Title	Duality in generalized homogeneous programming
Author(s)	Nozawa, Ryōhei
Citation	Osaka Journal of Mathematics. 1983, 20(4), p. 899-910
Version Type	VoR
URL	https://doi.org/10.18910/6313
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Nozawa, R.
Osaka J. Math.
20 (1983), 899–910

DUALITY IN GENERALIZED HOMOGENEOUS PROGRAMMING

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

RYÔHEI NOZAWA

(Received March 10, 1982)

1. Introduction with problem setting

Homogeneous programming problems were first studied by Eisenberg [1] in finite dimensional spaces and next by Schechter [7]. In this paper we shall be concerned with more generalized homogeneous programming problems and their duality relations.

More precisely, let X and Y be real linear spaces which are in duality with respect to a bilinear functional $\langle\cdot, \cdot\rangle_1$ and let Z and W be real linear spaces which are in duality with respect to a bilinear functional $\langle\cdot, \cdot\rangle_2$. Hereafter we denote $\langle\cdot, \cdot\rangle_1$ and $\langle\cdot, \cdot\rangle_2$ by $\langle\cdot, \cdot\rangle$ for simplicity. In this paper, we assume that each one of the paired spaces is assigned the weak topology unless otherwise stated. We denote by $\tau(X, Y)$ the Mackey topology on X. We also assume that the cones considered have their vertices at the origin of the space.

Let P and Q be closed convex cones in X and Z respectively and denote by P° and Q° the dual cones of P and Q. Let f be an extended real valued function on X which is lower semicontinuous and sublinear, i.e., the epigraph $\{(x, r)\in X\times R; f(x)\leq r\}$ of f is a closed convex cone or the empty set, and let g be an extended real valued function on W which is upper semicontinuous and superlinear, i.e., $-g$ is sublinear. Note that if f is finite at some point, then f does not take the value $-\infty$. Let Ψ be an extended real valued function on $X\times W$ such that $\Psi_x=\Psi(x, \cdot)$ is lower semicontinuous and sublinear on W for every fixed $x\in X$ and $\Psi_w=\Psi(\cdot, w)$ is upper semicontinuous and superlinear on X for every fixed $w\in W$. We assume that $\Psi(0, 0)=f(0)=g(0)=0$.

For the quintuple (Ψ, P, Q°, f, g), we consider the following generalized homogeneous programming problems (=HP) and its dual problem (=DHP):

(HP) Find $M=\inf\{f(x);\ x\in S\}$,

where $S=\{x\in P;\ g(w)\leq\Psi(x, w)\ \text{ for all } w\in Q^\circ\}$.

(DHP) Find $M^*=\sup\{g(w);\ w\in S^*\}$,

where $S^*=\{w\in Q^\circ;\ f(x)\geq\Psi(x, w)\ \text{ for all } x\in P\}$.

Here we use the convention that the infimum of a real function on the empty set $\emptyset$ is equal to $+\infty$.

Our aim is to find some conditions which assure that the above two problems have the same value and have optimal solutions. In the case where Ψ is continuous and bilinear, Schechter [7] investigated duality relations for these problems. In the next section, we introduce programming problems with constraints of convex processes studied in [5], and state some relations between those problems. In §3 and §4, we give main results. In §4, we deal with the case where Ψ is bilinear and improve a result in [7].

2. Reduction of HP and DHP

In order to obtain a convex process and its adjoint process from Ψ, we consider the following two sets:

$$\begin{aligned}\mathrm{dom}_X\Psi &= \{x\in X;\ \ \Psi(x, w) \text{ is finite for some } w\in W\}\ ,\\ \mathrm{dom}_W\Psi &= \{w\in W;\ \ \Psi(x, w) \text{ is finite for some } x\in X\}\ .\end{aligned}$$

If $x\in\mathrm{dom}_X\Psi$, then $\Psi(x, 0)=0$ and $\Psi(x, w)\neq-\infty$ for all $w\in W$. Thus $\mathrm{dom}_X\Psi=\{x\in X;\ \Psi(x,0)=0\}$ and this set is closed, since $\Psi(\cdot,0)$ is upper semicontinuous on X. If $w\in\mathrm{dom}_X\Psi$, then $\Psi(0, w)=0$ and $\Psi(x, w)\neq+\infty$ for all $x\in X$. Thus $\mathrm{dom}_W\Psi=\{w\in W;\ \Psi(0, w)=0\}$ and this set is closed. Note that $\Psi(x, w)$ is finite if and only if $x\in\mathrm{dom}_X\Psi$ and $w\in\mathrm{dom}_W\Psi$.

We recall the subdifferential $\partial f(0)$ of f and the superdifferential $\partial g(0)$ of g at the origins:

$$\begin{aligned}\partial f(0) &= \{y\in Y;\ \langle x, y\rangle\leq f(x)\quad \text{for all } x\in X\}\ ,\\ \partial g(0) &= \{z\in Z;\ \langle z, w\rangle\geq g(w)\quad \text{for all } w\in W\}\ .\end{aligned}$$

It is well-known that $\partial f(0)$ and $\partial g(0)$ are nonempty closed convex sets, and that $f(x)=\sup_{y\in\partial f(0)}\langle x, y\rangle$ for all $x\in X$ and $g(w)=\inf_{z\in\partial g(0)}\langle z, w\rangle$ for all $w\in W$. If f is $\tau(X, Y)$-continuous, then $\partial f(0)$ is weakly compact (cf. [5; Lemma 1]).

