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On Homotopy Type Problems of Special Kinds of Polyhedra I

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1. Introduction

It is one of the aims of modern topology to classify topological spaces by their homotopy types. Two spaces X and Y have the same homotopy type if there exist maps $f: X\rightarrow Y$ and $g: Y\rightarrow X$ such that gf and fg are homotopic to the identity maps $X\rightarrow X$ and $Y\rightarrow Y$ respectively. The problem of determining by means of invariants of X and Y whether X and Y are of the same homotopy type or not, is of great importance in modern topology. This general problem has not yet been solved. A number of particular results are well known.

In 1936 Witold Hurewicz solved in his famous paper [8]* the homotopy types of an n dimensional locally connected compact metric space aspherical in dimensions less than n, and of a locally connected compact metric space aspherical in dimensions greater than unity. After this, many endeavours have been made to solve this general problem by several modern topologists, J. H. C. Whitehead, R. H. Fox, S. C. Chang, and others. Among them the recent brilliant results of J. H. C. Whitehead [3], [4] and of S. C. Chang [6] have much to do with the present paper. Whitehead reported in [3] that two simply connected, 4 dimensional polyhedra are of the same homotopy type if and only if their cohomology rings are properly isomorphic. According to Whitehead, an arcwise connected polyhedron P is referred to as A_n^2-complex if dim. $P\leqq n+2$ and $\pi_i(P)=0$ for each $i<n$. Though the author is unfortunate enough to be inaccessible to [4] here, he is informed of Whitehead's far reaching results through the introduction of Chang's paper [6]. They are stated as follows. Two A_n^2-complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic. Chang introduced new numerical invariants called secondary torsions to characterize the homotopy type of an A_n^2-polyhedron together with the Betti numbers and coefficients of torsion. Furthermore he reduced a given A_n^2-complex to a reduced complex which consists of elementary A_n^2-polyhedra.

The main purpose of this paper is to determine the homotopy type

* The number in square bracket is referred to the bibliography listed at the end of this paper.
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of an A_n^3-complex P with vanishing $(n+1)$-st homotopy group of P. Throughout the whole paper we assume $n>3$. Let H^r $(r=0, n, n+1, n+2, n+3)$ be the r dimensional integral cohomology group and let $Sq_{n-2}: H^n(2k)\to H^{n+2}(2)$ and $Sq_{n-1}: H^{n+1}\to H^{n+3}(2)$ be Steenrod's squaring operations. Then, following J. H. C. Whitehead, we refer to $FH=H\{H^0, H^n, H^{n+1}, H^{n+2}, H^{n+3}, H^n(2k), H^{n+2}(2), H^{n+3}(2), \mu, \Delta, Sq_{n-2}, Sq_{n-1}\}$ as A_n^3-cohomology system. It will be shown in Theorem 1 that two such complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic. The method of proving this is analogous to that of Whitehead [3]. The reduction of such a given A_n^3-complex to a reduced complx is also shown. Before performing this, the author gives another elementary but elegant way of proving Chang's reduction of an A_n^2-complex to a reduced complex, which was pointed out for him by Gaishi Takeuti. The author would like to express his sincere gratitude to Professor G. Takeuti for his kind criticisms and encouragements.

2. A Spectrum

A brief sketch of the definition of spectrum of cohomology groups and related lemmas used in the sequel seems to be desirable for the convenience and the clearness of the applications in this paper. All the concepts and lemmas in this section are in [3]. Let a sequence $c=\{c^n\}$ $(n=0, 1, \ldots)$ of free abelian groups of finite rank be related by a "coboundary" homomorphism $\delta: c^n\to c^{n+1}$ for each n, such that $\delta\delta=0$. By an usual procedure, the n-dimensional cohomology group $H^n(m)$ with integers reduced mod. m can be defined in terms of C and δ. For integers $p>0$ and $q\geq 0$ two operations Δ_q, $\mu_{p,q}$ are defined such that

$$\Delta_q;\ H^n(q)H\to H^{n+1},$$

$$\mu_{p,q};\ H^n(q)\to H^n(p).$$

Let $x\in H^n(q)$ and let $x'\in x$. That is to say, x' is a cocycle mod. q. Then $\delta x'=qy'$, where y' is an $(n+1)$ absolute cocycle. We define $\Delta_q x=y$, *a* cohomolog class containing y'. Let $c=(p, q)$, then $\frac{p}{c}x'$ is a cocycle mod. p, and we define $\mu_{p,q}x$ as its cohomology class mod. p. It is easily verified that $\Delta_q x$ and $\mu_{p,q}x$ depend only on $x\in H^n(q)$ and not on the particular choice of $x'\in x$. They are obviously homomorphisms. The union of all the groups $H^n(m)$, for every integer $n\geq 0$ and for every integer $m\geq 0$, related by the homomorphisms Δ, μ, will be called the cohomology spectrum of the set of groups C, or merely spectrum of the groups C. We shall denote it by H. By a proper homomorphism $f: \bar{H}\to H$ of a spectrum H into a spectrum $\bar{H}$, we mean a transformation such that

i) $f|H^n(m):H^n(m)\to\overline{H}^n(m)$ is a homomorphism for all values of m, n,

ii) $f\Delta=\Delta f$ and $f\mu=\mu f$.

If f is a proper homomorphism and $f|H^n(m):H^n(m)\to\overline{H}^n(m)$ is an isomorphism onto for all values of m, n, f is called a proper isomorphism. Then a spectrum H is called to be properly isomorphic to a spectrum $\overline{H}$.

Let $Z^n(m)$ be a subgroup of C^n which consists of all the cocycles mod. m, and let j_m be a natural homomorphism of $Z^n(m)$ onto $H^n(m)$. We shall also use j_m to denote the natural homomorphism of cocyles mod. m, in $\overline{C}^n$ onto $\overline{H}^n(m)$. A cochain map $g:c^n\to\bar{c}^n$ foı every n, obviously induces a proper homomorphism of H into $\overline{H}$. Now let f be a given proper homomorphism of H into $\overline{H}$. If $f\ j_m a=j_m ga$ for any $a\in Z^n(m)$, a cochain map g is said to realize a proper homomorphism f.

Lemma 1. (WHITEHEAD [3], p. 57, Lemma 4) *Any proper homomorphism $f:H\to\overline{H}$ can be realized by a cochain map g.*

3. Two types of homorphisms

Let C^n of a sequence C be an n dimensional group of cochains of a finite simplicial complex K. Then two types of homomorphisms are defined among cohomology groups $H^n(m)$. One of them is a well known squaring homomorphism of Steenrod [7] and the other is $\mathfrak{q}_i$-homomorphism, which was introduced elsewhere [11] by N. Shimada and myself. For convenience, they are put down here. Steenrod showed that

if $p-i$ is odd, there exists the i-th square

$$Sq_i:\ H^p(m)\to {}_2H^{2p-i}(m),$$

and that

if $p-i$ is even and m is also even, the i-th square mod. 2 can be defined such that

$$Sq_i\ :\ H^p(m)\to H^{2p-i}(2).$$

This squaring operation will be used essentially in the sequel, while we shall not need the $\mathfrak{q}_i$-homomorphism except for cohomological properties in a reduced complex (refer to § 8).

If $p-i$ is odd and $m\geqq 0$ is an even integer, $\mathfrak{q}_i:H^p(m)\to{}_2H^{2p-i}$ can be defined as follows. Let $x\in H^p(m)$ and $x'\in x$. Since x' is a cocycle mod. m, we have $\delta x'=my'$. Putting $\delta'_m x'=\dfrac{1}{m}\delta x'=y'$, we have a $(2p-i)$ absolute cocycle

$$q'_i x'=x'\underset{i}{\smile}x'+mx'\underset{i+1}{\smile}\delta'_m x'+(-1)^p\frac{m^2}{2}\delta'_m x'\underset{i+2}{\smile}\delta'_m x'.$$

Notice that $q'_i x'=x'\underset{i}{\smile}x'$ in case $m=0$. If we define that $\mathfrak{q}_i x$ is a cohomology class containing a cocycle $q_i'x'$, it is verified that this definition does not depend on the choice of a representative x' of x. The spectrum H related by squaring operations, will be called the cohomology system, which is denoted by FH.

4. A_n^3-cohomology system

If a finite simplicial comylex K referred to in § 3, is an A_n^3-complex, some conditions are obviously assigned on its cohomology system. It is evident that

i) $H^i(m)=0$, for any m and $n>i>0$,

ii) $H^i(m)=0$, for any m and for each $i>n+3$,

iii) H^n contains no element of finite order,

iv) H^0 is cyclic infinite.

Thus, for the reasonable brevity we shall symbolize A_n^3-cohomology system by

$$FH = H\{H^0, H^n, H^{n+1}, H^{n+2}, H^{n+3}, H^n(2k), H^{n+2}(2), H^{n+3}(2), \Delta, \mu, Sq_{n-2}, Sq_{n-1}\} .$$

In this notation the operations Δ, μ are explained in § 2, and the other two operations are as follows:

$$Sq_{n-2} : H^n(2k) \rightarrow H^{n+2}(2) \text{ for every integer } k\geq 0,$$
$$Sq_{n-1} : H^{n+1} \rightarrow H^{n+3}(2) .$$

Let FH, $F\overline{H}$ be the cohomology systems of K, $\overline{K}$ respectively. By a proper homomorphism we mean the transformation $f: FH \rightarrow F\overline{H}$ such that

i) f is not the trivial homomorphism $FH \rightarrow 0$,

ii) f induces a proper homomorphism, as defined in § 2, of the spectra,

iii) $fSq_{n-2} = Sq_{n-2}f$ and $fSq_{n-1} = Sq_{n-1}f$,

iv) $f|H^0$ is an isomorphism onto.

If a proper homomorphism f induces a proper isomorphism onto of the spectra, f is called a proper isomorphism. Then FH is said to be properly isomorphic to $F\overline{H}$.

Let P be an $(n+3)$ dimensional finite connected simplicial complex such that $\pi_i(P)=0$ for each $i<n$ and $i=n+1$, and let us refer to such a complex as $\overline{A}_3^n$-complex. Then our theorems are:

Theorem 1. *Two $\overline{A}_n^3$-complexes are of the same homotopy type if and only if their cohomology systems are properly isomorphic.*

Theorem 2. *Let P, $\overline{P}$ be $\overline{A}_n^3$-complexes. Any proper homomorphism $f^* : FH(\overline{P}) \rightarrow FH(P)$ can be realized by at least one homotopy class of maps*

$f: P \rightarrow \bar{P}$. *That is to say, there exists a map* $f: P \rightarrow \bar{P}$ *such that the proper homomorphism induced by* f *is the same as* f^*.

It is verified as follows that Theorem 2 implies Theorem 1. Now let K and L be finite simply connected complexes of arbitrary dimensionality and let $f: K \rightarrow L$ be a map which induces an isomorphism of each cohomology group $H^n(L)$, with integral coefficients, onto the corresponding group $H^n(K)$. Then J. H. C. Whitehead proved in [2] that K and L are of the same homotopy type and f is a homotopy equivalence. If we use this, it is easily seen that Theorem 2 implies Theorem 1. In virtue of Theorem 2 there exists at least one map $f: P \rightarrow \bar{P}$ which induces the proper isomorphism $FH(\bar{P}) \rightarrow FH(P)$. If we utilize the above mentioned result of Whitehead, it is seen that P and $\bar{P}$ are of the same homotopy type, when their cohomology systems are properly isomorphic. The converse of this is obvious. If P and $\bar{P}$ have the same homotopy type, there exist maps $f: P \rightarrow \bar{P}$ and $g: \bar{P} \rightarrow P$ such that $fg \sim e'$ and $gf \sim e$, where e, e' denote the identical transformations of P, $\bar{P}$ respectively. Let us denote the proper homomorphisms induced by f, g by $f^*: FH(\bar{P}) \rightarrow FH(P)$ and $g^*: FH(P) \rightarrow FH(\bar{P})$ respectively. Since $f^*g^*: FH(P) \rightarrow FH(P)$ and $g^*f^*: FH(\bar{P}) \rightarrow FH(\bar{P})$ are proper isomorphisms, f^* is an proper isomorphism. Thus our aim is to prove Theorem 2.

