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ALMOST QF RINGS AND ALMOST QF* RINGS

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In this paper we assume that every ring $R$ is an associative ring with identity and $R$ is two-sided artinian. The author has defined almost projective modules and almost injective modules in [8], and by making use of the concept of almost projectives he has defined almost hereditary rings in [7], whose class contains that of hereditary rings and serial rings. Similarly to [7] we shall define an almost QF ring, which is a generalization of QF rings.

It is well known that an artinian ring $R$ is QF if and only if $R$ is self injective. Following this fact, if $R$ is almost injective as a right $R$-module, we call $R$ a right almost QF ring. Analogously we call $R$ a right almost QF* ring if every injective is right almost projective. On the other hand, the author studied rings with $(*)$ (resp. $(*)^*$) (see §1 for definitions) in [4]. K. Oshiro called such a ring a right H- (resp. co-H) ring in [10]. In this note we shall show that a right almost QF (resp. almost QF*) ring coincides with a right co-H (resp. H-) ring. In the final section we shall give a characterization of serial rings in terms of almost projectives and almost injectives.

In the forthcoming paper [9] we shall study certain conditions under which right almost QF rings are QF or serial.

1. Almost QF rings

In this paper we always assume that $R$ is a two-sided artinian ring with identity and that every module is a unitary right $R$-module. We use the same notations in [7]. We have studied almost hereditary rings in [7], i.e. $J$, the Jacobson radical of $R$, is right almost projective. We shall study, in this paper, some kind of the dual concept to almost hereditary rings (see Theorem 1 below). We call $R$ a right almost QF ring if $R$ is right almost injective as a right $R$-module [8]. We can define similarly a left almost QF ring. It is clear that $R$ is right almost QF if and only if every finitely generated projective $R$-module is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that $R$ is basic.

On the other hand, the author studied the two conditions in [3] and [4]. Let $M$ be an $R$-module. If $\text{MSoc}^1(R) \neq 0$ (resp. $\text{M Soc}^1(R) \neq 0$) then we call $M$ non-
small (resp. non-cosmall) [4], where Soc′(R) (resp. Soc′(R)) is the left (resp. right) socle of R.

(*) Every non-small module contains a non-zero injective module.

(**) Every non-cosmall module contains a projective direct summand.

K. Oshiro called rings with (*) (resp. (**)) right H (resp. right co-H) rings in [10]. Relating with those two concepts we have

**Theorem 1.** Let R be an artinian ring. Then the following are equivalent:
1) R is right almost QF.
2) The Jacobson radical J of R is almost injective as a right R-module.
3) R is a right co-H ring.

**Proof.** 1) «→ 2). This is clear from [8], Corollary 1* and Proposition 3.
1) «→ 3). We know, from the above implication, [8], Proposition 3, [4], Theorem 3.6 and [11], Theorem 4.1, that the structure of right almost QF rings coincides with that of right co-H rings.

**Corollary.** R is right almost QF if and only if R is right QF-2, QF-3 and every submodule containing a projective submodule of eR is local for any primitive idempotent e.

**Proof.** If R is right almost QF, R satisfies the conditions in the corollary by [8], Corollary 1* and Proposition 3. Conversely assume that R satisfies the conditions. Let A be a submodule of eR such that eR ⊃ A ⊃ fR, where e, f are primitive idempotents. Then A ≈ gR/B by assumption, where g is a primitive idempotent. Hence we obtain a natural epimorphism θ: gR → A. Put K = θ⁻¹(fR). Since fR is projective, K = B ⊕ K'. Further gR is uniform, and hence B = 0. Therefore A is projective, and R is right almost by Theorem 1 and [8], Proposition 3.

2. Almost QF² rings

In this section we shall study the dual concept to almost QF. If every indecomposable injective module is almost projective, we call R a right almost QF² ring. If every indecomposable injective module is local, we call R right QF-2². If a projective cover of every (indecomposable) injective module is injective, we call R right QF-3².

As the dual to Theorem 1, the following theorem is clear from [1], Theorem 2, [8], Theorem 1 and [10], Theorem 3.18.

**Theorem 2.** Let R be artinian. Then the following are equivalent:
1) R is right almost QF².
2) R is a right H-ring.

K. Oshiro [11] showed that R is right almost QF² if and only if R is left al-
most QF.

The following is dual to Corollary to Theorem 1.

**Proposition 1.** Let $R$ be an artinian ring. Then $R$ is right almost QF$^\sharp$ if and only if 1): $R$ is right QF-$2^\sharp$, 2): $R$ is right QF-$3^\sharp$ and 3): if $eR/A$ is injective for $A\neq 0$, then $eR/B$ is uniform for any $B\subset A$. 1) together with 2) is equivalent to 4): every indecomposable injective is a factor module of some local projective and injective module, where $e$ is a primitive idempotent.