Since Ψ_x is lower semicontinuous and sublinear on W, we can define the subdifferential $\partial\Psi_x(0)$ of Ψ_x at the origin for $x\in\mathrm{dom}_X\Psi$:

$$\partial\Psi_x(0) = \{z\in Z;\ \langle z, w\rangle\leq\Psi(x, w)\quad \text{for all } w\in W\}\ .$$

Now we define a set-valued mapping A from X to Z by

$$(2.1)\qquad Ax = \partial\Psi_x(0) \text{ if } x\in\mathrm{dom}_X\Psi, \text{ and } Ax = \emptyset \text{ if } x\notin\mathrm{dom}_X\Psi\ .$$

As an infinite version of [6; Theorem 39.4], we have

Proposition 1. *The mapping A is a closed convex process from X to Z, i.e.,* graph $A=\{(x, z);\ x\in\mathrm{dom}_X\Psi,\ z\in Ax\}$ *is a closed convex cone in $X\times Z$.*

Proof. It is easy to check that $tz\in A(tx)$ if $z\in Ax$ and $t>0$. Let $x_1, x_2\in \mathrm{dom}_X\Psi$, $z_1\in Ax_1$ and $z_2\in Ax_2$. Since $\Psi(x_1+x_2, 0)\geq\Psi(x_1, 0)+\Psi(x_2, 0)=0$ and $\Psi(\cdot, 0)$ does not take the value $+\infty$, $x_1+x_2\in\mathrm{dom}_X\Psi$. For all $w\in W$, $\Psi(x_1+x_2, w)\geq\Psi(x_1, w)+\Psi(x_2, w)\geq\langle z_1, w\rangle+\langle z_2, w\rangle=\langle z_1+z_2, w\rangle$. Thus $z_1+z_2\in A(x_1+x_2)$ and graph A is a convex cone.

Let $\{(x_\alpha, z_\alpha)\}$ be a net in graph A which converges to (x_0, z_0). Since $\mathrm{dom}_X\Psi$ is closed, $x_0\in\mathrm{dom}_X\Psi$. For all $w\in W$, $\Psi(x_0, w)\geq\limsup\Psi(x_\alpha, w)\geq\limsup\langle z_\alpha, w\rangle=\langle z_0, w\rangle$. Thus $z_0\in Ax_0$ and graph A is closed.

We regard A as a supremum oriented convex process (see [5] or [6]). Then the adjoint A^* of A is defined by $A^*w=\{y\in Y; \langle x, y\rangle\geq\langle z, w\rangle$ for all $(x, z)\in$ graph $A\}$.

Proposition 2. *$A^*w=\partial\Psi_w(0)=\{y\in Y; \langle x, y\rangle\geq\Psi(x, w)$ for all $x\in X\}$ if $w\in\mathrm{dom}_W\Psi$, and $A^*w=\emptyset$ if $w\notin\mathrm{dom}_W\Psi$.*

Proof. Note that $\Psi(x, w)=\sup_{z\in Ax}\langle z, w\rangle=\inf_{y\in\partial\Psi_w(0)}\langle x, y\rangle$ for all $x\in\mathrm{dom}_X\Psi$ and $w\in\mathrm{dom}_W\Psi$ (cf. [5; Lemma 1]). Let $w_0\in\mathrm{dom}_W\Psi$. If $y_0\in\partial\Psi_{w_0}(0)$, then $\langle x, y_0\rangle\geq\Psi(x, w_0)\geq\langle z, w_0\rangle$ for all $(x, z)\in$graph A. Thus $\partial\Psi_{w_0}(0)\subset A^*w_0$. Conversely if $y_0\in A^*w_0$, then $\langle x, y_0\rangle\geq\langle z, w_0\rangle$ for all $x\in\mathrm{dom}_X\Psi$ and $z\in Ax$. Thus $\langle x, y_0\rangle\geq\sup_{z\in Ax}\langle z, w_0\rangle=\Psi(x, w_0)$ for all $x\in\mathrm{dom}_X\Psi$. Since $\Psi(x, w_0)=-\infty$ if $x\notin\mathrm{dom}_X\Psi$, $\langle x, y_0\rangle\geq\Psi(x, w_0)$ for all $x\in X$. Therefore $y_0\in\partial\Psi_{w_0}(0)$ and $A^*w_0=\partial\Psi_{w_0}(0)$.

Let $w_0\notin\mathrm{dom}_W\Psi$. If $y_0\in A^*w_0$, then similarly we see that $\langle x, y_0\rangle\geq\sup_{z\in Ax}\langle z, w_0\rangle=\Psi(x, w_0)=+\infty$ for all $x\in\mathrm{dom}_X\Psi$. This is a contradiction, since $\mathrm{dom}_X\Psi$ is nonempty. Thus $A^*w_0=\emptyset$. This completes the proof.

Corollary. *If $x\in\mathrm{dom}_X\Psi$ or $w\in\mathrm{dom}_W\Psi$, then $\Psi(x, w)=\sup_{z\in Ax}\langle z, w\rangle=\inf_{y\in A^*w}\langle x, y\rangle$.*

In connection with HP and DHP, we consider the following extremum problems defined by the quintuple (A, P, Q, f, g):

(2.2) Find $\hat{M}=\inf\{f(x); x\in\hat{S}\}$,

where $\hat{S}=\{x\in P; (Ax-\partial g(0))\cap Q\neq\emptyset\}$.

