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Author(s)	Nagase, Michihiro; Yoshida, Manabu
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Osaka University

COMMUTATOR ESTIMATES AND A SHARP FORM OF GÅRDING'S INEQUALITY

MICHIHIRO NAGASE AND MANABU YOSHIDA

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0. Introduction

In the present paper we show a commutator estimate of pseudo-differential operators in the framework of $L^2(\mathbb{R}^n)$. As an application we give a sharp form of Gårding's inequality for sesqui-linear forms with coefficients in \mathcal{B}^2 . There has been similar kinds of commutator estimates. In [5], Kumano-go and Nagase obtain a result on commutator estimates and used it to show a sharp form of Gårding's inequality for sesqui-linear form defined by elliptic differential operators of the form

$$B[u, v] = \sum_{|\alpha| \leq m, |\beta| \leq m} (a_{\alpha\beta}(x) D_x^\alpha u, D_x^\beta v)$$

where the coefficients $a_{\alpha\beta}(x)$ are $\mathcal{B}^2(\mathbb{R}^n)$ functions.

In [3], Koshiya shows a sharp form of Gårding's inequality for the form

$$B[u, v] = (p(X, D_x)u, v)$$

where the symbol $p(x, \xi)$ of the operator $p(X, D_x)$ is \mathcal{B}^2 smooth in space variable x and homogeneous in covariable ξ , and used the sharp form of Gårding's inequality to the study of the stability of difference schemes for hyperbolic initial problems. On the other hand in [2], N. Jacob shows Gårding's inequality for the form

$$B[u, v] = \sum_{i,j=1}^m \int_{\mathbb{R}^n} \overline{a_{i,j}(x) Q_j(D) u(x)} P_i(D) v(x) dx$$

where $P_i(D)$ and $Q_j(D)$ are pseudo-differential operators, and $a_{i,j}(x)$ are non-smooth functions. The symbol class of the present paper is similar to the one in [2].

In section 1, as a preliminary we give definitions and fundamental facts of pseudo-differential operators. In section 2 we treat commutator estimates and give the main theorem relative to the commutator estimate. Finally in section 3 we give the sharp form of Gårding's inequalities for our class of operators.

1. Preliminaries

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be multi-integers. We denote

$$|\alpha| = \alpha_1 + \dots + \alpha_n$$

We denote n -dimensional partial differential operators by

$$\partial_\xi = \left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n}\right) \quad \text{and} \quad D_x = \frac{1}{i} \partial_x = \frac{1}{i} \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

Then for a function $f(x, \xi)$, we denote

$$\partial_\xi^\alpha D_x^\beta f(x, \xi) = f_{(\beta)}^{(\alpha)}(x, \xi)$$

and

$$\partial_\xi^\alpha D_x^\beta D_{x'}^{\beta'} f(x, \xi, x') = f_{(\beta, \beta')}^{(\alpha)}(x, \xi, x')$$

for a function $f(x, \xi, x')$. We denote by $\mathcal{B}^k = \mathcal{B}^k(\mathbb{R}^n)$ the set of k -times continuously differentiable functions on \mathbb{R}^n which are bounded with all upto k -th derivatives. We denote by $C_0^\infty(\mathbb{R}^n)$ the set of C^∞ -smooth functions with compact support. Moreover \mathcal{S} denotes the Schwartz space of rapidly decreasing functions on \mathbb{R}^n . Let λ be a real valued smooth function on \mathbb{R}^n satisfying

- (i) $\lambda(\xi) \geq 1$
- (ii) $|\lambda^{(\alpha)}(\xi)| \leq C_\alpha \lambda(\xi)^{1-|\alpha|}$

for any α . Then we say that the function $\lambda(\xi)$ is a basic weight function(see [5]).

Let $\lambda(\xi)$ be a basic weight function. Then we say that a function $p(x, \xi, x')$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ belongs to $S_{\rho, \delta, \lambda}^m$ if

$$|p_{(\beta, \beta')}^{(\alpha)}(x, \xi, x')| \leq C_{\alpha, \beta, \beta'} \lambda(\xi)^{m - \rho|\alpha| + \delta|\beta + \beta'|}$$

for any multi-integers α, β and β' . For any $p(x, \xi, x')$ in $S_{\rho, \delta, \lambda}^m$, we define the pseudo-differential operator $p(X, D_x, X')$ by

$$p(X, D_x, X')u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-x') \cdot \xi} p(x, \xi, x') u(x') dx' d\xi$$

for any u in \mathcal{S} . In the present paper the integrations \int are taken on \mathbb{R}^n . In particular if $p(x, \xi, x') \in S_{\rho, \delta, \lambda}^m$ is independent in x' , that is, $p(x, \xi) \in S_{\rho, \delta, \lambda}^m$, the operator $p(X, D_x)$ is defined, as usual, by

