Ozawa, S.
Osaka J. Math.
22 (1985), 639–655

# ASYMPTOTIC PROPERTY OF AN EIGENFUNCTION OF THE LAPLACIAN UNDER SINGULAR VARIATION OF DOMAINS —THE NEUMANN CONDITION—

SHIN OZAWA

(Received April 6, 1984)

## 1. Introduction

We consider a bounded domain $\Omega$ in $\boldsymbol{R}^2$ with smooth boundary $\gamma$. Let $B_\varepsilon$ be the $\varepsilon$-disk whose center is $\tilde{w}\in\Omega$. We put $\Omega_\varepsilon=\Omega\setminus\bar{B}_\varepsilon$. We consider the following eigenvalue problems (1.1) and (1.2):

$$(1.1)\qquad \begin{aligned} -\Delta_x u(x) &= \lambda(\varepsilon)u(x)\,, && x\in\Omega_\varepsilon\,,\\ u(x) &= 0\,, && x\in\gamma\,,\\ \frac{\partial u}{\partial\nu}(x) &= 0\,, && x\in\partial B_\varepsilon\,, \end{aligned}$$

where $\partial/\partial\nu$ denotes the derivative along the inner normal vector at $x$ with respect to the domain $\Omega_\varepsilon$.

$$(1.2)\qquad \begin{aligned} -\Delta_x u(x) &= \lambda u(x)\,, && x\in\Omega\,,\\ u(x) &= 0\,, && x\in\gamma\,. \end{aligned}$$

Let $0<\mu_1(\varepsilon)\leq\mu_2(\varepsilon)\leq\cdots$ be the eigenvalues of (1.1). Let $0<\mu_1\leq\mu_2\leq\cdots$ be the eigenvalues of (1.2). We arrange them repeatedly according to their multiplicities. Denote by $\{\varphi_j(\varepsilon)\}_{j=1}^\infty$ ($\{\varphi_j\}_{j=1}^\infty$, respectively) a complete orthonomal basis of $L^2(\Omega_\varepsilon)$ ($L^2(\Omega)$, respectively) consisting of eigenfunction of $-\Delta$ associated with $\{\mu_j(\varepsilon)\}_{j=1}^\infty$ ($\{\varphi_j\}_{j=1}^\infty$, respectively).

In this note we consider the following problem:
Problem. What can one say about asymptotic behaviour of $\varphi_j(\varepsilon)$ as $\varepsilon$ tends to zero?

It is well known that $\mu_j(\varepsilon)$ tends to $\mu_j$ as $\varepsilon$ tends to zero. See Rauch-Taylor [8], Ozawa [5]. As a consequence, $\mu_j(\varepsilon)$ is simple for small $\varepsilon>0$, if we assume that $\mu_j$ is simple. Thus $\varphi_j(\varepsilon)$ is uniquely determined up to the arbitratiness of multiplication by $+1$ or $-1$.

We have the following Theorem 1. Theorem 2 is our main result.

**Theorem 1.** *Fix j. Assume that $\mu_j$ is simple. Then, the following statements* (i) *and* (ii) *hold.*
(i) *We can choose $\varphi_j(\varepsilon)$ for $\varepsilon>0$ so that*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\varphi_j(\varepsilon))(x)\varphi_j(x)dx=1\,.$$

(ii) *If we choose $\varphi_j(\varepsilon)$ as in* (i), *then*

$$\|\varphi_j(\varepsilon)-\varphi_j\|_{L^\infty(\Omega_\varepsilon)}=0(\varepsilon)\,. \tag{1.3}$$

We introduce the polar coordinate $z-\tilde{w}=(r\cos\theta,\ r\sin\theta)$ to state the following

**Theorem 2.** *Fix j. Assume that $\mu_j$ is a simple eigenvalue. If $\varphi_j(\varepsilon)$ is chosen as in Theorem* 1, *then*

$$\begin{aligned}&\left(\frac{\partial}{\partial\theta}(\varphi_j(\varepsilon))\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)\\&=2\frac{\partial}{\partial r}\left(\varphi_j\left(r\cos\left(\theta+\frac{\pi}{2}\right),\ r\sin\left(\theta+\frac{\pi}{2}\right)\right)\right|_{r=0}+0(\varepsilon^{(1/2)-s})\end{aligned} \tag{1.4}$$

*for an arbitrary $s>0$.*

REMARK. 1) Proofs of Theorems 1 and 2 are given in the section 2.
2) The remainder estimates in (1.3) and (1.4) are not uniform with respect to $j$.
3) Theorems 1 and 2 prove the conjecture stated in the previous work [5] of the author.
4) The celebrated Hadamard variational formula (See Garabedian-Schiffer [4]) says that

$$\frac{\partial}{\partial\varepsilon}\mu_j(\varepsilon)=-\int_{\partial B_\varepsilon}(|\mathrm{grad}_z\,\varphi_j(\varepsilon)(z)|^2-\mu_j(\varepsilon)(\varphi_j(\varepsilon))(z)^2)d\sigma_z^\varepsilon\,, \tag{1.5}$$

holds when $\mu_j$ is simple, where $d\sigma_z^\varepsilon$ denotes the line element on $\partial B_\varepsilon$. If we apply Theorems 1 and 2 to (1.5), then

$$\frac{\partial}{\partial\varepsilon}\mu_j(\varepsilon)=0(\varepsilon)\,.$$

Hence $\mu_j(\varepsilon)-\mu_j=0(\varepsilon^2)$. Using (1.5) once more, we can prove that

$$\mu_j(\varepsilon)-\mu_j=-(2\pi|\mathrm{grad}\,\varphi_j(\tilde{w})^2-\pi\mu_j\varphi_j(\tilde{w})^2)\varepsilon^2+0(\varepsilon^{(5/2)-s})\,, \tag{1.6}$$

while we have already obtained in [5] much stronger result

$$\mu_j(\varepsilon)-\mu_j=-(2\pi|\mathrm{grad}\,\varphi_j(\tilde{w})|^2-\pi\mu_j\varphi_j(\tilde{w})^2)\varepsilon^2+0(\varepsilon^3|\log\varepsilon|^2)\,.$$

However, discussion in [5] was very complicated. Present proof via Hadamard's variational formula (1.5) is much simpler.

See Ozawa [6], [7], Figari-Orlandi-Teta [2] for other recent developments on the asymptotic behaviour of the eigenvalues of the Laplacian under singular variation of domains.

A part of this work was done while I stayed at Courant Institute of Mathematical Sciences. I here express my sincere thanks to C.I.M.S., Professor G. Papanicolaou and Ms. Vogelsang for their hospitality.