5. Reduction of A_n^2-complex to a reduced complex.

This section was proved by G. Takeuti. Before we perform this reduction, we give here some notations, definitions, and essential Lemmas for subsequent discussions.

Let X, R be topological spaces and let Y be a closed subset of X. Attaching X to R by a map $f: Y \rightarrow R$, we have a space $(R+X, f, Y)$, which may be simply denoted by $(R+X, f)$. More generally, we designate by $(R+X_1+\cdots+X_n, f_1, \ldots, f_n, Y_1, \ldots, Y_n)$ or merely $(R+X_1+\cdots+X_n, f_1, \ldots, f_n)$ a space attaching $X_i (i=1, \ldots, n)$ to R by a map $f_i: Y_i \rightarrow R$, where Y_i is a closed subset of X_i. In case where R is a space of a point O and $Y_i (i=1, \ldots, m)$ consists of a single point O_i of X_i, the space $(O+X_1+\cdots+X_m, f_1, \ldots, f_m)$ will be often denoted by $(O, X_1, \ldots, X_m, O_1, \ldots, O_m)$ or, as usually designated, by $X_1 \vee X_2 \vee \ldots \vee X_m$, where $f_i(O_i)=O$ is evidently assumed. Particularly, if X_i is an oriented n sphere S_i^n, the n-th homotopy group $\pi_n(S_1^n \vee \ldots \vee S_m^n)$ $(n \geq 1)$ of a space $S_1^n \vee \ldots \vee S_m^n$ may be regarded as the m-dimensional vector space with free base $\{S_1^m, \ldots, S_m^n\}$, where S_i^n denotes an element of the homotopy group as well as an n sphere. In the sequel we shall often use the notation $A \sim B$, when two spaces A, B have the same homotopy type.

Lemma 2. (J. H. C. WHITEHEAD [5], p. 239, Lemma 2) *If two spaces P, Q are of the same homotopy type and $f: P \to Q$ is a homotopy equivalence, and if a map $\alpha: \partial E^{n+1}=S^n \to P$ is given, $(P+E^{n+1}, \alpha)$ has the same homotopy type as $(Q+E^{n+1}, f\alpha)$, where E^{n+1} is an $(n+1)$ element.*

Let $\tilde{P}$ be a space attaching $E_i^{n+1}(i=1,\dots, m)$ to P one by one by a map $f_i: \partial E_i^{n+1} \to P$. Then the homotopy type of $\tilde{P}$ is completely determined by the homotopy elements β_i $(i=1, \dots, m)$ of $\pi_n(P)$ represented by maps f_i, so that without confusion we may represent $\tilde{P}$ by the symbol $(P; \beta_1, \dots, \beta_m)$. This is seen from Lemma 5, [3]. It is also verified that the following three operations, called elementary operations

i) $(\beta_1, \dots, \beta_i, \dots, \beta_m) \to (\beta_1, \dots, -\beta_i, \dots, \beta_m)$
ii) $(\beta_1, \dots, \beta_i, \dots, \beta_j, \dots, \beta_m) \to (\beta_1, \dots, \beta_j, \dots, \beta_i, \dots, \beta_m)$
iii) $(\beta_1, \dots, \beta_i, \dots, \beta_j, \dots, \beta_m) \to (\beta_1, \dots, \beta_i+\beta_j, \dots, \beta_j, \dots, \beta_m)$

do not alter the homotopy type of $\tilde{P}$. That is to say, we have

i) $(P; \beta_1, \dots, \beta_i, \dots, \beta_m) \sim (P; \beta_1, \dots, -\beta_i, \dots, \beta_m)$
ii) $(P; \beta_1, \dots, \beta_i, \dots, \beta_j, \dots, \beta_m) \sim (P; \beta_1, \dots, \beta_j, \dots, \beta_i, \dots, \beta_m)$
iii) $(P; \beta_1, \dots, \beta_i, \dots, \beta_j, \dots, \beta_m) \sim (P; \beta_1, \dots, \beta_i+\beta_j, \dots, \beta_j, \dots, \beta_m)$.

Given $P=(O; x_1, \dots, x_\rho; \alpha_1, \dots, \alpha_\lambda)$, where n spheres $x_i(i=1, \dots, \rho)$ have a point O in common and E_i^{n+1} $(i=1, \dots, \lambda)$ are attached to $x_1 \vee \cdots \vee x_\rho$ by the maps $f_i: \partial E_i^{n+1} \to x_1 \vee \cdots \vee x_\rho$, which represent the homotopy elements $\alpha_i = \sum_{j=1}^{\rho} c_{ij} x_j$. Consider two maps f, g between $P_0=(O; \bar{x}_1, \dots, x_\rho)$ and $Q_0=(O; \bar{x}_1, \dots, \bar{x}_\rho)$ such that $f: x_i \to \sum_{j=1}^{\rho}{}_{ij} a\bar{x}_j$ and $g: \bar{x}_j \to \sum_{k=1}^{\rho} b_{jk} x_k$, where (a_{ij}), (b_j) are reciprocal unimodular matrices. Then it is obvious that f, g are homotopy equivalences. In virtue of Lemma 2, we have

$$P \sim Q = (O; \bar{x}_1, \dots, \bar{x}_\rho; f\alpha_1, \dots, f\alpha_\lambda)$$

$$= (\bar{O}; \bar{x}_1, \dots, \bar{x}_\rho; \sum_{j=1}^{\rho} c_{1j} (\sum_{k=1}^{\rho} a_{jk}\bar{x}_k), \dots, \sum_{j=1}^{\rho} c_{\lambda j} (\sum_{k=1}^{\rho} a_j \bar{x}_k)).$$

To get Q from P is said to carry out the transformation $\bar{x}_j = \sum_{k=1}^{\rho} b_{jk} x_k$. Especially, when this transformation is an elementary transformation, Q is said to be made from P through an elementary operation with respect to x. In the sequel these terminologies will be often used.

Theorem 3. *Let $\{m_1, \dots, m_l\}$ be the invariant system of the n-th homology group $H_n(P)$ of a given A_n^1-complex P and let N be the $(n+1)$-th Betti number of P. Then we have*

$$P \sim Q_{m_1}^{n+1} + \cdots + Q_{m_l}^{n+1} + S_1^{n+1} + \cdots + S_N^{n+1},$$

where $Q_{m_i}^{n+1} = (x_i^n; m_i x_i^n)$ $(i=1, \dots, l)$ and $S_i^{n+1}(i=1, \dots, N)$ have a point in *common,* $x_i^n (i=1, \dots, l)$ *denoting* n *spheres.*

Proof. This theorem can be easily proved by a Theorem due to Hurewicz [8] and by Lemma 2.

Lemma 3. *Let* P *be a connected simply connected polyhedron.* $(P+x\,;\ \alpha+2^p mx)$ *denotes a complex* $P+x+e^{n+1}$, *where* e^{n+1} *is attached to* $P+x$ *by a map* $f: \partial e^{n+1} \to P+x$, *which represents an element* $\alpha+2^p mx$ *of* $\pi_n(P+x)$. $\alpha \in \pi_n(P)$, $2\alpha=0$, *and* x *is an* n *sphere. Moreover,* m *is odd and* p *is an integer. Then we have*

$$\{P+x\,;\ \alpha+2^p mx\} \sim \{P+y+z\,;\ \alpha+2^p y,\ mz\},$$

where y, z *are* n *spheres and* x, y, z *are attached to* P *at a point.*

Proof. It is obvious that $\{P+x\,;\ \alpha+2^p mx\} \sim \{P+x+x'\,;\ \alpha+2mx,\ x'\}$ where x' is an n sphere. Now we define homotopy equivalences f, g of two complexes $P+x+x'$, $P+y+z$ such that

i) $f|P = g|P$ is the identical map,
ii) $f(x) = Ay+Bz$,
iii) $f(x') = -2^p y+mz$,
iv) $g(y) = mx-Bx'$,
v) $g(z) = 2^p x+Ax'$,

where A, B are integers satisfying $mA+2^pB=1$. Then it is easily seen that $fg \sim gf \sim e$ (identical map). Applying Lemma 2 and elementary operations to the following arguments, we have

$$\begin{aligned}
\{P+x,\ \alpha+2^p mx\} &\sim \{P+x+x',\ \alpha+2^p mx,\ x'\} \\
&\sim \{P+y+z,\ \alpha+2^p m(Ay+Bz),\ -2^p y+mz\} \\
&= \{P+y+z,\ \alpha+2^p mAy+2^p mBz,\ -2^p y+mz\} \\
&= \{P+y+z,\ \alpha+2^p(mA+2^p B)y,\ -2^p y+mz\} \\
&= \{P+y+z,\ \alpha+2^p y,\ -2^p y+mz\} \\
&= \{P+y+z,\ \alpha+2^p y,\ \alpha+2^p y-2^p y+mz\} \\
&= \{p+y+z,\ \alpha+2^p y,\ \alpha+mz\}
\end{aligned}$$

If we put $\alpha+z=\bar{z}$, in virtue of Lemma 2 we have

$$\begin{aligned}
\{P+y+z\,;\ \alpha+2^p y,\ \alpha+mz\} &\sim \{P+y+\bar{z}\,;\ \alpha+2^p y,\ \alpha+m\alpha+m\bar{z}\} \\
&\sim \{P+y+\bar{z}\,;\ \alpha+2^p y,\ m\bar{z}\},
\end{aligned}$$

where $\bar{z}$ is an n sphere and $\alpha+m\alpha=0$ from $2\alpha=0$. This proves the Lemma 3.

Now we refer to the polyhedra of the following types as elementary A_n^2-polyhedra:

i) $Q_1^r = S^r \, (r = n, \ n+1, \ n+2)$,

ii) $Q_2 = S^n \cup e^{n+1}$, where e^{n+1} is attached to S^n by a map $f : \partial e^{n+1} \to S^n$ of degree odd,

iii) $Q_3 = S^n \cup e^{n+2}$, where e^{n+2} is attached to S^n by an essential map $f : \partial e^{n+2} \to S^n$,

iv) $Q_4 = (S^n \vee S^{n+1}) \cup e^{n+2}$, where e^{n+2} is attached to $S^n \vee S^{n+1}$ by a map $f : \partial e^{n+2} \to S^n \vee S^{n+1}$ of the form $a+b$; a is an essential map: $S^{n+1} \to S^n$ and b denotes a map: $S^{n+1} \to S^{n+1}$ of degree 2^p,

v) $Q_5 = S^n \cup e^{n+1} \cup e^{n+2}$, where e^{n+1} is attached to S^n by a map f: $\partial e^{n+1} \to S^n$ of degree 2^q and e^{n+2} is attached to S^n by an essential map: $\partial e^{n+2} \to S^n$,

vi) $Q_6 = (S^n \vee S^{n+1}) \cup e^{n+1} \cup e^{n+2}$, where e^{n+1} is attached to S^n by a map: $\partial e^{n+1} \to S^n$ of degree 2^q and e^{n+2} is attached to $S^n \vee S^{n+1}$ by a map of type iv),

vii) $Q_7 = S^n \cup e^{n+1}$, where e^{n+1} is attached to S^n by a map: $\partial e^{n+1} \to S^n$ of degree 2^q,

viii) $Q_8 = S^{n+1} \cup e^{n+2}$, where e^{n+2} is attached to S^{n+1} by a map: $\partial e^{n+2} \to S^{n+1}$ of degree odd,

ix) $Q_9 = S^{n+1} \cup e^{n+2}$, where e^{n+2} is attached to S^{n+1} by a map: $\partial e^{n+2} \to S^{n+1}$ of degree 2^p.