(2.3) Find $\hat{M}^*=\sup\{g(w); w\in\hat{S}^*\}$,

where $\hat{S}^*=\{w\in Q^\circ; (\partial f(0)-A^*w)\cap P^\circ\neq\emptyset\}$.

We have

Proposition 3. (1) *$\hat{S}\subset S$, $\hat{S}^*\subset S^*$ and $\hat{M}^*\leq M^*\leq M\leq\hat{M}$.*

(2) *If $Q+\partial g(0)-Ax$ is closed for every $x\in P$, then $\hat{S}=S$.*

(3) *If $P^\circ-\partial f(0)+A^*w$ is closed for every $w\in Q^\circ$, then $\hat{S}^*=S^*$.*

Proof. (1) Let $x\in\hat{S}$. Then there exist $z_1\in Ax$, $z_2\in\partial g(0)$ and $q\in Q$ such that $q=z_1-z_2$. For all $w\in Q^\circ$, $g(w)\leq\langle z_2, w\rangle=\langle z_1-q, w\rangle\leq\langle z_1, w\rangle\leq\Psi(x, w)$. Thus $x\in S$. Similarly we see that $\hat{S}^*\subset S^*$. It is easy to check that $M^*\leq M$. Therefore $\hat{M}^*\leq M^*\leq M\leq\hat{M}$.

(2) We assume that $x\in S$ and $x\notin\hat{S}$. Then $(Ax-\partial g(0))\cap Q=\emptyset$. If Ax is emtpy, then $\Psi(x, 0)=-\infty$. This is impossible since $\Psi(x, 0)\geq g(0)=0$. Thus Ax is nonempty. Since $0\notin Q+\partial g(0)-Ax$, by the separation theorem there exist $w_0\in W$ and $\mu>0$ such that $\langle q+z_1-z_2, w_0\rangle\geq\mu$ for all $q\in Q$, $z_1\in\partial g(0)$ and $z_2\in Ax$. Then $w_0\in Q^\circ$, $\langle z_1, w_0\rangle\geq\mu+\langle z_2, w_0\rangle$ and thus $g(w_0)=\inf_{z_1\in\partial g(0)}\langle z_1, w_0\rangle>\sup_{z_2\in Ax}\langle z_2, w_0\rangle=\Psi(x, w_0)$. This is a contradiction. Thus $\hat{S}\supset S$. By (1), we see that $\hat{S}=S$.

(3) By Proposition 2, we can similarly see that $\hat{S}^*=S^*$.

By the aid of Proposition 3, the following duality theorem for (2.2) and (2.3) is also applicable to HP and DHP in the case where f is $\tau(X, Y)$-continuous on X. See [5; Theorem 1].

Theorem A. *Assume that f is $\tau(X, Y)$-continuous on X and the following two conditions are satisfied:*

(2.4) *The set $G=\{(x, -z, f(x)+r); x\in\mathrm{dom}_X\Psi, z\in Ax, r\geq 0\}+(-P)\times(Q+\partial g(0))\times\{0\}$ is a closed subset of $X\times Z\times R$.*

(2.5) $\hat{S}\neq\emptyset$ or $\hat{S}^*\neq\emptyset$.

Then $\hat{M}=\hat{M}^*$. Furthermore if $\hat{M}$ is finite, then there exists $x_0\in\hat{S}$ such that $f(x_0)=\hat{M}$, i.e., problem (2.2) has an optimal solution.

3. First duality theorem

In this section, we establish a duality theorem by using the method of Rockafellar as in [7].

Theorem 1. *Assume that the following two conditions hold:*

(3.1) $\mathrm{dom}_X\Psi\supset P$ *or* $\mathrm{dom}_W\Psi\supset Q^\circ$.

(3.2) *There exists $w_0\in Q^\circ$ such that $g(w_0)\neq-\infty$ and the $\tau(Y, X)$-interior of $(\partial f(0)-P^\circ-A^*w_0)$ contains the origin. Then $\hat{M}^*=M^*=M$. Furthermore if $S\neq\emptyset$, then HP has an optimal solution.*

Proof. Condition (3.2) implies $\hat{M}^*\neq-\infty$. Since $\hat{M}^*\leq M^*\leq M$, we may assume that $\hat{M}^*$ is finite. We define a convex function Φ on $W\times Y$ by

$$\Phi(w, y)=-g(w)+\delta(w\,|\,Q^\circ)+\delta(y\,|\,\partial f(0)-P^\circ-A^*w),$$

where $\delta(w\,|\,Q^\circ)=0$ for $w\in Q^\circ$ and $\delta(w\,|\,Q^\circ)=+\infty$ for $w\notin Q^\circ$. Then $-\hat{M}^*=$