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) d\xi$$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is,

$$\hat{u}(\xi) = \int e^{-ix' \cdot \xi} u(x') dx'$$

For any functions $f(x)$ and $g(x)$ on \mathbb{R}^n , we define the inner product of $L^2(\mathbb{R}^n)$ by

$$(f, g) = \int f(x) \overline{g(x)} dx$$

and denote the usual L^2 norm of function $f(x)$ by

$$\|f\| = \left\{ \int |f(x)|^2 dx \right\}^{\frac{1}{2}}$$

For any real number s and $u \in \mathcal{S}$, we define the norm $\|u\|_{s,\lambda}$ by

$$\begin{aligned} \|u\|_{s,\lambda} &= \left\{ \int |\lambda(D_x)^s u(x)|^2 dx \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{1}{(2\pi)^n} \int |\lambda(\xi)^s \hat{u}(\xi)|^2 d\xi \right\}^{\frac{1}{2}} \end{aligned}$$

In particular if $s = 0$, the norm $\|\cdot\|_{s,\lambda}$ coincides with usual L^2 norm $\|\cdot\|$.

The space $H_{s,\lambda}$ is defined by the completion of the space \mathcal{S} by the norm $\|\cdot\|_{s,\lambda}$. It is not difficult to see that the space $H_{s,\lambda}$ is a Hilbert space.

Let s and m be real numbers. For a symbol $p(x, \xi, x')$ in $S_{\rho,\delta,\lambda}^m$, we have

$$\|p(X, D_x, X')u\|_{s,\lambda} \leq C \|u\|_{s+m,\lambda}$$

for any u in \mathcal{S} (see [5]).

2. Estimates of commutators

Let us consider commutators of pseudo-differential operators in $L^2(\mathbb{R}^n)$. The estimates is essential for the proof of sharp Gårding's inequality. However the estimates itself are interesting subject.

Let $0 \leq \delta < 1$ and $\lambda(\xi)$ be a basic weight function and $\phi(x)$ be an even function in $C_0^\infty(\mathbb{R}^n)$ satisfying $\int \phi(x) dx = 1$. For a function $b(x)$ on \mathbb{R}^n , we define

$$\tilde{b}(x, \xi) = \int \phi(z) b(x - \lambda(\xi)^{-\delta} z) dz$$

Then in [5], the following approximation theorem is shown.

Lemma 2.1. *If $b(x)$ is a bounded function, then $\tilde{b}(x, \xi)$ belongs to $S_{1,\delta,\lambda}^0$.*

(i) *If $b(x)$ is a function in \mathcal{B}^1 , then $\tilde{b}_{(\alpha)}(x, \xi)$ belongs to $S_{1,\delta,\lambda}^0$ for $|\alpha| \leq 1$ and we have*

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\| \leq C\|u\|_{-\delta,\lambda}$$

for any u in \mathcal{S} .

(ii) *If $b(x)$ is a function in \mathcal{B}^2 , then $\tilde{b}_{(\alpha)}(x, \xi)$ belongs to $S_{1,\delta,\lambda}^0$ for $|\alpha| \leq 2$ and we have*

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\| \leq C\|u\|_{-2\delta,\lambda}$$

for any u in \mathcal{S} .

From Lemma 2.1 we can prove the following lemma 2.2.

Lemma 2.2. (i) *If $b(x)$ is in \mathcal{B}^1 , then we have*

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\|_{\delta,\lambda} \leq C\|u\|$$

for any u in \mathcal{S} .

(ii) *If $b(x)$ is a \mathcal{B}^2 , then we have*

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\|_{2\delta,\lambda} \leq C\|u\|$$

for any u in \mathcal{S} .

Proof. We prove (i), and (ii) can be shown in a similar way.