Then we have

Theorem 4. *If P is an $(n+2)$ dimensional finite connected polyhedron which is aspherical in dimensions less than n, P is of the same homotopy type as a reduced complex which consists of a collection of elementary polyhedra of the above mentioned types, where the elementary polyhedra have a point in common.*

Proof. In virtue of Theorem 3, the $(n+1)$ skelton P^{n+1} of P has the same homotopy type as the complex

$$\{Q_{2^{q_1}}^{n+1} \vee Q_{2^{q_2}}^{n+1} \vee \cdots \vee Q_{2^{q_k}}^{n+1} \vee Q_{r_1}^{n+1} \vee \cdots \vee Q_{r_l}^{n+1} \vee S_1^n \vee \cdots \vee S_\lambda^n \vee S_1^{n+1} \vee \cdots \vee S_\mu^{n+1}\},$$

where $r_1, \ldots, r_l$ are odd, and $1 \leq q_1 \leq q_2 \leq \ldots \leq q_k$ are integers. Thus we have

$$P \sim \{Q_{2^{q_1}}^{n+1} \vee Q_{2^{q_2}}^{n+1} \vee \cdots \vee Q_{2^{q_k}}^{n+1} \vee \cdots \vee Q_{r_1}^{n+1} \vee \cdots \vee Q_{r_l}^{n+1} \vee S_1^n \vee \cdots \vee S_\lambda^n \vee S_1^{n+1} \vee \cdots \vee S_\mu^{n+1}; \ R_1, \ \ldots, \ R_m\},$$

where $R_1, \ldots, R_m$ are all the relations. Denoting by x_i the homotopy element represented by a map $S^{n+1} \to S_i^{n+1}$ of degree 1, we have

$$R_i = \sum_{j=1}^{\mu} \lambda_{ij} x_j + \alpha_i \qquad (i = 1, \ \ldots, \ m),$$

where α_i $(i=1, \ldots, m)$ are homotopy elements of $\pi_{n+1}(Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^n_\lambda)$. By elementary operations with respect to $\{S^{n+1}_1, \ldots, S^{n+1}_\mu\}$ and $\{R_1, \ldots, R_m\}$ we have

$$P \sim \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^{n+1}_\mu ; \alpha_1 + x_1, \ldots, \alpha_{i-1} + x_{i-1}, \alpha_i + b_i x_i, \ldots, \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_\kappa\} \\ + S^{n+1} + \cdots + S^{n+1},$$

where $b_{i+\nu}$ $(\nu=0, \ldots, j-i)$ are integers greater than unity, and γ_i $(i=1, \ldots, k)$ are homotopy elements of $\pi_{n+1}(Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^n)$. Since $\pi_{n+1}(Q^{n+1}_r)=0$ $(\lambda=1, \ldots, l)$, we have

$$P \sim \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_k}} \vee S^n_1 \vee \cdots \vee S^n_\lambda \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}_\mu ; \alpha_1 + x_1, \ldots, \\ \alpha_j + b_j x_j, \gamma_1, \ldots, \gamma_k\} + S^{n+1} + \cdots + S^{n+1} + Q^{n+1}_{r_1} + \cdots + Q^{n+1}_{r_l} .$$

Then it is obvious that we have

$$P \sim \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^{n+1}_\mu ; \alpha_i + b_i x_i, \ldots, \alpha_j + b_j x_j, r_1, \ldots \gamma_\rho\} + S^{n+1} + \cdots + S^{n+1} \\ + Q^{+1}_{r_1} + \cdots + Q^{n+1}_{r_l},$$

where γ_i $(i = 1, \ldots, \rho)$ are homotopy elements of $\pi_{n+1}(Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^n)$. Utilizing Lemma 3, and changing suffixes, we have

$$P \sim \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^{n+1}_\mu ; \alpha_1 + 2^{p_1} x_1, \ldots, \alpha_\mu + 2^{p_\mu} x_\mu, \gamma_1, \ldots, \gamma_\rho\} + S^{n+1} + \cdots + S^{n+1} \\ + Q^{n+1}_{r_1} + \cdots + Q^{n+1}_{r_\sigma} ,$$

so that it is sufficient for us to try to reduce $P_1 = \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_k}} \vee S^n_1 \vee \cdots \vee S^n_\lambda \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}_\mu ; \alpha_1 + 2^{p_1} x_1, \ldots, \alpha\ + 2^{p_\mu} x_\cdot, \gamma_1 \ldots \gamma_\rho\}$ to a normal form. Without loss of generality it may be assumed that $\gamma_1, \ldots, \gamma_\rho$ are linearly independent with respect to integer coefficients mod. 2. Here a number of S^{n+2} may be removed from bracket. If we denote by y_i $(i = 1, \ldots, \lambda)$ the homotopy element represented by a map $f_i : S^n \to S^n_i$ of degree 1 and by z_i $(i = 1, \ldots, \mu)$ the homotopy element represented by a map $f_i : S^n \to S^n_i \subset Q^{n+1}_{2^{q_i}}$ of degree 1, and if, for example,

$$\gamma_1 = (y_1 \eta) + \cdots + (z_i \eta) + \cdots ,$$

it is seen by means of the following operation

$$\begin{aligned} \overline{y}_1 &= y_1 + \cdots + z_i + \cdots \\ \overline{y}_i &= y_i \quad (i = 2, \ldots, \lambda) \end{aligned}$$

that $\overline{\gamma}_1 = (\overline{y}_1 \eta)$, where η denotes an essential map $S^{n+1} \to S^n$. If we change notations, and if $\{\alpha_{\tau_1}, \ldots, \alpha_{\tau_s} ; \gamma_{\sigma_1}, \ldots, \gamma_{\sigma_t}\}$ contain $\overline{\gamma}_1 = (\overline{y}_1 \eta)$, we change them, by elementary operations, to

$$\alpha_{\tau 1}+\bar{\gamma}_1, \dots, \alpha_{\tau_s}+\bar{\gamma}_1, \gamma_{\sigma 1}+\bar{\gamma}_1, \dots, \gamma_{\sigma_t}+\bar{\gamma}_1 .$$

Then we have

$$P_1 \sim \{ Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_k}} \vee \bar{S}^n_1 \vee \cdots \vee S^n_\lambda \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}_\mu ; \bar{\alpha}_1 + 2^{p_1}x_1, \dots, \bar{\alpha}_\mu + 2^{p_\mu}x_\mu, \bar{\gamma}_1, \dots, \bar{\gamma}_\rho \},$$

where $\bar{\alpha}_i (i = 1, \dots, \mu)$ and $\bar{\gamma}_i (i = 2, \dots, \rho)$ do not contain $(\bar{y}_1 \eta) = \bar{\gamma}_1$. It follows that

$$P_1 \sim \{ Q^{n+1}_{2^{q_1}} \vee \cdots \vee S^{n+1}_\lambda ; \bar{\alpha}_1 + 2^{p_1}x_1, \dots, \bar{\alpha}_\mu + 2^{p_\mu}x, \bar{\gamma}_2, \dots, \bar{\gamma}_\rho \} + (\bar{S}^n_1 \cup e^{n+2}),$$

where e^{n+2} is attached to $\bar{S}^n_1$ by an essential map: $\partial e^{n+2} \to \bar{S}^n_1$. By the same process all the γ_i involving at least one $(y_j \eta)$ may be deleted from the interior of the bracket together with S^n_j, so that, changing notations, we have

$$P_1 \sim \{ Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_k}} \vee S^n_1 \vee \cdots \vee S^n_\sigma \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}_\mu ; \alpha_1 + 2^{p_1}x_1, \dots, \alpha_\mu + 2^{p} x_\mu, \gamma_1, \dots, \gamma_\tau \} + (S^n \cup e^{n+2}) + \cdots + (S^n \cup e^{n+2}),$$ where $\gamma_1, \dots, \gamma_\tau$ do not contain any $(y_i \eta)$ $(i = 1, \dots, \sigma)$.

Putting

$$P_2 = \{ Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_k}} \vee S^n_1 \vee \cdots \vee S^n_\sigma \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}, \alpha_1 + 2^{p_1}x_1, \dots, \alpha + 2^{p_\mu}x_\mu, \gamma_1, \dots, \gamma_\tau \},$$

we proceed to reduce P_2 to a normal form. Let $p_1 \leq p_2 \leq \dots \leq p_\mu$ and let $\alpha_i + 2^{p_i}x_i$ be the term of the greatest p_j among all the terms containing at least one of $(y_j \eta)$ $(j = 1, \dots, \sigma)$. Then, for instance,

$$\alpha_i + 2^{p_i}x_i = (y_1 \eta) + \cdots + (z_i \eta) + \cdots + 2^{p_i}x_i .$$

If we carry out the operation

$$\begin{aligned} \bar{y}_1 &= y_1 + \cdots + z_i + \cdots + 2^{p_i}x_i \\ \bar{y}_\rho &= y_\rho \quad (\rho = 2, \dots, \sigma) \\ \bar{z}_\rho &= z_\rho \quad (\rho = 1, \dots, \kappa) \end{aligned}$$

we have $\bar{\alpha}_i = (\bar{y}_1 \eta)$. If there exist $\bar{\alpha}_j$ containing $(\bar{y}_1 \eta)$, we substract $(\bar{\alpha}_i + 2^{p_i}x_i)$ from $(\bar{\alpha}_j + 2^{p_j}x_j)$ by elementary operations. Then we have $(\bar{\alpha}_j + 2^{p_j}x_j) - (\bar{\alpha}_i + 2^{p_i}x_i) = (\bar{\alpha}_j - \bar{\alpha}_i) + 2^{p_j}(x_j - 2^{p_i - p_j}x_i)$, where $p_j \leq p_i$. Again by elementary operation

$$\begin{aligned} \bar{x}_j &= x_j - 2^{p_i - p_j}x_i \\ \bar{x}_\rho &= x_\rho \quad (\rho = 1, \dots, \hat{j}, \dots, \mu), \end{aligned}$$

it follows that the relation $\alpha_i + 2^{p_i}x_i$, an n sphere S^n_{i1}, and an $(n+1)$ sphere S^{n+1}_i are deleted from the contents interior the bracket $\{ \}$ and that

$$P_2 \sim \{ Q^{n+1}_{2^{q_1}} \vee \dots \vee Q^{n+1}_{2^{q_k}} \vee S^n_2 \vee \dots \vee S^n \vee S^{n+1}_1 \vee \dots \vee S^{n+1}_{i-1} \vee S^{n+1}_{i+1} \vee \dots \vee S^{n+1}_\mu ;$$

$\alpha_1+2^{p_1}x_1,\dots,\ \alpha_{i-1}+2^{p_{i-1}}x_{i-1},\ \alpha_{i+1}+2^{p_{i+1}}x_{i+1},\dots,\ \alpha_\mu+2^{p_\mu}x_\mu,\ \gamma_1,\dots,\gamma_\tau\}$
$+(S_1^n\vee S_i^{n+1})\cup e^{n+2}$, where $(S_1^n\vee S_i^{n+1})\cup e^{n+2}=Q_4$.

By the same procedure P_2 can be reduced to a complex

$P_3=\{Q^{n+1}_{2^{q_1}}\vee\dots\vee Q^{n+1}_{2^{q_k}}\vee S_1^{n+1}\vee\dots\vee S_\kappa^{n+1};\ \alpha_1+2^{p_1}x_1,\dots,\ \alpha_\kappa+2^{p}x_\kappa,\ \gamma_1,\dots,\gamma_\tau\}$
$+Q_4+\cdots+Q_4$.