## 2. Sketch of the proof

Let $G(x, y)$ be the Green function of the Laplacian in $\Omega$ under the Dirichlet condition on $\gamma$. Let $G_\varepsilon(x, y)$ be the Green function of the Laplacian in $\Omega_\varepsilon$ satisfying

$$\begin{aligned} -\Delta_x G_\varepsilon(x, y) &= \delta(x-y)\,, && x, y\in\Omega_\varepsilon \\ G_\varepsilon(x, y)_{|x\in\gamma} &= 0\,, && y\in\Omega_\varepsilon \\ \frac{\partial}{\partial\nu_x} G_\varepsilon(x, y)_{|x\in\partial B_\varepsilon} &= 0\,, && y\in\Omega_\varepsilon\,. \end{aligned}$$

Let $\boldsymbol{G}$ ($\boldsymbol{G}_\varepsilon$, respectively) be the bounded linear operator on $L^2(\Omega)$ ($L^2(\Omega_\varepsilon)$, respectively) defined by

$$(\boldsymbol{G}f)(x) = \int_\Omega G(x, y)f(y)dy\,,$$
$$(\boldsymbol{G}_\varepsilon g)(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x, y)g(y)dy\,,$$

respectively. Then, (1.1) and (1.2) are transformed into the problems

$$(\boldsymbol{G}_\varepsilon u)(x) = \lambda(\varepsilon)^{-1}u(x)$$
$$(\boldsymbol{G}v)(x) = \lambda^{-1}v(x)\,.$$

We want to compare $\boldsymbol{G}_\varepsilon$ and $\boldsymbol{G}$. It should be remarked that the Green operators $\boldsymbol{G}_\varepsilon$ and $\mathbf{G}$ act on different spaces $L^2(\Omega_\varepsilon)$ and $L^2(\Omega)$. One of technical difficulties arises from here.

In order to relate $\boldsymbol{G}_\varepsilon$ with $\boldsymbol{G}$, we introduce the operators $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$. To describe integral kernel of $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$, we put

$$\langle\nabla_w a(x, w), \nabla_w b(w, y)\rangle = \sum_{i=1}^{2} \frac{\partial}{\partial w_i} a(x, w) \frac{\partial}{\partial w_i} b(w, y)$$

for any $a$, $b\in C^1(\Omega\times\Omega\backslash(\Omega\times\Omega)_d)$, where $(\Omega\times\Omega)_d$ denotes the diagonal set of $\Omega\times\Omega$. Then, $\langle\nabla_w, \nabla_w\rangle$ is invariant under any orthogonal transformation of an orthonomal coordinates $(w_1, w_2)$. We define

$$r_\varepsilon(x, y; w) = G(x, y) + 2\pi\varepsilon^2\langle\nabla_w G(x, w), \nabla_w G(w, y)\rangle$$

and

$$r_\varepsilon(x, y) = r_\varepsilon(x, y; \tilde{w}).$$

Also we set

$$\tilde{r}_\varepsilon(x, y) = G(x, y) + 2\pi\varepsilon^2\langle\nabla_w G(x, w), \nabla_w G(w, y)\rangle_{|w=\tilde{w}}\xi_\varepsilon(x)\xi_\varepsilon(y),$$

where $\xi_\varepsilon \in C^\infty(\boldsymbol{R}^2)$ satisfies $0 \le \xi_\varepsilon(x) \le 1$, $\xi_\varepsilon(x) = 1$ for $x \in \boldsymbol{R}^2 \setminus \bar{B}_\varepsilon$ and $\xi_\varepsilon(x) = 0$ for $x \in B_{\varepsilon/2}$.

The operators $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$ are defined by

$$(\boldsymbol{R}_\varepsilon g)(x) = \int_{\Omega_\varepsilon} r_\varepsilon(x, y) g(y) dy, \qquad x \in \Omega_\varepsilon,$$

$$(\tilde{\boldsymbol{R}}_\varepsilon f)(x) = \int_\Omega \tilde{r}_\varepsilon(x, y) f(y) dy, \qquad x \in \Omega,$$

respectively. Roughly speaking, $\boldsymbol{R}_\varepsilon$ is a very good approximation of $\boldsymbol{G}_\varepsilon$. By definition it is not difficult to compare $\boldsymbol{R}_\varepsilon$ with $\tilde{\boldsymbol{R}}_\varepsilon$. Since $\tilde{\boldsymbol{R}}_\varepsilon$ acts on $L^2(\Omega)$ and not on $L^2(\Omega_\varepsilon)$, we can easily compare $\tilde{\boldsymbol{R}}_\varepsilon$ with $\boldsymbol{G}$. As a consequence we can compare $\boldsymbol{G}_\varepsilon$ with $\boldsymbol{G}$.

Proof of Theorems 1, 2 are divided into several steps.

First we show

$$|||\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon|||_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

for any fixed $s > 0$ as $\varepsilon$ tends to zero. Here $|||\ \ |||_{L^p(\Omega_\varepsilon)}$ denotes the operator norm on $L^p(\Omega_\varepsilon)$. This will be done in the section 4.

Second we consider $\tilde{\boldsymbol{R}}_\varepsilon$ as a perturbation of $\boldsymbol{G}$. We construct an approximate eigenfunction $\psi^*(\varepsilon)$ and an approximate eigenvalue $\lambda^*(\varepsilon)$ of $\tilde{\boldsymbol{R}}_\varepsilon$. Here $\lambda^*(\varepsilon)$, $\psi^*(\varepsilon)$ are explicitly constructed by usual perturbation method so that they satisfy

$$||(\tilde{\boldsymbol{R}}_\varepsilon - \lambda^*(\varepsilon))\psi^*(\varepsilon)||_{L^2(\Omega)} = 0(\varepsilon^4 |\log \varepsilon|^2)$$

and

$$||\psi^*(\varepsilon)||_{L^2(\Omega)} = 1 + 0(\varepsilon^2 |\log \varepsilon|).$$

Since $\lambda^*(\varepsilon)$ and $\psi^*(\varepsilon)$ are constructed by perturbation theory, $\lambda^*(\varepsilon)$ is close to $\mu_j$ and $\psi^*(\varepsilon)$ is close to $\varphi_j$.

A key step is to examine the following decomposition of $\varphi_j(\varepsilon)$.

$$\varphi_j(\varepsilon) = \sum_{k=1}^{3} J_k(\varepsilon),$$

where

$$J_1(\varepsilon) = \mu_j(\varepsilon)(\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)(\varphi_j(\varepsilon))$$
$$J_2(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\varphi_j(\varepsilon) - t_\varepsilon \chi_\varepsilon \psi^*(\varepsilon))$$
$$J_3(\varepsilon) = \mu_j(\varepsilon) t_\varepsilon \boldsymbol{R}_\varepsilon(\chi_\varepsilon \psi^*(\varepsilon)) .$$

Here $\chi_\varepsilon$ is the characteristic function of $\Omega_\varepsilon$ and

$$t_\varepsilon = \operatorname{sgn} \int_{\Omega_\varepsilon} (\varphi_j(\varepsilon))(x) \varphi_j(x) dx .$$