Proof. If $R$ is right almost QF$^\sharp$, we obtain the conditions 1), 2) and 3) by [8], Corollary 1$. Conversely we assume 1), 2) and 3). Let $eR/A$ be injective and $B\subset A$. Then $eR/B$ is uniform by 3). Take a diagram

$$
\begin{array}{c}
0 \to eR/B \to E(eR/B) \\
\downarrow = \mu \\
eR/A \end{array}
$$

where $i$ is the inclusion and $\nu$ is the natural epimorphism. Since $eR/A$ is injective, we have $\mu$ with $\mu i=\nu$. $\nu$ being an epimorphism, $E(eR/B)=eR/B+\mu^{-1}(0)$. Further $E(eR/B)$ is local by 1) and 3), and hence $E(eR/B)=eR/B$. Therefore $eR/A$ is almost projective by [8], Theorem 1$. Hence $R$ is right almost QF$^\sharp$.

If $R$ is hereditary and QF, then $R$ is semisimple. Concerning with this fact, we have

**Proposition 2.** Let $R$ be an artinian ring. Then the following are equivalent:

1) $R$ is serial.
2) $R$ is right almost QF$^\sharp$ and right co-serial. (cf. [11], Theorem 6.1.)
3) $R$ is right almost QF and right almost hereditary.
4) $R$ is right almost QF$^\sharp$ and right almost hereditary.
5) $R$ is left almost QF and right almost hereditary.

Proof. 1) $\to$ 2), 3), 4) and 5). This is clear from [7] and [10].

2) $\to$ 1). Let $f$ be a primitive idempotent and $E=E(fR)$. Then we may assume $E=e_1R\oplus \cdots \oplus e_sR \oplus g_1R/A_1 \oplus \cdots \oplus g_tR/A_t$, where the $e_jR$ and the $g_jR$ are injective and $A_j\neq 0$ for all $j$ by Proposition 1. Let $\theta$ be the natural epimorphism of $(\sum_{j=1}^s e_jR \oplus \sum_{j=1}^t g_jR)$ onto $E$ with $\theta^{-1}(0)=A_1 \oplus \cdots \oplus A_t$. Then since $fR$ is projective, $\theta^{-1}(fR)=P \oplus \theta^{-1}(0)$; $P \cong fR$. Set $E_1=\Sigma \oplus e_iR$ and $E_2=\Sigma \oplus g_jR$ and $\pi_1: E\to E_1$ the projection. Since $\theta^{-1}(0)$ is essential in $E_2$, $P \cap E_2=0$. Hence $fR \cong P \cong \pi_1(P) \subset E_1$. Next we shall show that every submodule of $E_1$ is standard. Take submodules $K_1 \subset L_i \subset e_iR$ for $i=1,2$ such that
\( \mu: L_1/K_1 \cong L_2/K_2. \) We may assume \( |e_i R/K_1| \leq |e_2 R/K_2|. \) Since \( e_i R \) is uniserial, we can suppose that \( e_i R/K_1 \) and \( e_2 R/K_2 \) are contained in \( F = E(L/K_1) \), which is also uniserial. We can extend \( \mu \) to an automorphism \( \mu^* \) of \( F. \) Since \( |e_i R/K_1| \leq |e_2 R/K_2| \), \( \mu^*(e_i R/K_1) \subseteq e_2 R/K_2. \) On the other hand, \( e_i R \) being projective, \( \mu^* \) is liftable to a homomorphism of \( e_i R \) into \( e_2 R. \) Hence \( f R \) is a standard submodule of \( E_i \) by [6], Lemma 5. Accordingly \( f R \subset \) some \( e_i R \) (isomorphically) for \( f R \) is local. Therefore \( f R \) is uniserial for any \( f_j \) and hence \( R \) is serial by [2], Theorem 5.4.

3) \( \rightarrow \) 1). If \( R \) is hereditary, \( R \) is a serial ring in the first category by Corollary to Theorem 1 and [7], Corollary 3. Assume that \( R \) is of the form (9) in [7], Theorem 2. Now we follow [7] for the notations. Then \( h_i R \) is not injective provided \( R \) is not serial by [7], Corollary 3. However \( h_i R \) must be contained in an injective and projective \( f R \) by Corollary to Theorem 1, which is impossible from the construction of \( R \) in [7].

4) \( \rightarrow \) 1). Let \( R \) be right almost hereditary and right almost QF*. We use the same notations as in [7]. If \( R \) is not serial, \( T_1 \neq 0 \) in [7], the figure (9). Then \( E(h_i R/h_i J) \) is a factor module of an injective and projective \( f R \) by Proposition 1, which is impossible by the structure of \( R. \) Hence \( R \) is serial.

4) \( \rightarrow \) 5). This is clear from the remark after Theorem 2 [11].

3. Serial rings

We have studied generalizations of QF rings in §§1 and 2. We shall consider, in this section, the remaining generalizations following previous sections.

**Theorem 3.** Let \( R \) be artinian. Then the following are equivalent:
1) Every almost projective is almost injective.
2) Every almost injective is almost projective.
3) \( R \) is serial.