If, for example, $\gamma_1=(z_1\eta)+\dots$, by elementary operations

$$\bar z_1=z_1+\cdots$$
$$\bar z_i=z_i\quad(i=2,\dots,\kappa)$$

we have $\bar\gamma_1=(\bar z_1\eta)$. Then, if $\bar\alpha_{\nu_1},\dots,\bar\alpha_{\nu_s};\ \bar\gamma_{i_1}\dots\bar\gamma_{i_t}$ contain $\bar\gamma_1$, we change them, by elementary operations, to

$$\bar\alpha_{\nu_1}+\bar\gamma_1,\dots,\ \bar\gamma_{i_1}+\bar\gamma_1,\dots\bar\gamma_{i_t}+\bar\gamma_1.$$

Then it is seen that, changing notations,

$P_3\sim\{Q^{n+1}_{2^{q_2}}\vee\dots\vee Q^{n+1}_{2^{q_k}}\vee S_1^{n+1}\vee\dots\vee S_k^{n+1};\ \alpha_1+2^{p_1}x_1,\dots,\alpha_\kappa+2^{p_\kappa}x_\kappa,\ \gamma_2,\dots,\gamma_\tau\}$
$+Q^{n+1}_{2^{q_1}}\cup e^{n+2}$,

where $Q^{n+1}_{2^{q_1}}\cup e^{n+2}=Q_5$. Repeating the same process and changing notation, we have

$P_3\sim\{Q^{n+1}_{2^{q_1}}\vee\dots\vee Q^{n+1}_{2^{q_2}}\vee S_1^{n+1}\vee\dots\vee S^{n+1},\ \alpha_1+2^{p_1}x_1,\dots,\ \alpha_\kappa+2^{p}x_\kappa,\}+Q_5+\dots$
$+Q_5$.

Now we arrive at the finial stage of reduction. Let $\alpha_i+2^{p_i}x_i$ be the term of the greatest p_i among the terms containing at least one of $(z_j\eta)$ $(j=1,\dots,l)$. If, for instance,

$$\alpha_i+2^{p_i}x_i=(z_1\eta)+\cdots+2^{p_i}x_i,$$

by the elementary operation

$$\bar z_1=z_1+\cdots$$
$$\bar z_i=z_i\quad(i=2,\dots,l)$$

we have $\bar\alpha_i=(\bar z_1\eta)$. If there exist some $\bar\alpha_j$ containing $(\bar z_1\eta)$, we substract $\bar\alpha_i+2^{p_i}x_i$ from $\bar\alpha_j+2^{p_j}x_j$ by elementary operation. Then we have $(\bar\alpha_j+2^{p_j}x_j)-(\bar\alpha_i+2^{p_i}x_i)=(\bar\alpha_j-\bar\alpha_i)+2^{p_j}(x_j-2^{p_i-p_j}x_i)$, where $p_j\leq p_i$. Again, by the elementary operation

$$\bar x_j=x_j-2^{p_i-p_j}x_i$$
$$\bar x_\rho=x_\rho\quad(\rho=1,\dots,\hat j,\dots,\mu)$$

It is seen that, changing notation,

$$P_4 = \{Q^{n+1}_{2^{q_1}} \vee \cdots \vee Q^{n+1}_{2^{q_2}} \vee S^{n+1}_1 \vee \cdots \vee S^{n+1}_\kappa ; \alpha_1 + 2^{p_1}x_1, \ldots , \alpha_\kappa + 2^{p_\kappa}x_\kappa\}$$
$$\sim \{\hat{Q}^{n+1}_{2^{q_1}} \vee Q^{n+1}_{2^{q_2}} \vee \cdots \vee Q^{n+1}_{2^{q_l}} \vee S^{n+1}_1 \vee \cdots \vee \hat{S}^{n+1}_i \vee \cdots \vee S^{n+1}_\kappa ;\ \alpha_1 + 2^{p}x_1, \ldots ,$$
$$\widehat{\alpha_i + 2^{p_i}x_i}, \cdots , \alpha_\kappa + 2^{p_\kappa}x_\kappa\} + (S^n \vee S^{n+1}) \cup e^{n+1} \cup e^{n+2},$$

where $(S^n \vee S^{n+1}) \cup e^{n+1} \cup e^{n+2} = Q_6$. Repeating the same process, we have $P_4 \sim Q_6 + \cdots + Q_6 + Q_7 + \cdots + Q_7 + Q_8 + \cdots + Q_8 + Q_9 + \cdots + Q_9 + S^{n+1} + \cdots + S^{n+1}$. This completes the proof.

6. Reduced complexes

$\bar{A}^n_3$-complexes K is referred to as a reduced complex when it satisfies the following conditions;

i) $K^0 = K^1 = \cdots = K^{n-1} = e^0$, a single point,

ii) $K^n = S^n_1 + \cdots + S^n_\kappa$, where n spheres S^n_i $(i = 1, \ldots \kappa,)$ are attached at a point e^0, and $S^n_i - e^0 = e^n_i (i = 1, \ldots \kappa)$ are n-cells,

iii) $K^{n+1} = K^n + e^{n+1}_1 + \cdots + e^{n+1}_k + e^{n+1}_{k+1} + \cdots + e^{n+1}_{k+l}$, where e^{n+1}_i $(i = 1, \ldots, k)$ is attached to S^n_i by a map f_i: $\partial e^{n+1}_i \rightarrow S^n_i$ of odd degree σ^i, and e^{n+1}_{k+i} $(i = 1, \ldots , l)$ is attached to S^n_{k+i} by a map f_i : $\partial e^{n+1}_{k+i} \rightarrow S^n_{k+i}$ of degree 2^{p_i},

iv) $K^{n+2} = K^{n+1} + e^{n+2}_{k+1} + \cdots + e^{n+2}_{k+l} + e^{n+2}_{k+l+1} + \cdots + e^{n+2}_\kappa \ + S^{n+2}_1 + \cdots + S^{n+2}_t$, where $e^{n+2}_{k+i} (i = 1, \ldots , l)$ is attached to S^n_{k+i} by an essential map: $\partial e^{n+2}_{k+i} \rightarrow S^n_{k+i}$, and $e^{n+2}_{k+l+i} (i = 1, \ldots , \kappa\text{-}k\text{-}l)$ is attached to S^n_{k+l+i} by an essential map: $\partial e^{n+2}_{k+l+i} \rightarrow S^n_{k+l+i}$.

v) If $K_0 = K^{n+2} - (e^{n+1}_1 \cup S^n_1 + \cdots + e^{n+1}_k \cup S^n_k)$, a finite number of $(n+3)$ cells $e^{n+3}_i (i = 1, \ldots , \alpha)$ are attached to K_0 by maps f_i : $\partial e^{n+3}_i \rightarrow K_0$.

Notice that $e^{n+1}_i \cup S^n_i$ $(i = 1, \ldots, k)$ are not bounded, and that n-spheres S^n_i $(i = 1, \ldots , \kappa)$ are all bounded. Of course, the case where $k = 0$, or $k = 0$, or $t = 0$, may be possible, but the most general reduced complex of $\bar{A}^3_n$-complex is the cell complex of the type just referred to above.

Theorem 5. *Any $\bar{A}^n_3$-complex P is of the same homotopy type as some reduced complex.*

Proof. Let P^{n+2} be the $(n+2)$ skelton of P, then $\pi_i(P^{n+2}) = 0$ for each $i < n$. In virtue of Theorem 4, P^{n+2} is of the same homotopy type as a cell complex Q^{n+2} consisting of a number of elementary A^2_n-complexes. It is evident that $\pi_{n+1}(P) \cong \pi_{n+1}(P^{n+2})$, and $\pi_{n+1}(Q^{n+2}) \cong \pi_{n+1}(P^{n+2})$, so that we have $\pi_{n+1}(Q^{n+2}) = 0$. By the recurrent use of a result of G. W. Whitehead [9] or a slight generalization of a lemma of Blakers and Massay [10], we have $\pi_{n+1}(Q^{n+2}) \cong \sum_{\lambda=1}^{9} \sum_{\mu} \pi_{n+1}(Q^\mu_\lambda)$, where the upper-suffix μ of Q^μ_λ, indicates the number of elementary polyhedra of the type Q_λ. It follows that if $\pi_{n+1}(Q^\mu_\lambda) \neq 0$, such polyhedra Q^μ_λ are deleted from Q^{n+2}. As $\pi_{n+1}(S^n) \cong I_2$ for $n > 2$, and $\pi_{n+1}(S^{n+1}) \cong I$, S^n, S^{n+1} must be deleted from Q^{n+2}.

It is verified that we have $\pi_{n+1}(Q_4) \cong I_{2^{p+1}}$. For we have $a+b=0$, so that $2b=0$. Thus the element represented by a map $S^{n+1} \rightarrow S^{n+1} \subset Q_4$ of degree 1, is the generator of $\pi_{n+1}(Q_4)$, whose order is $2 \cdot 2^p = 2^{p+1}$. Therefore all the Q_4^{μ} are deleted from Q^{n+2}. From the same arguments and from $\pi_{n+1}(e^{n+1} \cup S^n) \cong \pi_{n+1}(S^n)$, we have $\pi_{n+1}(Q_6) \cong I_{2^{p+1}}$, so that all the Q_4^{μ} are deleted from Q^{n+2}. Since $\pi_{n+1}(Q_7) \cong I_2$, $\pi_{n+1}(Q_8) \cong I_{\sigma}$, and $\pi_{n+1}(Q_9) \cong I_{2^p}$, all the Q_7^{μ}, Q_8^{μ}, $Q_9^{\mu}$are deleted from Q^{n+2}. From the verifications just referred to above and from the vanishing $(n+1)$ homotopy groups of S^{n+2}, Q_2, Q_3, Q_5, it is concluded that P^{n+2} is of the same homotopy type as K^{n+2} in the definition of a reduced complex. Let $f: P^{n+2} \rightarrow K^{n+2}$ be a homotopy equivalence, and let σ_i^{n+3} be an $(n+3)$ simplex of P. Then from Lemma 2, P is of the same homotopy type as a cell complex, $(n+3)$ simplexes $\sigma_i^{n+3} (i=1, \dots, \alpha)$ of which are attached to K^{n+2} by maps $f_i e$: $\partial \sigma_i^{n+3} \rightarrow K^{n+2}$, where e is the identical map of P. However, the element of $\pi_{n+2}(K^{n+2})$ represented by a map $f\,e$ may be regarded as an element of $\pi_{n+2}(K_0)$, so that from Lemma 5, [3], P is of the same homotopy type as a reduced complex defined above. This completes the proof.

7. $\pi_{n+2}(K^{n+2})$.