We can prove the following facts. Here $s$ is an arbitrary fixed positive constant:

$$\|J_1(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} + \|J_2(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) . \tag{2.1}$$

$$\|\mu_j(\varepsilon)^{-1} J_3(\varepsilon) - t_\varepsilon \mu_j \varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon) . \tag{2.2}$$

$$\max_{z \in \partial B_\varepsilon} |\operatorname{grad}_z (J_1(\varepsilon))(z)| = 0(\varepsilon^{(1/2)-s}) . \tag{2.3}$$

$$\max_{z \in \partial B_\varepsilon} |\operatorname{grad}_z (J_2(\varepsilon))(z)| = 0(\varepsilon^{2-s}) . \tag{2.4}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial \theta} (J_3(\varepsilon))(z)\right)_{|z=(\varepsilon \cos \theta,\, \varepsilon \sin \theta)} \\ &= 2t_\varepsilon \mu_j(\varepsilon) \mu_j^{-1} \left(\frac{\partial}{\partial r} (\varphi_j(r \cos(\theta + (\pi/2)),\, r \sin(\theta + (\pi/2)))\right)_{|r=0} + 0(\varepsilon^{1-s}) . \end{aligned} \tag{2.5}$$

These will be proved in the section 6.

Here we assume (2.1)～(2.5) and we would like to prove Theorems 1 and 2. From (2.1) and (2.2) we obtain

$$\|\varphi_j(\varepsilon) - t_\varepsilon \mu_j(\varepsilon) \mu_j^{-1} \varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon) . \tag{2.6}$$

It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned} &\mu_j(\varepsilon)^{-1} \left(\frac{\partial}{\partial \theta} (\varphi_j(\varepsilon))\right)(\varepsilon \cos \theta,\, \varepsilon \sin \theta) \\ &= 2t_\varepsilon \mu_j^{-1} \frac{\partial}{\partial r} (\varphi_j(r \cos(\theta + (\pi/2)),\, r \sin(\theta + (\pi/2)))_{|r=0} + 0(\varepsilon^{(1/2)-s}) . \end{aligned} \tag{2.7}$$

We put (2.6) and (2.7) into (1.6) and we obtain

$$\mu_j(\varepsilon) - \mu_j = 0(\varepsilon^2) . \tag{2.8}$$

This together with (2.6) proves Theorem 1. Theorem 2 follows from (2.7) and (2.8).

Thus, our effort to get Theorems 1, 2 will be concentrated on showing (2.1)～(2.5). This will be completed in the section 6.

Before going further, we explain the reason why $r_\varepsilon(x, y)$ approximates $G_\varepsilon(x, y)$ well. Put

$$q_\varepsilon(x, y) = r_\varepsilon(x, y) - G_\varepsilon(x, y)\,.$$

Then,

$$\begin{aligned} \Delta_x q_\varepsilon(x, y) &= 0\,, \qquad x, y \in \Omega_\varepsilon \\ q_\varepsilon(x, y) &= 0\,, \qquad x \in \gamma, y \in \Omega_\varepsilon \end{aligned}$$

and

$$\begin{aligned}(2.9)\qquad & \frac{\partial}{\partial \nu_x} q_\varepsilon(x, y)_{|x=(\varepsilon,0)} - \frac{\partial}{\partial x_1} G(x, y)_{|x=(\varepsilon,0)} \\ & \quad -2\pi\varepsilon^2 \frac{\partial}{\partial x_1} \langle \nabla_w S(x, w), \nabla_w G(w, y)\rangle_{|w=\tilde{w}=0, x=(\varepsilon,0)} \\ & = 2\pi\varepsilon^2 \frac{\partial}{\partial x_1}\Big(\frac{1}{2\pi}\frac{\partial}{\partial w_1} \log|x-w| \cdot \frac{\partial}{\partial w_1} G(w, y) \\ & \quad + \frac{1}{2\pi}\frac{\partial}{\partial w_2} \log|x-w| \cdot \frac{\partial}{\partial w_2} G(w, y)\Big)_{|w=\tilde{w}=0, x=(\varepsilon,0)} \\ & = -\frac{\partial}{\partial w_1} G(w, y)_{|w=\tilde{w}=0}\,, \end{aligned}$$

where $S(x, y) = G(x, y) + (1/2\pi)\log|x-y|$. And using (2.9) the $L^p(\Omega_\varepsilon)$-norm of the operator $\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon$ will be estimated in the section 4.

## 3. Preliminary lemmas

We recall the following:

**Lemma 1** (Ozawa [5]). *Assume that $u_\varepsilon \in C^\infty(\bar{\Omega}_\varepsilon)$ is harmonic in $\Omega_\varepsilon$, $u_\varepsilon(x) = 0$ for $x \in \gamma$ and*

$$\max\{|\partial u_\varepsilon(x)/\partial\nu|\,;\, x \in \partial B_\varepsilon\} = M\,.$$

*Then,*

$$|u_\varepsilon(x)| \le C\varepsilon M(1 + |\log(|x-w|/\varepsilon)|)\,, \qquad x \in \Omega_\varepsilon$$

*holds for a constant $C$ independent of $\varepsilon$.*

For any periodic function $\alpha(\theta)$ of $\theta \in [0, 2\pi]$ with the Fourier expansion

$$\alpha(\theta) = u_0 + \sum_{k=1}^{\infty} (u_k \sin k\theta + t_k \cos k\theta)\,,$$

we put

$$K_\vartheta(\alpha) = \sum_{k=1}^{\infty} k^\vartheta (u_k^2 + t_k^2)^{1/2}\,.$$

**Lemma 2.** *Consider the equation*

$$(3.1)\qquad \Delta v(x) = 0\,, \qquad x \in \boldsymbol{R}^2 \backslash \bar{B}_1$$

$$(3.2) \qquad \frac{\partial v}{\partial \nu}(x)_{|x=(\cos\theta, \sin\theta)} = \alpha(\theta)$$

*for given* $\alpha(\theta)$. *Then, there exists at least one solution* $v$ *of* (3.1), (3.2) *satisfying*

$$(3.3) \qquad |v(x)| \leq C \max_\theta |\alpha(\theta)|(1+|\log|x||)$$

*and*

$$(3.4) \qquad \max_{x\in\partial B_1} |\text{grad } v(x)| \leq C_\vartheta(\max_\theta |\alpha(\theta)|) K_\vartheta(\alpha)$$

*for* $\vartheta \in (1, \infty)$.

Proof. We know that

$$u_0^2 + \sum_{k=1}^{\infty} (u_k^2 + t_k^2) \leq 2\pi \max_\theta |\alpha(\theta)|^2 .$$

Put

$$v(x) = u_0 \log r + \sum_{k=1}^{\infty} (-k)^{-1}(u_k \sin k\theta + t_k \cos k\theta) r^{-k} .$$

Then, $v(x)$ satisfies (3.1), (3.2), (3.3) and (3.4). q.e.d.

