Kinoshita, Shin'ichi
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Alexander Polynomials as Isotopy Invariants, I

By Shin'ichi KINOSHITA

Introduction

As an isotopy invariant J. W. Alexander [1] introduced polynomials of knots, so-called the Alexander polynomials $\Delta(t)$ of knots. Recently R. H. Fox generalized this notion to the case of links and the polynomials $\Delta(t_1, \cdots, t_\mu)$ of links with multiplicity μ are called again the Alexander polynomials of links. The relation between the Alexander polynomials $\Delta(t)$ and $\Delta(t_1, \cdots, t_\mu)$ to the groups of knots and links, i.e. the fundamental groups of the complementary domain of knots and links, is studied by R. H. Fox [3] [4] by the use of his free differential calculus.

The notion of the Alexander polynomials is naturally extended to the more general cases. But the way of the extension does not seem to be unique in the method and in the subject. In this paper we shall treat the case of n-dimensional cycles $K^n (n \geqq 1)$ with integral coefficients in the $(n+2)$-dimensional sphere S^{n+2}. Of course we shall study them from the semi-linear stand point of view.

In §1 we define the Alexander polynomials of K^n in S^{n+2} by the use of free differential calculus. It should be remarked that according to R. H. Fox [4] not only one Alexander polynomial but the sequences of the Alexander polynomials are defined. In fact we have two sequences of the Alexander polynomials i.e. $\Delta^{(d)}(t_1, \cdots, t_\mu)$ and $\Delta^{(d)}(t)$. In §2 we give a presentation of the fundamental group of $S^{n+2} - |K^n|$ [1] and from this we are led to the Alexander polynomials. In §3 a theorem of $\Delta^{(1)}(t_1, \cdots, t_\mu)$ is proved, which is similar to that of G. Torres [7] for the case of links. In §4 we treat briefly the general cases of theorems of E. Artin [2] and H. Seifert [6]. In §5 we shall give an example of a linear graph, which will seem to be of interest to some readers.

§1.

1. Let K^n be an n-dimensional complex with integral coeffecients in the $(n+2)$-dimensional sphere $S^{n+2} (n \geqq 1)$. Further suppose that K^n is

1) K^n is a complex with integral coefficients and $|K^n|$ is a polyhedron.

a cycle. Of course the fundamental group $F(S^{n+2}-|K^n|)$ is an isotopy invariant of K^n in S^{n+2}.

Now suppose that $|K^n|$ consists of μ components $|K_1^n|, \cdots, |K_\mu^n|$. Let g be an element of $F(S^{n+2}-|K^n|)$ and $\tilde{g}$ a closed path which reprsents g. Put

$$(1) \qquad \lambda_j = \text{Link}\,(\tilde{g}, K_j^n)^{2)}. \qquad (j=1, \cdots, \mu)$$

Then λ_j is an integer and independent of the choice of $\tilde{g}$.

Let Z_j be an infinite cyclic group $(j=1, \cdots, \mu)$ and t_j a generator of Z_j. Put

$$Z^\mu = \Pi_{j=1}^{\mu} Z_j .$$

Further if we put

$$\varphi(g) = t_1^{\lambda_1} \cdots t_\mu^{\lambda_\mu}$$

for every $g \in F(S^{n+2}-|K^n|)$, then φ is a homomorphism of $F(S^{n+2}-|K^n|)$ into Z^μ. Let $\{g_0, g_1, \cdots, g_\alpha\}$ be a set of generators of $F(S^{n+2}-|K^n|)$ and $\{R_1, \cdots, R_\beta\}$ be a set of relators. Further let X be a free group of $\alpha+1$ generators. Then there exists a homomorphism ψ of X onto $F(S^{n+2}-|K^n|)$.

Using the Fox's free differential calculus [3] [4], we put

$$M = \left(\frac{\partial R_i}{\partial g_j}\right)^{\varphi\psi}.$$

The matrix M is the so-called *Alexander matrix* and its elements are polynomials of $t_1, \cdots, t_\mu$.