$\inf\{\Phi(w, 0);\ w\in W\}$. Let Φ^* be the conjugate function of Φ:

$$\Phi^*(z, x) = \sup\{\langle z, w\rangle+\langle x, y\rangle-\Phi(w, y);\ w\in W, y\in Y\}\ ,$$

for $z\in Z$ and $x\in X$. Then

$$\begin{aligned}\Phi^*(0, x) &= \sup\{\langle x, y\rangle+g(w);\ w\in Q^\circ\cap \mathrm{dom}_W\Psi,\ y\in\partial f(0)-P^\circ-A^*w\}\\ &= \sup\{\langle x, y_1\rangle-\langle x, y_2\rangle+(g(w)-\langle x, y_3\rangle);\\ &\qquad w\in Q^\circ\cap\mathrm{dom}_W\Psi,\ y_1\in\partial f(0),\ y_2\in P^\circ,\ y_3\in A^*w\}\ .\end{aligned}$$

In case $x\in S$, $-\langle x, y_2\rangle\leq 0$ and $g(w)-\langle x, y_3\rangle\leq 0$ so that $\Phi^*(0, x)=\sup_{y\in\partial f(0)}\langle x, y\rangle=f(x)$. In case $x\notin P$, $\sup_{y\in P^\circ}-\langle x, y\rangle=+\infty$ so that $\Phi^*(0, x)=+\infty$. We consider the case where $x\in P$ and $\Psi(x, \bar{w})<g(\bar{w})$ for some $\bar{w}\in Q^\circ$. If $\bar{w}\neq\mathrm{dom}_W\Psi$, then $x\in\mathrm{dom}_W\Psi$ by (3.1) so that $\Psi(x, \bar{w})=+\infty$. This is a contradiction. Therefore $\bar{w}\in\mathrm{dom}_W\Psi$ and there exists $\bar{y}\in A^*\bar{w}$ such that $\langle x, \bar{y}\rangle<g(\bar{w})$ by Corollary of Proposition 2. Since $t\bar{y}\in A^*(t\bar{w})$ for all $t>0$, we have $\Phi^*(0, x)=+\infty$. Thus $-M=\sup_{x\in X}-\Phi^*(0, x)$.

Condition (3.2) implies that $\Phi(w_0, y)$ is bounded above by $-g(w_0)$ in a $\tau(Y, X)$-neighborhood of 0. By [2; Proposition 2.5 in Chapter I], we see that $\Phi(w_0, y)$ is continuous in a $\tau(Y, X)$-neighborhood of 0. Thus by [2; Proposition 2.3 in Chapter III], we have $\hat{M}^*=M$ and HP has an optimal solution. Since $\hat{M}^*\leq M^*\leq M$, this completes the proof.

Now we examine condition (3.2). First we define a closed convex process $\tilde{A}$ from X to Z which is obtained by a modification of Ψ. We set $\tilde{\Psi}(x, w)=\Psi(x, w)$ if $x\in P$ and $w\in Q^\circ$, $\tilde{\Psi}(x, w)=+\infty$ if $x\in P$ and $w\notin Q^\circ$, and $\tilde{\Psi}(x, w)=-\infty$ if $x\notin P$. We define $\tilde{A}$ by replacing Ψ by $\tilde{\Psi}$ in (2.1).

Proposition 4. *Assume that Ψ is finite on $P\times Q^\circ$. If the $\tau(Y, X)$-interior* $\mathrm{int}(\partial f(0)-P^\circ)$ *of* $\partial f(0)-P^\circ$ *is nonempty, then the following three conditions are equivalent:*

(3.3) *There exists $w_0\in Q^\circ$ such that $A^*w_0\cap\mathrm{int}(\partial f(0)-P^\circ)\neq\emptyset$.*

(3.4) *There exists $w_0\in Q^\circ$ such that $\Psi(x, w_0)<f(x)$ for all $x\in P$ with $x\neq 0$.*

(3.5) *$x\in P$, $\tilde{A}x\cap Q\neq\emptyset$ and $f(x)\leq 0$ imply $x=0$.*

Proof. First we assume that (3.3) holds. Let $y_0\in A^*w_0\cap\mathrm{int}(\partial f(0)-P^\circ)$ and $x\in P$ with $x\neq 0$. Then there exist $y\in Y$ and $t>0$ such that $\langle x, y\rangle>0$ and $y_0+ty\in\partial f(0)-P^\circ$. Then $y_0+ty=y'-y''$ for some $y'\in\partial f(0)$ and $y''\in P^\circ$. We have $\Psi(x, w_0)\leq\langle x, y_0\rangle=\langle x, y'-y''-ty\rangle\leq\langle x, y'\rangle-t\langle x, y\rangle<f(x)$. Thus (3.4) holds.

Next we assume that (3.4) holds. Let x be an element in P such that $\tilde{A}x\cap Q\neq\emptyset$ and $f(x)\leq 0$. Then for $z\in\tilde{A}x\cap Q$, $\Psi(x, w_0)\geq\langle z, w_0\rangle\geq 0\geq f(x)$. Thus from (3.4) it follows that $x=0$.

Finally we assume that (3.5) holds. If (3.3) does not hold, then $A^*(Q^\circ)\cap \mathrm{int}(\partial f(0)-P^\circ)=\emptyset$. Then by the separation theorem, there exists $x_0\in X$ with $x_0\neq 0$ such that $\langle x_0, y'-y''\rangle\leq 0$ for all $y'\in\partial f(0)$ and $y''\in P^\circ$ and $\langle x_0, y\rangle\geq 0$ for all $w\in Q^\circ\cap \mathrm{dom}_W\Psi$ and $y\in A^*w$. From the first inequality, it follows that $x_0\in P^{\circ\circ}=P$ and $f(x_0)\leq 0$. By the second inequality, we have $\Psi(x_0, w)\geq 0$ for all $w\in Q^\circ$. Thus $0\in\tilde{A}x_0\cap Q$ and this is a contradiction. Hence (3.5) implies (3.3). This completes the proof.