For any u and v in \mathcal{S} , we have

$$(\{b(X) - \tilde{b}(X, D_x)\}u, v) = (u, \{\bar{b}(X) - \bar{b}(D_x, X')\}v)$$

where $\bar{b}(\xi, x') = \overline{\tilde{b}(x', \xi)}$. Then by using the asymptotic expansion formula of pseudo-differential operators (see, for example [4]), we have

$$\bar{b}(D_x, X') = \tilde{b}(X, D_x) + b_1(X, D_x)$$

where

$$b_1(x, \xi) = \sum_{j=1}^N \sum_{|\alpha|=j} \frac{1}{\alpha!} \tilde{b}_{(\alpha)}^{(j)}(x, \xi) + R_N(x, \xi) \quad \text{and} \quad R_N(x, \xi) \in S_{1,\delta,\lambda}^{-N(1-\delta)}$$

Since $b(x) \in \mathcal{B}^1$, we can see by Lemma 2.1 (i) that

$$\tilde{b}_{(\alpha)}^{(\alpha)}(x, \xi) \in S_{1, \delta, \lambda}^{-(1-\delta)|\alpha|-\delta} \quad \text{for } |\alpha| \neq 0$$

Hence taking N sufficiently large we can see

$$b_1(x, \xi) \in S_{1, \delta, \lambda}^{-1}$$

Using Lemma 2.1 (i) and the boundedness of pseudo-differential operators we have

$$\begin{aligned} \|\{\bar{b}(X) - \bar{b}(D_x, X')\}v\| &\leq C\|\{b(X) - \tilde{b}(X, D_x)\}v\| + \|b_1(X, D_x)v\| \\ &\leq C\|v\|_{-\delta, \lambda} \end{aligned}$$

Therefore by Schwarz inequality and duality argument of the spaces $H_{s, \lambda}$, we have the estimate. □

In order to show the main estimate in this section, the following theorem plays an essential role.

Theorem 2.3. *Let $b(x)$ be a function in \mathcal{B}^2 , and let $0 \leq \delta < 1$. For a basic weight function $\lambda(\xi)$ we define a symbol $\tilde{b}(x, \xi)$ by*

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

where $\phi(x)$ is an even function in \mathcal{S} with $\int \phi(x)dx = 1$. Then for any $s \in [0, 2\delta]$ we have

$$\|\{b(X) - \tilde{b}(X, D_x)\}u\|_{s, \lambda} \leq C\|u\|_{s-2\delta, \lambda}$$

for any u in \mathcal{S} .

Proof. For the proof, we use the three line theorem in complex analysis.

Let u and v be functions in \mathcal{S} and we consider the complex function

$$f(z) = (\lambda(D_x)^{2\delta(1-z)})\{b(X) - \tilde{b}(X, D_x)\lambda(D_x)^{2\delta z}u, v\}$$

Since u and v are in \mathcal{S} and $\lambda(\xi) \geq 1$, it is clear that the function $f(z)$ is holomorphic in the complex $z = \sigma + i\tau$ -plane \mathbb{C} . Since the symbol $\lambda(\xi)^z$ is in $S_{1, 0, \lambda}^{\text{Re}z}$, independent of x and $|\lambda(\xi)^{i\tau}| = 1$, we can see from the Lemma 2.1 and 2.2 that

$$\begin{aligned} |f(i\tau)| &\leq \|\lambda(D_x)^{2\delta}\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2i\delta\tau}u\| \|v\| \leq C\|u\| \|v\| \\ |f(1+i\tau)| &\leq \|\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2\delta(1+i\tau)}u\| \|v\| \leq C\|u\| \|v\| \end{aligned}$$

Hence from the three line theorem (see [7]), we have

$$|f(\sigma)| \leq C\|u\| \|v\|$$

for $0 < \sigma < 1$. Taking $\sigma = 1 - \frac{s}{2\delta}$, we have

$$|(\lambda(D_x)^s \{b(X) - \tilde{b}(X, D_x)\} \lambda(D_x)^{-s+2\delta} u, v)| \leq C\|u\| \|v\|$$

for any u and v in \mathcal{S} . Hence using the usual duality argument, we can get the inequality. □

Theorem 2.3 implies the following estimate.

Theorem 2.4. *Let $b(x)$ be a function in \mathcal{B}^2 and let $a(\xi)$ in $S_{1,0,\lambda}^s$ with $0 < s < 1$. Then we have*

$$(2.1) \quad \|[a(D_x), b(X)]u\|_{0,\lambda} \leq C\|u\|_{s-1,\lambda}$$

for any u in \mathcal{S} .

