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Osaka University
On Homotopy Type Problems of Special Kinds of Polyhedra I

Hiroshi Uehara

1. Introduction

It is one of the aims of modern topology to classify topological spaces by their homotopy types. Two spaces $X$ and $Y$ have the same homotopy type if there exist maps $f: X \to Y$ and $g: Y \to X$ such that $gf$ and $fg$ are homotopic to the identity maps $X \to X$ and $Y \to Y$ respectively. The problem of determining by means of invariants of $X$ and $Y$ whether $X$ and $Y$ are of the same homotopy type or not, is of great importance in modern topology. This general problem has not yet been solved. A number of particular results are well known.

In 1936 Witold Hurewicz solved in his famous paper [8] the homotopy types of an $n$ dimensional locally connected compact metric space aspherical in dimensions less than $n$, and of a locally connected compact metric space aspherical in dimensions greater than unity. After this, many endeavours have been made to solve this general problem by several modern topologists, J. H. C. Whitehead, R. H. Fox, S. C. Chang, and others. Among them the recent brilliant results of J. H. C. Whitehead [3], [4] and of S. C. Chang [6] have much to do with the present paper. Whitehead reported in [3] that two simply connected, 4 dimensional polyhedra are of the same homotopy type if and only if their cohomology rings are properly isomorphic. According to Whitehead, an arcwise connected polyhedron $P$ is referred to as $A^3_1$-complex if $\dim P \leq n+2$ and $\pi_i(P) = 0$ for each $i < n$. Though the author is unfortunate enough to be inaccessible to [4] here, he is informed of Whitehead's far reaching results through the introduction of Chang's paper [6]. They are stated as follows. Two $A^3_1$-complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic. Chang introduced new numerical invariants called secondary torsions to characterize the homotopy type of an $A^3_1$-polyhedron together with the Betti numbers and coefficients of torsion. Furthermore he reduced a given $A^3_1$-complex to a reduced complex which consists of elementary $A^3_1$-polyhedra.

The main purpose of this paper is to determine the homotopy type
of an $A^3_\alpha$-complex $P$ with vanishing $(n+1)$-st homotopy group of $P$. Throughout the whole paper we assume $n>3$. Let $H^r(r=0, n, n+1, n+2, n+3)$ be the $r$ dimensional integral cohomology group and let $Sq_{n-2}: H^n(2k) \rightarrow H^{n+2}(2)$ and $Sq_{n-1}: H^{n+1} \rightarrow H^{n+3}(2)$ be Steenrod's squaring operations. Then, following J. H. C. Whitehead, we refer to $FH=H^0, H^n, H^{n+1}, H^{n+2}, H^{n+3}, H^n(2k), H^{n+2}(2), H^{n+3}(2), \mu, \Delta, Sq_{n-2}, Sq_{n-1}$ as $A^3_\alpha$-cohomology system. It will be shown in Theorem 1 that two such complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic. The method of proving this is analogous to that of Whitehead [3]. The reduction of such a given $A^3_\alpha$-complex to a reduced complex is also shown. Before performing this, the author gives another elementary but elegant way of proving Chang's reduction of an $A^3_\alpha$-complex to a reduced complex, which was pointed out for him by Gaishi Takeuti. The author would like to express his sincere gratitude to Professor G. Takeuti for his kind criticisms and encouragements.

2. A Spectrum

A brief sketch of the definition of spectrum of cohomology groups and related lemmas used in the sequel seems to be desirable for the convenience and the clearness of the applications in this paper. All the concepts and lemmas in this section are in [3]. Let a sequence $c=\{c^n\}$ ($n=0, 1, \ldots$) of free abelian groups of finite rank be related by a "coboundary" homomorphism $\delta: c^n \rightarrow c^{n+1}$ for each $n$, such that $\delta \delta = 0$. By an usual procedure, the $n$-dimensional cohomology group $H^n(m)$ with integers reduced mod. $m$ can be defined in terms of $C$ and $\delta$. For integers $p>0$ and $q\geq 0$ two operations $\Delta, \mu_{p,q}$ are defined such that

$$\Delta_q: H^n(q) \rightarrow H^{n+1},$$

$$\mu_{p,q}: H^n(q) \rightarrow H^n(p).$$

Let $x \in H^n(q)$ and let $x' \in x$. That is to say, $x'$ is a cocycle mod. $q$. Then $\delta x' = qy'$, where $y'$ is an $(n+1)$ absolute cocycle. We define $\Delta y = y$, a cohomology class containing $y'$. Let $c=(p, q)$, then $\frac{p}{c}x'$ is a cocycle mod. $p$, and we define $\mu_{p,q}x$ as its cohomology class mod. $p$. It is easily verified that $\Delta x$ and $\mu_{p,q}x$ depend only on $x \in H^n(q)$ and not on the particular choice of $x' \in x$. They are obviously homomorphisms. The union of all the groups $H^n(m)$, for every integer $n\geq 0$ and for every integer $m\geq 0$, related by the homomorphisms $\Delta, \mu$, will be called the cohomology spectrum of the set of groups $C$, or merely spectrum of the groups $C$. We shall denote it by $H$. By a proper homomorphism $f: \overline{H} \rightarrow H$ of a spectrum $H$ into a spectrum $\overline{H}$, we mean a transformation such that
i) $f|H^n(m):H^n(m)\to \overline{H}^n(m)$ is a homomorphism for all values of $m, n,$
ii) $f\Delta = \Delta f \text{ and } f\mu = \mu f.$

If $f$ is a proper homomorphism and $f|H^n(m):H^n(m)\to \overline{H}^n(m)$ is an isomorphism onto for all values of $m, n,$ $f$ is called a proper isomorphism. Then a spectrum $H$ is called to be properly isomorphic to a spectrum $\overline{H}$.

Let $Z^n(m)$ be a subgroup of $C^n$ which consists of all the cocycles mod. $m,$ and let $j_m$ be a natural homomorphism of $Z^n(m)$ onto $H^n(m).$ We shall also use $j_m$ to denote the natural homomorphism of cocycles mod. $m,$ in $C^n$ onto $\overline{H}^n(m).$ A cochain map $g: C^n \to C^n$ for every $n,$ obviously induces a proper homomorphism of $H$ into $\overline{H}.$ Now let $f$ be a given proper homomorphism of $H$ into $\overline{H}.$ If $f j_m a = j_m ga$ for any $a \in Z^n(m),$ a cochain map $g$ is said to realize a proper homomorphism $f.$

**Lemma 1.** (Whitehead [3], p. 57, Lemma 4) Any proper homomorphism $f: H \to \overline{H}$ can be realized by a cochain map $g.$

### 3. Two types of homomorphisms

Let $C^n$ of a sequence $C$ be an $n$ dimensional group of cochains of a finite simplicial complex $K.$ Then two types of homomorphisms are defined among cohomology groups $H^n(m).$ One of them is a well known squaring homomorphism of Steenrod [7] and the other is $q_i$-homomorphism, which was introduced elsewhere [11] by N. Shimada and myself. For convenience, they are put down here. Steenrod showed that

if $p - i$ is odd, there exists the $i$-th square

$$Sq_i: H^p(m) \to H^{2p-1}(m),$$

and that

if $p - i$ is even and $m$ is also even, the $i$-th square mod. 2 can be defined such that

$$Sq_i: H^p(m) \to H^{2p-1}(2).$$

This squaring operation will be used essentially in the sequel, while we shall not need the $q_i$-homomorphism except for cohomological properties in a reduced complex (refer to § 8).

If $p - i$ is odd and $m \geq 0$ is an even integer, $q_i: H^n(m) \to \mathbb{Z}_{2p-i}$ can be defined as follows. Let $x \in H^n(m)$ and $x' \in x.$ Since $x'$ is a cocycle mod. $m,$ we have $\delta x' = my'.$ Putting $\delta_m x' = \frac{1}{m} \delta x' = y',$ we have a $(2p - i)$ absolute cocycle

$$q'_ix' = x' \cup x' + mx' \cup \delta_m x' + (-1)^{m^2} \frac{1}{2} \delta_m x' \cup \delta_m x'.$$

Notice that $q'_ix' = x' \cup x'$ in case $m = 0.$ If we define that $q_i x$ is a
cohomology class containing a cocycle \( q'x' \), it is verified that this definition does not depend on the choice of a representative \( x' \) of \( x \). The spectrum \( H \) related by squaring operations, will be called the cohomology system, which is denoted by \( FH \).

4. \( \mathbb{A}_n^3 \)-cohomology system

If a finite simplicial complex \( K \) referred to in §3, is an \( \mathbb{A}_n^3 \)-complex, some conditions are obviously assigned on its cohomology system. It is evident that

i) \( H'(m) = 0 \), for any \( m \) and \( n > i > 0 \),

ii) \( H'(m) = 0 \), for any \( m \) and for each \( i > n + 3 \),

iii) \( H^* \) contains no element of finite order,

iv) \( H^0 \) is cyclic infinite.

Thus, for the reasonable brevity we shall symbolize \( \mathbb{A}_n^3 \)-cohomology system by

\[
FH = H\{H^0, H^n, H^{n+1}, H^{n+2}, H^{n+3}, H^n(2k), H^{n+2}(2), H^{n+3}(2), \Delta, \mu, Sq_{n-2}, Sq_{n-1}\}.
\]

In this notation the operations \( \Delta, \mu \) are explained in §2, and the other two operations are as follows:

\[
Sq_{n-2} : H^n(2k) \to H^{n+2}(2) \text{ for every integer } k \geq 0,
\]

\[
Sq_{n-1} : H^{n+1} \to H^{n+3}(2).
\]

Let \( FH, F\bar{H} \) be the cohomology systems of \( K, \bar{K} \) respectively. By a proper homomorphism we mean the transformation \( f : FH \to F\bar{H} \) such that

i) \( f \) is not the trivial homomorphism \( FH \to 0 \),

ii) \( f \) induces a proper homomorphism, as defined in §2, of the spectra,

iii) \( fSq_{n-2} = Sq_{n-2}f \) and \( fSq_{n-1} = Sq_{n-1}f \),

iv) \( f|H^0 \) is an isomorphism onto.

If a proper homomorphism \( f \) induces a proper isomorphism onto of the spectra, \( f \) is called a proper isomorphism. Then \( FH \) is said to be properly isomorphic to \( F\bar{H} \).