If A is continuous and linear, it is easy to check that $\tilde{A}$ can be replaced by A in (3.5). Thus Proposition 4 is an improvement of [7; Lemma 3.2]. From Theorem 1 and Proposition 4, we have

Corollary. *Assume that g is finite on Q°, Ψ is finite on $P\times Q^\circ$ and $\tau(Y, X)$-interior of $\partial f(0)-P^\circ$ is nonempty. If* (3.5) *holds and $\hat{M}^*$ is finite, then $\hat{M}^*=M^*=M$ and HP has an optimal solution.*

We shall show that $\tilde{A}$ cannot be replaced by A in (3.5) in general.

EXAMPLE. We take $X=Y=R^2$, $Z=W=R^3$, $P=R^2_+=\{(x_1, x_2);\, x_1\geq 0,\, x_2\geq 0\}$ and $Q=\{(z_1, z_2, z_3);\, z_1\leq 0,\, z_2\leq 0,\, -\infty<z_3<+\infty\}$. We set $f(x)=-x_1x_2/(x_1+x_2)$ if $x=(x_1, x_2)\in P$ with $x\neq 0$, $f(x)=0$ if $x=0$, $f(x)=+\infty$ if $x\notin P$ and $g(w)=w_1+w_2$ for all $w=(w_1, w_2, w_3)\in W$. Then $P^\circ=\{(y_1, y_2);\, y_1\geq 0, y_2\geq 0\}$, $Q^\circ=\{(w_1, w_2, w_3);\, w_1\leq 0,\, w_2\leq 0,\, w_3=0\}$ and $\partial g(0)=\{(1, 1, 0)\}$. By the definition of $\partial f(0)$, $(y_1, y_2)\in\partial f(0)$ if and only if $-x_1x_2\geq(x_1y_1+x_2y_2)(x_1+x_2)$ for all positive numbers x_1 and x_2. By setting $t=x_1/x_2$, $(y_1, y_2)\in\partial f(0)$ if and only if $t^2y_1+t(y_1+y_2+1)+y_2\leq 0$ for all $t\geq 0$. From this we easily see that $\partial f(0)=\{(y_1, y_2)\in -R^2_+;\, y_1+y_2+1\leq 0$ or $4y_1y_2\geq(y_1+y_2+1)^2\}$.

Next we set $\Psi(x, w)=-2[(x_1^2w_2+x_2^2w_3)w_1]^{1/2}$ if $x=(x_1, x_2)\in R^2_+$ and $w=(w_1, w_2, w_3)\in -R^3_+$, $\Psi(x, w)=+\infty$ if $x\in R^2_+$ and $w\notin -R^3_+$, and $\Psi(x, w)=-\infty$ if $x\notin R^2_+$.

We show that $Ax=\{(z_1, z_2, z_3);\, z_1z_2\geq x_1^2,\, z_1z_3\geq x_2^2,\, z_1\geq 0\}$ if $x=(x_1, x_2)\in R^2_+$ and $Ax=\emptyset$ if $x\notin R^2_+$. Let $x_1\geq 0$ and $x_2\geq 0$. If $(z_1, z_2, z_3)\in Ax$, then $-2[(x_1^2w_2+x_2^2w_3)w_1]^{1/2}\geq w_1z_1+w_2z_2+w_3z_3$ for all negative numbers w_1, w_2 and w_3. We easily see that $z_1\geq 0$, $z_2\geq 0$ and $z_3\geq 0$. Furthermore we have $\psi(\alpha, \beta)=[(w_1z_1+w_2z_2+w_3z_3)^2-4(x_1^2w_2+x_2^2w_3)w_1]/w_1^2=\alpha^2z_2^2+2\alpha(z_1z_2-2x_1^2+z_1z_3\beta)+\beta^2z_3^2+2\beta(z_1z_3-2x_2^2)+z_1^2\geq 0$ where $\alpha=w_2/w_1$ and $\beta=w_3/w_1$. Since $\psi(\alpha, 0)\geq 0$ for all $\alpha\geq 0$, we have $z_1z_2\geq x_1^2$. Similarly $z_1z_3\geq x_2^2$. Conversely if z_1, z_2 and z_3 are nonnegative, $z_1z_2\geq x_1^2$ and $z_1z_3\geq x_2^2$, then

$$\begin{aligned}(w_1z_1+w_2z_2+w_3z_3)^2&\geq 4w_1w_2z_1z_2+4w_1w_3z_1z_3\\&\geq 4(x_1^2w_2+x_2^2w_3)w_1\end{aligned}$$

for all negative numbers w_1, w_2 and w_3, and thus $(z_1, z_2, z_3)\in Ax$.

Similarly we have $\tilde{A}x=\{(z_1, z_2, z_3);\, z_1z_2\geq x_1^2,\, z_1\geq 0\}$. Thus we see that $x\in P$ and $Ax\cap Q\neq\emptyset$ imply $x=0$, but condition (3.5) is not satisfied.

