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In this paper we assume that every ring $R$ is an associative ring with identity and $R$ is two-sided artinian. The author has defined almost projective modules and almost injective modules in [8], and by making use of the concept of almost projectives he has defined almost hereditary rings in [7], whose class contains that of hereditary rings and serial rings. Similarly to [7] we shall define an almost QF ring, which is a generalization of QF rings.

It is well known that an artinian ring $R$ is QF if and only if $R$ is self-injective. Following this fact, if $R$ is almost injective as a right $\Lambda$-module, we call $R$ a right almost QF ring. Analogously we call $R$ a right almost QF* ring if every injective is right almost projective. On the other hand, the author studied rings with $(\ast)$ (resp. $(\ast)^*$) (see §1 for definitions) in [4]. K. Oshiro called such a ring a right $H$- (resp. co-$H$) ring in [10]. In this note we shall show that a right almost QF (resp. almost QF*) ring coincides with a right co-$H$ (resp. $H$-) ring. In the final section we shall give a characterization of serial rings in terms of almost projectives and almost injectives.

In the forthcoming paper [9] we shall study certain conditions under which right almost QF rings are QF or serial.

1. Almost QF rings

In this paper we always assume that $R$ is a two-sided artinian ring with identity and that every module is a unitary right $R$-module. We use the same notations in [7]. We have studied almost hereditary rings in [7], i.e. $J$, the Jacobson radical of $R$, is right almost projective. We shall study, in this paper, some kind of the dual concept to almost hereditary rings (see Theorem 1 below). We call $R$ a right almost QF ring if $R$ is right almost injective as a right $R$-module [8]. We can define similarly a left almost QF ring. It is clear that $R$ is right almost QF if and only if every finitely generated projective $R$-module is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that $R$ is basic.

On the other hand, the author studied the two conditions in [3] and [4]. Let $M$ be an $R$-module. If $MSoc^{1}(R) \neq 0$ (resp. $M \text{ Soc}^{1}(R) \neq 0$) then we call $M$ non-
small (resp. non-cosmall) [4], where \( \text{Soc}'(R) \) (resp. \( \text{Soc}''(R) \)) is the left (resp. right) socle of \( R \).

\((*)\) Every non-small module contains a non-zero injective module.

\((*)^*\) Every non-cosmall module contains a projective direct summand.

K. Oshiro called rings with \((*)\) (resp. \((*)^*\)) right H (resp. right co-H) rings in [10]. Relating with those two concepts we have

**Theorem 1.** Let \( R \) be an artinian ring. Then the following are equivalent:

1) \( R \) is right almost QF.
2) The Jacobson radical \( J \) of \( R \) is almost injective as a right \( R \)-module.
3) \( R \) is a right co-H ring.

**Proof.** 1) \( \iff \) 2). This is clear from [8], Corollary 1\( ^* \) and Proposition 3.
1) \( \iff \) 3). We know, from the above implication, [8], Proposition 3, [4], Theorem 3.6 and [11], Theorem 4.1, that the structure of right almost QF rings coincides with that of right co-H rings.

**Corollary.** \( R \) is right almost QF if and only if \( R \) is right QF-2, QF-3 and every submodule containing a projective submodule of \( eR \) is local for any primitive idempotent \( e \).

**Proof.** If \( R \) is right almost QF, \( R \) satisfies the conditions in the corollary by [8], Corollary 1\( ^* \) and Proposition 3. Conversely assume that \( R \) satisfies the conditions. Let \( A \) be a submodule of \( eR \) such that \( eR \supset A \supset fR \), where \( e,f \) are primitive idempotents. Then \( A \approx gR/B \) by assumption, where \( g \) is a primitive idempotent. Hence we obtain a natural epimorphism \( \theta : gR \to A \). Put \( K = \theta^{-1}(fR) \). Since \( fR \) is projective, \( K = B \oplus K' \). Further \( gR \) is uniform, and hence \( B = 0 \). Therefore \( A \) is projective, and \( R \) is right almost by Theorem 1 and [8], Proposition 3.

2. Almost QF\( ^* \) rings

In this section we shall study the dual concept to almost QF. If every indecomposable injective module is almost projective, we call \( R \) a right almost QF\( ^* \) ring. If every indecomposable injective module is local, we call \( R \) right QF-2\( ^* \). If a projective cover of every (indecomposable) injective module is injective, we call \( R \) right QF-3\( ^* \).

As the dual to Theorem 1, the following theorem is clear from [1], Theorem 2, [8], Theorem 1 and [10], Theorem 3.18.

**Theorem 2.** Let \( R \) be artinian. Then the following are equivalent:

1) \( R \) is right almost QF\( ^* \).
2) \( R \) is a right H-ring.

K. Oshiro [11] showed that \( R \) is right almost QF\( ^* \) if and only if \( R \) is left al-
most QF.

The following is dual to Corollary to Theorem 1.