Proof. We take an even function $\phi(x)$ in $C_0^\infty(\mathbb{R}^n)$ such that $\int \phi(x)dx = 1$. For $b(x)$ we define a symbol $\tilde{b}(x, \xi)$ by

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

for $s < \delta = \frac{1+s}{2} (< 1)$. We write

$$\begin{aligned} [a(D_x), b(X)]u &= a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x) \\ &\quad + a(D_x)\tilde{b}(X, D_x)u(x) - \tilde{b}(X, D_x)a(D_x)u(x) \\ &\quad + \{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x) \end{aligned}$$

Since $a(\xi) \in S_{1,0,\lambda}^s$, we can see that the first term can be estimated by

$$\|a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)\|_{0,\lambda} \leq C\|\lambda(D_x)^s\{b(X) - \tilde{b}(X, D_x)\}u(x)\|_{0,\lambda}$$

and therefore by Theorem 2.3 we have

$$(2.2) \quad \begin{aligned} \|[a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)]\|_{0,\lambda} &\leq C\|u\|_{s-2\delta,\lambda} \\ &\leq C\|u\|_{s-1,\lambda} \end{aligned}$$

The third term is estimated by Lemma 2.1(ii) and we have

$$(2.3) \quad \begin{aligned} \|\{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x)\| &\leq C\|a(D_x)u\|_{-2\delta, \lambda} \\ &\leq C\|a(D_x)u\|_{s-1, \lambda} \end{aligned}$$

The second term is estimated by the usual asymptotic expansion formula for pseudo-differential operators (see [4]), that is, we have

$$a(D_x)\tilde{b}(X, D_x) = b_L(X, D_x)$$

and

$$\begin{aligned} b_L(x, \xi) &\sim \tilde{b}(x, \xi)a(\xi) + \sum_{j=1}^{\infty} b_j(x, \xi) \\ b_j(x, \xi) &= \sum_{|\alpha|=j} \frac{1}{a!} \tilde{b}_{(\alpha)}(x, \xi)a^{(\alpha)}(\xi) \end{aligned}$$

Since the symbols $\tilde{b}_{(\alpha)}(x, \xi)$ belong to $S_{1, \delta, \lambda}^0$ for $|\alpha| \leq 2$, we can see that $b_1(x, \xi) \in S_{1, \delta, \lambda}^{s-1}$ and $b_j(x, \xi) \in S_{1, \delta, \lambda}^{s-j+(j-2)\delta}$ for $j \geq 2$. Hence we can write

$$a(D_x)\tilde{b}(X, D_x)u - \tilde{b}(X, D_x)a(D_x)u = B_1(X, D_x)u$$

where $B_1(x, \xi)$ belongs to $S_{1, \delta, \lambda}^{s-1}$. Therefore we have

$$(2.4) \quad \begin{aligned} \|[a(D_x)\tilde{b}(X, D_x) - \tilde{b}(X, D_x)a(D_x)]u(x)\| &\leq \|B_1(X, D_x)u\| \\ &\leq C\|u(x)\|_{s-1, \lambda} \end{aligned}$$

From the estimates (2.2), (2.3) and (2.4), we have the estimate (2.1). □

In particular we have

Corollary 2.5. *Let $b(x)$ be a function in B^2 and let $a(\xi)$ be in $S_{1, 0, \lambda}^{\frac{1}{2}}$. Then we have*

$$\|[a(D_x), b(X)]u\|_{0, \lambda} \leq C\|u\|_{-\frac{1}{2}, \lambda}$$

for any u in \mathcal{S} .

REMARK 1. We note that if the basic weight function $\lambda(\xi) = \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$, we can get more general results for $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) than the ones in this section. Because in case of $\langle \xi \rangle$, we can use the kernel representations of the operators(see [5]). We can see sharper results in [1] than in [5] in the case $\lambda(\xi) = \langle \xi \rangle$.

3. A sharp form of Gårding's inequality

Let us begin with the following inequality, which we can say a sharp form of Gårding's inequality.

Theorem 3.1. *Let $\delta < 1$. We assume that a symbol $p(x, \xi)$ in $S_{1,\delta,\lambda}^m$ satisfies that $p_{(\beta)}(x, \xi)$ are in $S_{1,\delta,\lambda}^m$ for $|\beta| \leq 2$ and*

$$\operatorname{Re} p(x, \xi) \geq 0$$

for some constant c_0 . Then we have

$$\operatorname{Re} (p(X, D_x)u, u) \geq -C \|u\|_{\frac{m-1}{2}, \lambda}^2$$

for any u in \mathcal{S} .