We can easily see that $A^*w=\hat{A}^*w=\{(y_1,y_2);\ y_1\geq -2(w_1w_2)^{1/2}, y_2\geq 0\}$. Thus $M=M^*=\hat{M}^*=-1$. Since $x=(x_1, x_2)\in S$ if and only if $0\leq x_1\leq 1$ and $x_2\geq 0$, we see that HP has no optimal solution. Finally we note that all the conditions except (3.5) in Corollary hold.

REMARK. Fujimoto's result [3; Theorem 2.1] follows from Proposition 4.

4. Second duality theorem

In this section, we give another duality theorem under the assumption that Ψ is bilinear, $\Psi(x, \cdot)$ is continuous on W for every $x\in X$ and $\Psi(\cdot, w)$ is continuous on X for every $w\in W$. This assumption is equivalent to that the mapping A defined by (2.1) is continuous and linear.

For a closed convex subset C of X, we recall the asymptotic cone ac C of C:

$$\text{ac } C = \bigcap_{t>0} t(C-x), \text{ where } x\in C .$$

In connection with the asymptotic cone, we have two lemmas.

Lemma 1. *Let C and D be closed convex subsets of X. If C is locally compact and* ac $C\cap(-\text{ac } D)$ *is a linear subspace, then $C+D$ is closed.*

This lemma was proved by Zălinescu [8; Proposition 7] in the case where the projection of C to X/X' ($X'=\text{ac } C\cap(-\text{ac } D)$) is locally compact. It suffices to note that the projection of C is locally compact in this case.

Lemma 2. *Assume that $\{w\in Q^\circ;\ g(w)>-\infty\}$ is dense in Q°. Then* ac $\partial g(0)$ *is contained in Q. Furthermore if $Q+\partial g(0)$ is closed, then* $\text{ac}(Q+\partial g(0))=Q$.

Proof. If $Q+\partial g(0)$ is closed, then $\text{ac}(Q+\partial g(0))$ is well-defined. Let $z\in \text{ac}(Q+\partial g(0))$ and $z_0\in\partial g(0)$. Then $tz+z_0\in Q+\partial g(0)$ for all $t>0$. There exist $z_t\in\partial g(0)$ and $q_t\in Q$ such that $tz+z_0=z_t+q_t$. For all $w\in Q^\circ$ and $t>0$, $\langle tz+z_0, w\rangle=\langle z_t+q_t, w\rangle\geq\langle z_t, w\rangle\geq g(w)$. It follows that $\langle z, w\rangle\geq 0$ for all $w\in Q^\circ$ such that $g(w)>-\infty$ and hence for all $w\in Q^\circ$. Thus $z\in Q^{\circ\circ}=Q$. Since $\text{ac}(Q+\partial g(0))\supset Q$, $\text{ac}(Q+\partial g(0))=Q$. Similarly we can check that ac $\partial g(0)\subset Q$.

As the first step toward the second duality theorem, we prove

Lemma 3. *The equality $M=M^*$ holds if the following four conditions are fulfilled:*

(4.1) *P is locally compact and $Q+\partial g(0)$ is closed.*

(4.2) *f is $\tau(X, Y)$-continuous on X and g is finite on Q°.*

(4.3) *$x\in P$, $Ax\in Q$ and $f(x)\leq 0$ imply $x=0$.*

(4.4) $S \neq \emptyset$ *or* $S^* \neq \emptyset$.

Proof. We apply Theorem A to (A, P, Q, f, g). Since f is $\tau(X, Y)$-continuous, $\partial f(0)$ is weakly compact and thus $P^\circ - \partial f(0)$ is closed. By Proposition 3, we see that $S = \hat{S}$ and $S^* = \hat{S}^*$. From (4.4) it follows that condition (2.5) in Theorem A is satisfied.

We set $G_0 = \{(x, -Ax, f(x)+r); x \in X\ r \geq 0\}$. We show that the set $G = G_0 + (-P) \times (Q + \partial g(0)) \times \{0\}$ is closed. By the continuity of A and the lower semicontinuity of f, we easily check that $G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}$ is closed. By (4.2) and Lemma 2, we see that $\mathrm{ac}[G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] = G_0 + \{0\} \times \mathrm{ac}(Q + \partial g(0)) \times \{0\} = G_0 + \{0\} \times Q \times \{0\}$, and by (4.3), we see that $\mathrm{ac}[G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] \cap P \times \{0\} \times \{0\} = \{(0, 0, 0)\}$. From (4.1) and Lemma 1 it follows that $G = [G_0 + \{0\} \times (Q + \partial g(0)) \times \{0\}] + (-P) \times \{0\} \times \{0\}$ is closed and thus condition (2.4) also holds. Thus by Theorem A and Proposition 3, we see that $M = M^*$.