Now let d be an arbitrary integer. The greatest common factor $\Delta^{(d)}(t_1, \cdots, t_\mu)$ of the minor determinats of M of order $(\alpha+1)-d$, where $\alpha+1$ is the number of columns of M, is called the *d-th Alexander polynomial.* Of course $\Delta^{(d)}(t_1, \cdots, t_\mu)$ is determined only up to a factor $\pm t_1^{\lambda_1} \cdots t_\mu^{\lambda_\mu}$. It should be understood that $\Delta^{(d)}(t_1, \cdots, t_\mu)=1$ for $d \geqq \alpha+1$ and that $\Delta^{(d)}(t_1, \cdots, t_\mu)=0$ if M has fewer than $(\alpha+1)-d$ rows. If the group $F(S^{n+2}-|K^n|)$ and φ is fixed, then the sequence $\Delta^{(d)}(t_1, \cdots, t_\mu)$ $(d=0, 1, \cdots)$ is their invariant[3]. Therefore we have the following

Theorem 1. *The sequence* $\Delta^{(d)}(t_1, \cdots, t_\mu)$ *is an isotopy invariant of* K^n *in* S^{n+2}.

2. Let Z be an infinite cyclic group and t a generator of Z. If we put

$$\lambda = (\tilde{g}, K^n),$$

instead of (1), and put

2) Link $(\tilde{g}, K_j^n)$ is the linking number of $\tilde{g}$ and K_j^n.

3) See R. H. Fox [4].

$$\varphi'(g) = t^{\lambda}.$$

for every $g \in F(S^{n+2} - |K^n|)$, then we have another homomorphism φ' of $F(S^{n+2} - |K^n|)$ into Z. From this in the same way we have the Alexander matrix

$$M' = \left(\frac{\partial R_i}{\partial g_j}\right)^{\varphi'\psi}$$

and also a sequence $\Delta^{(d)}(t)$ of the Alexander polynomials. Of course $\Delta^{(d)}(t)$ is determined only up to a factor $\pm t^{\lambda}$. Just as Theorem 1 we have

Theorem 2. *The sequence of the Alexander polynomials $\Delta^{(d)}(t)$ is an isotopy invariant of K^n in S^{n+2}.*

REMARK. If $|K^n|$ is connected, then two sequences of the Alexander polynomials are the same.

§ 2.

1. Let K^n $(n \geqq 1)$ be an n-dimensional cycle with integral coefficients in S^{n+2}. Then we may suppose that K^n is contained in the $(n+2)$-dimensional Euclidean space E^{n+2}. Further we may assume that vertices of K^n are linearly independent, i. e., if $A_0, A_1, \cdots, A_{n+2}$ are $n+3$ vertices of K^n, then they are not cotained in a $(n+1)$-dimensional hyperplane of E^{n+2}. Then there is a projection p of K^n into an $(n+1)$-dimensional hyperplane E^{n+1} such that if $A_0, \cdots, A_{n+1}$ are vertices of K^n, then $p(A_0), \cdots, p(A_{n+1})$ are linearly independent in E^{n+1}. The projection of this kind will be called a regular projection of K^n.

$E^{n+1} - p(|K^n|)$ is decomposed in regions. We put

$$E^{n+1} - p(|K^n|) = G_0 \cup G_1 \cup \cdots \cup G_{\alpha}.$$

Let $B_0 \in G_0$, $B_1 \in G_1, \cdots, B_{\alpha} \in G_{\alpha}$. Further let P and Q be two points satisfying the following conditions:

(1) P and Q are contained in the different components of $E^{n+2} - E^{n+1}$,

(2) $(\overline{PB_i} \cup \overline{B_iQ}) \frown K^n_+ = 0$ for every $i\,(=0, 1, \cdots, \alpha)$[4].

Convensionally we may suppose that

$$\begin{aligned} E^{n+1} &= (x_1, \cdots, x_{n+1}, \ 0), \\ P &= (0, \ \cdots, 0, \quad 1), \\ Q &= (0, \ \cdots, 0, \ -1). \end{aligned}$$

2. Now we put

$$\tilde{g}_i = \overline{PB_i} \cup \overline{B_iQ} \cup \overline{QB_0} \cup \overline{B_0P}. \qquad (i = 0, 1, \cdots, \alpha)$$

4) $K^n_+ = \{ y \mid y \in \overline{xp(x)},\ x \in K^n \}$.