Let us consider K^{n+2} satisfying i), ii), iii), iv) in § 5. It is easily verified that

$$\pi_{n+2}(K^{n+2}) \cong \sum_{i=1}^{t} \pi_{n+2}(S_i^{n+2}) + \sum_{i=1}^{k} \pi_{n+2}(S_i^n \cup e_i^{n+1}) + \sum_{i=1}^{l} \pi_{n+2}(S_{k+i}^n \cup e_{k+i}^{n+1} \cup e_{k+i}^{n+2})$$
$$+ \sum_{i=1}^{\kappa-k-l} \pi_{n+2}(S_{k+l+i}^n \cup e_{k+l+i}^{n+2}), \quad \text{for } n>3.$$

It is also verified (for example, see [11]) that

$$\pi_{n+2}(S_i^n \cup e_i^{n+1}) = 0, \text{ for } i=1, \dots, k,$$
$$\pi_{n+2}(S_{k+l+i}^n \cup e_{k+l+i}^{n+2}) \cong I, \text{ for } i=1, \dots, \kappa-k-l,$$
$$\pi_{n+2}(S_{k+i}^n \cup e_{k+i}^{n+1} \cup e_{k+i}^{n+2}) \cong I+I_2, \text{ for } i=1, , \dots, l.$$

Now, let us denote by S_i^{n+2} the generator $\pi_{n+2}(S_i^{n+2})$, which is represented by a map $S^{n+2} \rightarrow S_i^{n+2}$ of degree 1. The generator ω_{k+l+i} of $\pi_{n+2}(S_{k+l+i}^n \cup e_{k+l+i}^{n+2})$ is represented by a map $\omega_{k+l+i}: S^{n+2} \rightarrow S_{k+l+i}^n \cup e_{k+l+i}^{n+2}$ as follows. Denote the northern hemisphere of S^{n+2} by $V_{\geq 0}^{n+2}$ and the southern hemisphere by $V_{\leq 0}^{n+2}$, then $S^{n+2} = V_{\geq 0}^{n+2} \cup V_{\leq 0}^{n+1}$ and the equator of S^{n+2} is represented by $V_{\geq 0}^{n+2} \cap V_{\leq 0}^{n+2} = S^{n+1}$. Then the partial map $\omega_{k+l+i} | V_{\geq 0}^{n+2} \cap V_{\leq 0}^{n+2}$ represents $2\eta: S^{n+1} = V_{\geq 0}^{n+2} \cap V_{\leq 0}^{n+2} \rightarrow S_{k+l+i}^n$, where η is an essential map of S^{n+1} onto S^n. Since $\omega_{k+l+i} | \partial V_{\leq 0}^{n+2}$ is inessential, we have an extended map: $V_{\leq 0}^{n+2} \rightarrow S_{k+l+i}^n$. From these considerations that $\partial e_{k+l+i}^{n+2} \rightarrow S_{k+l+i}^n$ represents an essential map η and that $\omega_{k+l+i} | V_{\geq 0}^{n+2}$ represents 2η, it follows that we have a map: $V^{n+2}_{\geq 0} \rightarrow S^n_{k+l+i} \cup e^{n+2}_{k+l+i}$ of degree two. It is proved in [11] that the map ω_{k+l+i} thus obtained represents the free generator of $\pi_{n+2}(S^n_{k+l+i} \cup e^{n+2}_{k+l+i})$. Notice that the free generator itself will be also denoted by ω_{k+l+i}. Next, the free generator of $\pi_{n+2}(S^n_{k+i} \cup e^{n+1}_{k+i} \cup e^{n+2}_{k+i})$ is represented by a map ω_{k+i}: $S^{n+2} \rightarrow S^n_{k+i} \cup e^{n+2}_{k+i}$ of the same property as referred to above, and the generator v_{k+i} $(i=1, \dots, l)$ of order two are represented by maps v_{k+i}: $S^{n+2} \rightarrow S^n_{k+i} \cup e^{n+1}_{k+i}$ as follows. Remember here that e^{n+1}_{k+i} is attached to S^n_{k+i} by a map: $\partial e^{n+1}_{k+i} \rightarrow S^n_{k+i}$ of degree 2^{p_i}. Let $T^{n+1} = S^n \cup e^{n+1}$, where e^{n+1} is attached by a map $\partial e^{n+1} \rightarrow S^n$ of degree 1. Construct a map $T^{n+1} \supset S^n \rightarrow S^n_{k+i}$ of degree 2^{p_i}, we can extend it to a map f_i: $T^{n+1} \rightarrow S^n_{k+i} \cup e^{n+1}_{k+i}$ such that e^{n+1} of T^{n+1} is mapped homeomorphically onto e^{n+1}_{k+i}. For e^{n+1}_{k+i} is attached to S^n_{k+i} by a map $\partial e^{n+1}_{k+i} \rightarrow S^n_{k+i}$ of degree 2^{p_i}. If the equator $V^{n+2}_{\geq 0} \cap V^{n+2}_{\leq 0} = S^{n+1}$ is mapped onto the n-sphere S^n of T^{n+1} by an essential map η and if we construct a suspension $\mathcal{E}(\eta)$ [13]: $V^{n+2}_{\geq 0} \rightarrow T^{n+1}$ of η, then $f_i \cdot \mathcal{E}(\eta) | \partial V^{n+2}_{\leq 0}$ is inessential because $2^{p_i}\eta = 0$. Thus $f_i \cdot \mathcal{E}(\eta) | \partial V^{n+2}_{\leq 0}$ can be extended to a map: $V^{n+2}_{\leq 0} \rightarrow S^n_{k+i}$. If we define $v_{k+i} | V^{n+2}_{\geq 0} = f_i \cdot \mathcal{E}(\eta)$ and $v_{k+i} | V^{n+2}_{\leq 0}$ is the map: $V^{n+2}_{\leq 0} \rightarrow S^n_{k+i}$ constructed above, it is verified that this map v_{k+i} is essential and of order two. Then in the reduced complex K we have the homotopy boundary

$$\beta e^{n+3}_i = \sum_{j=i}^{t} \bar{\lambda}_{ij} S^{n+2}_j + \sum_{j=k+1}^{k+l} \bar{\mu}_{ij}\, \omega_j + \sum_{j=k+1}^{k+l} \bar{\nu}_{ij}\, v_j + \sum_{j=k+l+1}^{\kappa} \bar{\gamma}_{ij}\, \omega_j \text{ (for each } i \leq \alpha).$$

If we carry out two operations, referred to in § 5, with respect to $\{S^{n+2}_1, \dots, S^{n+2}_t\}$ and $\{\beta e^{n+3}_1, \dots, \beta e^{n+3}_\alpha\}$, we have a slightly modified reduced complex L of the same homotopy type as K in § 6 such that

$L^{n+2} = K^{n+2}$, and

$$\beta e^{n+2}_i = \lambda_{ij} S^{n+2}_j + \sum_{j=k+1}^{k+l} \mu_{ij}\, \omega_i + \sum_{j=k+1}^{k+l} \nu_{ij}\, v_j + \sum_{j=k+l+1}^{\kappa} \gamma_{ij}\, \omega_j \text{ (for each } i \leq \alpha).$$

Notice that in this reduced complex L, e^{n+3}_i bounds only one $(n+2)$-sphere or bounds none. In the sequel we shall refer to L as a reduced complex.

8. Cohomological properties in a reduced complex L.

Referring to § 7, § 8, we have

i) e^n_{k+i} $(i=1, \dots, l)$ are cocycles mod 2^{p_i},

ii) e^n_i $(i=k+l+1, \dots, \kappa)$ are absolute cocycles,

iii) $\delta e^{n+2}_i = \sum_{j=1}^{\alpha} 2\mu_{ji}\, e^{n+3}_j$ $(i=k+1, \dots, k+l)$ and

$\delta e^{n+2}_i = \sum_{j=1}^{\alpha} 2\gamma_{ji}\, e^{n+3}_j$ $(i=k+l+1, \dots, \kappa)$,

iv) e_i^{n+1} $(i=k+1, \dots, k+l)$ are absolute cocycles.

It should be noted that "2" in the terms δe_i^{n+2} come from the degree of the maps ω and that e_i^{n+2} $(i=k+1, \dots, \kappa)$ are cocycles mod. 2. Then we have

Theorem 6.

a) $j_0 e_i^n \underset{n-2}{\smile} j_0 e_i^{n+2} = j_2 e_i^{n+2}$ $\quad (i=k+l+1, \dots, \kappa)$,

b) $j_{2^{\nu_i}} e_{k+i}^n \underset{n-2}{\smile} j_{2^{\nu_i}} e_{k+i}^n = j_2 e_{k+i}^{n+2}$ $\quad (i=1, \dots, l)$,

c) $j_0 e_i^{n+1} \underset{n-1}{\smile} j_0 e_i^{n+1} = \sum_{j=1}^{\alpha} \nu_{ji} j_2 e_j^{n+3}$ $\quad (i=k+1, \dots, k+l)$,

d) $\mathfrak{q}_{n-3} j_0 e_i^n = \sum_{j=1}^{\alpha} \gamma_{ji} j_0 e_j^{n+3}$ $\quad (i=k+l+1, \dots, \kappa)$,

e) $\mathfrak{q}_{n-3} j_{2^{\nu_i}} e_{k+i}^n = \sum_{j=i}^{\alpha} \mu^j{}_{,k+i} j_0 e_j^{n+3}$ $\quad (i=1, \dots, l)$.

Proof. Putting $M = L^{n+2}$ and considering the injection $\kappa: M \to L$, we have a proper homomorphism $\kappa^*: FH(L) \to FH(M)$ induced by κ. Put

$$j_0 e_i^n \underset{n-2}{\smile} j_0 e_i^n = \sum_{j=k+1}^{k+l} \varepsilon_j j_2 e_j^{n+2} + \sum_{j=k+l+1}^{\kappa} \varepsilon_j j_2 e_j^{n+2} \; (i=k+l+1, \dots, \kappa),$$

where $\varepsilon_i \equiv 0$ or 1 (mod. 2).

$$\kappa^*(j_0 e_i^n \underset{n-2}{\smile} j_0 e_i^n) = \sum_{j=k+1}^{k+l} \varepsilon_j \kappa^* j_2 e_j^{n+2} + \sum_{j=k+l}^{\kappa} \varepsilon_j \kappa^* j_2 e_j^{n+2}$$

$$\kappa^* j_0 e_i^n \underset{n-2}{\smile} \kappa^* j_0 e_i^n = \sum_j \varepsilon_j \kappa^* j_2 e_j^{n+2} + \sum_j \varepsilon_j \kappa^* j_2 e_j^{n+2}$$

$$j_0 e_i^n \underset{n-2}{\smile} j_0 e_i^n = \sum_{j=k+1}^{k+l} \varepsilon_j j_2 e_j^{n+2} + \sum_{j=k+l+1}^{\kappa} \varepsilon_j j_2 e_j^{n+2}$$

$$j_2 e_i^{n+2} = \sum_{j=k+1}^{k+l} \varepsilon_j j_2 e_i^{n+2} + \sum_{j=k+l+1}^{\kappa} \varepsilon_j j_2 e_j^{n+2}.$$

It follows that $\varepsilon_j \equiv 0$ $(j=k+1, \dots, k+l)$, $\varepsilon_j \equiv 0$ $(j=k+l+1, \dots, k+l+i-1, k+l+i+1, \dots, \kappa)$ and $\varepsilon_{k+l+i} \equiv 1$ (mod. 2). This proves Theorem 6 a).

Similary we have Theorem 6 b).

Let $M = S^{n+1} \cup e^{n+3}$, where e^{n+3} is attached to S^{n+1} by an essential map $\eta: \partial e^{n+3} \to S^{n+1}$. Then M is regarded as a cell complex composed of three cells, a point e^0, an $(n+1)$ cell e^{n+1}, and an $(n+3)$ cell e^{n+3}. Let us define a cellular map $\kappa: L \to M$ such that

i) $\kappa(S_i^{n+2}) = e^0$ $(i=1, \dots, t)$

ii) $\kappa(S_i^n \cup e_i^{n+1}) = e^0$ $(i=1, \dots, k)$,

iii) $\kappa(S_i^n \cup e_i^{n+2}) = e^0$ $\quad (i=k+l+1, \dots, \kappa)$

iv) $\kappa(S_i^n \cup e_i^{n+1} \cup e_i^{n+2})$
$= e^0$ $(i=k+1, \dots, k+i-1, k+i+1, \dots, k+l)$,

v) $\kappa(S_{k+i}^n \cup e_{k+i}^{n+1} \cup e_{k+i}^{n+2}) = S^{n+1}$,

vi) if $\nu_{j,k+i} \neq 0$, e_j^{n+3} is mapped by κ topologically onto e^{n+3}, and otherwise, e_j^{n+3} is mapped to e^0 by κ.

