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ALMOST QF RINGS AND ALMOST QF$^{\#}$ RINGS

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In this paper we assume that every ring R is an associative ring with identity and R is two-sided artinian. The author has defined almost projective modules and almost injective modules in [8], and by making use of the concept of almost projectives he has defined almost hereditary rings in [7], whose class contains that of hereditary rings and serial rings. Similarly to [7] we shall define an almost QF ring, which is a generalization of QF rings.

It is well known that an artinian ring R is QF if and only if R is self injective. Following this fact, if R is almost injective as a right R-module, we call R a right almost QF ring. Analogously we call R a right almost QF$^{\#}$ ring if every injective is right almost projective. On the other hand, the author studied rings with $(*)$ (resp. $(*)^*$) (see §1 for definitions) in [4]. K. Oshiro called such a ring a right H- (resp. co-H) ring in [10]. In this note we shall show that a right almost QF (resp. almost QF$^{\#}$) ring coincides with a right co-H (resp. H-) ring. In the final section we shall give a characterization of serial rings in terms of almost projectives and almost injectives.

In the forthcoming paper [9] we shall study certain conditions under which right almost QF rings are QF or serial.

1. Almost QF rings

In this paper we always assume that R is a two-sided artinian ring with identity and that every module is a unitary right R-module. We use the same notations in [7]. We have studied almost hereditary rings in [7], i.e. J, the Jacobson radical of R, is right almost projective. We shall study, in this paper, some kind of the dual concept to almost hereditary rings (see Theorem 1 below). We call R *a right almost* QF *ring* if R is right almost injective as a right R-module [8]. We can define similarly a *left almost* QF *ring*. It is clear that R is right almost QF if and only if every finitely generated projective R-module is right almost injective. Hence the concept of almost QF rings is presevred under Morita equivalence and we may assume that R is basic.

On the other hand, the author studied the two conditions in [3] and [4]. Let M be an R-module. If $M\mathrm{Soc}^l(R)\neq 0$ (resp. $M\,\mathrm{Soc}^r(R)\neq 0$) then we call M non-small (resp. non-cosmall) [4], where $\mathrm{Soc}^l(R)$ (resp. $\mathrm{Soc}^r(R)$) is the left (resp. right) socle of R.

(*) *Every non-small module contains a non-zero injective module.*

(*)* *Every non-cosmall module contains a projective direct summand.*

K. Oshiro called rings with (*) (resp. (*)*) right H (resp. right co-H) rings in [10]. Relating with those two concepts we have

Theorem 1. *Let R be an artinian ring. Then the following are equivalent:*

1) *R is right almost* QF.
2) *The Jacobson radical J of R is almost injective as a right R-module.*
3) *R is a right* co-H *ring.*

Proof. 1) $\leftrightarrow$ 2). This is clear from [8], Corollary $1^{\#}$ and Proposition 3.

1) $\leftrightarrow$ 3). We know, from the above implication, [8], Proposition 3, [4], Theorem 3.6 and [11], Theorem 4.1, that the structure of right almost QF rings coincides with that of right co-H rings.

Corollary. *R is right almost* QF *if and only if R is right* QF-2, QF-3 *and every submodule containing a projective submodule of eR is local for any primitive idempotent e.*

Proof. If R is right almost QF, R satisfies the conditions in the corollary by [8], Corollary $1^{\#}$ and Proposition 3. Conversely assume that R satisfies the conditions. Let A be a submodule of eR such that $eR\supset A\supset fR$, where e, f are primitive idempotents. Then $A\approx gR/B$ by assumption, where g is a primitive idempotent. Hence we obtain a natural epimorphism $\theta\colon gR\to A$. Put $K=\theta^{-1}(fR)$. Since fR is projective, $K=B\oplus K'$. Further gR is uniform, and hence $B=0$. Therefore A is projective, and R is right almost by Theorem 1 and [8], Proposition 3.

2. Almost QF$^{\#}$ rings

In this section we shall study the dual concept to almost QF. If every indecomposable injective module is almost projective, we call *R a right almost* QF$^{\#}$ *ring*. If every indecomposable injective module is loeal, we call *R right* QF-$2^{\#}$. If a projective cover of every (indecomposable) injective module is injective, we call *R right* QF-$3^{\#}$.

As the dual to Theorem 1, the following theorem is clear from [1], Theorem 2, [8], Theorem 1 and [10], Theorem 3.18.

Theorem 2. *Let R be artinian. Then the following are equivalent:*

1) *R is right almost* QF$^{\#}$.
2) *R is a right* H*-ring.*

K. Oshiro [11] showed that R is right almost QF$^{\#}$ if and only if R is left almost QF.

The following is dual to Corollary to Theorem 1.

