eigenvalue problem is given by

$$(4) \quad \begin{cases} \Delta \phi + \lambda \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p} \phi - p\lambda \frac{\int_{\Omega} e^v \phi dx}{\left(\int_{\Omega} e^v dx\right)^{p+1}} e^v = -\mu \phi & x \in \Omega, \\ \phi = 0 & x \in \partial\Omega. \end{cases}$$

Then, the Morse index $i = i(\lambda, v)$ and the radial Morse index $i_R = i_R(\lambda, v)$ denote the number of negative eigenvalues and that of radially symmetric eigenfunctions, respectively.

Theorem 3. *If Ω is the unit ball with $3 \leq n \leq 9$, $2/n \leq p \leq 1$ and $n \geq 10$, $p \leq 1$, respectively, then $i = i_R$ holds and $i = i(\lambda, v)$ increases by one at each bending point.*

The last theorem is on the annulus domain $A_a = \{x \in \mathbf{R}^n \mid a < |x| < 1\}$ with $a \in (0, 1)$. We deal with only radial solutions. Then we define the solution set by

$$\mathcal{C}_a = \{(\lambda, v) \mid v = v(|x|) \text{ is a classical solution of (1) for } \lambda > 0\}.$$

Theorem 4. *If Ω is the annulus domain A_a with $n \geq 3$ and $p \geq 1$, then \mathcal{C}_a is a one-dimensional open manifold and can be parametrized as*

$$\mathcal{C}_a = \{(\lambda(s), v(\cdot, s)) \mid 0 < s < +\infty\}$$

with the end points in $(\lambda, v) = (0, 0)$ and (λ, v) satisfying

$$\lim_{s \uparrow +\infty} \lambda(s) = +\infty \quad \text{and} \quad \lim_{s \uparrow +\infty} \sup_{a < x < 1} |v(x, s)| = +\infty.$$

This paper is composed of four sections. In §2, we treat a star-shaped domain and prove Theorem 1. Next in §3, we study the ball case and prove Theorems 2 and 3. Finally, §4 is on the annulus domain case and we prove Theorem 4.

2. Star-shaped domain

In this section, we assume that Ω is a star-shaped bounded domain with respect to the origin in \mathbf{R}^n with $n \geq 3$ with the smooth boundary $\partial\Omega$ and that ν is the outer unit normal vector.

Proof of Theorem 1. The first part of Theorem has already proven in [2], but we provide the proof for completeness. In fact we apply the Pohozaev identity ([16]) to (1).

(5)

$$\begin{aligned} \frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial v}{\partial \nu} \right)^2 ds &= \frac{n\lambda}{\left(\int_{\Omega} e^v dx \right)^p} \int_{\Omega} (e^v - 1) dx + \frac{2-n}{2} \frac{\lambda}{\left(\int_{\Omega} e^v dx \right)^p} \int_{\Omega} e^v v dx \\ &\leq \frac{n\lambda}{\left(\int_{\Omega} e^v dx \right)^{p-1}}, \end{aligned}$$

where ds is the area element of ∂B with standard metric.

On the other hand it follows from (1) that

$$\frac{\lambda}{\left(\int_{\Omega} e^v dx \right)^{p-1}} = \int_{\Omega} (-\Delta v) dx = \int_{\partial\Omega} \left(-\frac{\partial v}{\partial \nu} \right) ds,$$

and therefore we have

$$\frac{\lambda^2}{\left(\int_{\Omega} e^v dx \right)^{2(p-1)}} \leq \int_{\partial\Omega} (x \cdot \nu) \left(\frac{\partial v}{\partial \nu} \right)^2 ds \int_{\partial\Omega} \frac{1}{(x \cdot \nu)} ds.$$

Combining this inequality and (5), we have

$$\frac{\lambda^2}{\left(\int_{\Omega} e^v dx \right)^{2(p-1)}} \leq \frac{2n\lambda}{\left(\int_{\Omega} e^v dx \right)^{p-1}} \int_{\partial\Omega} \frac{1}{(x \cdot \nu)} ds.$$