It is verified in the following way that this map κ can be constructed. v) is constructed such that κ maps $S^n_{k+1} \cup e^{n+2}_{k+l}$ into e^0 and elsewhere topological. Let $\kappa' : c(M) \to c(L)$ be a cochain map induced by κ and let $\kappa^* : FH(M) \to FH(L)$ be a proper homomorphism induced by κ. In M we have

$$j_0 e^{n+1} \underset{n-1}{\smile} j_0 e^{n+1} = j_2 e^{n+3},$$

so that

$$\kappa^*(j_0 e^{n+1} \underset{n-1}{\smile} j_0 e^{n+1}) = \kappa^* j_2 e^{n+3}$$
$$\kappa^* j_0 e^{n+1} \underset{n-1}{\smile} \kappa^* j_0 e^{n+1} = j_2 \kappa' e^{n+3}$$
$$j_0 \kappa' e^{n+1} \underset{n-1}{\smile} j \kappa' e^{n+1} = j_2 \kappa' e^{n+3}$$

Since we have $\kappa' e^{n+1} = e^{n+1}_{k+i}$, and $\kappa' e^{n+3} = \sum_{j=1}^{\alpha} \nu_{j,\,k+i} e^{n+3}_j$, we have

$$j_0 e^{n+1}_{k+i} \underset{n-1}{\smile} j_0 e^{n+1}_{k+i} = \sum_{j=1}^{\alpha} \nu_{j,\,k+i} j_2 e^{n+3}_j .$$

This relation holds true for each $i = 1, \dots, l$, so that c) is completely established.

Though d), e) will not be used in the sequel, we prove them here for the completeness and the convenience of our discussions. They are essentially used in solving the $(n+3)$ extension cocycle and corresponding classification problem, which N. Shimada and I will discuss in our forthcoming paper [11]. From a) we have

$e^n_i \underset{n-2}{\smile} e^n_i = (-1)^n e^{n+2}_i + 2c^{n+2} + \delta c^{n+1}$ for each $i = k+l+1, \dots, \kappa$, where c^{n+2}, c^{n+1} are cochains. Considering the coboundary of both sides, we have

$$2(-1)^n e^n_i \underset{n-3}{\smile} e^n_i = 2(-1)^n \sum_{j=1}^{\alpha} \gamma_{ji} e^{n+3}_j + 2\delta c^{n+2}$$

$$e^n_i \underset{n-3}{\smile} e^n_i = \sum_{j=1}^{\alpha} \gamma_{ji} e^{n+3}_j + (-1)^n \delta c^{n+2}.$$

By the definition of $\mathfrak{q}_i$-operation, in case where $m = 0$, we have

$$q'_{n-3} e^n_i = \sum_j \gamma_{ji} e^{n+3}_j + (-1)^n \delta c^{n+2},$$

so that

$$\mathfrak{q}_{n-3} j_0 e^n_i = \sum_{j=1}^{\alpha} \gamma_{ji} j_0 e^{n+3}_j \text{ for each } i = k+l+1, \dots, \kappa .$$

This proves d).

The proof of e) is analogous to that of d). For the completeness of discussions we prove e).

From b),

$$e^n_{k+i} \underset{n-2}{\smile} e^n_{k+i} = (-1)^n e^{n+2}_{k+i} + 2c^{n+2} + \delta c^{n+1}$$
$$\delta(e^n_{k+i} \underset{n-2}{\smile} e^n_{k+i}) = (-1)^n \delta e^{n+2}_{k+i} + 2\delta c^{n+2}$$
$$2\cdot(-1)^n e^n_{k+i} \underset{n-3}{\smile} e^n_{k+i} + 2^{p_i} e^{n+1}_{k+i} \underset{n-2}{\smile} e^n_{k+i} + (-1)^n 2^{p_i} e^n_{k+i} \underset{n-2}{\smile} e^{n+1}_{k+i}$$
$$= 2(-1)^n \cdot \sum_{j=1}^{\alpha} \mu_{j,\,k+i} e^{n+3}_j + 2\delta c^{n+2} . \qquad \dots\dots\dots\dots\dots\dots \text{ i)}$$

Since $\delta(e^{n+1}_{k+i} \underset{n-1}{\smile} e^n_{k+i}) = (-1)^n e^n_{k+i} \underset{n-2}{\smile} e^{n+1}_{k+i} + (-1) e^{n+1}_{k+i} \underset{n-2}{\smile} e^n_{k+i} + 2^{p_i} e^{n+1}_{k+i} \underset{n-2}{\smile} e^{n+1}_{k+i}$, we have

$$e^{n+1}_{k+i} \underset{n-2}{\smile} e^n_{k+i} = (-1)^n e^n_{k+i} \underset{n-2}{\smile} e^{n+1}_{k+i} + 2^{p_i} e^{n+1}_{k+i} \underset{n-1}{\smile} e^{n+1}_{k+i} - \delta(e^n_{k+i} \underset{n-1}{\smile} e^{n+1}_{k+i}) . \qquad \text{ii)}$$

Substituting ii) for the term $e^{n+1}_{k+i} \underset{n-2}{\smile} e^n_{k+i}$ of i), we have

$$2\cdot(-1)^n e^n_{k+i} \underset{n-3}{\smile} e^n_{k+i} + 2(-1)^n 2^{p_i} e^n_{k+i} \underset{n-2}{\smile} e^{n+1}_{k+i} + 2^{p_i} e^{n+1}_{k+i} \underset{n-1}{\smile} e^{n+1}_{k+i} - 2^{p_i} \delta(e^{n+1}_{k+i} \underset{n-1}{\smile} e^n_{k+i})$$
$$= 2(-1)^n \sum_{j=1}^{\alpha} \mu_{j,\, i+k} e^{n+3}_j + 2\delta c^{n+2} .$$

Thus it follows that

$$e^n_{k+i} \underset{n-3}{\smile} e^n_{k+i} + 2^{p_i} e^n_{k+i} \underset{n-2}{\smile} \delta'_{2^{p_i}} e^n_{k+i} + (-1)^n \frac{2^{p_i}}{2} \delta_{2^{p'_i}} e^n_{k+i} \underset{n-1}{\smile} \delta_{2^{p'_i}} e^n_{k+i} \backsim \sum_{j=1}^{\alpha} \mu_{j,\, k+i}\, e^{n+3}_j$$

From the definition of $\mathfrak{q}_i$- operation it is proved that

$$q'_{n-3} e^n_i \backsim \sum_{j=1}^{\alpha} \eta_{ji} e^{n+3}_j \quad \text{for each } i = k+1, \dots, k+l .$$

This proves

$$\mathfrak{q}_{n-3}\, j_{2^{p_i}} e^n_{k+i} = \sum_{j=1}^{\alpha} \mu_{j,\, k+i}\, j_0 e^{n+3}_j \quad \text{for each } i = 1, \dots, l .$$

9. Proof of Theorem 2.

In virtue of Theorem 5 there exist reduced complexes L, $\overline{L}$ which are of the same homotopy type as P, $\overline{P}$ respectively. Let $u: L \to P$ and $v: P \to L$ be homotopy equivalences such that $vu \sim e$ and $uv \sim e$. If $u^*: FH(P) \to FH(L)$ and $v^*: FH(L) \to FH(P)$ are proper homomorphisms induced by u, v respectively, we have

$$u^* v^* = 1 \text{ and } v^* u^* = 1 ,$$

from $vu \sim e$ and $uv \sim e$. Suppose that $u^* f^*: FH(\overline{P}) \to FH(L)$ is realized by a map $h: L \to \overline{P}$. Then the proper homomorphism induced by the map $hv: P \to \overline{P}$, is $v^* h^* = v^*(u^* f^*) = f^*$, so that it is sufficient for us to prove this Theorem in case where two reduced complexes L, $\overline{L}$ take the place of two given complexes P, $\overline{P}$ respectively.

In virtue of Lemma 1 the proper homomorphism $\overline{H}(L) \to H(L)$ induced by $f^*: FH(\overline{L}) \to FH(L)$ is realized by a cochain map $g^*: c(\overline{L}) \to c(L)$.

If a chain map $g: c(L) \to c(\overline{L})$ dual to g^* [12] is realized by a cellular map $f: L \to \overline{L}$, the proper homomorphism induced by f is the given proper homomorphism f^*. Thus we intend to construct step by step a cellular map $f: L \to \overline{L}$, which realizes the chain map $g: c(L) \to c(\overline{L})$. In performing this, we utilize a lemma of J. H. C. Whitehead, which is of great importance and of use together with lemma 5, [3].

The lemma is stated as follows;

Lemma 4 (J. H. C. WHITEHEAD [3] Lemma 7).

Let K, L be simply connected complexes and let e^n be a principal cell, where $n>2$. Suppose that $g: c_r(K)\to c_r(L)$ be a chain map such that the map $g|c_r(K_0)(r=0, 1, \ldots)$ can be realized by a cellular map $f_0: K_0\to L$, where $K_0=K-e^n$. If $f_0\beta e^n=\beta g e^n$, then f_0 can be extended to a map $f: K\to L$, which realizes the chain map g.

Since $f^*|H^0(\bar{L})$ is an isomorphism onto, we have $g(e_0)=\bar{e}^0$. Thus $g|c_0(L)$ can be realized by a map $f: L^{n-1}=e^0\to \bar{e}^0=\bar{L}^{n-1}$. Next, let $g|c_n(L)$ be given such that $g(e_i^n)=\sum_{j=1}^{\kappa'} a_{ij}\bar{e}_j^n(i=1, \ldots, \kappa)$. Then a cellular map $f: (S_i^n, e^0)\to(\bar{L}^n, \bar{e}^0)$, for each $i=1, \ldots, \kappa$, can be constructed such that f represents a homotopy element $\sum_{j=1}^{\kappa'} a_{ij}\, \bar{S}_j^n$, where $\bar{S}_j^n$ denotes also a homotopy element represented by a map: $S^n\to\bar{S}_j^n$ of degree unity. Then it is obvious that the cellular map $f: L^n\to\bar{L}^n$ thus constructed realizes the chain map $g|c_n(L)$. Since $\pi_n(L^n)\approx H_n(L^n)$, and $\pi_n(\bar{L}^n)\approx H_n(\bar{L}^n)$, we identify elements corresponding by these isomorphisms. Then we have

$$\beta g e_i^{n+1}=\partial g e_i^{n+1}=g\partial e_i^{n+1}=f\partial e_i^{n+1}=f\partial e_i^{n+1} \text{ for each } i=1, \ldots k+l,$$

so that in virtue of Lemma 4 $g|c_{n+1}(L)$ can be realized by an extended cellular map $f: L^{n+1}\to\bar{L}^{n+1}$.

Now we are going to extend this cellular map $f: L^{n+1}\to\bar{L}^{n+1}$ to a map $f: L^{n+1}+e_i^{n+2}(t\geq i\geq 1)\to L^{n+2}$ such that this extended map f realizes the chain map $g|c(L^{n+1}+e_i^{n+2})$. If $ge_i^{n+2}=\sum_{p=1}^{t'} b_{ip}\bar{e}_p^{n+2}+\sum_{q=1}^{l'} b'_{i,\kappa'+q}\,\bar{e}_{\kappa'+q}^{n+2}+\sum_{r=\kappa'+l'+1}^{\kappa'} b''_{ir}\bar{e}_r^{n+2}$, we have $b'_{i,\kappa'+q}\equiv 0$ and $b''_{i,r}\equiv 0 \mod 2$. This is proved in the following way. Evidently we have

$$g^*\bar{e}_{\kappa'+q}^{n+2}=\cdots+b'_{i,\kappa'+q}e_i^{n+2}+\cdots \text{ for each } q=1, \ldots, l'. \tag{9.1}$$

From Lemma 1

$$j_2g^*\bar{e}_{\kappa'+q}^{n+2}=f^*j_2\bar{e}_{\kappa'+q}^{n+2}, \text{ for each } q=1, , \ldots l',$$

and from b), Theorem 6,

$$j_2\bar{e}_{\kappa'+q}^{n+2}=j_{2^{\nu_q}}\bar{e}_{\kappa'+q}^n\smile_{n-2} j_{2^{\nu_q}}\bar{e}_{\kappa'+q}^n \text{ for each } q=1, \ldots l'.$$

By the property of f^* we have

$$f^*j_2\bar{e}_{\kappa'+q}^{n+2}=f^*(j_{2^{\nu_q}}\bar{e}_{\kappa'+q}^n\smile_{n-2} j_{2^{\nu_q}}\bar{e}_{\kappa'+q}^n).$$

$$= f^* j_{2^{p_q}} \bar{e}^n_{k'+q} \underset{n-2}{\smile} f^* j_{2^{p_q}} \bar{e}^n_{k'+q} .$$

Again, from $f^* j_{2^{p_q}} = j_{2^{p_q}} g^*$ it is seen that

$$j_2 g^* \bar{e}^{n+2}_{k'+q} = j_{2^{p_q}} g^* \bar{e}^n_{k'+q} \underset{n-2}{\smile} j_{2^{p_q}} g^* \bar{e}^n_{k'+q} = \dots + b'_{i,k+q} j_2 e^{n+2}_i + \dots$$

$$j_{2^{p_q}} g^* \bar{e}^n_{k'+q} \underset{n-2}{\smile} j_{2^{p_q}} g^* \bar{e}^n_{k'+q} = j_2 (g^* \bar{e}^n_{k'+q} \underset{n-2}{\smile} g^* \bar{e}^n_{k'+q})$$
$$= j_2 \{ (\sum_{i=1}^{\kappa} a_{i,k'+q} e^n_i) \underset{n-2}{\smile} (\sum_{i=1}^{\kappa} a_{i,k'+q} e^n_i) \}$$
$$= j_2 (\sum_{j=k+1}^{\kappa} a^2_{j,k'+q} e^{n+2}_j) .$$

We have

$$\sum_{j=k+1}^{\kappa} a^i_{j,k'+q} j_2 e^{n+2}_j = \dots + b'_{i,k'+q} j_2 e^{n+2}_i + \dots (t \geq i \geq 1).$$

The left side of the last equation does not contain any $e^{n+2}_j (i = 1, \dots, t)$, so that we have

(9.2) $\qquad b'_{i,k'+q} \equiv 0$ mod. 2 for each $q = 1, \dots, l'$ and for each $i = 1, \dots, t$.