Proposition 1. *Let R be an artinian ring. Then R is right almost* $\mathrm{QF}^{\#}$ *if and only if* 1): *R is right* QF-$2^{\#}$, 2): *R is right* QF-$3^{\#}$ *and* 3): *if eR/A is injective for $A\neq 0$, then eR/B is uniform for any $B\subset A$.* 1) *together with* 2) *is equivalent to* 4): *every indecomposable injective is a factor module of some local projective and injective module, where e is a primitive idempotent.*

Proof. If R is right almost $\mathrm{QF}^{\#}$, we obtain the conditions 1), 2) and 3) by [8], Corollary $1^{\#}$. Conversely we assume 1), 2) and 3). Let eR/A be injective and $B\subset A$. Then eR/B is uniform by 3). Take a diagram

$$\begin{tikzcd} 0 \arrow[r] & eR/B \arrow[r, "i"] \arrow[d, "\nu"] & \mathrm{E}(eR/B) \arrow[dl, dashed, "\mu"] \\ & eR/A & \end{tikzcd}$$

where i is the inclusion and ν is the natural epimorphism. Since eR/A is injective, we have μ with $\mu i=\nu$. ν being an epimorphism, $\mathrm{E}(eR/B)=eR/B+\mu^{-1}(0)$. Further $\mathrm{E}(eR/B)$ is local by 1) and 3), and hence $\mathrm{E}(eR/B)=eR/B$. Therefore eR/A is almost projective by [8], Theorem $1^{\#}$. Hence R is right almost $\mathrm{QF}^{\#}$.

If R is hereditary and QF, then R is semisimple. Concerning with this fact, we have

Proposition 2. *Let R be an artinian ring. Then the following are equivalent:*

1) *R is serial.*
2) *R is right almost* $\mathrm{QF}^{\#}$ *and right co-serial.* (*cf.* [11], *Theorem* 6.1.)
3) *R is right almost* QF *and right almost hereditary.*
4) *R is right almost* $\mathrm{QF}^{\#}$ *and right almost hereditary.*
5) *R is left almost* QF *and right almost hereditary.*

Proof. 1) $\to$ 2), 3), 4) and 5). This is clear from [7] and [10].

2) $\to$ 1). Let f be a primitive idempotent and $E=\mathrm{E}(fR)$. Then we may assume $E=e_1R\oplus\cdots\oplus e_pR\oplus g_1R/A_1\oplus\cdots\oplus g_qR/A_q$, where the e_iR and the g_jR are injective and $A_j\neq 0$ for all j by Proposition 1. Let θ be the natural epimorphism of $(\sum_{i=1}^{p}\oplus e_iR\oplus\sum_{j=1}^{q}\oplus g_jR)$ onto E with $\theta^{-1}(0)=A_1\oplus\cdots\oplus A_q$. Then since fR is projective, $\theta^{-1}(fR)=P\oplus\theta^{-1}(0)$; $P\approx fR$. Set $E_1=\Sigma\oplus e_iR$ and $E_2=\Sigma\oplus g_jR$ and $\pi_1\colon E\to E_1$ the projection. Since $\theta^{-1}(0)$ is essential in E_2, $P\cap E_2=0$. Hence $fR\approx P\approx\pi_1(P)\subset E_1$. Next we shall show that every submodule of E_1 is standard. Take submodules $K_i\subset L_i\subset e_iR$ for $i=1, 2$ such that

μ: $L_1/K_1 \approx L_2/K_2$. We may assume $|e_1R/K_1| \leqq |e_2R/K_2|$. Since e_iR is uniserial, we can suppose that e_1R/K_1 and e_2R/K_2 are contained in $F=\mathrm{E}(L_1/K_1)$, which is also uniserial. We can extend μ to an automorphism μ^* of F. Since $|e_1R/K_1| \leqq |e_2R/K_2|$, $\mu^*(e_1R/K_1) \subset e_2R/K_2$. On the other hand, e_1R being projective, μ^* is liftable to a homomorphism of e_1R into e_2R. Hence fR is a standard submodule of E_1 by [6], Lemma 5. Accordingly $fR \subset$ some e_iR (isomorphically) for fR is local. Therefore fR is uniserial for any f, and hence R is serial by [2], Theorem 5.4.

3)$\to$1). If R is hereditary, R is a serial ring in the first category by Corollary to Theorem 1 and [7], Corollary 3. Assume that R is of the form (9) in [7], Theorem 2. Now we follow [7] for the notations. Then h_aR is not injective provided R is not serial by [7], Corollary 3. However h_aR must be contained in an injective and projective fR by Corollary to Theorem 1, which is impossible from the construction of R in [7].

4)$\to$1). Let R be right almost hereditary and right almost QF$^\sharp$. We use the same notations as in [7]. If R is not serial, $T_1 \neq 0$ in [7], the figure (9). Then $\mathrm{E}(h_1R/h_1J)$ is a factor module of an injective and projective fR by Proposition 1, which is impossible by the structure of R. Hence R is serial.

4)$\to$5). This is clear from the remark after Theorem 2 [11].

3. Serial rings

We have studied generalizations of QF rings in §§1 and 2. We shall consider, in this section, the remaining generalizations following previous sections.