**Lemma 3.** *Fix* $q \in (1/2, \infty)$. *Then, under the same assumption as in Lemma* 1,

$$\max_{x\in\partial B_\varepsilon} |\text{grad } u_\varepsilon(x)| \leq C\left(M + K_{2q}\left(\left(\frac{\partial u_\varepsilon}{\partial \nu}(z)\right)_{|z=(\varepsilon\cos\cdot,\, \varepsilon\sin\cdot)}\right)\right) .$$

Proof. In the following we write $(\varepsilon \cos\theta, \varepsilon \sin\theta) = \varepsilon e(\theta)$.

Applying the similarity transformation of coordinates to Lemma 1, we have the following:
There exists at least one solution of

$$\Delta v_\varepsilon(x) = 0 , \qquad x \in \boldsymbol{R}^2 \backslash \bar{B}_\varepsilon$$
$$\left(\frac{\partial v_\varepsilon}{\partial \nu_z}\right)(\varepsilon e(\theta)) = \left(\frac{\partial u_\varepsilon}{\partial \nu_z}\right)(\varepsilon e(\theta)) , \qquad \theta \in S^1 \ (=\partial B_1)$$

satisfying

$$|v_\varepsilon(x)|_{x\in\partial B_\varepsilon} \leq C\varepsilon \max_\theta \left|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\theta))\right| (1+|\log(|x-\tilde{w}|/\varepsilon|))$$

and

$$\max_\theta |\text{grad } v_\varepsilon(z)|_{z=\varepsilon e(\theta)} \leq C\left(\max_\theta \left|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\theta))\right| + K_{2q}\left(\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\cdot))\right)\right)$$

for $q \in (1/2, \infty)$.

Then, the function $v_\varepsilon$ may not satisfy $v_\varepsilon(x)=0$ for $x \in \gamma$. Overcome this difficulty, we apply the same argument as in Ozawa [5; Proposition 1], and we obtain the desired result. q.e.d.

We wish to replace the semi-norm $K_g(\alpha)$ by a Hölder norm. To do this we let $H^{q,2}(S^1)$ denote the $L^2$-Sobolev space of order $q$. Here $q$ may not be an integer. It is well known that

$$C_1\|\alpha\|_{H^{q,2}(S^1)}\leq\|\alpha\|_{L^2(S^1)}+K_{2q}(\alpha)$$
$$\leq C_2\|\alpha\|_{H^{q,2}(S^1)}$$

holds for a constant $C_1$, $C_2$ independent of $\alpha$ if $q\geq 0$. We know that $H^{q,2}(S^1)$-norm of $u$ is equivalent to the following norm:

$$\|u\|_{L^2(S^1)}+\left(\iint_{S^1\times S^1}|u(x)-u(y)|^2|x-y|^{-2q-1}dx\,dy\right)^{1/2}$$

when $0<q<1$. See, for example Adams [1]. Thus, we have

$$\|u\|_{H^{q,2}(S^1)}\leq C(\|u\|_{L^2(S^1)}+\|u\|_{C^{q+\sigma}(S^1)})$$

for any $\sigma>0$. Here $\|\ \|_{C^\mu(S^1)}$ denotes the usual Hölder norm on $S^1$.

We know the interpolation inequality

$$\|u\|_{C^\mu(S^1)}\leq C\|u\|_{C^0(S^1)}^{1-(\mu/\tilde{\mu})}\,\|u\|_{C^{\tilde{\mu}}(S^1)}^{(\mu/\tilde{\mu})}$$

for any $0<\mu\leq\tilde{\mu}<1$.

Summing up these facts, we get

$$K_{2q}(\alpha)\leq C(\|\alpha\|_{L^2(S^1)}+\|\alpha\|_{C^0(S^1)}^{1-(\xi'/\xi)}\|\alpha\|_{C^\xi(S^1)}^{(\xi'/\xi)})$$

for $q\in(1/2, 1)$, $1/2<\xi'<\xi<1$.

Applying this to Lemma 3 we get the following

Corollary 1. Fix $1/2<\xi'<\xi<1$. Under the assumption of Lemma 1,

$$\max_{x\in\partial B_\varepsilon}|\mathrm{grad}\, u_\varepsilon(x)|\leq C(M+M^{1-(\xi'/\xi)}L_\xi(\varepsilon)^{(\xi'/\xi)})\,. \tag{3.5}$$

Here

$$L_\xi(\varepsilon)=\left\|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(z)_{|z=\varepsilon e(\cdot)}\right\|_{C^\xi(S^1)}.$$

## 4. Approximate Green's function $r_\varepsilon(x, y)$

We use the following properties of the Green function frequently, so we here write them:

$$|G(x, y)|\leq C\,|\log|x-y||\tag{4.1}$$

$$|\nabla_x G(x, y)|\leq C\,|x-y|^{-1}\,.\tag{4.2}$$

Thus,

$$(4.3) \qquad |(\boldsymbol{G}f)(x)| \leq C\|f\|_{L^p(\Omega)} \qquad (p>1)$$

$$(4.4) \qquad |\operatorname{grad}_x(\boldsymbol{G}f)(x)| \leq C\|f\|_{L^p(\Omega)} \qquad (p>2)\,.$$

First we obtain the following

**Lemma 5.** *Let $p\in(2, \infty)$. Then, there exists a constant $C>0$ independent of $\varepsilon$ such that*

$$|||\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon|||_{L^p(\Omega_\varepsilon)} \leq C\varepsilon^{2-(2/p)}|\log\varepsilon|\,.$$

Proof. Fix $f\in C_0^\infty(\Omega_\varepsilon)$. Then $g_\varepsilon=(\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon)f$ satisfies $\Delta g_\varepsilon(x)=0$ for $x\in\Omega_\varepsilon$ and $g_\varepsilon(x)=0$ for $x\in\gamma$.

By (2.9) we have

$$(4.5) \qquad \frac{\partial}{\partial\nu}g_\varepsilon(x)\Big|_{x=(\varepsilon,0)}$$

$$=\frac{\partial}{\partial x_1}(\boldsymbol{G}f)(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}f)(w)+2\pi\varepsilon^2\frac{\partial}{\partial x_1}\langle\nabla_w S(x, w), \nabla_w(\boldsymbol{G}f)(w)\rangle$$

for $w=\tilde{w}$ $(=0)$.

By the Sobolev embedding theorem we have

$$(4.6) \qquad \|\boldsymbol{G}f\|_{C^{1+\alpha}(\Omega)} \leq C\|f\|_{L^p(\Omega_\varepsilon)}$$

if $\alpha=1-(2/p)$, $2<p<\infty$. Here $\|\ \|_{L^p(\Omega_\varepsilon)}$ denotes the $L^p(\Omega_\varepsilon)$-norm. Therefore, (4.5) and (4.6) imply

$$\max_{x\in\partial B_\varepsilon}\left|\frac{\partial}{\partial\nu}g_\varepsilon(x)\right| \leq C\varepsilon^{1-(2/p)}\|f\|_{L^p(\Omega_\varepsilon)}\,.$$

By Lemma 1 we get the desired result. q.e.d.