For the proof of this theorem 3.1 we use the following lemma

Lemma 3.2. (see [5]) *Let τ be a real number. Let $\psi(x)$ be a infinitely smooth function on \mathbb{R}^n , and let $\lambda(\xi)$ be a basic weight function. Then for any α we have*

$$\partial_\xi^\alpha \{ \psi(\lambda(\xi)^\tau x) \} = \sum_{|\alpha'| \leq |\alpha|} \phi_{\alpha', \alpha}(\xi) \{ \lambda(\xi)^\tau x \}^{\alpha'} \psi^{(\alpha')}(\lambda(\xi)^\tau x)$$

where $\phi_{\alpha', \alpha}(\xi)$ belong to $S_{\lambda, 1, 0}^{-|\alpha|}$ for all α' with $|\alpha'| \leq |\alpha|$.

Proof of Theorem 3.1. First we note that we can assume that the symbol $p(x, \xi)$ be real-valued. In fact, if a real-valued symbol $r(x, \xi) \in S_{1,\delta,\lambda}^m$ satisfies the same assumption of $p(x, \xi)$ in Theorem 3.1, then we have

$$\operatorname{Re}(ir(X, D_x)u, u) = \frac{1}{2} \operatorname{Im}(\{r(X, D_x) - r^*(X, D_x)\}u, u)$$

Since the symbol $r(x, \xi)$ is real-valued, by using the expansion formula for the symbol of the formal adjoint operator $r^*(X, D_x)$ we have

$$r^*(x, \xi) \sim r(x, \xi) + \sum_{j=1}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} r_{(\alpha)}^{(\alpha)}(x, \xi)$$

Therefore using the assumption we have

$$r(x, \xi) - r^*(x, \xi) = R(x, \xi) \in S_{1,\delta,\lambda}^{m-1}$$

From this relation we have

$$\begin{aligned} |\operatorname{Re}(ir(X, D_x)u, u)| &= \frac{1}{2} |\operatorname{Im}(\{r(X, D_x) - r^*(X, D_x)\}u, u)| \\ &= \frac{1}{2} |(R(X, D_x)u, u)| \\ &\leq C \|u\|_{\frac{m-1}{2}, \lambda}^2 \end{aligned}$$

Now we assume that the symbol $p(x, \xi)$ is real and non-negative. We take an even and real-valued function $\psi(x)$ in \mathcal{S} such that $\int \psi(x)^2 dx = 1$ and we put

$$p_G(x, \xi, x') = \int \psi(\lambda(\xi)^{\frac{1}{2}}(x-z)) \psi(\lambda(\xi)^{\frac{1}{2}}(x'-z)) p(z, \xi) dz \lambda(\xi)^{\frac{n}{2}}$$

Then using the lemma 3.2 we can see that $p_G(x, \xi, x')$ is in $S_{1, \frac{1}{2}, \lambda}^m$ and changing the order of integrations we have

$$\operatorname{Re}(p_G(X, D_x, X')w, w) \geq 0$$

for any $w \in \mathcal{S}$ (see [6]). Moreover by using the formula of simplified symbols in [4] for the operator with double symbols we can see that the operator $p_G(X, D_x, X')$ can be written asymptotically as

$$p_G(X, D_x, X') \sim \sum_{j=0}^{\infty} p_j(X, D_x)$$

where $p_j(x, \xi)$ is in $S_{1, \frac{1}{2}, \lambda}^{m-\frac{j}{2}}$ for any j and has the form

$$p_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} p_{G,(\alpha)}^{(\alpha)}(x, \xi, x)$$

In particular, $p_0(x, \xi)$ can be written as

$$\begin{aligned} p_0(x, \xi) &= p_G(x, \xi, x) \\ &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}(x-z))^2 p(z, \xi) dz \\ &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p(x-z, \xi) dz \\ &= \int \psi(z)^2 p(x - \lambda(\xi)^{-\frac{1}{2}}z, \xi) dz \end{aligned}$$

By using the Taylor expansion for the second expression, we have

$$p(x - z, \xi) = p(x, \xi) + \sum_{|\beta|=1} ip_{(\beta)}(x, \xi)z^\beta + R_2(z, x, \xi)$$

where the remainder term $R_2(z, x, \xi)$ is

$$R_2(z, x, \xi) = \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t)z^\beta p_{(\beta)}(x-tz, \xi) dt$$