Then $\tilde{g}_i$ is a simple closed curve in $S^{n+2}-|K^n|$. Let g_i be the element of $F(E^{n+2}-|K^n|)$ such that $\tilde{g}_i$ represents g_i. Then it is easy to see that $g_0, g_1, \cdots, g_\alpha$ generate $F(E^{n+2}-|K^n|)$. By definition $g_0=1$.

Let s_a and s_b be a pair of open n-dimensional simplices of K^n such that $p(s_a)\frown p(s_b)\neq 0$. Then $L=p(s_a)\frown p(s_b)$ is an $(n-1)$-dimensional open disk. Put

$$L^{ab}=\{x\,|\,p^{-1}(x)\frown(|K^n|-(s_a\cup s_b))=0,\ x\in L\}\,.$$

Then $L-L^{ab}$ is at most $(n-2)$-dimensional. Let $L_1^{ab}, \cdots, L_N^{ab}$ be components of L^{ab}. Then each $L_k^{ab}(k=1, \cdots, N)$ is the boundary of at most four regions. Let G_r, G_s, G_t, G_u be these four regions, where it may occur that some of them coincide. The position of G_r, G_s, G_t, G_u may be supposed as follows: Let s_a be the under simplex with regards to L_k^{ab} and s_b the over one. Of course we consider only a sufficiently small neighborhood of L_k^{ab}. Suppose that the normal vectors of $p(s_a)$ with sufficiently small absolute value are contained in G_r and G_s. Assume that G_r and G_u have an n-dimensional common boundary and that G_s and G_t have an n-dimensional common boundary in a sufficiently small neighborhood of L_k^{ab}.

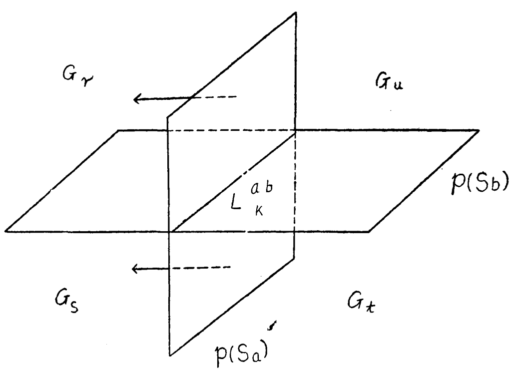

Fig. 1

Then it is easy to see that for each L_k^{ab} we have a relation

$$g_r g_s^{-1} g_t g_u^{-1}=1. \qquad (L_k^{ab})$$

It is easy to see that *a presentation of $F(E^{n+2}-|K^n|)$ is given as follows*:

Generators. $\quad g_0, g_1, \cdots, g_\alpha$.

Relations.
$$\begin{cases} \cdots\cdots\cdots\cdots \\ g_r g_s^{-1} g_t g_u^{-1}=1\,, \qquad (L_k^{ab}) \\ \cdots\cdots\cdots\cdots \\ g_0=1\,. \end{cases}$$

This is a kind of so-called over presentations.

3. Now we construct the Alexander matrix of the over presentation of $F(E^{n+2}-|K^n|)$. Suppose that $s_a\in|K_\gamma^n|$ and $s_b\in|K_\delta^n|$, where $|K_\gamma^n|$ and

$|K^n_\delta|$ are components of $|K^n|$. Suppose further that coefficients of s_a and s_b of K^n are β_a and β_b respectively.