The next lemma is stated in the introduction.

**Lemma 6.** *Fix $p\in(1, \infty]$. Then,*

$$|||\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon|||_{L^p(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

*holds for any fixed $s>0$ as $\varepsilon$ tends to zero.*

Proof. Assume that $p\in(1, \infty)$. Put $\boldsymbol{Q}_\varepsilon=\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon$. The operator $\boldsymbol{Q}_\varepsilon$ is self-adjoint on $L^2(\Omega_\varepsilon)$. Thus, we get

$$|||\boldsymbol{Q}_\varepsilon|||_{L^q(\Omega_\varepsilon)} = |||\boldsymbol{Q}_\varepsilon|||_{L^{q'}(\Omega_\varepsilon)} \qquad (q^{-1}+(q')^{-1}=1)\,.$$

By the Riesz-Thorin interpolation theorem we know that

$$|||\boldsymbol{Q}_\varepsilon|||_{L^p(\Omega_\varepsilon)} \leq |||\boldsymbol{Q}_\varepsilon|||_{L^q(\Omega_\varepsilon)}$$

for any $p \in (q', q)$, $q > 2$. We take sufficiently large $q > 2$ and apply Lemma 5. Then we have Lemma 6 for $p \neq 1, \infty$.

Assume that $p = \infty$. Then, we get Lemma 6 with $p = \infty$ by the same argument as in the proof of Lemma 5. q.e.d.

Now we wish to compare $\boldsymbol{R}_\varepsilon$ with $\tilde{\boldsymbol{R}}_\varepsilon$. We denote by $\hat{\chi}_\varepsilon$ the characteristic function of the set $B_\varepsilon$. Then, $\hat{\chi}_\varepsilon = 1 - \chi_\varepsilon$.

We have the following

**Lemma 7.** *Let $p \in (1, \infty)$, $q \in (2, \infty)$ and $r \in (2, \infty)$. Then, there exists a constant $C$ such that for any $v \in L^q(\Omega)$*

$$\begin{aligned} &||\tilde{\boldsymbol{R}}_\varepsilon v - \boldsymbol{R}_\varepsilon(\chi_\varepsilon v)||_{L^p(\Omega_\varepsilon)} \\ &\leq C(\varepsilon^{2-(2/q)} |\log \varepsilon| \, ||v||_{L^q(\Omega)} + \varepsilon^{(2/r')} |\log \varepsilon| \, ||v||_{L^r(B_\varepsilon)}) . \end{aligned}$$

Proof. Put $k_\varepsilon = \chi_\varepsilon \tilde{\boldsymbol{R}}_\varepsilon v - \boldsymbol{R}_\varepsilon(\chi_\varepsilon v)$. Then, $\Delta_x k_\varepsilon(x) = 0$ for $x \in \Omega_\varepsilon$ and $k_\varepsilon(x) = 0$ for $x \in \gamma$.

We have

$$\begin{aligned} (4.7) \qquad & \frac{\partial}{\partial \nu} k_\varepsilon(x)_{|x=(\varepsilon,0)} \\ &= \frac{\partial}{\partial x_1} (\boldsymbol{G}(\hat{\chi}_\varepsilon v))(x)_{|x=(\varepsilon,0)} - \frac{\partial}{\partial w_1} (\boldsymbol{G}(\hat{\chi}_\varepsilon \xi_\varepsilon v))(\tilde{w}) \\ &\quad + 2\pi\varepsilon^2 \frac{\partial}{\partial x_1} \langle \nabla_w S(x, w), \nabla_w \boldsymbol{G}(\hat{\chi}_\varepsilon \xi_\varepsilon v)(w) \rangle_{|x=(\varepsilon,0), w=\tilde{w}} . \end{aligned}$$

The first term minus the second term in the right hand side of (4.7) does not exceed

$$\varepsilon^\theta ||\boldsymbol{G}(\hat{\chi}_\varepsilon v)||_{C^{1+\theta}(\Omega)} + \left| \frac{\partial}{\partial w_1} (\boldsymbol{G}(\hat{\chi}_\varepsilon (1-\xi_\varepsilon)) v))(\tilde{w}) \right|$$

for $\theta \in (0, 1)$. By (4.2) we see that

$$\begin{aligned} &| \nabla_w \boldsymbol{G}(\hat{\chi}_\varepsilon \xi_\varepsilon v)(\tilde{w}) | + | \nabla_w \boldsymbol{G}(\hat{\chi}_\varepsilon (1-\xi_\varepsilon) v)(\tilde{w}) | \\ &\leq C \varepsilon^{(2/r')-1} ||v||_{L^r(B_\varepsilon)} , \end{aligned}$$

where $(r')^{-1} = 1 - r^{-1}$. Thus, Lemma 7 follows from these estimates and Lemma 1. q.e.d.

The following Lemma 8 asserts that $\varphi_j(\varepsilon)$ behaves well even in $L^p$ space as $\varepsilon$ goes to zero.

**Lemma 8.** *Fix $j$ and $p \in (1, \infty]$. Then,*

$$||\varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)} \leq C_p < \infty$$

*holds for a constant $C_p$ independent of $\varepsilon$.*

Proof. We devide $\varphi_j(\varepsilon)$ as follows:

$$\varphi_j(\varepsilon) = \mu_j(\varepsilon)^{-1}(\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)) + \mu_j(\varepsilon)^{-1}((\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)\varphi_j(\varepsilon))\,. \tag{4.8}$$

Rauch-Taylor [8] proved that

$$\lim_{\varepsilon \to 0} \mu_j(\varepsilon) = \mu_j\,. \tag{4.9}$$

By Lemma 6 we have

$$||\mu_j(\varepsilon)^{-1}(\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)\varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)} \leq 0(\varepsilon^{2-s})||\varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)}\,.$$

This together with (4.8) proves that

$$||\varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)} \leq C||\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)}\,.$$

By the definition of $\boldsymbol{R}_\varepsilon$ we have

$$||\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)} \leq C_p^*(1+\varepsilon|\log\varepsilon|^{1/2})||\varphi_j(\varepsilon)||_{L^2(\Omega_\varepsilon)}$$

for $p \in (1, \infty]$. Since $\varphi_j(\varepsilon)$ is a normalized eigenfunction we get the desired result. q.e.d.