Hence since $p \leq 1$ and $v > 0$ in Ω , we have

$$\lambda \leq \frac{2n}{\left(\int_{\Omega} e^v dx \right)^{1-p}} \int_{\partial\Omega} \frac{1}{(x \cdot \nu)} ds \leq \frac{2n}{|\Omega|^{1-p}} \int_{\partial\Omega} \frac{1}{(x \cdot \nu)} ds,$$

where $|\Omega|$ is the measure of Ω . This gives $\bar{\lambda}$ in the statement.

We denote by \mathcal{C}_1 the branch of solutions of (1) starting from $(\lambda, v) = (0, 0)$. Supposing that \mathcal{C}_1 is bounded, we prove unboundedness of the component \mathcal{C}_1 by contradiction by making use of the standard degree argument similar to [18]. We provide the proof for completeness and proceed it in the same method as in [19]. Putting

$$F(\lambda, v) = \Delta v + \lambda \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p},$$

we apply the implicit function theorem. Then there exists a solution (λ, v) for $0 < \lambda \ll 1$. \mathcal{C}_1 is compact from the assumption and the existence of the upper bound $\bar{\lambda} < +\infty$. Because

$$\mathcal{C} \cap \{\lambda = 0\} = \{(0, 0)\},$$

we can take open set \mathcal{U} containing \mathcal{C}_1 with the properties

$$\partial\mathcal{U}_\lambda \cap \mathcal{C} = \emptyset \quad \text{and} \quad \mathcal{U}_\lambda = \emptyset \quad \text{for } \lambda \gg 1,$$

and $\mathcal{U}_\lambda \cap \mathcal{C}$ is composed of the solution of (1) with $\mu_1(\lambda, v) > 0$ for $0 < \lambda \ll 1$, where

$$\mathcal{U}_\lambda = \{v \in C(\bar{\Omega}) \mid (\lambda, v) \in \mathcal{U}\}.$$

In Banach space $C(\bar{\Omega})$, Leray-Schauder degree $d(\Psi_\lambda, 0, \mathcal{U}_\lambda)$ is taken for any $\lambda > 0$, where $\Psi_\lambda = I_{C(\bar{\Omega})} - \Phi_\lambda$ with

$$\Phi_\lambda(v) = (-\Delta)^{-1} \lambda \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p}.$$

From the homotopy invariance ([18]), $d(\Psi_\lambda, 0, \mathcal{U}_\lambda)$ is independent of $\lambda > 0$. However by existence and nonexistence of the solution of (1), we have

$$\begin{cases} d(\Psi_\lambda, 0, \mathcal{U}_\lambda) = 0 & \text{for } \lambda \gg 1, \\ d(\Psi_\lambda, 0, \mathcal{U}_\lambda) = 1 & \text{for } 0 < \lambda \ll 1, \end{cases}$$

which is a contradiction. □

3. Ball case

In this section, we assume that $\Omega = B$, where $B = \{x \in \mathbf{R}^n \mid |x| < 1\}$.

Proof of Theorem 2. According to [6], any solution of (1) is radially symmetric. Hence we have

$$\begin{cases} (r^{n-1}v')' + \lambda r^{n-1} \frac{e^v}{\left(\int_{\Omega} e^v dx\right)^p} = 0 & \text{for } r > 0, \\ v(1) = 0, \quad v'(0) = 0, \end{cases}$$

where

$$v = v(r) \quad \text{for } r = |x|.$$

We begin with the parametrization of the solution set \mathcal{C} , following [7], [13], [14] and [15]. In fact, any solution is obtained as a solution of the initial value problem

$$(6) \quad \begin{cases} (r^{n-1}v')' + \sigma r^{n-1}e^v = 0 & \text{for } r > 0, \\ v(0) = A, \quad v'(0) = 0, \end{cases}$$

with a certain positive constant A . Through the Emden transformation

$$(7) \quad v(r) = w(t) - 2t + A, \quad r = \left\{ \frac{2(n-2)}{\sigma e^A} \right\}^{1/2} e^t,$$