Let \( P \) be an \((n+3)\) dimensional finite connected simplicial complex such that \( \pi_i(P) = 0 \) for each \( i < n \) and \( i = n + 1 \), and let us refer to such a complex as \( \mathbb{A}_n^3 \)-complex. Then our theorems are:

**Theorem 1.** Two \( \mathbb{A}_n^3 \)-complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic.

**Theorem 2.** Let \( P, \bar{P} \) be \( \mathbb{A}_n^3 \)-complexes. Any proper homomorphism \( f^* : FH(\bar{P}) \to FH(P) \) can be realized by at least one homotopy class of maps.
f : \overline{P} \to P. That is to say, there exists a map f : P \to \overline{P} such that the proper homomorphism induced by f is the same as f*.

It is verified as follows that Theorem 2 implies Theorem 1. Now let K and L be finite simply connected complexes of arbitrary dimensionality and let f : K \to L be a map which induces an isomorphism of each cohomology group H^n(L), with integral coefficients, onto the corresponding group H^n(K). Then J. H. C. Whitehead proved in [2] that K and L are of the same homotopy type and f is a homotopy equivalence. If we use this, it is easily seen that Theorem 2 implies Theorem 1. In virtue of Theorem 2 there exists at least one map f : P \to \overline{P} which induces the proper isomorphism FH(\overline{P}) \to FH(P). If we utilize the above mentioned result of Whitehead, it is seen that P and \overline{P} are of the same homotopy type, when their cohomology systems are properly isomorphic. The converse of this is obvious. If P and \overline{P} have the same homotopy type, there exist maps f : P \to \overline{P} and g : \overline{P} \to P such that fg \sim e' and gf \sim e, where e, e' denote the identical transformations of P, \overline{P} respectively. Let us denote the proper homomorphisms induced by f, g by f* : FH(\overline{P}) \to FH(P) and g* : FH(P) \to FH(\overline{P}) respectively. Since f*g* : FH(P) \to FH(\overline{P}) are proper isomorphisms, f* is an proper isomorphism. Thus our aim is to prove Theorem 2.

5. Reduction of A^n-complex to a reduced complex.

This section was proved by G. Takeuti. Before we perform this reduction, we give here some notations, definitions, and essential Lemmas for subsequent discussions.

Let X, R be topological spaces and let Y be a closed subset of X. Attaching X to R by a map f : Y \to R, we have a space (R + X, f, Y), which may be simply denoted by (R + X, f). More generally, we designate by \(R + X_1 + \cdots + X_n, f_1, ... , f_n, Y_1, ..., Y_n\) or merely \(R + X_1 + \cdots + X_n, f_1, ... , f_n\) a space attaching \(X_i(i=1, ..., n)\) to R by a map \(f_i : Y_i \to R\), where \(Y_i\) is a closed subset of \(X_i\). In case where R is a space of a point O and \(Y_i(i=1, ..., m)\) consists of a single point \(O_i\) of \(X_i\), the space \((O + X_1 + \cdots + X_n, f_1, ..., f_m)\) will be often denoted by \((O, X_1, ..., X_n, O_1, ..., O_m)\) or, as usually designated, by \(X_1 \lor X_2 \lor ... \lor X_m\), where \(f_*(O_i)=O\) is evidently assumed. Particularly, if \(X_i\) is an oriented n sphere \(S^n_i\), the n-th homotopy group \(\pi_n(S^n_1 \lor \cdots \lor S^n_m)(n \geq 1)\) of a space \(S^n_1 \lor \cdots \lor S^n_m\) may be regarded as the m-dimensional vector space with free base \(\{S^n_1, ..., S^n_m\}\), where \(S^n_i\) denotes an element of the homotopy group as well as an n sphere. In the sequel we shall often use the notation \(A \sim B\), when two spaces A, B have the same homotopy type.
Lemma 2. (J. H. C. Whitehead [5], p. 239, Lemma 2) If two spaces \( P, Q \) are of the same homotopy type and \( f : P \to Q \) is a homotopy equivalence, and if a map \( \alpha : \partial E^{n+1} = S^n \to P \) is given, \( (P + E^{n+1}, \alpha) \) has the same homotopy type as \( (Q + E^{n+1}, f\alpha) \), where \( E^{n+1} \) is an \((n+1)\) element.

Let \( \tilde{P} \) be a space attaching \( E_t^{n+1}(i=1, \ldots, m) \) to \( P \) one by one by a map \( f_i : \partial E_t^{n+1} \to P \). Then the homotopy type of \( \tilde{P} \) is completely determined by the homotopy elements \( \beta_i(i=1, \ldots, m) \) of \( \pi_n(P) \) represented by maps \( f_i \), so that without confusion we may represent \( \tilde{P} \) by the symbol \( (P; \beta_1, \ldots, \beta_m) \). This is seen from Lemma 5, [3]. It is also verified that the following three operations, called elementary operations

\[
\begin{align*}
&\text{i)} \quad (P; \beta_1, \ldots, \beta_t, \ldots, \beta_m) \rightarrow (P; \beta_1, \ldots, -\beta_t, \ldots, \beta_m) \\
&\text{ii)} \quad (P; \beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_m) \rightarrow (P; \beta_1, \ldots, -\beta_i, \ldots, \beta_j, \ldots, \beta_m) \\
&\text{iii)} \quad (P; \beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_m) \rightarrow (P; \beta_1, \ldots, \beta_i + \beta_j, \ldots, \beta_j, \ldots, \beta_m)
\end{align*}
\]

do not alter the homotopy type of \( \tilde{P} \). That is to say, we have

\[
\begin{align*}
&\text{i)} \quad (P; \beta_1, \ldots, \beta_t, \ldots, \beta_m) \sim (P; \beta_1, \ldots, -\beta_t, \ldots, \beta_m) \\
&\text{ii)} \quad (P; \beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_m) \sim (P; \beta_1, \ldots, -\beta_i, \ldots, \beta_j, \ldots, \beta_m) \\
&\text{iii)} \quad (P; \beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_m) \sim (P; \beta_1, \ldots, \beta_i + \beta_j, \ldots, \beta_j, \ldots, \beta_m).
\end{align*}
\]

Given \( P = (O; x_1, \ldots, x_p; \alpha_1, \ldots, \alpha_s) \), where \( n \) spheres \( x_i(i=1, \ldots, p) \) have a point \( O \) in common and \( E_t^{n+1}(i=1, \ldots, \lambda) \) are attached to \( x_1 \vee \cdots \vee x_p \) by the maps \( f_1 : \partial E_t^{n+1} \to x_1 \vee \cdots \vee x_p \), which represent the homotopy elements \( \alpha_i = \sum_{j=1}^p c_i x_j \). Consider two maps \( f, g \) between \( P_0 = (O; x_1, \ldots, x_p) \) and \( Q_0 = (O; \bar{x}_1, \ldots, \bar{x}_p) \) such that \( f : x_i \to \sum_{j=1}^p a_{ij} x_j \) and \( g : \bar{x}_j \to \sum_{k=1}^p b_{jk} \bar{x}_k \), where \((a_{ij}), (b_{jk})\) are reciprocal unimodular matrices. Then it is obvious that \( f, g \) are homotopy equivalences. In virtue of Lemma 2, we have

\[
P \sim Q = (O; \bar{x}_1, \ldots, \bar{x}_p; f\alpha_1, \ldots, f\alpha_s)
\]

\[
= (O; \bar{x}_1, \ldots, \bar{x}_p; \sum_{j=1}^p c_{1j} (\sum_{k=1}^p a_{jk} \bar{x}_k), \ldots, \sum_{j=1}^p c_{sj} (\sum_{k=1}^p a_{jk} \bar{x}_k)).
\]

To get \( Q \) from \( P \) is said to carry out the transformation \( \bar{x}_j = \sum_{k=1}^p b_{jk} x_k \).

Especially, when this transformation is an elementary transformation, \( Q \) is said to be made from \( P \) through an elementary operation with respect to \( x \). In the sequel these terminologies will be often used.

Theorem 3. Let \( \{m_1, \ldots, m_s\} \) be the invariant system of the \( n \)-th homology group \( H_n(P) \) of a given \( A_1 \)-complex \( P \) and let \( N \) be the \((n+1)\)-th Betti number of \( P \). Then we have

\[
P \sim Q^{m_1+1} + \cdots + Q^{m_s+1} + S^{s+1} + \cdots + S^{s+1},
\]

where \( Q^{m_i} = (x_i^n; m_i x_i^i) \) \((i=1, \ldots, l)\) and \( S_i^{s+1}(i=1, \ldots, N) \) have a point in
On Homotopy Type Problems of Special Kinds of Polyhedra

common, \(x_i(i=1, ..., l)\) denoting \(n\) spheres.

Proof. This theorem can be easily proved by a Theorem due to Hurewicz [8] and by Lemma 2.

**Lemma 3.** Let \(P\) be a connected simply connected polyhedron. \((P+x;\alpha+2^nx)\) denotes a complex \(P+x+e^{n+1}\), where \(e^{n+1}\) is attached to \(P+x\) by a map \(f: \partial e^{n+1} \to P+x\), which represents an element \(\alpha+2^nmx\) of \(\pi_n(P+x)\). \(\alpha \in \pi_n(P), 2\alpha=0\), and \(x\) is an \(n\) sphere. Moreover, \(m\) is odd and \(p\) is an integer. Then we have

\[
\{P+x; \alpha+2^nmx\} \sim \{P+y+z; \alpha+2^ny, mz\},
\]

where \(y, z\) are \(n\) spheres and \(x, y, z\) are attached to \(P\) at a point.