## 5. An approximate eigenfunction of $\tilde{\boldsymbol{R}}_\varepsilon$

Let $\boldsymbol{G}_w$ denote the functional $v(x) \mapsto (\boldsymbol{G}v)(w)$. Put

$$A(\varepsilon)\colon v \mapsto 2\pi\langle \nabla_w G(\cdot, w), \nabla_w \boldsymbol{G}_w(\xi_\varepsilon v)\rangle|_{w=\tilde{w}}\,.$$

Then, $\tilde{\boldsymbol{R}}_\varepsilon = \boldsymbol{G} + \varepsilon^2 A(\varepsilon)$. We wish to construct an approximate eigenvalue $\lambda^*(\varepsilon)$ and an approximate eigenfunction $\psi^*(\varepsilon)$ of $\tilde{\boldsymbol{R}}_\varepsilon$ in such a way that

$$||(\tilde{\boldsymbol{R}}_\varepsilon - \lambda^*(\varepsilon))\psi^*(\varepsilon)||_{L^2(\Omega)} = o(\varepsilon^2) \tag{5.1}$$

and

$$||\psi^*(\varepsilon)||_{L^2(\Omega)} = 1 + O(\varepsilon^2|\log\varepsilon|) \tag{5.2}$$

By virtue of perturbation theory, we may take

$$\lambda^*(\varepsilon) = \mu_j^{-1} + \varepsilon^2\lambda(\varepsilon)\,,$$

where $\lambda(\varepsilon) = (A(\varepsilon)\varphi_j, \varphi_j)_{L^2}$. Here $(\ ,\ )_{L^2}$ denotes the inner product on $L^2(\Omega)$. And we may assume that $\psi^*(\varepsilon)$ is of the form

$$\psi^*(\varepsilon) = \varphi_j + \varepsilon^2\psi(\varepsilon)\,,$$

where $\psi(\varepsilon)$ should satisfy (5.3) and (5.4):

$$(5.3)\qquad (\boldsymbol{G}-\mu_j^{-1})\psi(\varepsilon)=(\lambda(\varepsilon)-A(\varepsilon))\varphi_j$$

$$(5.4)\qquad \int_\Omega (\psi(\varepsilon))(x)\varphi_j(x)dx=0\,.$$

Note that $\boldsymbol{G}$ is a compact operator and that the right hand side of (5.3) is orthogonal to $\varphi_j$. Thus, the unique solution $\psi(\varepsilon)$ of (5.3), (5.4) exists.
We see that

$$(5.5)\qquad (\tilde{\boldsymbol{R}}_\varepsilon-\lambda^*(\varepsilon))\psi^*(\varepsilon)=\varepsilon^4(A(\varepsilon)-\lambda(\varepsilon))\psi(\varepsilon)\,.$$

To estimate the left hand sides of (5.1) and (5.2), we need the following

**Lemma 9.** *For a constant $C$ independent of $\varepsilon$, we have*

$$(5.6)\qquad |||A(\varepsilon)|||_{L^p(\Omega)}\le C\varepsilon^{(2-p)/p}|\log\varepsilon|^{1/2}\,,\qquad (p>2)$$

$$(5.7)\qquad |||A(\varepsilon)|||_{L^2(\Omega)}\le C|\log\varepsilon|$$

*and*

$$||\psi(\varepsilon)||_{L^p(\Omega)}\le C\varepsilon^{(2-p)/p}|\log\varepsilon|^{1/2}\,,\qquad (p>2)$$

$$||\psi(\varepsilon)||_{L^2(\Omega)}\le C|\log\varepsilon|\,.$$

Proof. By a Hölder inequality and (4.1) we obtain (5.6) and (5.7). Using (5.7) we have

$$\begin{aligned}||(\lambda(\varepsilon)-A(\varepsilon))\varphi_j||_{L^2(\Omega)}&\le C'|||A(\varepsilon)|||_{L^2(\Omega)}\\ &\le C|\log\varepsilon|\,.\end{aligned}$$

Thus, by virtue of the Fredholm theory we obtain a bound for $L^2(\Omega)$-norm of $\psi(\varepsilon)$. Similarly we get $L^p$ estimates. q.e.d.

By (5.5) and Lemma 9 we have the following fact, which is stronger than (5.1).

**Lemma 10.** *For a constant $C$ independent of $\varepsilon$*

$$(5.8)\qquad ||(\tilde{\boldsymbol{R}}_\varepsilon-\lambda^*(\varepsilon))\psi^*(\varepsilon)||_{L^2(\Omega)}\le C\varepsilon^4|\log\varepsilon|^2\,.$$

Since $\boldsymbol{G}_\varepsilon$ is approximated by $\boldsymbol{R}_\varepsilon$ (Lemma 6) and $\boldsymbol{R}_\varepsilon$ is approximated by $\tilde{\boldsymbol{R}}_\varepsilon$ (Lemma 7), we may consider $\psi^*(\varepsilon)$ as an approximate eigenfunction of $\boldsymbol{G}_\varepsilon$. More precisely we have

**Lemma 11.** *For a constant $C$ independent of $\varepsilon$*

$$(5.9)\qquad ||(\boldsymbol{G}_\varepsilon-\lambda^*(\varepsilon))(\chi_\varepsilon\psi^*(\varepsilon))||_{L^2(\Omega_\varepsilon)}=0(\varepsilon^{2-s})$$

*holds, where $s$ being an arbitrary fixed positive constant.*

Proof. We see that the left hand side of (5.6) does not exceed

$$(5.10)\qquad \|(\boldsymbol{G}_\varepsilon-\boldsymbol{R}_\varepsilon)(\chi_\varepsilon\psi^*(\varepsilon))\|_{L^2(\Omega_\varepsilon)}+\|\tilde{\boldsymbol{R}}_\varepsilon\psi^*(\varepsilon)-\boldsymbol{R}_\varepsilon(\chi_\varepsilon\psi^*(\varepsilon))\|_{L^2(\Omega_\varepsilon)} +\|(\tilde{\boldsymbol{R}}_\varepsilon-\lambda^*(\varepsilon))\psi^*(\varepsilon)\|_{L^2(\Omega_\varepsilon)}\,.$$

The last term is estimated by Lemma 10. By Lemma 7, the second term of (5.10) does not exceed

$$C\varepsilon^{2-(2/q)}|\log\varepsilon|\,\|\psi^*(\varepsilon)\|_{L^q(\Omega)}+C\varepsilon^{(2/r')}|\log\varepsilon|\,\|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)}\,.$$

We see from the definition of $\psi^*(\varepsilon)$ that

$$\|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)}\le\|\varphi_j\|_{L^r(B_\varepsilon)}+\varepsilon^2\|\psi(\varepsilon)\|_{L^r(B_\varepsilon)}\,.$$

We apply Lemma 9 to this and we have

$$\|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)}\le C(\varepsilon^{3/r}+\varepsilon^{2+(2-r)/r}|\log\varepsilon|^{1/2})$$

for $r>2$. Thus, the second term of (5.10) is $0(\varepsilon^{2-s})$. The first term of (5.10) is also $0(\varepsilon^{2-s})$, since we have Lemma 6 and $\|\psi^*(\varepsilon)\|_{L^2(\Omega)}=0(1)$. Summing up these facts we obtain (5.9). q.e.d.

The next Lemma states that $\mu_j(\varepsilon)$ is close to $\lambda^*(\varepsilon)$ and $\varphi_j(\varepsilon)$ is close to $\chi_\varepsilon\psi^*(\varepsilon)$.