**Proof.** 3) \( \rightarrow \) 1) and 2). This is clear from [7], Figure (2) and [8], Corollary 1 and Corollary 1*.

1) or 2) \( \rightarrow \) 3). Let \( \{g_j R\} \) be the set of indecomposable, projective and injective modules, which is not empty by 1) or 2) and [8], and rename \( \{g_j R\} = \{e_1 R, \ldots, e_k R, f_1 R, \ldots, f_q R\} \), where the \( e_i R \) are uniserial and the \( f_j R \) are not. Assume \( q > 0 \), i.e. \( R \) is not serial. We shall find the set of non-projective, non-injective, non-uniserial and indecomposable almost projectives. Since \( f_j R \) is not uniserial, there exists an integer \( k_j \) such that \( f_j R/Soc_{k_j}(f_j R) \) is injective for all \( 0 \leq r < k_j \) and \( f_j R/Soc_{k_j}(f_j R) \) is not injective. Then \( f_j R/Soc_{k_j}(f_j R) \) is non-projective, non-injective, non-uniserial and almost projective module by [8], Corollary 1. Further \( f_j R/Soc_{k_i}(f_i R) \cong f_i R/Soc_{k_i}(f_j R) \) for \( i \neq j \). Therefore since \( e_i R \) is uniserial, we obtain just \( q \) non-projective, non-injective, non-uniserial almost projective modules
\{f_j R / \text{Soc}_{k_j}(f_j R)\}_{j \leq q}

by [8], Theorem 1. On the other hand, if \( f_i J \) is projective, \( f_i J \cong g_i R \) for some primitive idempotent \( g \) (note that \( f_i R \) is uniform). Continuing this argument, we may suppose
\[ f_i R \supset f_i J \supset \cdots \supset f_i J'_{i-1}(i \leq q), f_i J'_{i-1} \cong f_i h_i R \] and \( f_i r_i J \) is not projective, where the \( f_i s \) are primitive idempotents.

Hence since \( f_i R \) is not uniserial, non-projective, non-injective, non-uniserial almost injective is of \( f_i J^s (= f_i r_i J) \) for \( s \leq q \) by [8], Theorem 1'. Further \( f_i J'^s \cong f_i J' t \) for \( s \neq t \), since \( f_i R \) and \( f_i J \) are indecomposable and injective. Hence we obtain just \( q \) non-projective, non-injective, non-uniserial and almost injective modules
\[ \{f_i J'^s\}_{i \leq q}. \]

From 1) we shall show that \( f_i r_i J \) is local. By 1) \( f_j R / \text{Soc}_{k_j}(f_j R) \) is almost injective. Hence \( \{f_j R / \text{Soc}_{k_j}(f_j R)\}_{i \leq q} = \{f_j J'^s\}_{i \leq q} \) up to isomorphism. Therefore the \( f_j J'^s = f_j r_j J \) are local. On the other hand, since every indecomposable projective \( g R \) is almost injective by assumption, \( g R \cong e_i J^t \) for some \( t \) or \( g R \cong f_i J'^t \) for some \( t' \leq r_i - 1 \) by [8], Corollary 1'. Hence \( g J \) is local from the above. Therefore \( R \) is right serial by [5], Proposition 1, and \( R \) is serial by [10], Theorem 6.1.

Assume 2). Then \( \{f_j R / \text{Soc}_{k_j}(f_j R)\}_{i \leq q} = \{f_j J'^t\}_{i \leq q} \) as above. Hence the \( f_i R / \text{Soc}_{k_i}(f_i R) \) are uniform and \( R \) is co-serial by [8], Corollary 1 and [5], Proposition 1'. Therefore \( R \) is serial by Proposition 2.

**Proposition 3.** Let \( R \) be artinian. Then the following are equivalent:
1) Every almost projective is injective.
2) Every almost injective is projective.
3) \( R \) is semi-simple.

Proof. Assume 1). Then \( R \) is QF. Since \( e R / \text{Soc}(e R) \) is almost projective by [8], Theorem 1, \( e R / \text{Soc}(e R) \) is injective by 1). Hence \( e R / \text{Soc}(e R) \) is projective. Therefore \( \text{Soc}(e R) = e R \). Remaining parts are also clear.

The final type is

**Proposition 4.** Let \( R \) be as above. Then the following are equivalent:
1) Every almost projective is projective.
2) Every almost injective is injective.
3) \( R \) is a direct sum of semi-simple rings and rings whose every projective is never injective.

Proof. 1) \(\rightarrow\) 3). Let \( R \) be basic and \( e R \) injective. Then \( e R / \text{Soc}(e R) \) is almost projective by [8], Theorem 1, and hence \( e R / \text{Soc}(e R) \) is projective by 1). Therefore \( e R \) is simple and \( R = e R \oplus (1 - e) R \) as rings. Thus we obtain 3).
The remaining implications are also clear.

References


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