Proof. It is obvious that \(\{P+x; \alpha+2^nmx\} \sim \{P+x+x'; \alpha+2mx, x'\}\)
where \(x'\) is an \(n\) sphere. Now we define homotopy equivalences \(f, g\) of two complexes \(P+x+x', P+y+z\) such that

i) \(f|P = g|P\) is the identical map,

ii) \(f(x) = Ay+Bz\),

iii) \(f(x') = -2^ny+mz\),

iv) \(g(y) = mx-Bx'\),

v) \(g(z) = 2^n+x+Ax'\),

where \(A, B\) are integers satisfying \(mA+2^nB=1\). Then it is easily seen that \(fg \sim gf \sim e\) (identical map). Applying Lemma 2 and elementary operations to the following arguments, we have

\[
\{P+x, \alpha+2^nmx\} \sim \{P+x+x', \alpha+2^nx, x'\}
\sim \{P+y+z, \alpha+2^nm(Ay+Bz), -2^ny+mz\}
\sim \{P+y+z, \alpha+2^n(Ay+2^nmBz), -2^ny+mz\}
\sim \{P+y+z, \alpha+2^n(A+2^nB)y, -2^ny+mz\}
\sim \{P+y+z, \alpha+2^n(y, -2^ny+mz\}
\sim \{P+y+z, \alpha+2^n(y, \alpha+2^ny-2^ny+mz\}
\sim \{P+y+z, \alpha+2^ny, \alpha+mx\}
\]

If we put \(\alpha+z=\bar{z}\), in virtue of Lemma 2 we have

\[
\{P+y+z; \alpha+2^ny, \alpha+mx\} \sim \{P+y+z; \alpha+2^ny, \alpha+mx+\bar{m}z\}
\sim \{P+y+z; \alpha+2^ny, m\bar{z}\},
\]

where \(\bar{z}\) is an \(n\) sphere and \(\alpha+mx=0\) from \(2\alpha=0\). This proves the Lemma 3.

Now we refer to the polyhedra of the following types as elementary \(A^n\)-polyhedra:
\[ Q_i = S^n (r = n, n+1, n+2), \]

**ii)** \[ Q_2 = S^n \cup e^{n+1}, \] where \( e^{n+1} \) is attached to \( S^n \) by a map \( f : e^{n+1} \to S^n \) of degree odd,

**iii)** \[ Q_3 = S^n \cup e^{n+2}, \] where \( e^{n+2} \) is attached to \( S^n \) by an essential map \( f : e^{n+2} \to S^n \).

**iv)** \[ Q_4 = (S^n \cup S^{n+1}) \cup e^{n+2}, \] where \( e^{n+2} \) is attached to \( S^n \cup S^{n+1} \) by a map \( f : e^{n+2} \to S^n \cup S^{n+1} \) of the form \( a + b; a \) is an essential map : \( S^{n+1} \to S^n \) and \( b \) denotes a map : \( S^{n+1} \to S^n \) of degree 2.

**v)** \[ Q_5 = S^n \cup e^{n+1} \cup e^{n+2}, \] where \( e^{n+1} \) is attached to \( S^n \) by a map \( f : e^{n+1} \to S^n \) of degree odd and \( e^{n+2} \) is attached to \( S^n \) by an essential map : \( e^{n+2} \to S^n \).

**vi)** \[ Q_6 = (S^n \cup S^{n+1}) \cup e^{n+1} \cup e^{n+2}, \] where \( e^{n+1} \) is attached to \( S^n \) by a map : \( e^{n+1} \to S^n \) of degree 2 and \( e^{n+2} \) is attached to \( S^n \) by a map of type iv).

**vii)** \[ Q_7 = S^n \cup e^{n+1}, \] where \( e^{n+1} \) is attached to \( S^n \) by a map : \( e^{n+1} \to S^n \) of degree 2.

**viii)** \[ Q_8 = S^{n+1} \cup e^{n+2}, \] where \( e^{n+2} \) is attached to \( S^{n+1} \) by a map : \( e^{n+2} \to S^{n+1} \) of degree even.

**ix)** \[ Q_9 = S^{n+1} \cup e^{n+2}, \] where \( e^{n+2} \) is attached to \( S^{n+1} \) by a map : \( e^{n+2} \to S^{n+1} \) of degree 2.

Then we have

**Theorem 4.** If \( P \) is an \((n+2)\) dimensional finite connected polyhedron which is aspherical in dimensions less than \( n \), \( P \) is of the same homotopy type as a reduced complex which consists of a collection of elementary polyhedra of the above mentioned types, where the elementary polyhedra have a point in common.

Proof. In virtue of Theorem 3, the \((n+1)\) skelton \( P^{n+1} \) of \( P \) has the same homotopy type as the complex

\[ \{ Q_1^{n+1} \cup Q_2^{n+1} \cup \ldots \cup Q_{r_1}^{n+1} \cup Q_{r_2}^{n+1} \cup \ldots \cup Q_{r_1}^{n+1} \cup S_1^n \cup \ldots \cup S_1^n \cup S_1^{n+1} \cup \ldots \cup S_1^{n+1} \}, \]

where \( r_1, \ldots, r_t \) are odd, and \( 1 \leq q_1 \leq q_2 \leq \ldots \leq q_k \) are integers. Thus we have

\[ P \sim \{ Q_1^{n+1} \cup Q_2^{n+1} \cup \ldots \cup Q_{r_1}^{n+1} \cup \ldots \cup Q_{r_1}^{n+1} \cup S_1^n \cup \ldots \cup S_1^n \cup S_1^{n+1} \cup \ldots \cup S_1^{n+1} \cup S_2^{n+1} \}; R_1, \ldots, R_m, \]

where \( R_1, \ldots, R_m \) are all the relations. Denoting by \( x_i \) the homotopy element represented by a map \( S^{n+1} \to S_1^{n+1} \) of degree 1, we have

\[ R_i = \sum_{j=1}^m \lambda_{i,j} x_j + \alpha_i \quad (i = 1, \ldots, m), \]
where $\alpha_i (i=1, \ldots, m)$ are homotopy elements of $\pi_{n+1}(Q_{2i}^{n+1} \vee \cdots \vee S^n)$. By elementary operations with respect to $\{S_1^{n+1}, \ldots, S_m^{n+1}\}$ and $\{R_1, \ldots, R_m\}$ we have

$$P \sim \{Q_{1}^{n+1} \vee \cdots \vee S_1^{n+1};\alpha_i + x_i, \ldots, \alpha_{i-1} + x_{i-1}, \alpha_i + b_i x_i, \ldots, \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^n,$$

where $b_{i+j} (\nu=0, \ldots, j-i)$ are integers greater than unity, and $\gamma_i (i=1, \ldots, k)$ are homotopy elements of $\pi_{n+1}(Q_{2i+1}^{n+1} \vee \cdots \vee S^n)$. Since $\pi_{n+1}(Q_{2i+1}^{n+1})=0 (\lambda=1, \ldots, l)$, we have

$$P \sim \{Q_{2i+1}^{n+1} \vee \cdots \vee Q_{2j+1}^{n+1} \vee S_1^{n+1} \vee \cdots \vee S_k^{n+1} \vee S_{1}^{n+1} \vee \cdots \vee S_{m}^{n+1};\alpha_i + x_i, \ldots, \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^n + Q_{r_1}^{n+1} + \cdots + Q_{r_t}^{n+1}.$$

Then it is obvious that we have

$$P \sim \{Q_{2i+1}^{n+1} \vee \cdots \vee Q_{2j+1}^{n+1} \vee S_1^{n+1} \vee \cdots \vee S_k^{n+1} \vee S_{1}^{n+1} \vee \cdots \vee S_{m}^{n+1};\alpha_i + x_i, \ldots, \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^n + Q_{r_1}^{n+1} + \cdots + Q_{r_t}^{n+1},$$

where $\gamma_i (i=1, \ldots, \rho)$ are homotopy elements of $\pi_{n+1}(Q_{2i+1}^{n+1} \vee \cdots \vee S^n)$. Utilizing Lemma 3, and changing suffixes, we have

$$P \sim \{Q_{2i+1}^{n+1} \vee \cdots \vee S_k^{n+1};\alpha_i + x_i, \ldots, \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^n + Q_{r_1}^{n+1} + \cdots + Q_{r_2}^{n+1},$$

so that it is sufficient for us to try to reduce

$$P_1 = \{Q_{2i+1}^{n+1} \vee \cdots \vee Q_{2j+1}^{n+1} \vee S_1^{n+1} \vee \cdots \vee S_k^{n+1} \vee S_{1}^{n+1} \vee \cdots \vee S_{m}^{n+1};\alpha_i + x_i, \ldots, \alpha_j + 2^n x_i, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^n + Q_{r_1}^{n+1} + \cdots + Q_{r_2}^{n+1},$$

$x_i, \gamma_1, \ldots, \gamma_k$ to a normal form. Without loss of generality it may be assumed that $\gamma_1, \ldots, \gamma_k$ are linearly independent with respect to integer coefficients mod. 2. Here a number of $S^{n+2}$ may be removed from bracket. If we denote by $y_i (i=1, \ldots, \lambda)$ the homotopy element represented by a map $f_i: S^n \to S^n_i$ of degree 1 and by $z_i (i=1, \ldots, \mu)$ the homotopy element represented by a map $f_i: S^n \to Q_{2i+1}^{n+1}$ of degree 1, and if, for example,

$$\gamma_1 = (y_1 \eta) + \cdots + (z_i \eta) + \cdots,$$

it is seen by means of the following operation

$$\overline{y}_i = y_i + z_i + \cdots,$$

$$\overline{y}_i = y_i \quad (i=2, \ldots, \lambda)$$

that $\overline{\gamma}_1 = (\overline{y}_1 \eta)$, where $\eta$ denotes an essential map $S^{n+1} \to S^n$. If we change notations, and if $\{\alpha_{\tau_1}, \ldots, \alpha_{\tau_2}; \gamma_{\sigma_1}, \ldots, \gamma_{\sigma_2}\}$ contain $\overline{\gamma}_1 = (\overline{y}_1 \eta)$, we change them, by elementary operations, to
\[ \alpha_1 + \gamma_1, \ldots, \alpha_{\tau_1} + \gamma_1, \gamma_{\alpha_1} + \gamma_1, \ldots, \gamma_{\tau_1} + \gamma_1. \]

Then we have
\[ P_1 \sim \{ Q_{2^1}^{n+1} \vee \ldots \vee Q_{2^k}^{n+1} \vee S_1^n \vee \ldots \vee S_1^{n+1} \vee \ldots \vee S_1^{n+1} ; \alpha_1 + 2^n x_1, \ldots, \alpha_{\mu} + 2^n x_{\mu}, \gamma_1, \ldots, \gamma_{\rho} \}, \]
where \( \bar{\alpha}_i (i = 1, \ldots, \mu) \) and \( \bar{\gamma}_i (i = 2, \ldots, \rho) \) do not contain \( (y_1, \eta) = \gamma_1 \).