Since $\psi(z)$ is an even function we see that

$$\int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 z^\beta dz = 0$$

for $|\beta| = 1$. Therefore we have

$$\begin{aligned} p_0(x, \xi) &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p(x, \xi) dz + \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 R_2(z, x, \xi) dz \\ &= p(x, \xi) + \lambda(\xi)^{\frac{n}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t) \int z^\beta \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p_{(\beta)}(x-tz, \xi) dz \\ &= p(x, \xi) + r_2(x, \xi) \end{aligned}$$

From the assumption of the symbol $p(x, \xi)$, the symbols $p_{(\beta)}(x - tz, \xi)$ belong to $S_{\lambda,1,\delta}^m$ for $|\beta| = 2$. Hence using Lemma 3.2, we can see that

$$r_2(x, \xi) = \lambda(\xi)^{\frac{n}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t) \int z^\beta \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p_{(\beta)}(x-tz, \xi) dz$$

is in $S_{\lambda,1,\delta}^{m-1}$. Thus we can write

$$p_0(x, \xi) = p(x, \xi) + r_2(x, \xi)$$

with symbol $r_2(x, \xi)$ in $S_{\lambda,1,\delta}^{m-1}$.

Similarly for $|\alpha| = 1$, since

$$p_{G,(\alpha)}^{(\alpha)}(x, \xi, x) = \partial_\xi^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z) \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z) p(x-z, \xi) dz \}$$

we can see that

$$p_{G,(\alpha)}^{(\alpha)}(x, \xi, x) = \partial_\xi^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z) \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z) \{ p(x, \xi) + R_1(z, x, \xi) \} dz \}$$

where the remainder term $R_1(z, x, \xi)$ is

$$R_1(z, x, \xi) = \sum_{|\beta|=1} i \int_0^1 (1-t)z^\beta p_{(\beta)}(x-tz, \xi)dt$$

Since $\int \psi(z)\psi^{(\alpha)}(z)dz = 0$ for $|\alpha| = 1$, we can see that

$$\begin{aligned} p_{G,(\alpha)}^{(\alpha)}(x, \xi, x) &= \partial_\xi^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)R_1(z, x, \xi) dz \} \\ &= \sum_{|\beta|=1} i \int_0^1 (1-t)dt \\ &\quad \times \partial_\xi^\alpha \{ \lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)z^\beta p_{(\beta)}(x-tz, \xi) dz \} \end{aligned}$$

for $|\alpha| = 1$. In a similar way to the estimate of $R_2(z, x, \xi)$, we can see from Lemma 3.2 and the assumption of the symbol $p(x, \xi)$ that $p_{1,\alpha}(x, \xi) = p_{G,(\alpha)}^{(\alpha)}(x, \xi, x)$ belongs to $S_{\lambda,1,\delta}^{m-1}$ for $|\alpha| = 1$. Therefore we can see that

$$p_1(x, \xi) \in S_{\lambda,1,\delta}^{m-1}$$

Thus we can write

$$p_G(X, D_x, X') = p(X, D_x) + R(X, D_x) + Q(X, D_x)$$

where $Q(x, \xi) \in S_{\lambda,1,\delta}^{m-1}$ and $R(x, \xi) \in S_{\lambda,1,\frac{1}{2}}^{m-1}$. Now from the L^2 -boundedness theorems and the algebra of pseudo-differential operators with symbols in $\cup_{m \in \mathbb{R}} S_{\lambda,1,\delta}^m$ for any δ with $0 \leq \delta < 1$, we can see that

$$\begin{aligned} |(Q(X, D_x)u, u)| &\leq \|Q(X, D_x)u\|_{-\frac{m+1}{2}, \lambda} \|u\|_{\frac{m-1}{2}, \lambda} \leq C \|u\|_{\frac{m-1}{2}, \lambda}^2 \\ |(R(X, D_x)u, u)| &\leq \|R(X, D_x)u\|_{-\frac{m+1}{2}, \lambda} \|u\|_{\frac{m-1}{2}, \lambda} \leq C \|u\|_{\frac{m-1}{2}, \lambda}^2 \end{aligned}$$

Therefore we have

$$\begin{aligned} \text{Re}(p(X, D_x)u, u) &= \text{Re}(p_G(X, D_x, X')u, u) - \text{Re}(R(X, D_x)u, u) - \text{Re}(Q(X, D_x)u, u) \\ &\geq -|(Q(X, D_x)u, u)| - |(R(X, D_x)u, u)| \\ &\geq -C \|u\|_{\frac{m-1}{2}, \lambda}^2 \end{aligned}$$