**Proposition 1.** Let $R$ be an artinian ring. Then $R$ is right almost QF$^*$ if and only if 1): $R$ is right QF-2*, 2): $R$ is right QF-3* and 3): if $eR/A$ is injective for $A=\varnothing$, then $eR/B$ is uniform for any $B \subset A$. 1) together with 2) is equivalent to 4): every indecomposable injective is a factor module of some local projective and injective module, where $e$ is a primitive idempotent.

Proof. If $R$ is right almost QF$^*$, we obtain the conditions 1), 2) and 3) by [8], Corollary 1*. Conversely we assume 1), 2) and 3). Let $eR/A$ be injective and $B \subset A$. Then $eR/B$ is uniform by 3). Take a diagram

$$
\begin{array}{ccc}
0 & \rightarrow & eR/B \\
& \downarrow & \\
& \mu & \rightarrow \ E(eR/B)
\end{array}
$$

where $i$ is the inclusion and $\nu$ is the natural epimorphism. Since $eR/A$ is injective, we have $\mu_i = \nu$. $\nu$ being an epimorphism, $E(eR/B) = eR/B + \mu^{-1}(0)$. Further $E(eR/B)$ is local by 1) and 3), and hence $E(eR/B) = eR/B$. Therefore $eR/A$ is almost projective by [8], Theorem 1*. Hence $R$ is right almost QF$^*$.

If $R$ is hereditary and QF, then $R$ is semisimple. Concerning with this fact, we have

**Proposition 2.** Let $R$ be an artinian ring. Then the following are equivalent:

1) $R$ is serial.
2) $R$ is right almost QF$^*$ and right co-serial. (cf. [11], Theorem 6.1.)
3) $R$ is right almost QF and right almost hereditary.
4) $R$ is right almost QF$^*$ and right almost hereditary.
5) $R$ is left almost QF and right almost hereditary.

Proof. 1) $\rightarrow$ 2), 3), 4) and 5). This is clear from [7] and [10].

2) $\rightarrow$ 1). Let $f$ be a primitive idempotent and $E=E(fR)$. Then we may assume $E=e_iR \oplus \cdots \oplus e_sR \oplus \sum_j g_jR/A_j \oplus \cdots \oplus \sum_j g_sR/A_s$, where the $e_iR$ and the $g_jR$ are injective and $A_j \neq \varnothing$ for all $j$ by Proposition 1. Let $\theta$ be the natural epimorphism of $(\sum_j e_1R \oplus \sum_j g_jR)$ onto $E$ with $\theta^{-1}(0)=A_1 \oplus \cdots \oplus A_s$. Then since $fR$ is projective, $\theta^{-1}(fR)=P \oplus \theta^{-1}(0)$; $P \approx fR$. Set $E_1=E \oplus e_iR$ and $E_2=E \oplus g_jR$ and $\pi_i: E \rightarrow E_1$ the projection. Since $\theta^{-1}(0)$ is essential in $E_2$, $P \cap E_2=0$. Hence $fR \approx P \approx \pi_i(P) \subset E_i$. Next we shall show that every submodule of $E_i$ is standard. Take submodules $K \subset L \subset e_iR$ for $i=1, 2$ such that
We may assume $|e_1R/K_1| \leq |e_2R/K_2|$. Since $e_1R$ is uniserial, we can suppose that $e_1R/K_1$ and $e_2R/K_2$ are contained in $F=E(L_1/K_1)$, which is also uniserial. We can extend $\mu$ to an automorphism $\mu^*$ of $F$. Since $|e_1R/K_1| \leq |e_2R/K_2|$, $\mu^*(e_1R/K_1) \subseteq e_2R/K_2$. On the other hand, $e_1R$ being projective, $\mu^*$ is liftable to a homomorphism of $e_1R$ into $e_2R$. Hence $fR$ is a standard submodule of $E_1$ by [6], Lemma 5. Accordingly $fR \subset$ some $e_iR$ (isomorphically) for $fR$ is local. Therefore $fR$ is uniserial for any $f$, and hence $R$ is serial by [2], Theorem 5.4.

3) $\rightarrow$ 1). If $R$ is hereditary, $R$ is a serial ring in the first category by Corollary to Theorem 1 and [7], Corollary 3. Assume that $R$ is of the form (9) in [7], Theorem 2. Now we follow [7] for the notations. Then $h_1R$ is not injective provided $R$ is not serial by [7], Corollary 3. However $h_1R$ must be contained in an injective and projective $fR$ by Corollary to Theorem 1, which is impossible from the construction of $R$ in [7].

4) $\rightarrow$ 1). Let $R$ be right almost hereditary and right almost QF*. We use the same notations as in [7]. If $R$ is not serial, $T_1 \neq 0$ in [7], the figure (9). Then $E(h_1R/h_1J)$ is a factor module of an injective and projective $fR$ by Proposition 1, which is impossible by the structure of $R$. Hence $R$ is serial.

4) $\rightarrow$ 5). This is clear from the remark after Theorem 2 [11].

3. Serial rings

We have studied generalizations of QF rings in §§1 and 2. We shall consider, in this section, the remaining generalizations following previous sections.