(6) is reduced to the autonomous ordinary differential equation

$$(8) \quad \begin{cases} \ddot{w} + (n-2)\dot{w} + 2(n-2)(e^w - 1) = 0 \\ \lim_{t \rightarrow -\infty} (w(t) - 2t) = \lim_{t \rightarrow -\infty} e^{-t}(\dot{w}(t) - 2) = 0. \end{cases}$$

Then there exists a unique global solution $w = w(t)$ of (8) by [14]. The orbit $\mathcal{O} = \{(w(t), z(t)) = (w(t), \dot{w}(t)) \mid t \in \mathbf{R}\}$ starts at $t = -\infty$ along and below the line $z = 2$ with $w = -\infty$, and approaches the origin $(0, 0)$ as $t \rightarrow +\infty$. If $3 \leq n \leq 9$, it proceeds clockwise in $\{(w, z) \mid z < 2\}$, crosses infinitely many times z - and w -axes alternately, while it keeps to stay in $\{(w, z) \mid w < 0, 0 < z < 2\}$ in the case of $n \geq 10$. Through the Emden transformation (8), the boundary condition in (1) is converted to

$$w(\tau) - 2\tau + A = 0$$

with

$$\left\{ \frac{2(n-2)}{\sigma e^A} \right\}^{1/2} e^\tau = 1.$$

Therefore for any $\tau \in \mathbf{R}$, (σ_τ, v_τ) defined by

$$(9) \quad v_\tau(r) = w(t) - 2t - \{w(\tau) - 2\tau\} = w(\log r + \tau) - w(\tau) - 2 \log r$$

with $r = e^{t-\tau}$ and

$$(10) \quad A_\tau = 2\tau - w(\tau), \quad \sigma_\tau = 2(n-2)e^{2\tau-A} = 2(n-2)e^{w(\tau)}$$

satisfies (1). Conversely every solution of (1) can be expressed in the form of (9) and (10). According to [7], [13], [14] and [19], the total set \mathcal{S} of the solution (σ, v)

of (2) is homeomorphic to \mathcal{O} through the relation (7) with the constants A, σ determined by (10). This means that \mathcal{C} is homeomorphic to \mathcal{O} . Each point of \mathcal{O} is given as $(w(\tau), z(\tau))$ and hence \mathcal{C} is parametrized by $\tau \in \mathbf{R}$. Then we have

$$(11) \quad \lambda(\tau) = \omega_n^p 2^{1-p} (n-2)^{1-p} (2-z(\tau))^p e^{(1-p)w(\tau)}.$$

In fact, putting $K = \int_B e^v dx$ we have

$$\lambda^{1/p-1} = K^{1-p} \sigma^{1/p-1}$$

because of $K^p = \lambda/\sigma$. By (10), we have

$$\lambda^{1/p} = \lambda K^{1-p} \sigma^{1/p-1} = \lambda K^{1-p} (2(n-2)e^{w(\tau)})^{1/p-1}.$$

Integrating (1) over B , we have

$$-\lambda K^{1-p} = \omega_n v'(1) = \omega_n (z(\tau) - 2)$$

by (7). Finally combining two equations, we have the desired one. Hence the behaviour of $\tau = +\infty$ follows at once. On the other hand $\lim_{\tau \rightarrow -\infty} v_\tau = 0$ and $\lim_{\tau \rightarrow -\infty} \sigma_\tau = 0$ imply that $\lim_{\tau \rightarrow -\infty} \lambda_\tau = 0$. The rest follows.