□

If a function $\lambda(\xi)$ is a basic weight function, then we can see that for $0 < \rho \leq 1$ the fractional power $\lambda(\xi)^\rho$ is also a basic weight function. Using this fact and Theorem 3.1 we have

Corollary 3.3. *Let $0 \leq \delta < \rho \leq 1$. We assume that a symbol $p(x, \xi)$ in $S_{\rho, \delta, \lambda}^m$ satisfies that $p_{(\beta)}(x, \xi)$ are in $S_{\rho, \delta, \lambda}^m$ for $|\beta| \leq 2$ and*

$$\operatorname{Re} p(x, \xi) \geq 0$$

Then we have

$$\operatorname{Re} (p(X, D_x)u, u) \geq -C\|u\|_{\frac{m-\rho}{2}, \lambda}^2$$

for any u in \mathcal{S} .

Proof. Since $\lambda(\xi)^\rho$ is a basic weight function, we see that

$$S_{\rho, \delta, \lambda}^m = S_{1, \frac{\delta}{\rho}, \lambda^\rho}^{\frac{m}{\rho}}$$

Hence we can see that the symbol $p(x, \xi)$ in Corollary satisfies the assumptions of the one in Theorem 3.1 as the class of symbols in $S_{1, \frac{\delta}{\rho}, \lambda^\rho}^{\frac{m}{\rho}}$. Therefore we see that

$$\begin{aligned} \operatorname{Re} (p(X, D_x)u, u) &\geq -C\|u\|_{(\frac{m}{\rho}-1)/2, \lambda^\rho}^2 \\ &= -C\|u\|_{\frac{m-\rho}{2}, \lambda}^2 \end{aligned}$$

□

REMARK 2. Let $0 < \rho < 1$. Then we can show a similar sharp form of Garding's inequality to Theorem 3.1, under the assumption that the symbol $p(x, \xi)$ belongs to $S_{\rho, \rho, \lambda}^m$ and $p_{(\beta)}(x, \xi)$ belongs to $S_{\rho, \rho, \lambda}^m$ for any β with $|\beta| \leq 2$, by using the similar approximation $p_G(x, \xi, x')$ defined by

$$p_G(x, \xi, x') = \int \psi(\lambda(\xi)^{\frac{\rho}{2}}(x-z))\psi(\lambda(\xi)^{\frac{\rho}{2}}(x'-z))p(z, \xi)dz\lambda(\xi)^{\frac{n\rho}{2}}$$

and L^2 -boundedness theorem(Theorem of Calderon and Vaillancourt, see [4]) of operators with symbols in $S_{\rho, \rho, \lambda}^0$.

Now using the commutator estimates in section 2 we can show the following sharp form of Garding's inequality.

Theorem 3.4. *Let $a_j(\xi)$ and $c_j(\xi)$ be in $S_{1, 0, \lambda}^m$ and let $b_j(x)$ be \mathcal{B}^2 functions for $j = 1, \dots, N$. We assume that*

$$\operatorname{Re} \sum_{j=1}^N a_j(\xi)b_j(x)c_j(\xi) \geq 0$$

Then there exists a positive constant C such that

$$\operatorname{Re} \sum_{j=1}^m (b_j(X)a_j(D_x)u, c_j(D_x)u) \geq -C\|u\|_{m-\frac{1}{2},\lambda}^2$$

for any $u \in \mathcal{S}$.

Proof. We set $\tilde{a}_j(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}a_j(\xi)$ and $\tilde{c}_j(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}c_j(\xi)$. From the assumption we see that $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S_{1,0,\lambda}^{\frac{1}{2}}$. So writing

$$\sum_{j=1}^N (b_j(X)a_j(D_x)u, c_j(D_x)u) = \sum_{j=1}^N (b_j(X)\tilde{a}_j(D_x)\lambda(D_x)^{m-\frac{1}{2}}u, \tilde{c}_j(D_x)\lambda(D_x)^{m-\frac{1}{2}}u)$$

we put

$$\begin{aligned} \sum_{j=1}^N (b_j(X)a_j(D_x)u, c_j(D_x)u) &= \sum_{j=1}^N (b_j(X)\tilde{c}_j(D_x)\tilde{a}_j(D_x)v, v) \\ &\quad + \sum_{j=1}^N ([\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v, v) \\ (3.1) \qquad \qquad \qquad &= I + II \end{aligned}$$

where $v = \lambda(D_x)^{m-\frac{1}{2}}u$. Since $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S_{1,0,\lambda}^{\frac{1}{2}}$, using the commutator estimate in Corollary 2.5 we can see that

$$\|[\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v\| \leq C\|v\| = C\|u\|_{m-\frac{1}{2},\lambda}$$