**Theorem 3.** Let $R$ be artinian. Then the following are equivalent:

1) Every almost projective is almost injective.

2) Every almost injective is almost projective.

3) $R$ is serial.

Proof. 3) $\rightarrow$ 1) and 2). This is clear from [7], Figure (2) and [8], Corollary 1 and Corollary 1*.

1) or 2) $\rightarrow$ 3). Let $\{g_iR\}$ be the set of indecomposable, projective and injective modules, which is not empty by 1) or 2) and [8], and rename $\{g_iR\} = \{e_1R, \ldots, e_pR, f_1R, \ldots, f_4R\}$, where the $e_iR$ are uniserial and the $f_jR$ are not. Assume $q > 0$, i.e. $R$ is not serial. We shall find the set of non-projective, non-injective, non-uniserial and indecomposable almost projectives. Since $f_jR$ is not uniserial, there exists an integer $k_j$ such that $f_jR/\text{Soc}_s(f_jR)$ is injective for all $0 \leq r < k_j$ and $f_jR/\text{Soc}_{k_j}(f_jR)$ is not injective. Then $f_jR/\text{Soc}_{k_j}(f_jR)$ is non-projective, non-injective, non-uniserial and almost projective module by [8], Corollary 1. Further $f_jR/\text{Soc}_{k_j}(f_jR) \neq f_iR/\text{Soc}_{k_i}(f_iR)$ for $i \neq j$. Therefore since $e_iR$ is uniserial, we obtain just $q$ non-projective, non-injective, non-uniserial almost projective modules.
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\{f_j R/\text{Soc}_k(f_j R)\}_{j \leq q}

by [8], Theorem 1. On the other hand, if \(f_i J\) is projective, \(f_i J \approx g R\) for some primitive idempotent \(g\) (note that \(f_i R\) is uniform). Continuing this arument, we may suppose

\[ f_i R \supset f_i J \supset \cdots \supset f_i J^{r-1}(i \leq q), f_i J^{r-1} \approx f_i h R \text{ and } f_i r_i J \text{ is not projective, where} \]

the \(f_is\) are primitive idempotents.

Hence since \(f_i R\) is not uniserial, non-projective, non-injective, non-uniserial almost injective is of \(f_i J^{t'}(= f_{r-t} J)\) for \(s \leq q\) by [8], Theorem 1'. Further \(f_i J^{t'} \approx f_i J'\) for \(s = t\) since \(f_i R\) and \(f_i R\) are indecomposable and injective. Hence we obtain just \(q\) non-projective, non-injective, non-uniserial and almost injective modules

\[ \{f_i J'\}_{j \leq q}. \]

From 1) we shall show that \(f_j r_j J\) is local. By 1) \(f_j R/\text{Soc}_k(f_j R)\) is almost injective. Hence \(\{f_j R/\text{Soc}_k(f_j R)\}_{j \leq q} = \{f_j J'\}_{j \leq q}\) up to isomorphism. Therefore the \(f_j J' = f_{r_j} J\) are local. On the other hand, since every indecomposable projective \(e R\) is almost injective by assumption, \(g R \approx e_i J'\) for some \(t\) or \(g R \approx f_i J^{t'}\) for some \(t' \leq r_i - 1\) by [8], Corollary 1'. Hence \(g J\) is local from the above. Therefore \(R\) is right serial by [5], Proposition 1, and \(R\) is serial by [10], Theorem 6.1.

Assume 2). Then \(\{f_j R/\text{Soc}_k(f_j R)\}_{j \leq q} = \{f_j J'\}_{j \leq q}\) as above. Hence the \(f_j J'/\text{Soc} J/f_j R\) are uniform and \(R\) is co-serial by [8], Corollary 1 and [5], Proposition 1'. Therefore \(R\) is serial by Proposition 2.

**Proposition 3.** Let \(R\) be artinian. Then the following are equivalent:

1) Every almost projective is injective.
2) Every almost injective is projective.
3) \(R\) is semi-simple.

Proof. Assume 1). Then \(R\) is QF. Since \(e R/\text{Soc}(e R)\) is almost projective by [8], Theorem 1, \(e R/\text{Soc}(e R)\) is injective by 1). Hence \(e R/\text{Soc}(e R)\) is projective. Therefore \(\text{Soc}(e R) = e R\). Remaining parts are also clear.

The final type is

**Proposition 4.** Let \(R\) be as above. Then the following are equivalent:

1) Every almost projective is projective.
2) Every almost injective is injective.
3) \(R\) is a direct sum of semi-simple rings and rings whose every projective is never injective.

Proof. 1) \(\rightarrow\) 3). Let \(R\) be basic and \(e R\) injective. Then \(e R/\text{Soc}(e R)\) is almost projective by [8], Theorem 1, and hence \(e R/\text{Soc}(e R)\) is projective by 1). Therefore \(e R\) is simple and \(R = e R \oplus (1 - e) R\) as rings. Thus we obtain 3).
The remaining implications are also clear.

References


[9] \textit{Almost QF rings with }J^3=0\textit{, to appear.}


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