In the use of (11), we have

$$\dot{\lambda}(\tau) = \omega_n^p 2^{1-p} (n-2)^{1-p} (2-z(\tau))^{p-1} e^{(1-p)w(\tau)} \{(1-p)z(\tau)(2-z(\tau)) - pz(\tau)\}.$$

If $n \geq 10$ and $p \leq 1$, $\dot{\lambda}(\tau) > 0$ for all $\tau \in \mathbf{R}$ which proves the statement. Next we concentrate on the case of $3 \leq n \leq 9$ and $2/n \leq p \leq 1$. To do so, we put $g(\tau) = (1-p)z(\tau)(2-z(\tau)) - pz(\tau)$. Then we have

$$(12) \quad \dot{g}(\tau) = 2(1-p)(2-z(\tau))\dot{z}(\tau) + (pn-2)\dot{z}(\tau) + 2p(n-2)e^{w(\tau)}z(\tau).$$

Let \mathcal{O}_k ($k \geq 2$) denote the successive points w -axis and $z = -2(e^w - 1)$ crossed by the orbit \mathcal{O} in $w-z$ plane in order. Moreover we set $\mathcal{O}_1 = (-\infty, 2)$. Then $\dot{\lambda}(\tau) > 0$ and $\dot{\lambda}(\tau) < 0$ on the arc $\mathcal{O}_{4k-3}\mathcal{O}_{4k-2}$ and $\mathcal{O}_{4k-1}\mathcal{O}_{4k}$, respectively for $k \geq 1$. On the other hand $\dot{g}(\tau) < 0$ and $\dot{g}(\tau) > 0$ on the arc $\mathcal{O}_{4k-2}\mathcal{O}_{4k-1}$ and $\mathcal{O}_{4k}\mathcal{O}_{4k+1}$, respectively for $k \geq 1$. Hence there exists a unique point $\tilde{\mathcal{O}} = (w(\tau), z(\tau))$ on the every arc $\mathcal{O}_{4k-2}\mathcal{O}_{4k-1}$ and $\mathcal{O}_{4k}\mathcal{O}_{4k+1}$ respectively for $k \geq 1$ such that $\dot{\lambda}(\tau) = 0$. The proof is complete. \square

We proceed the proof of Theorem 3 in the same argument and computation as in [15].

Proof of Theorem 3. As far as we consider negative eigenvalues in (4), the corresponding eigenfunctions are radially symmetric ([12], [13]). Hence $i(\lambda, v) = i_R(\lambda, v)$ for

$(\lambda, v) \in \mathcal{C}$. We denote by μ_τ^l the l -th eigenvalue of (4) in $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ corresponding to radially symmetric eigenfunctions. Any of them is simple. If $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ is the turning point of \mathcal{C} , then there exists $l \geq 1$ such that $\mu_\tau^l = 0$ by the implicit function theorem. On the contrary, $\mu_\tau^l = 0$ for some $l \geq 1$ at $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ implies that it is a turning point by the bifurcation theorem from the critical point of odd multiplicity ([17], [18]). Since $\mu_{-\tau}^1 > 0$ for sufficiently large $\tau > 0$, we have $i(\lambda, v) = 0$ for $(\lambda(\tau), v(\tau)) \in \mathcal{C}$ for $n \geq 10$ and $p \leq 1$.

For $3 \leq n \leq 9$ and $2/n \leq p \leq 1$, let $T_k = (\lambda(\tau_k), v(\tau_k))$ for $\tau_1 < \tau_2 < \dots$ denote the turning point of \mathcal{C} . Then we have $\mu_{\tau_k}^l = 0$ for some $l \geq 1$ and we have only to show that $\dot{\mu}_{\tau=\tau_k}^l < 0$ for all $k \geq 1$.