First we suppose that the normal vectors of $p(s_b)$ with sufficiently small absolute value are contained in G_r and G_u in a sufficiently small neighborhood of L^{ab}_k. If

$$\varphi(g_t) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma} \cdots t_\delta^{\lambda_\delta} \cdots t_\mu^{\lambda_\mu},$$

then

$$\varphi(g_r) = t_1^{\lambda_1} \cdots t_\gamma^{\delta_\gamma+\beta_a} \cdots t_\delta^{\lambda_\delta+\beta_b} \cdots t_\mu^{\lambda_\mu},$$
$$\varphi(g_s) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma+\beta_a} \cdots t_\delta^{\lambda_\delta} \cdots t_\mu^{\lambda_\mu},$$
$$\varphi(g_u) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma} \cdots t_\delta^{\lambda_\delta+\beta_b} \cdots t_\mu^{\lambda_\mu}.$$

Put $R^{ab}_k = g_r g_s^{-1} g_t g_u^{-1}$. Then

$$\left(\frac{\partial}{\partial g_r} R^{ab}_k\right)^{\varphi\psi} = 1,$$
$$\left(\frac{\partial}{\partial g_s} R^{ab}_k\right)^{\varphi\psi} = (-g_r g_s^{-1})^\varphi = -t_\delta^{\beta_b},$$
$$\left(\frac{\partial}{\partial g_t} R^{ab}_k\right)^{\varphi\psi} = (g_r g_s^{-1})^\varphi = t_\delta^{\beta_b},$$
$$\left(\frac{\partial}{\partial g_u} R^{ab}_k\right)^{\varphi\psi} = (-g_r g_s^{-1} g_t g_u^{-1})^\varphi = -1.$$

Secondly we suppose that the normal vectors of s_b with sufficiently small absolute value are contained in G_s and G_t in a sufficiently small neighborhood of L^{ab}_k. If

$$\varphi(g_u) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma} \cdots t_\delta^{\lambda_\delta} \cdots t_\mu^{\lambda_\mu},$$

then

$$\varphi(g_r) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma+\beta_a} \cdots t_\delta^{\lambda_\delta} \cdots t_\mu^{\lambda_\mu},$$
$$\varphi(g_s) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma+\beta_a} \cdots t_\delta^{\lambda_\delta+\beta_b} \cdots t_\mu^{\lambda_\mu},$$
$$\varphi(g_t) = t_1^{\lambda_1} \cdots t_\gamma^{\lambda_\gamma} \cdots t_\delta^{\lambda_\delta+\beta_b} \cdots t_\mu^{\lambda_\mu}.$$

Then

$$\left(\frac{\partial}{\partial g_r} R^{ab}_k\right)^{\varphi\psi} = 1,$$
$$\left(\frac{\partial}{\partial g_s} R^{ab}_k\right)^{\varphi\psi} = (-g_r g_s^{-1})^\varphi = -t_\delta^{-\beta_b},$$
$$\left(\frac{\partial}{\partial g_t} R^{ab}_k\right)^{\varphi\psi} = (g_r g_s^{-1})^\varphi = t_\delta^{-\beta_b},$$
$$\left(\frac{\partial}{\partial g_u} R^{ab}_k\right)^{\varphi\psi} = (-g_r g_s^{-1} g_t g_u^{-1})^\varphi = -1.$$

§ 3.

Let K^n be an n-dimensional cycle with integral coefficients in S^{n+2}. Let M be the Alexander matrix obtained from the over presentation (in § 2) of $F(S^{n+2}-|K^n|)$. Further let $\varphi(g_i)=t_1^{\lambda_{i1}}\cdots t_\mu^{\lambda_{i\mu}}$ $(i=0,\cdots,\alpha)$. Put

$$M=\begin{matrix} & G_0 & G_1 \cdots G_i \cdots G_\alpha \\ & \left(\begin{matrix} 1 \\ \xi_0 \end{matrix}\right. & \left.\begin{matrix} 0 \cdots 0 \cdots 0 \\ \xi_1 \cdots \xi_i \cdots \xi_\alpha \end{matrix}\right) \end{matrix}$$

Then $\sum_{i=1}^{\alpha}\xi_i=0$. Further it is easy to see that

$$\sum_{i=0}^{\alpha} t_1^{\lambda_{i1}}\cdots t_\mu^{\lambda_{i\mu}}\xi_i=0\,.$$