**Lemma 12.** *Under the same assumption as in Theorem* 1

$$(5.11)\qquad \lambda^*(\varepsilon)-\mu_j(\varepsilon)=0(\varepsilon^{2-s})$$

*and*

$$(5.12)\qquad \|\varphi_j(\varepsilon)-t_\varepsilon\chi_\varepsilon\psi^*(\varepsilon)\|_{L^2(\Omega_\varepsilon)}=0(\varepsilon^{2-s})$$

*hold.*

Proof. We know from (5.9) and a spectral theory of compact self-adjoint operator that there exists at least one eigenvalue $\lambda_*(\varepsilon)$ of $\boldsymbol{G}_\varepsilon$ satisfying

$$\lambda_*(\varepsilon)-\lambda^*(\varepsilon)=0(\varepsilon^{2-s})\,.$$

Rauch-Taylor [8] showed that $\mu_k(\varepsilon)$ tends to $\mu_k$ as $\varepsilon$ tends to zero for any $k$. Thus, we get $\lambda_*(\varepsilon)=\mu_j(\varepsilon)^{-1}$.

By the eigenfunction expansion

$$\boldsymbol{G}_\varepsilon f=\sum_{k=1}^{\infty}\mu_k(\varepsilon)^{-1}\langle\varphi_k(\varepsilon),f\rangle\varphi_k(\varepsilon)\,,$$

we have

$$\|(\boldsymbol{G}_\varepsilon-\lambda^*(\varepsilon))(\chi_\varepsilon\psi^*(\varepsilon))\|^2_{L^2(\Omega_\varepsilon)} =\sum_{k=1}^{\infty}|\mu_k(\varepsilon)^{-1}-\lambda^*(\varepsilon)|^2|\langle\varphi_k(\varepsilon),\chi_\varepsilon\psi^*(\varepsilon)\rangle|^2\,.$$

Since $\lambda^*(\varepsilon)\to\mu_j^{-1}$ and $\mu_k(\varepsilon)^{-1}\to\mu_k^{-1}$ as $\varepsilon\to 0$, we have

$$\sum_{k=1,k\neq j}^{\infty}|\langle\varphi_k(\varepsilon),\ \chi_\varepsilon\psi^*(\varepsilon)\rangle|^2=0(\varepsilon^{4-2s})\,.$$

This implies

$$\|\chi_\varepsilon\psi^*(\varepsilon)-\langle\varphi_j(\varepsilon\ ),\ \chi_\varepsilon\psi^*(\varepsilon)\rangle\varphi_j(\varepsilon)\|_{L(\Omega_\varepsilon)}=0(\varepsilon^{2-s})\,.$$

Thus,

$$|\langle\varphi_j(\varepsilon),\ \chi_\varepsilon\psi^*(\varepsilon)\rangle^2-1|=0(\varepsilon^{4-2s})$$

and we obtain (5.12). q.e.d.

## 6. Proof of (2.1)~(2.5)

In this section we shall complete the proof of Theorems 1, 2 by giving proofs of (2.1)~(2.5).

Recall the definition of $J_k(\varepsilon)$.

$$\begin{aligned}J_1(\varepsilon)&=\mu_j(\varepsilon)(\boldsymbol{G}_\varepsilon-\boldsymbol{R}_\varepsilon)(\varphi_j(\varepsilon))\\ J_2(\varepsilon)&=\mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\varphi_j(\varepsilon)-\chi_\varepsilon\psi^*(\varepsilon))\\ J_3(\varepsilon)&=\mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\chi_\varepsilon\psi^*(\varepsilon))\,.\end{aligned}$$

Here we should state that we choose $\varphi_j(\varepsilon)$ so that $t_\varepsilon=1$, because we see in the final part of the section 5 that $t_\varepsilon^2=1$ for small $\varepsilon>0$.

**Lemma 13.** *Fix an arbitrary* $s>0$. *Then,*

$$\|J_1(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}=0(\varepsilon^{2-s})$$

*and* (2.3) *hold.*

Proof. Let $\tilde{\varphi}_j(\varepsilon)$ be the extension of $\varphi_j(\varepsilon)$ to $\Omega$ putting its value zero on $B_\varepsilon$. We know that $J_1(\varepsilon)$ is harmonic in $\Omega_\varepsilon$ and zero on $\gamma$. We have

$$\begin{aligned}(6.1)\qquad &\mu_1(\varepsilon)\frac{\partial}{\partial\nu_z}(J_1(\varepsilon))(z)_{|z=\varepsilon e(\theta)}\\ &=\frac{\partial}{\partial r}((\boldsymbol{G}\tilde{\varphi}_j(\varepsilon)))(r\cos\theta,\ r\sin\theta)_{|r=\varepsilon}\\ &\quad-\frac{\partial}{\partial r}((\boldsymbol{G}\tilde{\varphi}_j(\varepsilon))(r\cos\theta,\ r\sin\theta))_{|r=0}\\ &\quad+2\pi\varepsilon^2\left(\frac{\partial}{\partial r}\langle\nabla_w S(x,\ w),\ \nabla_w(\boldsymbol{G}\tilde{\varphi}_j(\varepsilon))(w)\rangle_{|x=\varepsilon e(\theta),w=\tilde{w}}\right.\,.\end{aligned}$$

Thus, by the same argument as in the proof of Lemma 5 we have

$$(6.2)\qquad \max_{x\in\partial B_\varepsilon}\left|\frac{\partial}{\partial\nu}(J_1(\varepsilon))(x)\right|\leq C\varepsilon^{1-(2/p)}\|\varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)}$$

for $p>2$. By Lemma 8 we see that (6.2) does not exceed $C'\varepsilon^{1-(2/p)}$. This fact together with Lemma 1 show that

$$||J_1(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}).$$

We now wish to apply Corollary 1 to $J_1(\varepsilon)$ to prove (2.3). We know that $S(x, w) \in C^\infty(\Omega)$. Then, $C^\xi(S^1)$ norm of the third term in the right hand side of (6.1) (considering it as a function of $\theta$) does not exceed $C$. Here we used (4.4) and Lemma 8. By the fact

$$||\boldsymbol{G}f||_{C^{1+\xi}(\Omega)} \leq C||f||_{L^\infty(\Omega)} \qquad (\xi<1)$$

we see that the $C^\xi(S^1)$ norm of the first and the second term in the right hand side of (6.1) do not exceed $C_\xi''$ for $\xi<1$. From Corollary 1 we obtain

$$\max_{z\in\partial B_\varepsilon} |\mathrm{grad}_z\,(J_1(\varepsilon))(z)| \leq C(\varepsilon^{1-s} + C_\xi(\varepsilon^{1-s})^{(1-(\xi/\xi'))}). \tag{6.3}$$

We take $\xi'>1/2$, $\xi<1$ such that $|\xi'-1/2|+|\xi-1|$ is sufficiently small and we get (2.3). q.e.d.