Differentiating (1) with respect to τ , we have

$$(13) \quad \begin{cases} \Delta \dot{v} + \dot{\lambda} \frac{e^v}{(\int_B e^v dx)^p} + \lambda \frac{e^v \dot{v}}{(\int_B e^v dx)^p} - \lambda p \frac{\int_B e^v \dot{v} dx}{(\int_B e^v dx)^{p+1}} e^v = 0 & x \in B, \\ \dot{v} = 0 & x \in \partial B, \end{cases}$$

and hence

$$\begin{cases} \Delta \dot{v}_k + \dot{\lambda}_k \frac{e^{v_k} \dot{v}_k}{(\int_B e^{v_k} dx)^p} - \dot{\lambda}_k p \frac{\int_B e^{v_k} \dot{v}_k dx}{(\int_B e^{v_k} dx)^{p+1}} e^{v_k} = 0 & x \in B, \\ \dot{v}_k = 0 & x \in \partial B, \end{cases}$$

for $v_k = v(\cdot, \tau_k)$. Then we have

$$\dot{v}(r, \tau) = \dot{w}(\log r + \tau) - \dot{w}(\tau) \neq 0,$$

and therefore, \dot{v}_k is an eigenfunction of (4) corresponding to $\mu = \mu_{\tau_k}^l = 0$. Then, the standard perturbation theory ([8]) guarantees the existence of $\phi = \phi(\cdot, \tau)$ and $\mu = \mu(\tau)$ satisfying (4), $\phi(\cdot, \tau_k) = \dot{v}_k$, and $\mu(\tau_k) = \mu_{\tau_k}^k = 0$. Differentiating (4) and (13) with respect to τ , subtracting each other with $\tau = \tau_k$, multiplying by \dot{v} and integrating it over B , we have

$$(14) \quad \ddot{\lambda} \frac{\int_B e^v \dot{v} dx}{(\int_B e^v dx)^p} = \dot{\mu} \int_B \dot{v}^2 dx,$$

where $\dot{\mu} = \dot{\mu}(\tau_k)$, $\ddot{\lambda} = \ddot{\lambda}(\tau_k)$, $v = v(\cdot, \tau_k)$ and $\dot{v} = \dot{v}(\cdot, \tau_k)$. As is stated in the proof of Theorem 2,

$$(15) \quad \dot{\lambda}(\tau) = \omega_n^p 2^{1-p} (2 - z(\tau))^{p-1} e^{(1-p)w(\tau)} g(\tau) = \omega_n^p \sigma(\tau)^{1-p} (2 - z(\tau))^{p-1} g(\tau),$$

where $g(\tau) = (1 - p)z(\tau)(2 - z(\tau)) - pz(\tau)$. Differentiating (15), we have

$$\begin{aligned} \ddot{\lambda}(\tau) &= \omega_n^p \sigma(\tau)^{1-p} (2 - z(\tau))^{p-2} \\ &\quad \times \{(1 - p)(-z(\tau)^2 + 2z(\tau) + \dot{z}(\tau))g(\tau) + (2 - z(\tau))\dot{g}(\tau)\}. \end{aligned}$$

Since $\dot{\lambda}(\tau) = 0$ at $\tau = \tau_k$, it holds that $g(\tau_k) = 0$, namely,

$$(1-p)z(\tau_k)(2-z(\tau_k)) - p\dot{z}(\tau_k) = 0$$

from (15). We have $\lambda = \omega_n^p(2-z)^p\sigma^{1-p}$ from (10) and (11),

$$\dot{\lambda} = \omega_n^p(2-z)^{p-1}\sigma^{1-p}\{(2-z)\sigma^{-1}\dot{\sigma}(1-p) - pz\}$$

by differentiating, and

$$\sigma\dot{z} = \frac{(2-z)(1-p)}{p}\dot{\sigma}$$

at $\tau = \tau_k$. Hence we have

$$(16) \quad \begin{aligned} \ddot{\lambda}(\tau) &= \omega_n^p\sigma(\tau)^{-p}(2-z(\tau))^{p-1} \\ &\times \left\{ \frac{2(p-1)^2(2-z(\tau))^2}{p} + \frac{(pn-2)(1-p)(2-z(\tau))}{p} + p\sigma(\tau) \right\} \dot{\sigma}(\tau) \end{aligned}$$