Hence

$$\sum_{i=0}^{\alpha}(t_1^{\lambda_{i1}}\cdots t_\mu^{\lambda_{i\mu}}-1)\xi_i=0\,.$$

Then

$$\begin{pmatrix} 1 & 0\cdots\hat{0}\cdots 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \xi_0 & \xi_1\cdots\hat{\xi}_j\cdots\xi_k(t_1^{\lambda_{k1}}\cdots t_\mu^{\lambda_{k\mu}}-1)\cdots\xi_\alpha \end{pmatrix}^{5)}$$

$$=\begin{pmatrix} 1 & 0\cdots\hat{0}\cdots 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \xi_0 & \xi_1\cdots\hat{\xi}_j\cdots\xi_j(t_1^{\lambda_{j1}}\cdots t_\mu^{\lambda_{j\mu}}-1)\cdots\xi_\alpha \end{pmatrix}.$$

Now put

$$M_i=\begin{pmatrix} 1 & 0\cdots\hat{0}\cdots 0 \\ \xi_0 & \xi_1\cdots\hat{\xi}_i\cdots\xi_\alpha \end{pmatrix}.$$

Let $\Delta_i(t_1,\cdots,t_\mu)$ be the greatest common factor of determinants of order α of M_i. If $t_1^{\lambda_{i1}}\cdots t_\mu^{\lambda_{i\mu}}-1=0$, then all the determinants of order α of M_i are equal to 0. From now on we consider the terms $t_1^{\lambda_{i1}}\cdots t_\mu^{\lambda_{i\mu}}-1\neq 0$. Then $(t_1^{\lambda_{j1}}\cdots t_\mu^{\lambda_{j\mu}}-1)$ divides $\Delta_j(t_1,\cdots,t_\mu)$ $(t_1^{\lambda_{k1}}\cdots t_\mu^{\lambda_{k\mu}}-1)$ for $k=1,\cdots,\mu$ and therefore it must divide $\Delta_j(t_1,\cdots,t_\mu)$ $\delta(t_1,\cdots,t_\mu)$, where $\delta(t_1,\cdots,t_\mu)$ is the greatest common factor of $t_1^{\lambda_{11}}\cdots t_\mu^{\lambda_{1\mu}}-1,\cdots,t_1^{\lambda_{\alpha1}}\cdots t_\mu^{\lambda_{\alpha\mu}}-1$. But it is easy to see that

$$\frac{\Delta_j\delta}{t_1^{\lambda_{j1}}\cdots t_\mu^{\lambda_{j\mu}}-1}\equiv\frac{\Delta_k\delta}{t_1^{\lambda_{k1}}\cdots t_\mu^{\lambda_{k\mu}}-1}\,. \qquad (k=1,\cdots,\alpha)$$

Then the common value of $\dfrac{\Delta_k\delta}{t_1^{\lambda_{k1}}\cdots t_\mu^{\lambda_{k\mu}}-1}$ $(k=1,\cdots,\alpha)$ is $\Delta^{(1)}(t_1,\cdots,t_\mu)$.

Therefore

5) In the proof we use the notation $\hat{\xi}$ mening delation of ξ.

Theorem 3.[6] $\Delta_j(t_1, \cdots, t_\mu) \equiv \Delta^{(1)}(t_1, \cdots, t_\mu) \dfrac{(t_1^{\lambda_{j_1}} \cdots t_\mu^{\lambda_{j_\mu}} - 1)}{\delta(t_1, \cdots, t_\mu)}$.

It K^n is an orientable manifold with multipicity $\mu = 1$, then $\delta(t) = t-1$. If K^n is that with $\mu \geqq 2$, then $\delta(t_1, \cdots, t_\mu) = 1$.

§ 4.

Let E_+^{n+2} be the subset of E^{n+2} such that $(x_1, \cdots, x_{n+2}) \in E^{n+2}$ if and only if $x_{n+2} > 0$. Let K^n be an n-dimensional complex in E_+^{n+2} and K^{n+1} be the $(n+1)$-dimensional complex in E^{n+3} obtained from K^n by the rotation about the axis $x_{n+2} = 0$ in E^{n+2}. Then we have the following

Theorem 4. *$F(E_+^{n+2} - |K^n|)$ is isomorphic to $F(E^{n+3} - |K^{n+1}|)$.*

This theorem was proved by E. Artin [2] for $n = 1$. The proof for arbitrary n will be done similarly.