Theorem 3. *Let R be artinian. Then the following are equivalent:*

1) *Every almost projective is almost injective.*
2) *Every almost injective is almost projective.*
3) *R is serial.*

Proof. 3)$\to$1) and 2). This is clear from [7], Figure (2) and [8], Corollary 1 and Corollary 1$^\sharp$.

1) or 2)$\to$3). Let $\{g_iR\}$ be the set of indecomposable, projective and injective modules, which is not empty by 1) or 2) and [8], and rename $\{g_iR\}=\{e_1R, \cdots, e_pR, f_1R, \cdots, f_qR\}$, where the e_iR are uniserial and the f_jR are not. Assume $q>0$, i.e. R is not serial. We shall find the set of non-projective, non-injective, non-uniserial and indecomposable almost projectives. Since f_jR is not uniserial, there exists an integer k_j such that $f_jR/\mathrm{Soc}_r(f_jR)$ is injective for all $o \leqq r < k_j$ and $f_jR/\mathrm{Soc}_{k_j}(f_jR)$ is not injective. Then $f_jR/\mathrm{Soc}_{k_j}(f_jR)$ is non-projective, non-injective, non-uniserial and almost projective module by [8], Corollary 1. Further $f_jR/\mathrm{Soc}_{k_j}(f_jR) \not\approx f_iR/\mathrm{Soc}_{k_i}(f_iR)$ for $i \neq j$. Therefore since e_iR is uniserial, we obtain just q non-projective, non-injective, non-uniserial almost projective modules

$$\{f_jR/\mathrm{Soc}_{k_j}(f_jR)\}_{j\leqq q}$$

by [8], Theorem 1. On the other hand, if f_iJ is projective, $f_iJ\approx gR$ for some pimitive idempotent g (note that f_iR is uniform). Continuing this arument, we may suppose

$f_iR\supset f_iJ\supset\cdots\supset f_iJ^{r_i-1}(i\leqq q)$, $f_iJ^{k-1}\approx f_{ik}R$ and $f_{ir_i}J$ is not projective, where the f_{is} are primitive idempotents.

Hence since f_sR is not uniserial, non-projective, non-injective, non-uniseial almost injective is of $f_sJ^{r_s}(=f_{sr_s}J)$ for $s\leqq q$ by [8], Theorem 1^*. Further $f_sJ^{r_s}\not\approx f_tJ^{r_t}$ for $s\neq t$, since f_sR and f_tR are indecomposable and injective. Hence we obtain just q non-projective, non-injective, non-uniserial and almost injective modules

$$\{f_jJ^{r_j}\}_{j\leqq q}\,.$$

Fiom 1) we shall show that $f_{jr_j}J$ is local. By 1) $f_jR/\mathrm{Soc}_{k_j}(f_jR)$ is almost injective. Hence $\{f_iR/\mathrm{Soc}_{k_i}(f_iR)\}_{i\leqq q}=\{f_sJ^{r_s}\}_{s\leqq q}$ up to isomorphism. Therefore the $f_sJ^{r_s}=f_{sr_s}J$ are local. On the other hand, since every indecomposable projective gR is almost injective by assumption, $gR\approx e_iJ^t$ for some t or $gR\approx f_iJ^{t'}$ for some $t'\leqq r_i-1$ by [8], Corollary 1^*. Hence gJ is local from the above. Therefore R is right serial by [5], Proposition 1, and R is serial by [10], Theorem 6.1.

Assume 2). Then $\{f_iR/\mathrm{Soc}_{k_i}(f_iR)\}_{i\leqq q}=\{f_sJ^{r_s}\}_{s\leqq q}$ as above. Hence the $f_iR/\mathrm{Soc}_{k_i}(f_iR)$ are uniform and R is co-serial by [8], Corollary 1 and [5], Proposition 1′. Therefore R is serial by Proposition 2.

Proposition 3. *Let R be artinian. Then the following are equivalent:*

1) *Every almost projective is injective.*
2) *Every almost injective is projective.*
3) *R is semi-simple.*

Proof. Assume 1). Then R is QF. Since $eR/\mathrm{Soc}(eR)$ is almost projective by [8], Theorem 1, $eR/\mathrm{Soc}(eR)$ is injective by 1). Hence $eR/\mathrm{Soc}(eR)$ is projective. Therefore $\mathrm{Soc}(eR)=eR$. Remaining parts are also clear.

The final type is

Proposition 4. *Let R be as above. Then the following are equivalent:*

1) *Every c lmost projective is projective.*
2) *Every almost injective is injective.*
3) *R is a direct sum of semi-simple rings and rings whose every projective is never injective.*

Proof. 1)→3). Let R be basic and eR injective. Then $eR/\mathrm{Soc}(eR)$ is almost projecive by [8], Theorem 1, and hence $eR/\mathrm{Soc}(eR)$ is projective by 1). Therefore eR is simple and $R=eR\oplus(1-e)R$ as rings. Thus we obtain 3).

The remaining implications are also clear.

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