at $\tau = \tau_k$. Since $\sigma = \lambda / (\int_B e^v dx)^p$, it holds that

$$(17) \quad \dot{\sigma} = \frac{\dot{\lambda}}{(\int_B e^v dx)^p} - \lambda p \frac{\int_B e^v \dot{v} dx}{(\int_B e^v dx)^{p+1}}.$$

Hence we have

$$\begin{aligned} \dot{\mu} \int_B \dot{v}^2 dx &= -\lambda p \frac{(\int_B e^v \dot{v} dx)^2}{(\int_B e^v dx)^{2p+1}} \omega_n^p \sigma(\tau)^{-p} (2-z(\tau))^{p-1} \\ &\times \left\{ \frac{2(p-1)^2(2-z(\tau))^2}{p} + \frac{(pn-2)(1-p)(2-z(\tau))}{p} + p\sigma(\tau) \right\} \end{aligned}$$

at $\tau = \tau_k$ from (14) and (16). Let $\int_B e^v \dot{v} dx = 0$. Then (17) means that $\dot{\lambda} = 0$ and $\dot{\sigma} = 0$ vanish at $\tau = \tau_k$ simultaneously. However it is impossible from (10) and (11). Finally we have $\dot{\mu} < 0$ at $\tau = \tau_k$. \square

4. Annulus domain

In this section, we assume that $\Omega = A_a = \{x \in \mathbf{R}^n \mid a < |x| < 1\}$ with $a \in (0, 1)$ and consider radially symmetric solutions of (1).

We cite known results in [14] for the case of $3 \leq n \leq 9$. Radial solutions of (2) satisfy

$$(18) \quad \begin{cases} (r^{n-1}v')' + \sigma r^{n-1}e^v = 0 & \text{for } a < r < 1, \\ v(a) = v(1) = 0. \end{cases}$$

We can continue the solution of (18) up to $r = +0$ satisfying

$$\lim_{r \downarrow 0} \left(v'(r) - \frac{L}{r^{n-1}} \right) = 0$$

and

$$\lim_{r \downarrow 0} \left(v(r) + \frac{L}{(n-2)r^{n-2}} \right) = M$$

for some $L, M > 0$. Through the modified Emden transformation

$$v(r) = w(t) - 2t + M, \quad r = Be^t$$

with

$$B = \left\{ \frac{2(n-2)}{\sigma e^M} \right\}^{1/2},$$

(18) is reduced to the autonomous ordinary differential equation

$$(19) \quad \begin{cases} \ddot{w} + (n-2)\dot{w} + 2(n-2)(e^w - 1) = 0, \\ \lim_{t \rightarrow -\infty} (w(t) - 2t + \alpha e^{-(n-2)t}) = 0, \\ \lim_{t \rightarrow -\infty} e^{-t} (\dot{w}(t) - 2 - \alpha(n-2)e^{-(n-2)t}) = 0, \end{cases}$$

where $\alpha = LB^{-(n-2)}/(n-2)$.

Then there exists a unique global solution $w = w_\alpha(t)$ of (19) for every $\alpha > 0$. The orbit $\mathcal{O}_\alpha = \{(w(t), z(t)) = (w_\alpha(t), \dot{w}_\alpha(t)) \mid t \in \mathbf{R}\}$ starts at $t = -\infty$ above the line $z = 2$ with $w = -\infty$, and approaches the origin $(0, 0)$ as $t \rightarrow +\infty$. Then the family of orbits $\{\mathcal{O}_\alpha\}_{\alpha \geq 0}$ forms a foliation, that is, $\mathcal{O}_\alpha \cap \mathcal{O}_\beta = \emptyset$ if $\alpha \neq \beta$. Every orbit \mathcal{O}_α with $\alpha > 0$ crosses the line $z = 2$ just once. Moreover if $\alpha > \beta > 0$, \mathcal{O}_α lies in the left side of \mathcal{O}_β as t increases. For every point $(\eta_0, \zeta_0) \in \Sigma \equiv \{(w, z) \mid w > 0, z > 2\}$, there is a unique nonnegative α such that $(\eta_0, \zeta_0) \in \mathcal{O}_\alpha$.