Hence the second term II of (3.1) can be estimated by

$$|II| \leq \sum_{j=1}^N |([\tilde{c}_j(D_x), b_j(X)]\tilde{a}_j(D_x)v, v)| \leq C\|u\|_{m-\frac{1}{2},\lambda}^2$$

Now we consider the operator

$$p(X, D_x) = \sum_{j=1}^N b_j(X)\tilde{c}_j(D_x)\tilde{a}_j(D_x)$$

with symbol

$$p(x, \xi) = \sum_{j=1}^N b_j(x)\tilde{c}_j(\xi)\tilde{a}_j(\xi)$$

For the symbol $p(x, \xi)$ we define a new symbol $\tilde{p}(x, \xi)$ by

$$\begin{aligned} \tilde{p}(x, \xi) &= \int \phi(y)p(x - \lambda(\xi)^{-\frac{1}{2}}y, \xi)dy \\ &= \int \phi(\lambda(\xi)^{\frac{1}{2}}(x - y))p(y, \xi)dy\lambda(\xi)^{\frac{n}{2}} \end{aligned}$$

where $\phi(x)$ is a non-negative function in $C_0^\infty(\mathbb{R}^n)$ with $\int \phi(x)dx = 1$. Then by Lemma 2.1 (ii) we can see that the symbol $\tilde{p}(x, \xi)$ belongs to $S_{1, \frac{1}{2}, \lambda}^1$, $\tilde{p}_{(\beta)}(x, \xi)$ belongs to $S_{1, \frac{1}{2}, \lambda}^1$ for any $|\beta| \leq 2$ and satisfies

$$\|\{p(X, D_x) - \tilde{p}(X, D_x)\}v\| \leq \sum_{j=1}^N \|\{b_j(X) - \tilde{b}_j(X, D_x)\}\tilde{c}_j(D_x)\tilde{a}_j(D_x)v\|$$

where

$$\tilde{b}_j(x, \xi) = \int \phi(y)b_j(x - \lambda(\xi)^{-\frac{1}{2}}y)dy$$

Then by Lemma 2.1 (ii) we have

$$\|\{b_j(X) - \tilde{b}_j(X, D_x)\}w\| \leq C\|w\|_{-1, \lambda}$$

for any $w \in \mathcal{S}$ and therefore we have

$$\begin{aligned} \|\{p(X, D_x) - \tilde{p}(X, D_x)\}v\| &\leq C \sum_{j=1}^N \|\tilde{c}_j(D_x)\tilde{a}_j(D_x)v\|_{-1, \lambda} \\ (3.2) \qquad \qquad \qquad &\leq C\|v\| \end{aligned}$$

Moreover $\tilde{p}(x, \xi)$ satisfies

$$\text{Re } \tilde{p}(x, \xi) \geq 0$$

Thus the symbol $\tilde{p}(x, \xi)$ satisfies the assumption in the Theorem 3.1 with $m = 1$ and $\delta = \frac{1}{2}$. Therefore we have

$$\text{Re } (\tilde{p}(X, D_x)v, v) \geq -C\|u\|_{\frac{1}{2}, \lambda}^2$$

From (3.1) we see

$$\begin{aligned} \text{Re } \sum_{j=1}^N (b_j(X)a_j(D_x)u, c_j(D_x)u) &= \text{Re}\{I + II\} \\ &= \text{Re}(\tilde{p}(X, D_x)v, v) + \text{Re}(\{p(X, D_x) - \text{Re}(\tilde{p}(X, D_x))\}v, v) + II \\ &\geq -\|\{p(X, D_x) - \text{Re}(\tilde{p}(X, D_x))\}v\| \cdot \|v\| - |II| \\ &\geq -C\|v\|^2 \end{aligned}$$

Hence we have the theorem. □

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M. Nagase
Department of Mathematics, Graduate School of Science,
Osaka University, Toyonaka,
560, Japan
e-mail: nagase@math.wani.osaka-u.ac.jp

M. Yoshida
Isogo engineering center, Toshiba Co.,
Yokohama, Japan
e-mail : manabu@rdec.iec.toshiba.co.jp