Through analogous arguments we have

(9 3) $\qquad b''_{i,r} \equiv 0$ mod. 2 for $r = k' + l' + 1, \dots, \kappa'$ and for $i = 1, \dots, t$.

From (9.2) and (9.3) it is easily seen that

$$\beta g e^{n+2}_i = \sum_{p=1}^{t'} b_{i,p} \beta \bar{e}^{n+2}_p + \sum_{q=1}^{l'} b'_{i,k'+q} \beta \bar{e}^{n+2}_{k+q} + \sum_{r=k'+l'+i}^{\kappa'} b''_{i,r} \beta \bar{e}^{n+2}_r$$
$$= 0 + \sum_{q=1}^{l'} b'_{i,k'+q} (\bar{\alpha}_{k'+q} \eta) + \sum_{r=k'+l'+1}^{\kappa'} b''_{i,r} (\bar{\alpha}_r \eta) = 0 ,$$

where $\eta: S^{n+1} \to S^n$ denotes an essential map $\bar{\alpha}_i: S^n \to \bar{S}^n_i$ $(i = k'+1, \dots, \kappa')$ of degree unity, and $(\bar{\alpha}_i \eta)(i = k'+1, \dots, \kappa')$ are homotopy elements represented by maps $\bar{\alpha}_i \eta$: $S^{n+1} \xrightarrow{\eta} S^n \xrightarrow{\bar{\alpha}_i} S^n_i$. On the other hands we have

$$f \beta e^{n+2}_i = 0 \text{ for each } i = 1, \dots, t,$$

so that

$$\beta g e^{n+2}_i = f \beta e^{n+2}_i \text{ for each } i = 1, \dots, t .$$

This shows the existence of the desired extended map $f: L^n + \sum_{i=1}^{t} e^{n+2}_i \to \bar{L}^{n+2}$.

In the next place we intend to extend f to a map $f: L^{n+2} \to \bar{L}^{n+2}$ such that f realizes the chain map $g | c_{n+2}(L)$. It is easily seen that

(9.4) $f\beta e_i^{n+2}=\sum_{q=1}^{\kappa'} a_{i,q}(\overline{\alpha_q\eta})=\sum_{q=k'+1}^{\kappa'} a_{i,q}(\overline{\alpha_q\eta})$ for each $i=k+1,\dots,\kappa$, for $\pi_{n+1}(\bar{S}_i^n\cup\bar{e}_i^{n+1})=0$ $(i=1,\dots,k')$. Putting

$$(9.5)\qquad ge_i^{n+2}=\sum_{p=1}^{t'} c_{i,p}\bar{e}_p^{n+2}+\sum_{q=1}^{l'} c'_{i,k'+q}\bar{e}_{k'+q}^{n+2}+\sum_{r=k'+l'+1}^{\kappa'} c''_{i,r}\bar{e}_r^{n+2},$$ we have

$$(9.6)\qquad \beta ge_i^{n+2}=\sum_{q=1}^{l'} c'_{i,k'+q}(\overline{\alpha_{k'+q}\eta})+\sum_{r=k'+l'+1}^{\kappa'} c''_{i,r}(\overline{\alpha_r\eta}).$$

If the following relations

$$(\mathrm{A})\begin{cases}\text{i)}\quad a_{i,k'+q}\equiv c'_{i,k'+q} \quad \text{mod. } 2 \quad \begin{pmatrix} i=k+1,\dots,k+l \\ q=1,\dots,l' \end{pmatrix}\\[2ex] \text{ii)}\quad a_{i,q}\equiv c''_{i,q} \quad \text{mod. } 2 \quad \begin{pmatrix} i=k+1,\dots,k+l \\ q=k'+l'+1,\dots,\kappa' \end{pmatrix}\end{cases}$$

$$(\mathrm{B})\begin{cases}\text{i)}\quad a_{i,k'+q}\equiv c'_{i,k'+q} \quad \text{mod. } 2 \quad \begin{pmatrix} i=k+l+1,\dots,\kappa \\ q=1,\dots,l' \end{pmatrix}\\[2ex] \text{ii)}\quad a_{i,q}\equiv c''_{i,q} \quad \text{mod. } 2 \quad \begin{pmatrix} i=k+l+1,\dots,\kappa \\ q=k'+l'+1,\dots,\kappa' \end{pmatrix}\end{cases}$$

are proved, we have $f\beta e_i^{n+2}=\beta ge_i^{n+2}$ from (9.4) and (9.6).

From (9.1) and (9.5) it is seen that

$$(9.7)\qquad g^*\bar{e}_{k'+q}^{n+2}=\sum_{i=1}^{t} b'_{i,k'+q}e_i^{n+2}+\sum_{i=k+1}^{\kappa} c'_{i,k'+q}e_i^{n+2}$$ for each $q=1,\dots,l'$.

It is also verified that

$$\begin{aligned} j_2g^*\bar{e}_{k'+q}^{n+2} &= f^*j_2\bar{e}_{k'+q}^{n+2}=f^*(j_{2^{p_q}}\bar{e}_{k'+q}^n \underset{n-2}{\smile} j_{2^{p_q}}\bar{e}_{k'+q}^n)\\ &= f^*j_{2^{p_q}}\bar{e}_{k'+q}^n \underset{n-2}{\smile} f^*j_{2^{p_q}}\bar{e}_{k'+q}^n\\ &= j_{2^{p_q}}g^*\bar{e}_{k'+q}^n \underset{n-2}{\smile} j_{2^{p_q}}g^*\bar{e}_{k'+q}^n\\ &= j_2(g^*\bar{e}_{k'+q}^n \underset{n-2}{\smile} g^*\bar{e}_{k'+q}^n)\\ &= j_2\{(\sum_{i=1}^{\kappa} a_{i,k'+q}e_i^n)\underset{n-2}{\smile}(\sum_{i=1}^{\kappa} a_{i,k'+q}e_i^n)\}\\ (9.8)\qquad &= \sum_{j=k+1}^{\kappa} a_{j,k'+q}^2 j_2e_j^{n+2}\,. \end{aligned}$$

From (9.7) and (9.8) we have

$$c'_{i,k'+q}\equiv a_{i,k+q}^2\equiv a_{i,k'+q} \quad \text{mod. } 2 \quad \begin{pmatrix} i=k+1,\dots,k+l,\dots,\kappa \\ q=1,\dots,l' \end{pmatrix}$$

This proves (A) i) and (B) i). By analogous arguments we have (A) ii) and (B) ii). Thus, in virtue of Lemma 4 there exists the desired map $f: L^{n+2}\to\bar{L}^{n+2}$.

Now we are at the last stage of proving this theorem. *An easy example shows that* $f\beta e_i^{n+3}=\beta g e_i^{n+3}$ *(for each* $i=1,\ \dots,\ \alpha$*) is not always possible, so that we shall modify the map* f*, which has been established in* L^{n+2}*, to a map* f_0*. In this modification of* f *we notice that* $f_0|L^{n+1}=f$ *and* f *is modified in all the* $(n+2)$ *cells of* L. From the last part of § 7 we have

$$(9.9)\qquad \beta e_i^{n+3}=\lambda_{ij}S_j^{n+2}+\sum_{j=k+1}^{k+l}\mu_{i,j}\omega^j+\sum_{j=k+1}^{k+l}\nu_{i,j}v_j+\sum_{j=k+l+1}^{\kappa}\gamma_{i,j}\omega_j \quad (i=1,\dots\alpha)$$

From (9.1) we have

$$ge_i^{n+2}=\sum_{p=1}^{t'}b_{i,p}\bar{e}_p^{n+2}+\sum_{q=1}^{l'}b'_{i,k'+q}\bar{e}_{k'+q}^{n+2}+\sum_{r=k'+l'+1}^{\kappa'}b''_{i,r}\bar{e}_r^{n+2}\quad (i=1,\dots,t).$$

Then we may define $f_0|S_i^{n+2}$ (for each $i=1,\ \dots,\ t$) such that

$$(9.10)\qquad f_0(S_i^{n+2})=\sum_{p=1}^{t'}b'_{i,p}\bar{S}_p^{n+2}+\sum_{q=1}^{l'}\frac{b'_{i,k'+q}}{2}\bar{\omega}_{k'+q}+\sum_{r=k'+l'+1}^{\kappa'}\frac{b''_{i,r}}{2}\bar{\omega}_r,\ *$$

where $b'_{i,k'+q}\equiv 0$, and $b''_{i,r}\equiv 0$ mod. 2, are utilized here. *This modification does not alter* g. From (9.5) we have

$$ge_j^{n+2}=\sum_{p=1}^{t'}c_{j,p}\bar{e}_p^{n+2}+\sum_{q=1}^{l'}c'_{j,k'+q}\bar{e}_{k'+q}^{n+2}+\sum_{r=k'+l'+1}^{\kappa'}c''_{j,r}\bar{e}_r^{n+2}\quad (j=k+1,\dots,\kappa).$$

Thus it can be also defined that

$$(9.11)\quad f_0\omega_j=\sum_{p=1}^{t'}2c_{j,p}\bar{S}_p^{n+2}+\sum_{q=1}^{l'}c'_{j,k'+q}\bar{\omega}_{k'+q}+\sum_{r=k'+l'+1}^{\kappa'}c''_{j,r}\bar{\omega}_r{}^{**}\ (j=k+1,\dots,\kappa).$$

If we define $ge_j^{n+1}=\sum_{p=1}^{k'+l'}\theta_{j,p}\bar{e}_p^{n+1}$, we have

$$(9.13)\qquad f_0v_j=\sum_{p=k'+1}^{k'+l'}\theta_{j,p}\bar{v}_p\quad (j=k+1,\dots,k+l)$$

From (9.9), (9.10), (9.11). and (9.13), it follows that

$$\begin{aligned} f_0\beta e_i^{n+3} &= \sum_{p=1}^{t'}\Big(\lambda_{ij}b_{i,p}+2\sum_{j=k+1}^{k+l}\mu_{i,j}c_{j,p}+2\sum_{j=k+l+1}^{\kappa}\gamma_{i,j}c_{jp}\Big)\bar{S}_p^{n+2}\\ &+\sum_{q=1}^{l'}\Big(\lambda_{ij}\frac{b_{j,k'+q}}{2}+\sum_{j=k+1}^{k+l}\mu_{i,j}c'_{j,k'+q}+\sum_{j=k+l+1}^{\kappa}\gamma_{i,j}c'_{j,k'+q}\Big)\bar{\omega}_{k'+q}\\ (9.14)\qquad &+\sum_{r=k'+l'+1}^{\kappa'}\Big(\lambda_{ij}\frac{b_{j,r}}{2}+\sum_{j=k+1}^{k+l}\mu_{i,j}c''_{j,r}+\sum_{j=k+l+1}^{\kappa}\gamma_{i,j}c''_{j,r}\Big)\bar{\omega}_r \end{aligned}$$

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* As we have often referred to, it should be noticed that homology and homotopy are distinguished adequately according as the place where they are used.