Let $w_\alpha = w_\alpha(t)$ be the solution of (19). Then every point (w, z) on \mathcal{O}_α determines the time t so that the boundary condition

$$v(a) = v(1) = 0$$

is converted into

$$(20) \quad w(t^-) - 2t^- + M = 0 \quad \text{and} \quad w(t^+) - 2t^+ + M = 0$$

for

$$(21) \quad t^- = \log \frac{a}{B} \quad \text{and} \quad t^+ = \log \frac{1}{B}.$$

Henceforward we omit the subscripts α and so on unless there is any confusion. Further w^\pm and z^\pm denote $w_\alpha(t^\pm)$ and $z_\alpha(t^\pm)$, respectively. From (20) and (21), it holds that

$$(22) \quad w^+ - w^- = -2 \log a \quad \text{and} \quad t^+ - t^- = -\log a.$$

Conversely, if there exists a pair of points $P^\pm(w^\pm, z^\pm)$ on O_α satisfying (22), we have a radial solution $v = v(r)$ for (18) with some positive constant σ . In fact, we define B, M, L, σ and $v(r)$ as

$$\begin{aligned} B &= ae^{-t^-} = e^{-t^+}, \\ M &= 2t^- - w^- = 2t^+ - w^+, \\ L &= \alpha(n-2)a^{n-2}e^{-(n-2)t^-} = \alpha(n-2)e^{-(n-2)t^+}, \\ \sigma &= 2(n-2)e^{w^+} = 2(n-2)a^{-2}e^{w^-}, \\ v(r) &= w(t) - 2t + M. \end{aligned}$$

Therefore the structure of the solution of (18) is reduced to that of pairs of P^\pm on O_α . We call $\{P_\alpha^\pm\}$ the boundary pair on O_α associated with the annulus A_α in \mathbf{R}^n .

For every $a \in (0, 1)$, there exists a unique pair of points $P_\alpha^\pm = P^\pm(w_\alpha^\pm, z_\alpha^\pm)$ satisfying (22) on each orbit O_α ($\alpha > 0$). The points P_α^+ and P_α^- lie below and above the line $z = 2$ respectively, and further, these points P_α^\pm depend on α continuously. Conversely for each point $(w_\alpha(t), z_\alpha(t))$ on O_α with $z_\alpha(t) < 2$ ($z_\alpha(t) > 2$), there exists a unique $a^* = a^*(t) \in (0, 1)$ ($a_* = a_*(t) \in (0, 1)$) such that

$$\begin{aligned} w_\alpha(t) - w_\alpha(t + \log a^*) &= -2 \log a^*, \\ (w_\alpha(t - \log a_*) - w_\alpha(t)) &= -2 \log a_*. \end{aligned}$$

Hence we have only to study $K_\alpha = \{P_\alpha^-(w_\alpha^-, z_\alpha^-) \mid \alpha > 0\}$ to get the structure of solution of (18). The set K_α forms a continuous curve in \mathbf{R}^2 , which is homeomorphic to \mathbf{R} . Now we have two lemmas.