We have the following

**Lemma 14.** *Fix an arbitrary* $s>0$. *Then*

$$||J_2(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) \tag{6.4}$$

*and* (2.4) *hold.*

Proof. Put $\chi_\varepsilon = \varphi_j(\varepsilon) - \chi_\varepsilon \psi^*(\varepsilon)$. Then, $J_2(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon \kappa_\varepsilon$. By the definition of $\boldsymbol{R}_\varepsilon$ and (4.2), (4.3) and (4.4) we have

$$||J_2(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} \leq C(||\kappa_\varepsilon||_{L^2(\Omega_\varepsilon)} + \varepsilon||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)}) \tag{6.5}$$

for $p\in(2, \infty)$. Lemma 8 asserts that

$$||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)} \leq C', \qquad p\in(2, \infty), \tag{6.6}$$

while Lemma 12 gives us the estimate

$$||\kappa_\varepsilon||_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}). \tag{6.7}$$

Let $s'$ be an arbitrary fixed number. Then, by the Riesz-Thorin interpolation theorem we get

$$||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)} = 0(\varepsilon^{2-s'}) \tag{6.8}$$

for $p>2$ close to 2. Thus, (6.4) is proved by (6.5), (6.6) and (6.7).

By the definition of $J_2(\varepsilon)$,

(6.9) $$|\partial_{x_i}\partial_{x_j}G(x, y)| \leq C|x-y|^{-2}$$

and (4.4) we have

$$\max_{z\in\partial B_\varepsilon}|\mathrm{grad}_z\,(J_2(\varepsilon))(z)| \leq C\|\kappa_\varepsilon\|_{L^p(\Omega_\varepsilon)}$$

for $p\in(2, \infty)$. Thus, (2.4) is proved by (6.8). q.e.d.

Finally we have the following

**Lemma 15.** *Fix an arbitrary $s>0$. Then,* (2.2) *and* (2.5) *hold.*

Proof. We see that $\mu_j(\varepsilon)^{-1}J_3(\varepsilon)$ can be written as $\Pi(\varepsilon)+\Pi'(\varepsilon)$. Here

$$\Pi(\varepsilon) = \boldsymbol{G}\varphi_j+2\pi\varepsilon^2\langle\nabla_w G(\cdot, w), \nabla_w \boldsymbol{G}(\chi_\varepsilon\varphi_j)(w)\rangle_{|w=\tilde{w}}$$

and

$$\Pi'(\varepsilon) = \boldsymbol{G}((\chi_\varepsilon-1)\varphi_j)+\varepsilon^2\boldsymbol{G}(\chi_\varepsilon\psi(\varepsilon)) +2\pi\varepsilon^4\langle\nabla_w G(\cdot, w), \nabla_w \boldsymbol{G}(\chi_\varepsilon\psi(\varepsilon))(w)\rangle_{|w=\tilde{w}}\,.$$

We have

(6.10) $$\|\Pi'(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} \leq C(\|\varphi_j\|_{L^p(B_\varepsilon)}+\varepsilon^2\|\psi(\varepsilon)\|_{L^r(\Omega)})$$

for $p>1$, $r>2$. Thus, (6.10) is estimated by Lemma 9 and we get

(6.11) $$\|\Pi'(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)}=0(\varepsilon^{2-s})$$

for any $s>0$.

On the other hand, by (4.4) we have

(6.12) $$\|\Pi(\varepsilon)-\mu_j^{-1}\varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon)\,.$$

Thus, (6.11) and (6.12) imply (2.2).

We wish to show (2.5). By (4.4) and (6.9) we see that $\max\{|\mathrm{grad}_z\,(\Pi'(\varepsilon))(z)|\,;\, z\in\partial B_\varepsilon\}$ does not exceed

$$C(\|\varphi_j\|_{L^r(B_\varepsilon)}+\varepsilon^2\|\psi(\varepsilon)\|_{L^r(\Omega)})$$

for $r>2$. Thus,

(6.13) $$\max_{z\in\partial B_\varepsilon}|\mathrm{grad}_z\,(\Pi'(\varepsilon))(z)| = 0(\varepsilon^{1-s})$$

by Lemma 9. By the similar calculation as in (2.9) we see that

(6.14) $$\Big(\frac{\partial}{\partial\theta}(2\pi\varepsilon^2\langle\nabla_w G(\cdot, w), \nabla_w(\boldsymbol{G}\varphi_j)(w)_{|w=\tilde{w}})\Big)(\varepsilon\cos\theta, \varepsilon\sin\theta) = \mu_j^{-1}\Big(\frac{\partial}{\partial\theta}\varphi_j\Big)(\varepsilon\cos\theta, \varepsilon\sin\theta)+0(\varepsilon^2)|\nabla_w(\boldsymbol{G}\varphi_j)(\tilde{w})|\,.$$

Thus,

$$\left(\frac{\partial}{\partial\theta}(\Pi(\varepsilon)-\mu_j^{-1}\varphi_j)\right)(\varepsilon\cos\theta,\,\varepsilon\sin\theta) \tag{6.15}$$

$$=\mu_j^{-1}\left(\frac{\partial}{\partial\theta}\varphi_j\right)(\varepsilon\cos\theta,\,\varepsilon\sin\theta)+0(\varepsilon^2)\|\varphi_j\|_{L^r(\Omega)}$$

$$+0(1)|\nabla_w(G(\hat{\chi}_\varepsilon\varphi_j))(\tilde{w})|$$

for $r>2$. Thus, by Lemma 9, (4.4), (6.15) and

$$\left(\frac{\partial}{\partial\theta}\varphi_j\right)(\varepsilon\cos\theta,\,\varepsilon\sin\theta) \tag{6.16}$$

$$=\frac{\theta}{\partial r}(\varphi_j(r\cos(\theta+(\pi/2)),\,r\sin(\theta+(\pi/2)))_{|r=0}+0(\varepsilon)\,,$$

we get (2.5). q.e.d.

We have thus proved all of (2.1)~(2.5) which were stated in the section 2. Therefore our proofs of Theorem 1 and 2 are complete.

---

## References


[1] R. Adams: Sobolev spaces, Academic Press, New York, 1975.

[2] R. Figari, E. Orlandi and S. Teta: *The Laplacian in regions with many small obstacles, Fluctuation around the limit operator*, preprint (1984).

[3] P.R. Garabedian: Partial differential equations, J. Wiley and Sons, Inc., New York, 1964.

[4] P.R. Garabedian and M. Schiffer: *Convexity of domain functionals,* J. Analyse Math. **2** (1952–53), 281–369.

[5] S. Ozawa: *Spectra of domains with small spherical Neumann boundary,* J. Fac. Sci. Univ. Tokyo Sec. IA, **30** (1983), 259–277.

[6] S. Ozawa: *Electrostatic capacity and eigenvalues of the Laplacian*, Ibid. **30** (1983), 53–62.

[7] S. Ozawa: *Random media and eigenvalues of the Laplacian*, Comm. Math. Phys. **94** (1984), 421–437.

[8] J. Rauch and M. Taylor: *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal. **18** (1975), 27–59.



Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560, Japan