It follows that
\[ P_1 \sim \{ Q_{2^1}^{n+1} \vee \ldots \vee S_1^{n+1} ; \alpha_1 + 2^n x_1, \ldots, \alpha_{\mu} + 2^n x_{\mu}, \bar{\gamma}_2, \ldots, \bar{\gamma}_\rho \} \cup (S_1^n \cup e^{n+2}), \]
where \( e^{n+2} \) is attached to \( S_1^n \) by an essential map: \( \exists e^{n+2} \to S_1^n \). By the same process all the \( \gamma_i \) involving at least one \( (y, \eta) \) may be deleted from the interior of the bracket together with \( S_1^n \), so that, changing notations, we have
\[ P_1 \sim \{ Q_{2^1}^{n+1} \vee \ldots \vee S_1^{n+1} ; \alpha_1 + 2^n x_1, \ldots, \alpha_{\mu} + 2^n x_{\mu}, \gamma_1, \ldots, \gamma_{\tau} \} \cup (S_1^n \cup e^{n+2}), \]
where \( \gamma_1, \ldots, \gamma_{\tau} \) do not contain any \( (y_i, \eta) (i = 1, \ldots, \sigma) \).

Putting
\[ P_2 = \{ Q_{2^1}^{n+1} \vee \ldots \vee S_1^{n+1} \vee \ldots \vee S_2^{n+1} \vee \ldots \vee S_2^{n+1} ; \alpha_1 + 2^n x_1, \ldots, \alpha_{\mu} + 2^n x_{\mu}, \gamma_1, \ldots, \gamma_{\tau} \}, \]
we proceed to reduce \( P_2 \) to a normal form. Let \( p_1 \leq p_2 \leq \ldots \leq p_\mu \) and let \( \alpha_i + 2^n x_i \) be the term of the greatest \( p_j \) among all the terms containing at least one of \( (y, \eta) (j = 1, \ldots, \sigma) \). Then, for instance,
\[ \alpha_i + 2^n x_i = (y_1, \eta) + \cdots +(z_i, \eta) + \cdots + 2^n x_i. \]

If we carry out the operation
\[ \bar{y}_1 = y_1 + \cdots + z_i + \cdots + 2^n x_i, \]
\[ \bar{y}_\rho = y_\rho \quad (\rho = 2, \ldots, \sigma) \]
\[ \bar{z}_\rho = z_\rho \quad (\rho = 1, \ldots, \kappa) \]
we have \( \bar{\alpha}_i = (\bar{y}_1, \eta) \). If there exist \( \bar{\alpha}_i \) containing \( (\bar{y}_1, \eta) \), we substract \( (\alpha_i + 2^n x_i) \) from \( (\alpha_i + 2^n x_i) \) by elementary operations. Then we have
\[ (\alpha_i + 2^n x_i) - (\bar{\alpha}_i + 2^n x_i) = (\alpha_i - \bar{\alpha}_i) + 2^n(x_i - 2^{n-p} x_i), \]
where \( p_1 \leq p_i \). Again by elementary operation
\[ \bar{x}_j = x_j - 2^{n-p} x_i \]
\[ \bar{x}_\rho = x_\rho \quad (\rho = 1, \ldots, \mu) \]
it follows that the relation \( \alpha_i + 2^n x_i \), an \( n \) sphere \( S_1^n \), and an \( (n+1) \) sphere \( S_1^{n+1} \) are deleted from the contents interior the bracket \{ \} and that
\[ P_2 \sim \{ Q_{2^1}^{n+1} \vee \ldots \vee Q_{2^k}^{n+1} \vee S_2^n \vee \ldots \vee S_2^{n+1} \vee \ldots \vee S_2^{n+1} \vee \ldots \vee S_\mu^{n+1} \vee \ldots \vee S_\mu^{n+1} \}; \]
\[ \alpha_1 + 2^n x_1, ..., \alpha_{i-1} + 2^n x_{i-1}, \alpha_{i+1} + 2^n x_{i+1}, ..., \alpha_{\mu} + 2^n x_{\mu}, \gamma_1, ..., \gamma_r \]
\[ + (S^n \cup S^{n+1}_t) \cup e^{n+2}, \text{ where } (S^n \cup S^{n+1}_t) \cup e^{n+2} = Q_4. \]

By the same procedure, \( P_2 \) can be reduced to a complex
\[ P_3 = \{ Q^{n+1}_{2k-1} \cup ... \cup Q^{n+1}_{2k} \cap S^{n+1}_1 \cup ... \cup S^{n+1}_l; \alpha_1 + 2^n x_1, ..., \alpha_x + 2^n x_x, \gamma_1, ..., \gamma_r \} \]
\[ + Q_i + ... + Q_4. \]

If, for example, \( \gamma_1 = (z_1 \eta) + ... \), by elementary operations
\[ \bar{z}_i = z_i + ... \]
\[ \bar{z}_i = z_i \quad (i = 2, ..., \kappa) \]
we have \( \bar{\gamma}_1 = (\bar{z}_1 \eta) \). Then, if \( \bar{\alpha}_1, ..., \bar{\alpha}_x; \bar{\gamma}_1, ..., \bar{\gamma}_r \) contain \( \bar{\gamma}_1 \), we change them, by elementary operations, to
\[ \bar{\alpha}_1 + \bar{\gamma}_1, ..., \bar{\gamma}_1 + \bar{\gamma}_1, ..., \bar{\gamma}_r + \bar{\gamma}_1. \]

Then it is seen that, changing notations,
\[ P_3 \sim \{ Q^{n+1}_{2k-1} \cup ... \cup Q^{n+1}_{2k} \cap S^{n+1}_1 \cup ... \cup S^{n+1}_l; \alpha_1 + 2^n x_1, ..., \alpha_x + 2^n x_x, \gamma_2, ..., \gamma_r \} \]
\[ + Q^{n+1}_{2k} \cup e^{n+2}, \]
where \( Q^{n+1}_{2k} \cup e^{n+2} = Q_5 \). Repeating the same process and changing notation, we have
\[ P_3 \sim \{ Q^{n+1}_{2k-1} \cup ... \cup Q^{n+1}_{2k} \cap S^{n+1}_1 \cup ... \cup S^{n+1}_l, \alpha_1 + 2^n x_1, ..., \alpha_x + 2^n x_x \} + Q_5 + ... \]
\[ + Q_5. \]

Now we arrive at the final stage of reduction. Let \( \alpha_i + 2^n x_i \) be the term of the greatest \( p_i \) among the terms containing at least one of \( (z_i \eta) \) \((j = 1, ..., l)\). If, for instance,
\[ \alpha_i + 2^n x_i = (z_1 \eta) + ... + 2^n x_i, \]
by the elementary operation
\[ \bar{z}_1 = z_1 + ... \]
\[ \bar{z}_i = z_i \quad (i = 2, ..., l) \]
we have \( \bar{\alpha}_i = (\bar{z}_1 \eta) \). If there exist some \( \bar{\alpha}_j \) containing \( (\bar{z}_1 \eta) \), we subtract \( \bar{\alpha}_i + 2^n x_i \) from \( \bar{\alpha}_j + 2^n x_j \) by elementary operation. Then we have \( (\bar{\alpha}_j + 2^n x_j) \]
\[ - (\bar{\alpha}_i + 2^n x_i) = (\bar{\alpha}_j - \bar{\alpha}_i) + 2^n (x_j - 2^n x_i), \text{ where } p_j \leq p_i. \]
Again, by the elementary operation
\[ \bar{x}_j = x_j - 2^n x_i \]
\[ \bar{x}_i = x_i \quad (\rho = 1, ..., \hat{j}, ..., \mu) \]

It is seen that, changing notation,
where \((S_n \lor S^{n+1}) \lor e^{n+1} \lor e^{n+2} = Q_6\). Repeating the same process, we have
\[ P_4 \sim Q_6 + \cdots + Q_6 + Q_7 + \cdots + Q_7 + Q_8 + \cdots + Q_8 + S^{n+1} + \cdots + S^{n+1}. \]
This completes the proof.

6. Reduced complexes

\(\mathcal{A}_g\)-complexes

\(K\) is referred to as a reduced complex when it satisfies the following conditions:

i) \(K^0 = K^1 = \cdots = K^{n-1} = e^0\), a single point,

ii) \(K^n = S^n + \cdots + S^n\), where \(n\) spheres \(S^n(i = 1, \ldots, \kappa)\) are attached at a point \(e^0\), and \(S^n_t - e^0 = e^n_t(i = 1, \ldots, \kappa)\) are \(n\)-cells,

iii) \(K^{n+1} = K^n + e^{n+1}_1 + \cdots + e^{n+1}_1 + \cdots + e^{n+1}_l\), where \(e^{n+1}_i(i = 1, \ldots, k)\) is attached to \(S^n_{i+1}\) by a map \(f_i : e^{n+1}_i \to S^n_{i+1}\) of odd degree \(\sigma_i\), and \(e^{n+1}_i(i = 1, \ldots, l)\) is attached to \(S^n_{i+1}\) by a map \(f_i : e^{n+1}_i \to S^n_{i+1}\) of degree \(2\sigma_i\),

iv) \(K^{n+2} = K^{n+1} + e^{n+2}_1 + \cdots + e^{n+2}_1 + \cdots + e^{n+2}_l + S^{n+2} + \cdots + S^{n+2}\), where \(e^{n+2}_i(i = 1, \ldots, l)\) is attached to \(S^n_{i+1}\) by an essential map: \(e^{n+2}_i \to S^n_{i+1}\), and \(e^{n+2}_i(i = 1, \ldots, \kappa-k-l)\) is attached to \(S^n_{i+1}\) by an essential map: \(e^{n+2}_i \to S^n_{i+1}\).

v) If \(K_0 = K^{n+2} - (e^{n+1}_1 \cup S^n_1 + \cdots + e^{n+1}_1 \cup S^n_\kappa)\), a finite number of \((n+3)\) cells \(e^{n+2}_i(i = 1, \ldots, \alpha)\) are attached to \(K_0\) by maps \(f_i : e^{n+2}_i \to K_0\).

Notice that \(e^{n+1}_i \cup S^n_i(i = 1, \ldots, k)\) are not bounded, and that \(n\)-spheres \(S^n_i(i = 1, \ldots, \kappa)\) are all bounded. Of course, the case where \(k = 0\), or \(k = 0\), or \(t = 0\), may be possible, but the most general reduced complex of \(\mathcal{A}_g\)-complex is the cell complex of the type just referred to above.