** From the following reasons we can modify f to f_0. Let us denote $S_i^n\cup e_i^{n+2}$ by Π_i, then there exists a map $\varphi:\Pi\to\Pi_1\vee\Pi_2$ such that φ maps S^n of Π to $S_1^n\vee S_2^n$ with degree (1,1) Besides this, let a map $\psi:\Pi\to\Pi$ be given such that ψ maps S^n of Π to $\tilde{S}^n$ of $\tilde{\Pi}$ with degree a, and $a\equiv c$ mod. 2. Then we can construct a map $\tilde{\psi}:\Pi\to\tilde{\Pi}$ by modifying $\tilde{\psi}$ such that $\tilde{\psi}|S^n=\psi|S^n$ and $\tilde{\psi}$ maps e^{n+2} of Π to $\tilde{e}^{n+2}$ of $\tilde{\Pi}$ with degree c. We use, in the above modification of f to f_0, the relations (A), (B) on the previous page and also $\pi_{n+2}(S^n\cup e^{n+1})=0$, where e^{n+1} is attached to S^n by a map $\partial e^{n+1}\to S^n$ of degree odd. It is clear that this modification does not alter the realization of a chain map g.

$$+\sum_{p=k'+1}^{k+l'}\sum_{j=k+1}^{k+l}\nu_{i,j}\theta_{j,p}\bar{v}_p$$

Putting

$$(9.15)\qquad ge_i^{n+3}=\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{e}_p^{n+3}$$

we have

$$\beta ge_i^{n+3}=\sum_{p=1}^{\alpha'}\rho_{ip}(\bar{\lambda}_{pq}\bar{S}_q^{n+2}+\sum_{q=k'+1}^{k'+l'}\bar{\mu}_{p,q}\bar{\omega}_q+\sum_{q=k'+1}^{k'+l'}\nu_{p,q}\bar{v}_q+\sum_{q=k'+l'+1}^{\kappa'}\bar{\gamma}_{p,q}\bar{\omega}^q$$

$$(9.16)\ldots\ldots=\sum_{p=1}^{\alpha'}(\rho_{i,p}\bar{\lambda}_{pq})\bar{S}_q^{n+2}+\sum_{q=k'+1}^{k'+l'}\left(\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{\mu}_{p,q}\right)\bar{\omega}_q+\sum_{q=k'+1}^{k'+l'}\left(\sum_{p=1}^{\alpha'}\rho_{i,p}\,\bar{\nu}_{p,q}\right)\bar{v}_q$$

$$+\sum_{q=k'+l'+1}^{\kappa}\left(\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{\gamma}_{p,q}\right)\bar{\omega}^q$$

Proving the following relations

i) $\lambda_{ij}b_{j,p}+\sum_{j=k+1}^{k+l}2\mu_{i,j}c_{j,p}+\sum_{j=k+l+1}^{\kappa}2\gamma_{i,j}c_{j,p}=\rho_{i,j}\bar{\lambda}_{j,p}\quad(p=1,\ldots,t')$,

ii) $\lambda_{ij}\dfrac{b'_{j,k'+q}}{2}+\sum_{j=k+1}^{k+l}\mu_{i,j}c'_{j,k'+q}+\sum_{j=k+l+1}^{\kappa}\gamma_{i,j}c'_{j,k'+q}$

$$=\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{\mu}_{p,k'+q}\quad(q=1,\ldots,l'),$$

iii) $\lambda_{ij}\dfrac{b''_{j,r}}{2}+\sum_{j=k+l+1}^{k+l}\mu_{i,j}c''_{j,r}+\sum_{j=k+1}^{\kappa}\gamma_{i,j}c''_{j,r}$

$$=\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{\gamma}_{p,r}\quad(\gamma=k'+l'+1,\ldots,\kappa'),$$

iv) $\sum_{j=k+1}^{k+l}\theta_{j,p}\nu_{i,j}\equiv\sum_{j=1}^{\alpha'}\rho_{i,j}\bar{\nu}_{j,p}\quad\text{mod. 2.}\quad(p=k'+1,\ldots,k'+l')$,

we have $f\beta e_i^{n+3}=\beta ge_i^{n+3}$ (for each $i=1,\ldots,\alpha$) from (9.14) and (9.16). From (9.1), (9.5) we have

$$g^*\bar{e}_p^{n+2}=\sum_{i=1}^{t}b_{i,p}e_i^{n+2}+\sum_{i=k+1}^{\kappa}c_{i,p}e_i^{n+2}\quad\text{for each } p=1,\ldots,t'.$$

Taking the coboundary of both sides of this equation,

$$\delta g^*\bar{e}_p^{n+2}=\sum_{i=1}^{t}(b_{i,p}\lambda_{ji})e_j^{n+3}+\sum_{q=1}^{\alpha}\left(\sum_{i=k+1}^{k+l}2c_{i,p}\mu_{q,i}\right)e_q^{n+3}+\sum_{q=1}^{\alpha}\left(\sum_{i=k+l+1}^{\kappa}2c_{i,p}\gamma_{q,i}\right)e_q^{n+3}.$$

It is also seen that

$$\delta g^*\bar{e}_p^{n+2}=g^*\delta\bar{e}_p^{n+2}=g^*(\bar{\lambda}_{jp}\bar{e}_j^{n+3})=\bar{\lambda}_{jp}g^*\bar{e}_j^{n+3}$$
$$=\bar{\lambda}_{jp}\left(\sum_{i=1}^{\alpha}\rho_{i,j}e_i^{n+3}\right)=\sum_{i=1}^{\alpha}\left(\bar{\lambda}_{jp}\rho_{i,j}\right)e_i^{n+3}.$$

It follows that

$$\bar{\lambda}_{j,p}\rho_{i,j}=\lambda_{i,j}b_{j,p}+\sum_{j=k+1}^{k+l}2c_{j,p}\mu_{i,j}+\sum_{j=k+l+1}^{\kappa}2c_{j,p}\gamma_{i,j}\quad(p=1,\ldots,t').$$

This proves i). Again, from (9. 1) and (9. 5) we have

$$g^*\bar{e}^{n+2}_{k'+q} = \sum_{i=1}^{t} b'_{i,k'+q} e_i^{n+2} + \sum_{j=k+1}^{k+l} c'_{j,k'+q} e_j^{n+2} + \sum_{j=k+l+1}^{\kappa} c'_{j,k+q} e_j^{n+2}.$$

Considering the coboundary, we have

$$\delta g^*\bar{e}^{n+2}_{k'+q} = \sum_{i=1}^{t} (b'_{i,k'+q}\lambda_{ji}) e_j^{n+3} + \sum_{i=1}^{\alpha}\Big(\sum_{j=k+l+1}^{k+1} 2c'_{j,k+q}\mu_{i,j}\Big) e_i^{n+3}$$
$$+\sum_{i=1}^{\alpha}\Big(\sum_{j=k+l+1}^{\kappa} 2c'_{j,k+1}\gamma_{i,j}\Big) e_i^{n+3}.$$

It is seen that

$$\delta g^*\bar{e}^{n+2}_{k'+q} = g^*\delta\bar{e}^{n+2}_{k'+q} = g^*\Big(2\sum_{p=1}^{\alpha'}\bar{\mu}_{p,k'+q}\bar{e}_p^{n+3}\Big) = \sum_{p=1}^{\alpha'} 2\bar{\mu}_{p,k'+q} g^*\bar{e}_p^{n+3}$$
$$=\sum_{i=1}^{\alpha}\Big(\sum_{p=1}^{\alpha'} 2\rho_{i,p}\mu_{p,k'+q}\Big) e_i^{n+3}.$$

Thus it is concluded that

$$\sum_{p=1}^{\alpha'}\rho_{i,p}\mu_{p,k'+q} = \lambda_{ij}\frac{b_{j,k'+q}}{2} + \sum_{j=1}^{k+l}\mu_{i,j}c'_{j,k'+q} + \sum_{j=k+l+1}^{k}\gamma_{i,j}c'_{j,k+q}.$$

This proves ii). Similarly iii) can be proved. Lastly we proceed to prove iv). From (9. 15) we have

$$g^*\bar{e}_p^{n+3} = \sum_{i=1}^{\alpha}\rho_{i,p}\, e_i^{n+3}.$$

Thus

$$\sum_{p=1}^{\alpha'} g^*\bar{\nu}_{p,q}\bar{e}_p^{n+3} = \sum_{i=1}^{\alpha}\Big(\sum_{p=1}^{\alpha'}\rho_{i,p}\bar{\nu}_{p,q}\Big) e_i^{n+3} (q = k'+1, \dots, k'+l'),$$

(9. 17) $$\sum_{p=1}^{\alpha'} j_2 g^*\bar{\nu}_{p,q}\bar{e}_p^{n+3} = \sum_{i=1}^{\alpha}\Big(\sum_{i=1}^{\alpha'}\rho_{i,p}\bar{\nu}_{p,q}\Big) j_2 e_i^{n+3}.$$

$$\sum_{p=1}^{\alpha'} j_2 g^*\nu_{p,q}\bar{e}_p^{n+3} = f^*\sum_{p=1}^{\alpha'}\bar{\nu}_{p,q} j_2\bar{e}_p^{n+3} = f^*\big(j_0\bar{e}^{n+1}_{k'+j}\underset{n-1}{\smile} j_0\bar{e}^{n+1}_{k'+j}\big) \quad (j = 1, \dots, l')$$
$$= f^*j_0\bar{e}^{n+1}_{k'+j}\underset{n-1}{\smile} f^*j_0\bar{e}^{n+1}_{k'+j}$$
$$= j_0 g^*\bar{e}^{n+1}_{k'+j}\underset{n-1}{\smile} j_0 g^*\bar{e}^{n+1}_{k'+j}$$
$$= j_0\Big(\sum_{i=1}^{k+1}\theta_{i,k'+j}e_i^{n+1}\Big)\underset{n-1}{\smile} j_0\Big(\sum_{i=1}^{k+l}\theta_{i,k'+j}e_i^{n+1}\Big) \quad \text{from (9. 12)}$$
$$= \sum_{i=1}^{k+l}\theta^2_{i,k'+j}\Big(\sum_{p=1}^{\alpha}\nu_{p,i} j_2 e_p^{n+3}\Big) \quad \text{from iii) of Theorem 6}$$

(9. 18) $$= \sum_{p=1}^{\alpha}\Big(\sum_{i=1}^{k+l}\theta^2_{i,k'+j}\nu_{p,i}\Big) j_2 e_p^{n+3}.$$

From (9.17) and (9.18) we have

$$\sum_{j=1}^{\alpha'}\rho_{i,j}\nu_{j,p} \equiv \sum_{j=k+1}^{k+l}\theta_{j,p}^2\nu_{i,j} \equiv \sum_{j=k+1}^{k+l}\theta_{j,p}\nu_{i,j} \quad \text{mod. } 2 \ (p = k'+1, \dots, k'+l').$$

This completes the proof.

Added in proof: I could read [4], and I hope, I shall come back soon to some subjects related to this paper (refer to my paper of the same title in this issue).

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The paper inaccessible to us here, is indicated by *

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