Lemma 1 ([14], Lemmas 4.8 and 4.9). *For any fixed $a \in (0, 1)$, we have*

$$\begin{cases} \lim_{\alpha \rightarrow 0} w_\alpha^+ = \lim_{\alpha \rightarrow 0} w_\alpha^- = -\infty, \\ \lim_{\alpha \rightarrow 0} z_\alpha^+ = \lim_{\alpha \rightarrow 0} z_\alpha^- = 2, \\ \lim_{\alpha \rightarrow 0} \|v_\alpha\|_{C(\overline{\Omega})} = 0, \\ \lim_{\alpha \rightarrow 0} \sigma_\alpha = 0, \end{cases}$$

where $P_\alpha^\pm(w_\alpha^\pm, z_\alpha^\pm)$ are the pair of boundary points on O_α associated with A_α .

Lemma 2 ([14], Remark 6.1). *For any fixed $a \in (0, 1)$, we have*

$$\begin{cases} \lim_{\alpha \rightarrow +\infty} w_\alpha^+ = \lim_{\alpha \rightarrow +\infty} w_\alpha^- = -\infty, \\ \lim_{\alpha \rightarrow +\infty} \|v_\alpha\|_{C(\overline{A}_a)} = +\infty, \\ \lim_{\alpha \rightarrow +\infty} \sigma_\alpha = 0, \end{cases}$$

where $P_\alpha^\pm(w_\alpha^\pm, z_\alpha^\pm)$ are the pair of boundary points on O_α associated with A_a .

Proof of Theorem 4. We concentrate on the case of $3 \leq n \leq 9$. The behaviour of λ_α and v_α as $\alpha \rightarrow +0$ follows from Lemma 1, that is,

$$\lim_{\alpha \rightarrow 0} \lambda_\alpha = \lim_{\alpha \rightarrow 0} \sigma_\alpha \left(\int_{A_a} e^{v_\alpha} dx \right)^p = 0 \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \|v_\alpha\|_{C(\overline{A}_a)} = 0.$$

Integrating (1) over A_a , we have

$$(23) \quad \lambda = \omega_n^p (2(n-2)a^{-2}e^{w^-})^{1-p} (a^{n-2}(z^- - 2) - (z^+ - 2))$$

in the same way as we deduce (11). Now it holds that

$$\lim_{\alpha \rightarrow +\infty} z_\alpha^- = +\infty.$$

In fact, we assume that $z_\alpha^- < M$ for any $\alpha \in \mathbf{R}$, where M is a positive constant owing to $z_\alpha^- > 2$. Since K_a is homeomorphic to \mathbf{R} , there is a constant $K > 0$ such that $w_\alpha^- < K$ for any $\alpha \in \mathbf{R}$ by Lemma 2. By setting $Q = \{(w, z) \mid w < K, 2 < z < M\}$, it holds that $O_\beta \cap \overline{Q} = \emptyset$ for some $\beta \in \mathbf{R}$. Actually, for $(w, z) = (K, M)$ there is $\alpha > 0$ such that $(K, M) \in O_\alpha$ because of $(K, M) \in \Sigma \subset \bigcup_{\alpha \geq 0} O_\alpha$. Since z - and w -coordinate are decreasing and increasing with respect to t respectively in $\{(w, z) \mid w > g(z), z > 0\}$, $O_\alpha \cap \overline{Q} = (K, M)$. Hence if we put $\beta = \alpha + \delta$ for any $\delta > 0$, we have $O_\beta \cap \overline{Q} = \emptyset$ because O_β lies in the left side of O_α as t increases. The points $\{(w_\alpha^-, z_\alpha^-) \mid \alpha > 0\}$ on O_β don't satisfy $w_\alpha^- < K, z_\alpha^- < M$ simultaneously, which is a contradiction. Therefore we have $\lim_{\alpha \rightarrow +\infty} z_\alpha^- = +\infty$. Finally from (23), $z_\alpha^+ < 2$ and $p \geq 1$, we have $\lim_{\alpha \rightarrow +\infty} \lambda_\alpha = +\infty$. In the case of $n \geq 10$, we change Σ and K by $\{(w, z) \mid w < 0, z > 2\}$ and a negative constant, respectively. \square

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