**Theorem 5.** Any \(\mathcal{A}_g\)-complex \(P\) is of the same homotopy type as some reduced complex.

**Proof.** Let \(P^{n+2}\) be the \((n+2)\) skeleton of \(P\), then \(\pi_0(P^{n+2}) = 0\) for each \(i < n\). In virtue of Theorem 4, \(P^{n+2}\) is of the same homotopy type as a cell complex \(Q^{n+2}\) consisting of a number of elementary \(\mathcal{A}_g\)-complexes. It is evident that \(\pi_{n+1}(P^{n+2}) \approx \pi_{n+1}(P^{n+2})\), and \(\pi_{n+1}(Q^{n+2}) \approx \pi_{n+1}(P^{n+2})\), so that we have \(\pi_{n+1}(Q^{n+2}) = 0\). By the recurrent use of a result of G.W. Whitehead [9] or a slight generalization of a lemma of Blakers and Massay [10], we have \(\pi_{n+1}(Q^{n+2}) \approx \sum_{\alpha = 1}^\rho \pi_{n+1}(Q^n)\), where the upper-suffix \(\mu\) of \(Q^n\), indicates the number of elementary polyhedra of the type \(Q^n\). It follows that if \(\pi_{n+1}(Q^n) = 0\), such polyhedra \(Q^n\) are deleted from \(Q^{n+2}\). As \(\pi_{n+1}(S^n) \approx I_2\) for \(n > 2\), and \(\pi_{n+1}(S^{n+1}) \approx I, S^n, S^{n+1}\) must be deleted from \(Q^{n+2}\).
It is verified that we have $\pi_{n+1}(Q_4) \cong I_{2^n+1}$. For we have $a+b = 0$, so that $2b = 0$. Thus the element represented by a map $S^{n+1} \to S^{n+1} \subseteq Q_4$ of degree 1, is the generator of $\pi_{n+1}(Q_4)$, whose order is $2 \cdot 2^n = 2^{n+1}$. Therefore all the $Q_i$ are deleted from $Q^{n+2}$. From the same arguments and from $\pi_{n+1}(e^{n+1} \cup S^m) \cong \pi_{n+1}(S^m)$, we have $\pi_{n+1}(Q_6) \cong I_{2^n+1}$, so that all the $Q_i$ are deleted from $Q^{n+2}$. Since $\pi_{n+1}(Q_7) \cong I_2$, $\pi_{n+1}(Q_8) \cong I_2$, and $\pi_{n+1}(Q_9) \cong I_2$, all the $Q_i^6$, $Q_i^5$, $Q_i^2$ are deleted from $Q^{n+2}$. From the verifications just referred to above and from the vanishing $(n+1)$ homotopy groups of $S^{n+2}$, $Q_2$, $Q_3$, $Q_5$, it is concluded that $P^{n+2}$ is of the same homotopy type as $K^{n+2}$ in the definition of a reduced complex. Let $f : P^{n+2} \to K^{n+2}$ be a homotopy equivalence, and let $\sigma_i^{n+3}$ be an $(n+3)$ simplex of $P$. Then from Lemma 2, $P$ is of the same homotopy type as a cell complex, $(n+3)$ simplexes $\sigma_i^{n+3}(i = 1, \ldots, \alpha)$ of which are attached to $K^{n+2}$ by maps $f, e : \partial \sigma_i^{n+3} \to K^{n+2}$, where $e$ is the identical map of $P$. However, the element of $\pi_{n+2}(K^{n+2})$ represented by a map $f$ may be regarded as an element of $\pi_{n+2}(K_0)$, so that from Lemma 5, [3], $P$ is of the same homotopy type as a reduced complex defined above. This completes the proof.

7. $\pi_{n+2}(K^{n+2})$.

Let us consider $K^{n+2}$ satisfying i), ii), iii), iv) in § 5. It is easily verified that

$$
\pi_{n+2}(K^{n+2}) \cong \sum_{i=1}^k \pi_{n+2}(S_{i+2}) + \sum_{i=1}^k \pi_{n+2}(S_{i} \cup e^{n+1}_{i}) + \sum_{i=1}^l \pi_{n+2}(S_{i} \cup e^{n+1}_{i+1} \cup e^{n+2}_{i+1}) + \sum_{i=1}^{k-l} \pi_{n+2}(S_{i} \cup e^{n+1}_{i+1} \cup e^{n+2}_{i+1}), \text{ for } n \geq 3.
$$

It is also verified (for example, see [11]) that

$$
\pi_{n+2}(S_{i} \cup e^{n+1}_{i}) = 0, \text{ for } i = 1, \ldots, k,
$$

$$
\pi_{n+2}(S_{k+i+1} \cup e^{n+3}_{k+i}) = I, \text{ for } i = 1, \ldots, k - n - l,
$$

$$
\pi_{n+2}(S_{k+i} \cup e^{n+1}_{k+i} \cup e^{n+2}_{k+i}) = I + I_2, \text{ for } i = 1, \ldots, l.
$$

Now, let us denote by $S_{i}^{n+2}$ the generator $\pi_{n+2}(S_{i}^{n+2})$ represented by a map $S^{n+2} \to S_{i}^{n+2}$ of degree 1. The generator $\omega_{k+i+1} \in \pi_{n+2}(S_{k+i+1} \cup e^{n+2}_{k+i+1})$ is represented by a map $\omega_{k+i+1} : S^{n+2} \to S_{k+i+1} \cup e^{n+2}_{k+i+1}$ as follows. Denote the northern hemisphere of $S^{n+2}$ by $V_{g}^{n+2}$ and the southern hemisphere by $V_{s}^{n+2}$, then $S^{n+2} = V_{g}^{n+2} \cup V_{s}^{n+2}$ and the equator of $S^{n+2}$ is represented by $V_{e}^{n+2} \cap V_{s}^{n+2} \cong S^{n+1}$. Then the partial map $\omega_{k+i+1} | V_{g}^{n+2} \cap V_{s}^{n+2}$ represents $2\eta : S^{n+2} \to S_{k+i+1} \cup e^{n+2}_{k+i+1}$, where $\eta$ is an essential map of $S^{n+1}$ onto $S^{n+1}$. Since $\omega_{k+i+1} | \partial V_{s}^{n+2}$ is inessential, we have an extended map $V_{g}^{n+2} \to S_{k+i+1}$. From these considerations that $\partial e^{n+2}_{k+i+1} \to S_{k+i+1}$ represents an essential map $\eta$ and that $\omega_{k+i+1} | V_{g}^{n+2}$ represents $2\eta$, it follows
that we have a map: $\nu_{k+l+1}^{n+2} \rightarrow S^{n+2}_{k+l+1} \cup e_{k+l+1}^{n+2}$ of degree two. It is proved in [11] that the map $\nu_{k+l+1}^{n+2}$ thus obtained represents the free generator of $\pi_{n+2}(S^{n+2}_{k+l+1} \cup e_{k+l+1}^{n+2})$. Notice that the free generator itself will be also denoted by $\nu_{k+l+1}^{n+2}$. Next, the free generator of $\pi_{n+2}(S^{n+2}_{k+l+1} \cup e_{k+l+1}^{n+2})$ is represented by a map $\nu_{k+l+1}^{n+2} : S^{n+2}_{k+l+1} \cup e_{k+l+1}^{n+2}$ of the same property as referred to above, and the generator $\nu_{k+l+1}^{n+2}(i = 1, \ldots, l)$ of order two are represented by maps $\nu_{k+l+1}^{n+2} : S^{n+2}_{k+l+1} \cup e_{k+l+1}^{n+2}$ as follows. Remember here that $e_{k+l+1}^{n+2}$ is attached to $S^{n+2}_{k+l+1}$ by a map $de_{k+l+1}^{n+2} \rightarrow S^{n+2}_{k+l+1}$ of degree 2.

8. Cohomological properties in a reduced complex $L$.

Referring to §7, §8, we have

i) $e_{k+l}^{n}(i = 1, \ldots, l)$ are cocycles mod 2,

ii) $e_{k+l}^{n}(i = k+l+1, \ldots, \kappa)$ are absolute cocycles,

iii) $\delta e_{k+l}^{n+2} = \sum_{j=1}^{k} 2\mu_j e_{k+l}^{n+3}(i = k+1, \ldots, k+l)$ and

\[ \delta e_{k+l}^{n+2} = \sum_{j=1}^{k} \gamma_j e_{k+l}^{n+3}(i = k+l+1, \ldots, \kappa), \]
iv) \( e_i^{n+1} = 0 \) for \( i = k+1, \ldots, k+l \) are absolute cocycles.

It should be noted that "2" in the terms \( \delta e_i^{n+2} \) come from the degree of the maps \( \omega \) and that \( e_i^{n+2} = 0 \) for \( i = k+1, \ldots, \kappa \) are cocycles mod. 2. Then we have

**Theorem 6.**

a) \( j_0 e_i^{n-2} = j_0 e_i^{n-2} + \sum_{j=k+1}^{k+l} e_i j_2 e_j^{n+2} \) for \( i = k+l+1, \ldots, \kappa \),

b) \( j_2 e_i^{n-2} = j_2 e_i^{n-2} \) for \( i = 1, \ldots, l \),

c) \( j_0 e_i^{n+1} = j_0 e_i^{n+1} + \sum_{j=1}^{n} \nu_j j_2 e_j^{n+3} \) for \( i = k+1, \ldots, k+l \),

d) \( q_{n-3} j_0 e_i^{n} = \sum_{j=1}^{n} \gamma_j j_0 e_j^{n+3} \) for \( i = k+l+1, \ldots, \kappa \),

e) \( q_{n-3} j_2 e_i^{n+1} = \sum_{j=1}^{n} \mu_j, k+i j_0 e_j^{n+3} \) for \( i = 1, \ldots, l \).

**Proof.** Putting \( M = L_i^{n+2} \) and considering the injection \( \kappa : M \to L \), we have a proper homomorphism \( \kappa^* : \text{FH}(L) \to \text{FH}(M) \) induced by \( \kappa \). Put

\[
j_0 e_i^{n-2} = \sum_{j=k+1}^{k+l} e_i j_2 e_j^{n+2} + \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2} (i = k+l+1, \ldots, \kappa),
\]

where \( e_i \equiv 0 \) or 1 (mod. 2).

\[
\kappa^*(j_0 e_i^{n-2}) = \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2} + \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2},
\]

\[
\kappa^*(j_0 e_i^{n-2} e_i^{n-2}) = \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2} + \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2},
\]

\[
\kappa^*(j_0 e_i^{n-2} e_i^{n-2}) = \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2} + \sum_{j=k+1}^{k+l} e_j j_2 e_j^{n+2}.
\]

It follows that \( e_i \equiv 0 \) for \( i = k+1, \ldots, k+l \), \( e_j \equiv 0 \) for \( i = k+l+1, \ldots, k+l+i-1, k+l+i+1, \ldots, \kappa \) and \( e_{k+i+1} \equiv 1 \) (mod. 2). This proves Theorem 6 a).

Similarly we have Theorem 6 b).

Let \( M = S^{n+1} \cup S^{n+3} \), where \( S^{n+3} \) is attached to \( S^{n+1} \) by an essential map \( \eta : \partial e^{n+3} \to S^{n+1} \). Then \( M \) is regarded as a cell complex composed of three cells, a point \( e^0 \), an \((n+1)\) cell \( e^{n+1} \), and an \((n+3)\) cell \( e^{n+3} \). Let us define a cellular map \( \kappa : L \to M \) such that

i) \( \kappa(S_i^{n+2}) = e^0 \) for \( i = 1, \ldots, t \),

ii) \( \kappa(S_i^{n+1}) = e^0 \) for \( i = 1, \ldots, k \),

iii) \( \kappa(S_i^{n+2}) = e^0 \) for \( i = k+1, \ldots, \kappa \)

iv) \( \kappa(S_i^{n+1} \cup e_i^{n+2}) = e^0 \) for \( i = k+1, \ldots, k+i-1, k+i+1, \ldots, k+l \),

v) \( \kappa(S_i^{n+1} \cup e_i^{n+2} \cup e_i^{n+3}) = S_i^{n+1} \),

vi) if \( \nu_{j+k+l} = 0 \), \( e_i^{n+3} \) is mapped by \( \kappa \) topologically onto \( e_i^{n+3} \), and otherwise, \( e_i^{n+3} \) is mapped to \( e^0 \) by \( \kappa \).
It is verified in the following way that this map \( \kappa \) can be constructed. v) is constructed such that \( \kappa \) maps \( S_{n+1}^e \cap e_{k+1}^{n+1} \) into \( e^0 \) and elsewhere topological. Let \( \kappa': c(M) \to c(L) \) be a cochain map induced by \( \kappa \) and let \( \kappa^*: FH(M) \to FH(L) \) be a proper homomorphism induced by \( \kappa \). In \( M \) we have
\[
j_0e^{n+1} \cup n_0e^{n+1} = j_2e^{n+3},
\]
so that
\[
k^*(j_0e^{n+1} \cup j_0e^{n+1}) = k^*j_2e^{n+3}
\]
\[
k^*j_0e^{n+1} = k^*j_0e^{n+1} = j_2e^{n+3}
\]
\[
j_0k'e^{n+1} = j_0k'e^{n+1} = j_0k'e^{n+3}
\]
Since we have \( k'e^{n+1} = e_{b+1}^{n+1} \) and \( k'e^{n+3} = \sum_{j=1}^n \nu_{j+1}e_{j+1}^{n+1} \), we have
\[
j_0e^{n+1} \cup j_0e^{n+1} = \sum_{j=1}^n \nu_{j+1}e_{j+1}^{n+3}.
\]
This relation holds true for each \( i = 1, \ldots, l \), so that c) is completely established.

Though d), e) will not be used in the sequel, we prove them here for the completeness and the convenience of our discussions. They are essentially used in solving the \((n+3)\) extension cocycle and corresponding classification problem, which N. Shimada and I will discuss in our forthcoming paper [11]. From a) we have
\[
e_n^0 \cap e_n^0 = (-1)^{n+2}+2c^{n+2}+dc^{n+1} \text{ for each } i = k+l+1, \ldots, k,
\]
e_{n+1}^0, e_{n+1}^1 \text{ are cochains. Considering the coboundary of both sides, we have}
\[
2(-1)^n e_{n+3}^0 = 2(-1)^n \sum_{j=1}^n \gamma_{j+1}^{n+3} + 2d c^{n+2}
\]
\[
e_n^0 \cap e_n^0 = \sum_{j=1}^n \gamma_{j+1}^{n+3} + (-1)^d c^{n+2}.
\]
By the definition of \( q \)-operation, in case where \( m = 0 \), we have
\[
q_{n-1}e_n^0 = \sum_{j=1} \gamma_{j+1}^{n+3} + (-1)^n d c^{n+2},
\]
sot that
\[
q_{n-1}j_0e_n^0 = \sum_{j=1}^n \gamma_{j+1}^{n+1} \text{ for each } i = k+l+1, \ldots, k.
\]
This proves d).

The proof of e) is analogous to that of d). For the completeness of discussions we prove e).

From b),
\[
e_{k+l}^n \cap e_{k+l}^n = (-1)^n e_{k+l}^n + 2c^{n+2} + dc^{n+1}
\]
\[
\delta(e_{k+l}^n \cap e_{k+l}^n) = (-1)^n d e_{k+l}^n + 2d c^{n+2}
\]
\[
2(-1)^n e_{k+l}^n \cap e_{k+l}^n + \delta(e_{k+l}^n \cup e_{k+l}^n) = 2(-1)^n \sum_{j=1}^n \mu_{j+1} e_{j+1}^{n+1} + 2d c^{n+2}.
\]

\[i)\]
Since \( \delta(e_{k+1}^{n+1} \cup e_{k+1}^n) = (-1)^n e_{k+1}^n \cup e_{k+1}^{n+1} + (-1) e_{k+1}^n \cup e_{k+1}^{n+1} + 2^k e_{k+1}^{n+1} \cup e_{k+1}^n \),
we have
\[
e_{k+1}^n \cup e_{k+1}^n = (-1)^n e_{k+1}^n \cup e_{k+1}^{n+1} + 2^k e_{k+1}^{n+1} \cup e_{k+1}^n - \delta(e_{k+1}^{n+1} \cup e_{k+1}^n).
\]
Substituting ii) for the term \( e_{k+1}^n \cup e_{k+1}^{n+1} \) of i), we have
\[
2(-1)^n e_{k+1}^n \cup e_{k+1}^{n+1} + 2(-1)^n 2^k e_{k+1}^n \cup e_{k+1}^{n+1} + 2^k e_{k+1}^{n+1} \cup e_{k+1}^n - 2^k \delta(e_{k+1}^{n+1} \cup e_{k+1}^n)
= 2(-1)^n \sum_{j=1}^n \mu_j e_{j}^{n+3} + 2 \delta e_{n+2}.
\]
Thus it follows that
\[
e_{k+1}^n \cup e_{k+1}^n + 2^k e_{k+1}^n \cup e_{k+1}^{n+1} \delta_2 \mu e_{k+1}^n + (-1)^n 2^k \delta_2 \mu e_{k+1}^n \cup e_{k+1}^{n+1} \cup e_{k+1}^n \cup e_{k+1}^{n+1}
= \sum_{j=1}^n \mu_j e_{j}^{n+3}.
\]
From the definition of \( q_n \) operation it is proved that
\[
q_{n-3} e_i \Rightarrow \sum_{j=1}^n \eta_{n} e_{j}^{n+3} \text{ for each } i = k+1, \ldots, k+l.
\]
This proves
\[
q_{n-3} \delta_2 \mu e_{k+1}^n = \sum_{j=1}^n \mu_j e_{j}^{n+3} \text{ for each } i = 1, \ldots, l.
\]


In virtue of Theorem 5 there exist reduced complexes \( L, \overline{L} \) which are of the same homotopy type as \( P, \overline{P} \) respectively. Let \( u : L \rightarrow P \) and \( v : P \rightarrow L \) be homotopy equivalences such that \( vu \sim e \) and \( uv \sim e \). If \( u^* : FH(P) \rightarrow FH(L) \) and \( v^* : FH(L) \rightarrow FH(P) \) are proper homomorphisms induced by \( u, v \) respectively, we have
\[
u^* v^* = 1 \text{ and } v^* u^* = 1,
\]
from \( vu \sim e \) and \( uv \sim e \). Suppose that \( w^* f^* : FH(\overline{P}) \rightarrow FH(L) \) is realized by a map \( h : L \rightarrow \overline{P} \). Then the proper homomorphism induced by the map \( hv : P \rightarrow \overline{P} \), is \( v^* h^* = v^*(w^* f^*) = f^* \), so that it is sufficient for us to prove this theorem in case where two reduced complexes \( L, \overline{L} \) take the place of two given complexes \( P, \overline{P} \) respectively.

In virtue of Lemma 1 the proper homomorphism \( H(L) \rightarrow H(L) \) induced by \( f^* : FH(\overline{L}) \rightarrow FH(L) \) is realized by a cochain map \( g^* : c(\overline{L}) \rightarrow c(L) \).

If a chain map \( g : c(L) \rightarrow c(\overline{L}) \) dual to \( g^* [12] \) is realized by a cellular map \( f : L \rightarrow \overline{L} \), the proper homomorphism induced by \( f \) is the given proper homomorphism \( f^* \). Thus we intend to construct step by step a cellular map \( f : L \rightarrow \overline{L} \), which realizes the chain map \( g : c(L) \rightarrow c(\overline{L}) \). In performing this, we utilize a lemma of J. H. C. Whitehead, which is
of great importance and of use together with lemma 5, [3].

The lemma is stated as follows;


Let $K, L$ be simply connected complexes and let $e^n$ be a principal cell, where $n > 2$. Suppose that $g : c_\ast(K) \to c_\ast(L)$ be a chain map such that the map $g|_c(K_0)(r = 0, 1, \ldots)$ can be realized by a cellular map $f_0 : K_0 \to L_0$ where $K_0 = K - e^n$. If $f_0 \beta e^n = \beta g e^n$, then $f_0$ can be extended to a map $f : K \to L$, which realizes the chain map $g$.

Since $f_* | H^n(L)$ is an isomorphism onto, we have $g(e_0) = \bar{e}^0$. Thus $g|_c_0(L)$ can be realized by a map $f : L^{n-1} = e^0 \to \bar{e}^0 = \bar{L}^{n-1}$. Next, let $g|_c_{n}(L)$ be given such that $g(e_i^n) = \sum_{j=1}^{k'} a_{ij} \bar{e}^l_j (i = 1, \ldots, \kappa)$. Then a cellular map $f : (S^n, e^0) \to (\bar{L}^n, \bar{e}^0)$, for each $i = 1, \ldots, \kappa$, can be constructed such that $f$ represents a homotopy element $\sum_{j=1}^{k'} a_{ij} S^n_j$, where $S^n_j$ denotes also a homotopy element represented by a map $: S^n \to S^n_j$ of degree unity. Then it is obvious that the cellular map $f : L^n \to \bar{L}^n$ thus constructed realizes the chain map $g|_c_{n}(L)$. Since $\pi_n(L^n) \approx H_n(L^n)$, and $\pi_n(\bar{L}^n) \approx H_n(\bar{L}^n)$, we identify elements corresponding by these isomorphisms. Then we have

$$
\beta g e_i^{n+1} = \partial g e_i^{n+1} = g \partial e_i^{n+1} = f \partial e_i^{n+1} = f \partial e_i^{n+1} \text{ for each } i = 1, \ldots, k + l,
$$

so that in virtue of Lemma 4 $g|_c_{n+1}(L)$ can be realized by an extended cellular map $f : L^{n+1} \to \bar{L}^{n+1}$.

Now we are going to extend this cellular map $f : L^{n+1} \to \bar{L}^{n+1}$ to a map $f : L^{n+1} + e_i^{n+2}(t \geq i \geq 1) \to L^{n+2}$ such that this extended map $f$ realizes the chain map $g|_c(L^{n+1} + e_i^{n+2})$. If $g e_i^{n+2} = \sum_{p=1}^{t'} b_{i,p} \bar{e}_p^{n+2} + \sum_{q=1}^{t'} b_{i, q, k+q} \bar{e}_q^{n+2} + \sum_{p=k'+l+1}^{k'} b_{i,p} \bar{e}_p^{n+2}$, we have $b_{i, k+q} = 0$ and $b_{i,p} = 0 \mod. 2$. This is proved in the following way. Evidently we have

$$(9.1) \quad g^q \bar{e}_i^{n+2} = \ldots + b_{i,k+q} \bar{e}_i^{n+2} + \ldots \text{ for each } q = 1, \ldots, t'$$

From Lemma 1

$$j_2 g^q \bar{e}_i^{n+2} = f^q j_2 \bar{e}_i^{n+2}, \text{ for each } q = 1, \ldots, t'$$

and from b), Theorem 6,

$$j_2 \bar{e}_i^{n+2} = j_2 \bar{e}_i^{n+2} \cup j_2 \bar{e}_i^{n+2} \text{ for each } q = 1, \ldots, t'$$

By the property of $f^*$ we have

$$f^q j_2 \bar{e}_i^{n+2} = f^q (j_2 \bar{e}_i^{n+2} \cup j_2 \bar{e}_i^{n+2})$$
Again, from \( f^* j_2^* g^* \) it is seen that
\[
j_2^* g^* e_{k+q}^{n+2} = j_2^* g^* e_{k+q}^{n+2} \cup j_2^* g^* e_{k+q}^{n+2} = \ldots + b_{i, k+q}^* j_2 e_i^{n+2} + \ldots
\]
\[
= j_2^* \left( \sum_{i=1}^t a_i e_i^{n+2} \right) \cup ( \sum_{i=1}^t a_i e_i^{n+2} )
\]
\[
= j_2^* \left( \sum_{j=k+1}^t a_j^{n+2} e_j^{n+2} \right).
\]

We have
\[
\sum_{i=1}^t a_{i'} e_i^{n+2} + b_{i, k+q}^* j_2 e_i^{n+2} = \ldots + b_{i, k+q}^* j_2 e_i^{n+2} + \ldots (t \geq i \geq 1).
\]
The left side of the last equation does not contain any \( e_i^{n+2} (i = 1, \ldots, t) \), so that we have
\[
(9.2) \quad b_{i, k+q}^* \equiv 0 \mod. 2 \text{ for each } q = 1, \ldots, t' \text{ and for each } i = 1, \ldots, t.
\]
Through analogous arguments we have
\[
(9.3) \quad b_i^{n+2} \equiv 0 \mod. 2 \text{ for } r = k' + t' + 1, \ldots, t' \text{ and for } i = 1, \ldots, t.
\]

From (9.2) and (9.3) it is easily seen that
\[
\beta e_i^{n+2} = \sum_{p=1}^t b_{i, p} \beta e_p^{n+2} + \sum_{q=1}^t b_{i, k+q}^* \beta e_{k+q}^{n+2} + \sum_{r=k+t'+1}^t b_{i, r}^* \beta e_r^{n+2}
\]
\[
= 0 + \sum_{q=1}^t b_{i, k+q}^* (\alpha e_{k+q}^*) + \sum_{r=k+t'+1}^t b_{i, r}^* (\alpha) = 0,
\]
where \( \eta : S^{n+1} \rightarrow S^n \) denotes an essential map \( \alpha_i : S^n \rightarrow S_i^n (i = k' + 1, \ldots, t') \) of degree unity, and \( (\alpha, \eta) i = k' + 1, \ldots, t' \) are homotopy elements represented by maps \( \alpha_i, \eta : S^{n+1} \rightarrow S^n \). On the other hand we have
\[
f \beta e_i^{n+2} = 0 \text{ for each } i = 1, \ldots, t,
\]
so that
\[
\beta e_i^{n+2} = f \beta e_i^{n+2} \text{ for each } i = 1, \ldots, t.
\]
This shows the existence of the desired extended map \( f : L^n + \sum_{i=1}^t e_i^{n+2} \rightarrow L^{n+2} \).

In the next place we intend to extend \( f \) to a map \( f : L^{n+2} \rightarrow L^{n+2} \) such that \( f \) realizes the chain map \( g | e_{n+2}^*(L) \). It is easily seen that
(9.4) \[ f(\beta e^+_{i+k} = \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q) = \sum_{q=\kappa+1}^{\alpha_i} \alpha_i, q \alpha_i q \] for each \( i = k + 1, \ldots, \kappa \), for \( \pi_{k+1}(S_i \cup e_{i+i}^+) = 0 \) (\( i = 1, \ldots, k' \)). Putting

\[ \beta e^+_{i+k} = \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q + \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q \]

(9.5) \[ ge^+_{i+k} \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q + \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q \]

(9.6) \[ \beta ge^+_{i+k} \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q + \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q \]

If the following relations

(A) \[ \begin{align*}
\text{i)} & \quad \alpha_i, k+i = \alpha_i, k+i + q \mod 2 \quad (i = k + 1, \ldots, k + l) \\
\text{ii)} & \quad \alpha_i, q = \alpha_i, q \mod 2 \quad (i = k + 1, \ldots, k + l) \\
\end{align*} \]

(B) \[ \begin{align*}
\text{i)} & \quad \alpha_i, k+i = \alpha_i, k+i + q \mod 2 \quad (i = k + 1, \ldots, k + l) \\
\text{ii)} & \quad \alpha_i, q = \alpha_i, q \mod 2 \quad (i = k + 1, \ldots, k + l) \\
\end{align*} \]

are proved, we have \( f(\beta e^+_{i+k} = \beta e^+_{i+k} \) from (9.4) and (9.6).

From (9.1) and (9.5) it is seen that

\[ \sum_{i=1}^{\alpha_i} b_i \beta e^+_{i+k} \]

(9.7) \[ ge^+_{i+k} \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q + \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q \] for each \( q = 1, \ldots, \kappa \).

It is also verified that

\[ j_2g^+_{e^+_{i+k}} = f^*(j_2^e_{e^+_{i+k}} + \sum_{q=\kappa+1}^{\alpha_i} \alpha_i, q \alpha_i q) \]

(9.8) \[ \sum_{q=1}^{\alpha_i} \alpha_i, q \alpha_i q \]

From (9.7) and (9.8) we have

\[ \alpha_i, k+i = 1 \quad (i = k + 1, \ldots, k + l, \ldots, \kappa) \]

This proves (A) i) and (B) i). By analogous arguments we have (A) ii) and (B) ii). Thus, in virtue of Lemma 4 there exists the desired map \( f : L^+ \rightarrow L^+ \).
Now we are at the last stage of proving this theorem. An easy example shows that \( f \beta_{e_l+3} = \beta g e_{l+3} \) (for each \( i = 1, \ldots, \alpha \)) is not always possible, so that we shall modify the map \( f \), which has been established, in \( L^{n+2} \), to a map \( f_0 \). In this modification of \( f \) we notice that \( f_0|L^{n+2} = f \) and \( f \) is modified in all the \((n+2)\) cells of \( L \). From the last part of \( \S 7 \) we have

\[
\beta e_{l+3} = \lambda_{l+j} S_{l+2}^{n+2} + \sum_{j+k+1}^{k+l} \mu_{l,j} e_{l+2}^{n+2} + \sum_{j+k+l+1}^{\kappa} \gamma_{l,j} e_{l+2}^{n+2} \quad (i = 1, \ldots, \alpha)
\]

From (9.1) we have

\[
ge_{l+2}^{n+2} = \sum_{p-1}^{t'} b_{l,p} e_{l+2}^{n+2} + \sum_{q=1}^{t'} b_{l,q} e_{l+2}^{n+2} + \sum_{r=k+r+1}^{t'} b_{l,r} e_{l+2}^{n+2} \quad (i = 1, \ldots, t).
\]

Then we may define \( f_0|S^{n+2} \) (for each \( i = 1, \ldots, t \)) such that

\[
f_0(S^{n+2}) = \sum_{p=1}^{t'} b_{l,p}^{n+2} + \sum_{q=1}^{t'} b_{l,q}^{n+2} + \sum_{r=k+r+1}^{t'} \frac{b_{l,r}}{2} \quad (i = 1, \ldots, t).
\]

where \( b_{l,q} = 0 \), and \( b_{l,r} = 0 \) mod. 2, are utilized here. This modification does not alter \( g \). From (9.5) we have

\[
ge_{l+2}^{n+2} = \sum_{p=1}^{t'} c_{l,p} e_{l+2}^{n+2} + \sum_{q=1}^{t'} c_{l,q} e_{l+2}^{n+2} + \sum_{r=k+r+1}^{t'} c_{l,r} e_{l+2}^{n+2} \quad (j = k+1, \ldots, \kappa).
\]

Thus it can be also defined that

\[
f_0 \omega_j = \sum_{p=1}^{t'} \theta_{l,p} e_{l+2}^{n+2} \quad (j = k+1, \ldots, k+l)
\]

If we define \( g e_{l+1}^{n+2} = \sum_{p=1}^{t'} \theta_{l,p} e_p^{n+2} \), we have

\[
f_0 \omega_j = \sum_{p=1}^{t'} \omega_{p} \quad (j = k+1, \ldots, k+l)
\]

From (9.9), (9.10), (9.11), and (9.13), it follows that

\[
f_0 \beta e_{l+3} = \sum_{p=1}^{t'} (\lambda_{l+p} b_{l,p} + \sum_{j+k+1}^{k+l} \mu_{l,j} e_{l+2}^{n+2} + \sum_{j+k+l+1}^{\kappa} \gamma_{l,j} e_{l+2}^{n+2}) e_{l+2}^{n+2} \]

\[
+ \sum_{q=1}^{t'} (\lambda_{l+p} b_{l,q}^{n+2} + \sum_{j+k+1}^{k+l} \mu_{l,j} e_{l+2}^{n+2} + \sum_{j+k+l+1}^{\kappa} \gamma_{l,j} e_{l+2}^{n+2}) \quad (j = 1, \ldots, t)
\]

\[
(9.14)
\]

* As we have often referred to, it should be noticed that homology and homotopy are distinguished adequately according as the place where they are used.

** From the following reasons we can modify \( f \) to \( f_0 \). Let us denote \( S_{l}^{n} \cup e^{n+2} \) by \( \Pi_l \), then there exists a map \( \psi: \Pi_l \to \Pi_{l-1} \cup \Pi_{l-2} \) such that \( \psi \) maps \( S_{n} \) of \( \Pi_{l-1} \) to \( S_{n} \) of \( S_{l}^{n} \) of \( \Pi_{l-2} \) with degree \( (1,1) \). Besides this, let a map \( \nu: \Pi \to \Pi \) be given such that \( \nu \) maps \( S_{n} \) of \( \Pi \) to \( S_{n} \) of \( \Pi \) with degree \( a \), and \( a = 0 \) mod. 2. Then we can construct a map \( \phi: \Pi \to \Pi \) by modifying \( \psi \) such that \( \phi|S_{n} = \psi|S_{n} \) and \( \phi \) maps \( e^{n+2} \) of \( \Pi \to e^{n+2} \) of \( S_{n} \) with degree \( c \). We use, in the above modification of \( f \) to \( f_0 \), the relations (A), (B) on the previous page and also \( \tau_{n+1} e^{n+1} = 0 \), where \( e^{n+1} \) is attached to \( S_{n} \) by a map \( \partial e^{n+1} \to S_{n} \) of degree odd. It is clear that this modification does not alter the realization of \( a \) chain map \( g \).
Putting (9.15)
\[ g^{n+3} = \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{e}_{n+3}^p \]
we have
\[ \beta g^{n+3} = \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\lambda}_{p,q} \bar{g}^{n+2} + \sum_{q=k'+1}^{k'+l} \mu^{q,r} \bar{\lambda}_{q,r} \bar{g}^{n+2} + \sum_{q=k'+1}^{k'+l} \mu^{q,r} \bar{\lambda}_{q,r} \bar{g}^{n+2} + \sum_{q=k'+1}^{k'+l} \mu^{q,r} \bar{\lambda}_{q,r} \bar{g}^{n+2} \]
(9.16)..... = \sum_{p=1}^{\alpha'} (\rho_{i,p} \bar{\lambda}_{p,q}) \bar{g}^{n+2} + \sum_{q=k'+1}^{k'+l} \left( \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\lambda}_{p,q} \bar{g}^{n+2} \right) \bar{\omega}_q + \sum_{q=k'+1}^{k'+l} \left( \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\lambda}_{p,q} \bar{g}^{n+2} \right) \bar{\omega}_q

Proving the following relations

1) \( \lambda_{ij} b_{j,p} + \sum_{j=k+1}^{k+l} 2 \mu_{t,i} c_{j,p} + \sum_{j=k+1}^{k+l} 2 \gamma_{t,i} c_{j,p} = \rho_{i,p} \bar{\lambda}_{j,p} \) \( (p = 1, ..., t') \),

2) \( \frac{\lambda_{ij}}{2} b_{j,k+q} + \sum_{j=k+1}^{k+l} \mu_{t,i} c_{j,k+q} + \sum_{j=k+1}^{k+l} \gamma_{t,i} c_{j,k+q} = \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\mu}_{p,k+q} \) \( (q = 1, ..., l') \),

3) \( \frac{\lambda_{ij}}{2} b_{j,r} + \sum_{j=k+1}^{k+l} \mu_{t,i} c_{j,r} + \sum_{j=k+1}^{k+l} \gamma_{t,i} c_{j,r} = \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\gamma}_{p,r} \) \( (q = k'+l+1, ..., k') \),

4) \( \sum_{j=k+1}^{k+l} \theta_{j,p} \mu_{i,j} = \sum_{p=1}^{\alpha'} \rho_{i,p} \bar{\mu}_{p,j} \) mod. 2. \( (n = k'+1, ..., k'+l') \),

we have \( f \beta e^{n+3} = \beta g^{n+3} \) (for each \( i = 1, ..., \alpha \)) from (9.14) and (9.16).

From (9.1), (9.5) we have
\[ g^{n+2}_p = \sum_{t=1}^{t} b_{i,p} e^{n+2}_t + \sum_{t=k+1}^{t} c_{i,p} e^{n+2}_t \] for each \( p = 1, ..., t' \).

Taking the coboundary of both sides of this equation,
\[ \delta g^{n+2}_p = \sum_{t=1}^{t} (b_{i,p} \lambda_{i,j}) e^{n+3}_j + \sum_{q=1}^{q} \left( \sum_{t=k+1}^{t} 2 c_{i,p} \mu_{i,j} \right) e^{n+3}_q + \sum_{q=1}^{q} \left( \sum_{t=k+1}^{t} 2 c_{i,p} \gamma_{i,j} \right) e^{n+3}_q \]
It is also seen that
\[ \delta g^{n+2}_p = g^{n+2}_p \delta e^{n+3}_j = g^{n+2}_p (\lambda_{i,j} e^{n+3}_j) = \lambda_{i,j} g^{n+2}_p e^{n+3}_j \]
\[ = \lambda_{i,j} \left( \sum_{t=1}^{t} (b_{i,p} \mu_{i,j}) \right) e^{n+3}_t \]
It follows that
\[ \lambda_{i,j} \rho_{i,j} = \lambda_{i,j} b_{j,p} + \sum_{j=k+1}^{k+l} 2 c_{i,p} \mu_{i,j} + \sum_{j=k+1}^{k+l} 2 c_{i,p} \gamma_{i,j} \] \( (p = 1, ..., t') \).
This proves i). Again, from (9.1) and (9.5) we have
\[ g^k e_{l+k+q}^{n+2} = \sum_{k=1}^{t} b_{l+k+q} e_{l+k+q}^{n+2} + \sum_{j=1}^{k+l} c_{j+k+q} e_{j+k+q}^{n+2} + \sum_{j=1}^{k+l} c_{j+k+q} e_{j+k+q}^{n+2}. \]

Considering the coboundary, we have
\[ \delta g^k e_{l+k+q}^{n+3} = \sum_{k=1}^{t} (b_{l+k+q} + c_{j+k+q}) e_{l+k+q}^{n+3} + \sum_{k=1}^{k+l} (c_{j+k+q} + \alpha) e_{j+k+q}^{n+3}. \]

It is seen that
\[ \delta g^k e_{l+k+q}^{n+3} = g^k \delta e_{l+k+q}^{n+3} = g^k \left( 2 \sum_{p=1}^{t} \mu_{p+k+q} e_{p+k+q}^{n+3} + 2 \lambda_{l+k+q} e_{l+k+q}^{n+3} \right) = \sum_{p=1}^{t} 2 \mu_{p+k+q} g^k e_{l+k+q}^{n+3} \]
\[ = \sum_{p=1}^{t} \left( \sum_{p=1}^{t} 2 \mu_{p+k+q} \right) e_{l+k+q}^{n+3}. \]

Thus it is concluded that
\[ \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} = \sum_{p=1}^{t} \left( \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} \right) \]
\[ = \sum_{l=1}^{n} \rho_{l+k+q} e_{l+k+q}^{n+3}. \]

This proves ii). Similarly iii) can be proved. Lastly we proceed to prove iv). From (9.15) we have
\[ g^k e_{l+k+q}^{n+3} = \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3}. \]

Thus
\[ \sum_{p=1}^{t} g^k e_{l+k+q}^{n+3} = \sum_{p=1}^{t} \left( \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} \right) \]
\[ = \sum_{l=1}^{n} \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3}. \]

(9.17) \[ \sum_{p=1}^{t} j_2 g^k e_{l+k+q}^{n+3} = \sum_{p=1}^{t} \left( \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} \right) j_2 e_{l+k+q}^{n+3}. \]

(9.18) \[ = \sum_{l=1}^{n} \sum_{p=1}^{t} \theta_{l+k+q} e_{l+k+q}^{n+3} + \sum_{l=1}^{n} \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} \]
\[ = \sum_{l=1}^{n} \sum_{p=1}^{t} \theta_{l+k+q} e_{l+k+q}^{n+3} + \sum_{p=1}^{t} \rho_{l+k+q} e_{l+k+q}^{n+3} \]
\[ = \sum_{l=1}^{n} \sum_{p=1}^{t} \theta_{l+k+q} e_{l+k+q}^{n+3}. \]
From (9.17) and (9.18) we have
\[ \sum_{j=1}^{p} \rho_{i,j} \nu_{j,p} = \sum_{j=k+1}^{k+t} \theta_{j,p} \nu_{i,j} \equiv \sum_{j=k+1}^{k+t} \theta_{j,p} \nu_{i,j} \pmod{2} \quad (p = k' + 1, \ldots, k' + l'). \]

This completes the proof.

Added in proof: I could read [4], and I hope, I shall come back soon to some subjects related to this paper (refer to my paper of the same title in this issue).

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Bibliography

The paper inaccessible to us here, is indicated by *