As for the existence of an optimal solution fcr HP, we obtain

Lemma 4. *Assume that* (4.1) *and* (4.3) *are satisfied. If g is finite on Q° and $S \neq \emptyset$, then HP has an optimal solution.*

Proof. We may assume that $M \neq +\infty$. Let $\{x_\alpha\} \subset S$ be a net such that $\{f(x_\alpha)\}$ converges to M. Since P is locally compact, there exists a neighborhood U of the origin of X such that $P \cap U$ is compact. We set

$$K = \{x \in P \cap U;\ x \notin 2^{-1} U^i\},$$

where U^i is the interior of U. Then there exist $t_\alpha > 0$ and $\bar{x}_\alpha \in K$ such that $x_\alpha = t_\alpha \bar{x}_\alpha$. Since K is compact, there exists a subnet of $\{\bar{x}_\alpha\}$ which converges to an element $\bar{x} \in K$. We may assume that $\{\bar{x}_\alpha\}$ converges to $\bar{x}$. We show that there exists a subnet of $\{t_\alpha\}$ which converges to a real number $t_0 \geq 0$. Otherwise, $\lim t_\alpha = +\infty$. Let $z_0 \in \partial g(0)$ and $s > 0$. Then $sA(t_\alpha^{-1} x_\alpha) + (1 - s t_\alpha^{-1}) z_0 \in Q + \partial g(0)$ for all α such that $s t_\alpha^{-1} < 1$. Since $Q + \partial g(0)$ is closed, $\lim \{sA(t_\alpha^{-1} x_\alpha) + (1 - s t_\alpha^{-1}) z_0\} = sA\bar{x} + z_0 \in Q + \partial g(0)$. Thus we see $A\bar{x} \in \mathrm{ac}(Q + \partial g(0)) = Q$ by Lemma 1. Since $f(\bar{x}) \leq \liminf f(\bar{x}_\alpha) = \liminf t_\alpha^{-1} f(x_\alpha) \leq 0$, from (4.3) it follows that $\bar{x} = 0$. This is a contradiction, since $0 \notin K$. Thus $\{t_\alpha\}$ contains a convergent subnet. Denote the subnet by $\{t_\alpha\}$ again and let t_0 be its limit. Then $\lim x_\alpha = \lim t_\alpha \bar{x}_\alpha = t_0 \bar{x}$ and $M = \lim f(x_\alpha) \geq f(t_0 \bar{x})$. Since S is closed, $t_0 \bar{x} \in S$ and $f(t_0 \bar{x}) \geq M$. Thus $M = f(t_0 \bar{x})$ and HP has an optimal solution. This completes the proof.

Now we prove the second duality theorem.

Theorem 2. *Assume that* (4.3), (4.4) *in Lemma* 3 *and the following* (4.1′) *and* (4.2′) *are satisfied:*

(4.1′) *P is locally compact and $P^\circ - \partial f(0)$, $Q + \partial g(0)$ are closed.*

(4.2′) *g is finite on Q°.*

Then $M=M^*$. Furthermore if $S\neq\emptyset$, then HP has an optimal solution.

Proof. For arbitrary $y_1, \cdots, y_k \in \partial f(0)$, the function $h(x)=\max\{\langle x, y_j\rangle; j=1, \cdots, k\}$ is continuous and sublinear on X. By J we denote the set of all such functions. Then J is directed by a natural ordering and increases to f at each point in X. For each $h\in J$, we set

$$M_h = \inf\{h(x);\, x\in S\}\ ,$$
$$S_h^* = \{w\in Q^\circ;\, \langle Ax, w\rangle \leq h(x) \quad \text{for all } x\in P\}\ ,$$
$$M_h^* = \sup\{g(w);\, w\in S_h^*\}\ .$$

By Proposition 3, we see that $S_h^*=\{w\in Q^\circ; A^*w\in\partial h(0)-P^\circ\}$ and $S^*=\{w\in Q^\circ; A^*w\in\partial f(0)-P^\circ\}$. Since $\{\partial h(0); h\in J\}$ is an increasing net of sets and $\cup_{h\in J}\partial h(0)=\partial f(0)$, $\{S_h^*; h\in J\}$ increases to S^* and $\lim_{h\in J}M_h^*=M^*$.

In order to show that $\lim_{h\in J}M_h=M$ and $M_h=M_h^*$ for all sufficiently large $h\in J$, we examine condition (4.3). Condition (4.3) is equivalent to the condition that $f(x)>0$ for all $x\in P\cap A^{-1}(Q)$ such that $x\neq 0$. Let K be the set as in the proof of Lemma 4. Then $f(x)>0$ for all $x\in K\cap A^{-1}(Q)$. Since f is lower semicontinuous and $K\cap A^{-1}(Q)$ is compact, $\inf_{x\in K\cap A^{-1}(Q)}f(x)>\mu$ for $\mu>0$. For any $x_0\in K\cap A^{-1}(Q)$ there exists $y_0\in\partial f(0)$ such that $\langle x_0, y_0\rangle>\mu$, since $\sup_{x\in\partial f(0)}\langle x, y\rangle=f(x)$ for all $x\in X$. Since $\langle\cdot, y_0\rangle$ is continuous on X, there exists a neighborhood V_0 of x_0 such that $\langle x, y_0\rangle>\mu$ for all $x\in V_0$. Since $K\cap A^{-1}(Q)$ is compact, there exist $x_1, \cdots, x_n\in K$, $y_1^0, \cdots, y_n^0\in\partial f(0)$ and $V_1, \cdots, V_n$ such that V_j is a neighborhood of x_j, $\langle x, y_j^0\rangle>\mu$ for each j and $x\in V_j$, and $\cup_{j=1}^n V_j\supset K\cap A^{-1}(Q)$. Then $h_0(x)=\max\{\langle x, y_j^0\rangle; j=1, \cdots, n\}>\mu$ on $K\cap A^{-1}(Q)$. We set $J'=\{h\in J; h\geq h_0\}$. Then $x\in P$, $Ax\in Q$ and $h(x)\leq 0$ imply $x=0$ for all $h\in J'$.