If K^n is a cycle with integral coefficients, then K^{n+1} is also a cycle with the same one. Clearly we have

Theorem 5. *$\Delta^{(d)}(t_1, \cdots, t_\mu)$ and $\Delta^{(d)}(t)$ of K^n in E_+^{n+2} are equal to that of K^{n+1} in E^{n+3}, respectively.*

Let $f(t)$ be a pclynomial satisfying the following conditions:

(1) $|f(1)| = 1$,

(2) The coefficients of $f(t)$ are symmetric.

Then it was proved by H. Seifert [6] that there exists a knot whose Alexander polynomial $\Delta^{(1)}(t)$ is equal to $f(t)$. Using Theorem 2, we can see the following

Theorem 6. *Let $f(t)$ be a polynomial satisfying the following conditions:*

(1) $|f(1)| = 1$,

(2) *The coefficients of $f(t)$ are symmetric.*

Then there exists a connected n-dimensional manifold in E^{n+2}, whose Alexander polynomial $\Delta^{(1)}(t)$ is equal to $f(t)$.

Further by a remark of E. Artin [2] a connected manifold can be replaced by S^n.

Recently the above Seifert's theorem is extended to the case of links by F. Hosokawa [5]. Of course his theorem can be generalized along the same way.

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6) By the same way it can be proved that

$$\Delta_j(t) \equiv \Delta^{(1)}(t) \frac{t^{\lambda_j} - 1}{\delta(t)}.$$

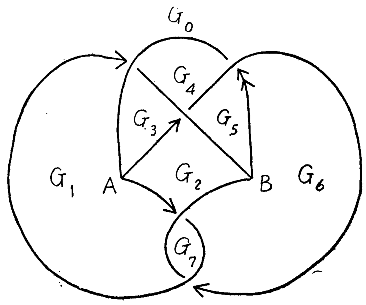


Fig. 2

§5.

Let K_0 be a linear graph as shown in Fig. 2, where two simple arc from A to B have the coefficients 1 and other one simple arc from B to A has the coefficient 2. Then the presentation of $F(S^3-|K_0|)$ is as follows:

Generators. $g_0, g_1, \cdots, g_7$.

Relations.

$$g_0 = 1, \qquad g_4g_5^{-1}g_6g_0^{-1} = 1,$$
$$g_4g_3^{-1}g_2g_5^{-1} = 1, \qquad g_1g_0^{-1}g_6g_7^{-1} = 1,$$
$$g_6g_2^{-1}g_1g_7^{-1} = 1, \qquad g_4g_0^{-1}g_1g_3^{-1} = 1.$$

The homomorphism φ maps $g_0 \to 1$, $g_1 \to t^{-1}$, $g_2 \to 1$, $g_3 \to t$, $g_4 \to t^2$, $g_5 \to t$, $g_6 \to t^{-1}$, $g_7 \to 1$. Therefore the Alexander matrix is given by the following one:

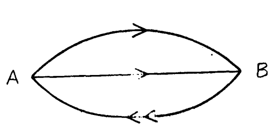
g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	0	0	0	0	0	0	0
-1	0	0	0	1	$-t$	t	0
0	0	t	$-t$	1	-1	0	0
$-t^{-1}$	1	0	0	0	0	t^{-1}	-1
0	t^{-1}	$-t^{-1}$	0	0	0	1	-1
$-t^2$	t^2	0	-1	1	0	0	0

From this it follows that

$$\Delta^{(2)}(t) \equiv t^2-t+1\,.$$

On the other hand if K_1 is a trivial θ-curve (as shown in Fig. 3), then

$$\Delta^{(2)}(t) \equiv 1\,.$$



Therefore K_0 and K_1 are not isotopic. It should be remarked that three simple closed curves constructed from K_0 are trivial knots.

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