First we assume that $S\neq\emptyset$. Then from Lemmas 3 and 4 it follows that there exists a net $\{x_h; h\in J'\}\subset S$ such that $h(x_h)=M_h=M_h^*$ for all $h\in J'$. If $\lim_{h\in J'}h(x_h)=+\infty$, then $M=M^*=+\infty$. So we may assume that $\lim_{h\in J'}h(x_h)$ is finite. Then as in the proof of Lemma 4, we see that there exists a subnet of $\{x_h\}$ which converges to an element $x_0\in S$. We may assume that $\{x_h; h\in J'\}$ converges to x_0. If we fix an arbitrary $h_1\in J'$, then $h_1(x_0)=\lim_{h\in J'}h_1(x_h)\leq\lim_{h\in J'}h(x_h)=\lim_{h\in J'}M_h^*=M^*$. Thus $M\leq f(x_0)=\sup_{h\in J'}h(x_0)\leq M^*$. Since $M\geq M^*$, $M=M^*=f(x_0)$.

Next we assume that $S=\emptyset$ and $S^*\neq\emptyset$. Then $M_h=M_h^*=+\infty$ for $h\in J'$ such that $S_h^*\neq\emptyset$. Since $M_h^*\leq M^*\leq M$, we have $M=M^*=+\infty$. This completes the proof.

In the finite dimensional case, we can omit condition (4.1′) in Theorem 2. To prove this we prepare

Lemma 5. *Assume that X, Y, Z and W are all finite dimensional spaces and set* $\operatorname{dom} g = \{w \in W; g(w)$ *is finite*$\}$. *Then* $(\operatorname{dom} g)^\circ + \partial g(0)$ *is closed. Similarly* $(\operatorname{dom} f)^\circ - \partial f(0)$ *is also closed.*

Proof. Let $z \in (\operatorname{dom} g)^\circ$ and $z_0 \in \partial g(0)$. Since $\langle tz + z_0, w \rangle \geq \langle z_0, w \rangle \geq g(w)$ for all $w \in \operatorname{dom} g$ and $t > 0$, $z \in \operatorname{ac} \partial g(0)$. From Lemma 2 it follows that $\operatorname{ac} \partial g(0) = (\operatorname{dom} g)^\circ$. Hence $(-\operatorname{dom} g)^\circ \cap \operatorname{ac} \partial g(0) = (-\operatorname{dom} g)^\circ \cap (\operatorname{dom} g)^\circ$ and this is a linear subspace. Since Z and W are finite dimensional, $\partial g(0)$ is locally compact, so that we see by Lemma 1 that $(\operatorname{dom} g)^\circ + \partial g(0)$ is closed. The last statement can be similarly proved.

Corollary. *Assume that X, Y, Z and W are finite dimensional spaces and that conditions* (4.2′), (4.3) *and* (4.4) *are satisfied. Then $M = M^*$. Furthermore if $S \neq \emptyset$, then HP has an optimal solution.*

Proof. Let $\tilde{P}$ be the closure of $\{x \in P; f(x) < +\infty\}$. We set $\tilde{f}(x) = f(x)$ for $x \in \tilde{P}$, $\tilde{f}(x) = +\infty$ for $x \notin \tilde{P}$, $\tilde{g}(w) = g(w)$ for $u \in Q^\circ$ and $\tilde{g}(w) = -\infty$ for $w \notin Q^\circ$. Then $\tilde{P}$ is the closure of $\operatorname{dom} \tilde{f}$ and Q° is $\operatorname{dom} \tilde{g}$. Thus by Lemma 5, we see that $\tilde{P}^\circ - \partial \tilde{f}(0)$ and $Q + \partial \tilde{g}(0)$ are closed. By applying Theorem 2 to $(A, \tilde{P}, Q, \tilde{f}, \tilde{g})$, we complete the proof.

This is a more precise version of Corollary of Theorem 1 and an improvement of [7; Theorem 3.1]. In this corollary the assumption that all spaces are finite dimensional cannot be omitted. See [4; Example 3.1]. By the following example, we observe that condition (4.2′) cannot be omitted either.

Example. We take $X = Y = Z = W = R^2$, $P = R^2_+$ and $Q = \{(0, 0)\}$. We set $\Psi(x, w) = x_1 w_1 + x_2 w_2$ and $f(x) = x_1$ for $x = (x_1, x_2) \in X$ and $w = (w_1, w_2) \in W$, $g(w) = 2(w_1 w_2)^{1/2}$ for $(w_1, w_2) \in R^2_+$ and $g(w) = -\infty$ for $(w_1, w_2) \notin R^2_+$. Then A is the identity mapping from X to Z so that condition (4.3) is satisfied. Since $x = (x_1, x_2) \in S$ if and only if $x_1 x_2 \geq 1$ and $x_1 > 0$, $M = \inf\{x_1; x \in S\} = 0$ and HP has no optimal solution.

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Added in proof. Recently the author noticed that Lemma I was also proved in [9] J. Gwinner: *Closed images of convex multivalued mapping in linear topological spaces with applications*, J. Math. Anal. Appl. **60** (1977), 75–86.

Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan